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A New Minimal Point Theorem in Product Spaces

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Abstract. We derive a minimal point theorem for a subset A in a cone in product spaces under a weak assumption concerning the boundedness of the considered set A. Using this result we improve two vectorial variants of Ekeland's variational principle. Finally, a new characterization of well-based cones is given.

Keywords: *Minimal point theorems, Ekeland's variational principle, well-based cones*

AMS subject classification: 49 J 40, 90 C 29, 46 A 40, 90 C 26

Assume that (X, d) is a complete metric space, Y is a separated locally convex space, Y^* is its topological dual, $K \subset Y$ is a convex cone, i.e. $K + K \subset K$ and $[0, \infty) \cdot K \subset K$,

$$K^+ = \big\{ y^* \in Y^* : \langle y, y^* \rangle \geq 0 \ \text{ for all } \ y \in K \big\}$$

is the dual cone of K and

$$K^\# = \big\{ y^* \in Y^* : \langle y, y^* \rangle > 0 \ \text{ for all } \ y \in K \setminus \{0\} \big\}.$$

In this note we suppose that K is pointed, i.e. $K \cap (-K) = \{0\}$. The cone K determines an order relation on Y, denoted in the sequel by $\leq_K$; so, for $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. It is well known that "$\leq_K$" is reflexive, transitive and antisymmetric. Let $k^0 \in K \setminus \{0\}$; using the element k^0 we introduce an order relation on $X \times Y$, denoted by "$\preceq_{k^0}$", in the following manner:

$$(x_1, y_1) \preceq_{k^0} (x_2, y_2) \qquad \text{iff} \qquad y_1 + k^0 d(x_1, x_2) \leq_K y_2.$$

Note that "$\preceq_{k^0}$" is reflexive, transitive and antisymmetric. That is, our notations are those of [3].

The essential idea for the derivation of a minimal point theorem (cf. [2, 8]) in general product spaces $X \times Y$, as well as of the vectorial Ekeland principle, consists in including the ordering cone $K \subset Y$ in a "larger" cone $B \subset Y$: $K \setminus \{0\} \subset \operatorname{int} B$. We will use B to define a suitable functional $z_B : Y \to \mathbb{R}$. Moreover, we will replace the usual boundedness condition of the projection $P_Y A$ of A onto Y by a weaker one.

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Theorem 1. *Assume that there exists a proper convex cone $B \subset Y$ such that $K \setminus \{0\} \subset \operatorname{int} B$. Suppose that the set $A \subset X \times Y$ satisfies the condition*

(H1) *for every $\preceq_{k^0}$-decreasing sequence $((x_n, y_n)) \subset A$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x, y) \in A$ and $(x, y) \preceq_{k^0} (x_n, y_n)$ for every $n \in \mathbb{N}$*

and that $P_Y(A) \cap (\widetilde{y} - \operatorname{int} B) = \emptyset$ for some $\widetilde{y} \in Y$. Then for every $(x_0, y_0) \in A$ there exists $(\overline{x}, \overline{y}) \in A$, minimal with respect to $\preceq_{k^0}$, such that $(\overline{x}, \overline{y}) \preceq_{k^0} (x_0, y_0)$.

Proof. Let

$$z_B : Y \to \mathbb{R}, \qquad z_B(y) = \inf\{t \in \mathbb{R} : y \in tk^0 - \operatorname{cl} B\}.$$

By [3: Lemma 7], z_B is a continuous sublinear function such that $z_B(y+tk^0) = z_B(y)+t$ for all $t \in \mathbb{R}$ and $y \in Y$, and for every $\lambda \in \mathbb{R}$

$$\begin{aligned}\{y \in Y : z_B(y) \le \lambda\} &= \lambda k^0 - \operatorname{cl} B \\ \{y \in Y : z_B(y) < \lambda\} &= \lambda k^0 - \operatorname{int} B.\end{aligned}$$

Moreover, if $y_2 - y_1 \in K \setminus \{0\}$, then $z_B(y_1) < z_B(y_2)$. Observe that for $(x, y) \in A$ we have that $z_B(y - \widetilde{y}) \ge 0$. Otherwise for some $(x, y) \in A$ we have $z_B(y - \widetilde{y}) < 0$. It follows that there exists $\lambda > 0$ such that $y - \widetilde{y} \in -\lambda k^0 - \operatorname{cl} B$. Hence

$$y \in \widetilde{y} - (\lambda k^0 + \operatorname{cl} B) \subset \widetilde{y} - (\operatorname{int} B + \operatorname{cl} B) \subset \widetilde{y} - \operatorname{int} B$$

which is a contradiction. Since $0 \le z_B(y - \widetilde{y}) \le z_B(y) + z_B(-\widetilde{y})$, it follows that z_B is bounded from below on $P_Y(A)$. Let us construct a sequence $((x_n, y_n))_{n \ge 0} \subset A$ as follows: having $(x_n, y_n) \in A$ we take $(x_{n+1}, y_{n+1}) \in A$, $(x_{n+1}, y_{n+1}) \preceq_{k^0} (x_n, y_n)$, such that

$$z_B(y_{n+1}) \le \inf\Big\{z_B(y) : (x, y) \in A \text{ and } (x, y) \preceq_{k^0} (x_n, y_n)\Big\} + \frac{1}{n+1}.$$

Of course, the sequence $((x_n, y_n))$ is $\preceq_{k^0}$-decreasing. It follows that

$$y_{n+p} + k^0 d(x_{n+p}, x_n) \le_K y_n \qquad \forall\, n, p \in \mathbb{N}^*$$

so that

$$d(x_{n+p}, x_n) \le z_B(y_n) - z_B(y_{n+p}) \le \frac{1}{n} \quad \forall\, n, p \in \mathbb{N}^*.$$

It follows that (x_n) is a Cauchy sequence in the complete metric space (X, d), and so (x_n) is convergent to some $\bar{x} \in X$. By condition (H1) there exists $\overline{y} \in Y$ such that $(\overline{x}, \overline{y}) \in A$ and $(\bar{x}, \bar{y}) \preceq_{k^0} (x_n, y_n)$ for every $n \in \mathbb{N}$.

Let us show that $(\overline{x}, \overline{y})$ is the desired element. Indeed, $(\bar{x}, \bar{y}) \preceq_{k^0} (x_0, y_0)$. Suppose that $(x', y') \in A$ is such that $(x', y') \preceq_{k^0} (\bar{x}, \bar{y})$ $(\preceq_{k^0} (x_n, y_n)$ for every $n \in \mathbb{N})$. Thus $z_B(y') + d(x', \overline{x}) \le z_B(\overline{y})$, whence

$$d(x', \bar{x}) \le z_B(\bar{y}) - z_B(y') \le z_B(y_n) - z_B(y') \le \frac{1}{n} \qquad \forall\, n \ge 1.$$

It follows that $d(x', \bar{x}) = z_B(\bar{y}) - z_B(y') = 0$. Hence $x' = \overline{x}$. As $y' \le_K \overline{y}$, if $y' \ne \overline{y}$, then $\overline{y} - y' \in K \setminus \{0\}$, whence $z_B(y') < z_B(\bar{y})$, which is a contradiction. Therefore $(x', y') = (\bar{x}, \bar{y})$ ∎

Comparing with [3: Theorem 4], note that the present condition on K is stronger (because in this case $K^{\#} \neq \emptyset$), while the condition on A is weaker (A may be not contained in a half-space). Note that when K and k^0 are as in Theorem 1, Corollaries 2 and 3 from [3] may be improved. In the next result $Y^\bullet = Y \cup \{\infty\}$ with $\infty \notin Y$; we consider that $y \leq_K \infty$ for every $y \in Y$. We consider also a function $f : X \to Y^\bullet$ and $\operatorname{dom} f = \{x \in X : f(x) \neq \infty\}$.

In the following corollary we derive a variational principle of Ekeland's type for objective functions which take values in a general space Y (cf. [2, 3, 5 - 7]) under a weaker assumption with respect to the usual lower semicontinuity. For the case $Y = \mathbb{R}$, assumption (H4) in Corollary 2 is fulfilled for decreasingly semicontinuous real-valued functions as in the paper [4].

Corollary 2. *Let $f : X \to Y^\bullet$. Assume that there exists a proper convex cone $B \subset Y$ such that $K \setminus \{0\} \subset \operatorname{int} B$ and $f(X) \cap (\widetilde{y} - B) = \emptyset$ for some $\widetilde{y} \in Y$. Also, suppose that*

(H3) $\{x' \in X : f(x') + k^0 d(x', x) \leq_K f(x)\}$ *is closed for every* $x \in X$

or

(H4) *for every sequence $(x_n) \subset \operatorname{dom} f$ with $x_n \to x$ and $(f(x_n))$ $\leq_K$-decreasing, $f(x) \leq_K f(x_n)$ for every $n \in \mathbb{N}$, and K is closed in the direction k^0.*

Then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in X$ such that

$$f(\overline{x}) + k^0 d(\overline{x}, x_0) \leq_K f(x_0)$$

and

$$\forall\, x \in X : \quad f(x) + k^0 d(\overline{x}, x) \leq_K f(\overline{x}) \quad \Longrightarrow \quad x = \overline{x}.$$

We say that K is *closed in the direction* k^0 if $K \cap (y - \mathbb{R}_+ k^0)$ is closed for every $y \in K$. The proof of Corollary 2 is similar to those of Corollaries 2 and 3 in [3].

As mentioned in [3], condition (H1) is verified if K is a well based convex cone, Y is a Banach space and A is closed. As usually (cf. [1]), a convex set S is said to be a *base* for a convex cone $K \subset Y$ if

$$K = \mathbb{R}_+ S = \{\lambda y : \lambda \geq 0 \text{ and } y \in S\} \qquad \text{and} \qquad 0 \notin \operatorname{cl} S.$$

The cone K is called *well based* if K has a bounded base S. Concerning well based convex cones in normed spaces we have the following characterization.

Proposition 3. *Let Y be a normed vector space and $K \subset Y$ a proper convex cone. Then K is well based if and only if there exist $k^0 \in K$ and $z^* \in K^+$ such that $\langle k^0, z^* \rangle > 0$ and*

$$K \cap S_1 \subset k^0 + \{y \in Y : \langle y, z^* \rangle > 0\}$$

where $S_1 = \{y \in Y : ||y|| = 1\}$ is the unit sphere in Y.

Proof. Suppose first that K is well based with bounded base S; therefore $0 \notin \operatorname{cl} S$ and $K = [0, \infty) \cdot S$. Then there exists $z^* \in Y^*$ such that $1 \leq \langle y, z^* \rangle$ for all $y \in S$. Consider $\widetilde{S} := \{k \in K : \langle k, z^* \rangle = 1\}$. It follows that $\widetilde{S}$ is a base of K; moreover, since

$\widetilde{S} \subset [0,1] \cdot S$, $\widetilde{S}$ is also bounded. Taking $k^1 \in K \setminus \{0\}$ we have $K \cap S_1 \subset \lambda k^1 + B_+$ for some $\lambda > 0$, where $B_+ = \{y \in Y : \langle y, z^* \rangle > 0\}$. Otherwise

$$\forall\, n \in \mathbb{N}^* \;\exists\, k_n \in K \cap S_1 : \qquad k_n \notin \frac{1}{n} k^1 + B_+.$$

Therefore $\langle k_n, z^* \rangle \leq \frac{1}{n} \langle k^1, z^* \rangle$ for every $n \geq 1$. But, because $\widetilde{S}$ is a base, $k_n = \lambda_n b_n$ with $\lambda_n > 0$ and $b_n \in \widetilde{S}$; it follows that $1 = ||k_n|| = \lambda_n ||b_n|| \leq \lambda_n M$ with $M > 0$ (because $\widetilde{S}$ is bounded). Therefore

$$M^{-1} \leq \lambda_n = \langle \lambda_n b_n, z^* \rangle = \langle k_n, z^* \rangle \leq n^{-1} \langle k^1, z^* \rangle \qquad \forall\, n \in \mathbb{N}^*$$

whence $M^{-1} \leq 0$, which is a contradiction. Thus there exists $\lambda > 0$ such that $K \cap S_1 \subset \lambda k^1 + B_+$. Taking $k^0 := \lambda k^1$ the conclusion follows.

Suppose now that $K \cap S_1 \subset k^0 + B_+$ for some $k^0 \in K$ and $z^* \in K^+$ with $\langle k^0, z^* \rangle = c > 0$, where B_+ is defined as above. Consider $S = \{k \in K : \langle k, z^* \rangle = 1\}$. Let $k \in K \setminus \{0\}$. Then $\|k\|^{-1} k = k^0 + y$ for some $y \in B_+$. It follows that $\langle k, z^* \rangle > c\|k\| > 0$; therefore $z^* \in K^\#$ and so $k \in (0,\infty) \cdot S$. Since $\operatorname{cl} S \subset \{y \in Y : \langle k, z^* \rangle = 1\}$, we have that S is a base of K. Let now $y \in S$ $(\subset K)$. Then $\|y\|^{-1} y \in K \cap S_1$. There exists $z \in B_+$ such that $\|y\|^{-1} y = k^0 + z$. We get

$$1 = \langle y, z^* \rangle = \|y\| \langle k^0 + z, z^* \rangle \geq c\|y\|$$

whence $\|y\| \leq c^{-1}$. Therefore S is bounded, and so K is well-based ∎

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