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On Operators and Elementary Functions in Clifford Analysis

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Dedicated to Prof. L. von Wolfersdorf on the occasion of his 65th birthday

Abstract. In this paper a survey on the construction of Clifford regular elementary functions by Fueter's mapping is given. Furthermore, using a suitable decomposition of the Dirac operator an application of the $\overline{\partial}$-problem is lined out.

Keywords: *Clifford regular elementary functions, Fueter's method, $\overline{\partial}$-problem*

AMS subject classification: 30 G 35

1. Indroduction

The effective application of methods of Clifford analysis to partial differential equations needs systems of elementary functions which are Clifford regular and still satisfy most of the properties which we know from the complex plane. The reason for writing this paper is to show presumed users of Clifford-valued elementary functions the considerable similarity to classical complex elementary functions and to reduce the shyness in applying higher-dimensional elementary functions. In this paper we will give a review on Fueter's method and will construct some important Clifford regular elementary functions which knowledge seems to be useful in transform analysis.

We will mention here important authors which deliver contributions in this field. In his early papers [5 - 7] R. Fueter, a follower of D. Hilbert, formulated a method to transfer complex analytic functions to Clifford regular ones. Later M. Sce [18], A. Sudbury [22], M. Imaeda [9], F. Sommen [20, 21], P. Lounesto and P. Bergh [15], G. Jank and F. Sommen [11], H. Leutwiler [12 - 14], M. Marinov [16] and K. Nono [17] made attempts in generalizing classical elementary functions in a hypercomplex sense. Further, the reader can find contributions to this subject in [1, 2, 4].

Using the power series expansion of an exponential function with a paravector argument we deduce a "full" class of so-called radially regular elementary functions including a paravector-valued logarithm. Most of the expected properties could be maintained. We continue with a suitable decomposition of the Dirac operator and study the operators which there occur. In this part we use results of J. de Graaf [3] and N. van Acker [23]. On the basis of papers by M. Sce [19] and T. Qian [18] we are able to describe

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at least in the case of quaternions the transform of radially regular function to quaternionic regular functions. Finally, we apply these results for the construction of kernel functions of the $\overline{\partial}$-problem.

2. Preliminaries

Let $\mathbb{R}^{0,n}$ be the anti-Euclidean space with the basis $\{e_1, ..., e_n\}$ and the quadratic form $Q(x) = -\sum_{i=1}^n x_i^2$. We consider the 2^n-dimensional real Clifford algebra $C\ell_{0,n}$ which is generated by $\{e_1, ..., e_n\}$ and contains copies of $\mathbb{R}$ and $\mathbb{R}^{0,n}$. The multiplication rule is defined by $e_ie_j+e_je_i = -2\delta_{ij}$. In $C\ell_{0,n}$ a basis is given by $e_0, e_1, \ldots, e_n,\ e_1e_2, \ldots e_{n-1}e_n, \ldots, e_1 \cdots e_n$. Elements of the type $x = x_{1\cdots k}e_{\ell_1} \cdots e_{\ell_k}\quad (1 \leq \ell_1 < \ldots < \ell_k \leq n)$ are called *k-vectors.*

For k-vektors the *conjugation* is defined by $\overline{x} = (-1)^{\frac{k(k+1)}{2}} x$. Elements $x = x_0 + \underline{x}$ with $\underline{x} = \sum_{i=1}^n x_ie_i$ are called *paravectors.* Furthermore, x_0 is called *scalar part,* $x_0 := \operatorname{Sc} x$, $\underline{x}$ is called *vector part,* $\underline{x} := \operatorname{Vec} x$, and $x - x_0$ is called *imaginary part,* $x - x_0 := \operatorname{Im} x$. Obviously, $x\overline{x} = \sum_{i=1}^n x_i^2$ for $x = x_0 + \underline{x}$. Denote $\omega(x) := \underline{x}/|\underline{x}| \in S^n$, where S^n is the unit sphere in $\mathbb{R}^n$.

3. Paravector-valued elementary functions

Let x be a paravector in $C\ell_{0,n}$. The following statement can be easily shown.

Proposition 3.1. *For an arbitrary $\varepsilon > 0$ it is always possible to find a sufficiently large number N such that for any $r, s > N$*

$$\left| \sum_{k=r}^{s} \frac{x^k}{k!} \right| \leq \sum_{k=r}^{s} \frac{K^k |x|^k}{k!} < \varepsilon \tag{3.1}$$

holds where K is a constant which only depends on n and satisfies the inequality $|xy| < K(n)|x|\,|y|$.

Inequality (3.1) gives us the possibility to define elementary functions similarly to the case of one complex variable. First of all we have to introduce an exponential function:

Definition 3.2. For a paravector $x \in C\ell_{0,n}$ the exponential function e^x is defined by $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Similarly to the complex case one can prove the following properties.

Theorem 3.3. *Let x be a paravector in $C\ell_{0,n}$. Then we have:*

(i) $e^x = e^{x_0}(\cos|\underline{x}| + \omega(x)\sin|\underline{x}|)$.

(ii) $e^x = \lim_{m\to\infty}\left(1 + \frac{x}{m}\right)^m$.

(iii) $e^{x+y} = e^x e^y$ *if* $xy = yx$.

Proof. We will only prove property (ii). Because of $\left(\frac{1}{k!} - \frac{\binom{m}{k}}{m^k}\right) > 0$ we obtain

$$\left|e^x - \left(1 + \frac{x}{m}\right)^m\right| \leq \sum_{k=0}^{\infty} \left(\frac{1}{k!} - \frac{\binom{m}{k}}{m^k}\right)(K\,|x|)^k = e^{|Kx|} - \left(1 + \frac{|Kx|}{m}\right)^m.$$

For $m \to \infty$ this difference tends to zero ∎

Corollary 3.4. *Furthermore, we have:*

(i) $e^x \neq 0$.

(ii) $|e^x| = e^{x_0}$.

(iii) $e^{-x}e^x = 1$.

(iv) $e^{k\underline{x}} = (e^{\underline{x}})^k$ (Moivre's Formula)

(v) $e^{\omega(x)\pi} = -1$.

Proof. This is a straightforward consequence of Theorem 3.3 ∎

Let x be a paravector in $C\ell_{0,k}$. Then hyperbolic and trigonometric functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

and, for $|\underline{x}| \neq 0$,

$$\sin x = \frac{e^{x\omega(x)} - e^{-x\omega(x)}}{2}\,\omega(x) \qquad \text{and} \qquad \cos x = \frac{e^{x\omega(x)} + e^{-x\omega(x)}}{2}.$$

Corollary 3.5. *Immediately the representations*

$$\begin{aligned}
\cosh x &= \cos hx_0 \cos|\underline{x}| + \omega(x)\sin hx_0 \sin|\underline{x}| \\
\sinh x &= \sin hx_0 \cos|\underline{x}| + \omega(x)\cos hx_0 \sin|\underline{x}| \\
\cos x &= \cos x_0 \cos h|\underline{x}| + \omega(x)\sin h|\underline{x}| \sin x_0 \\
\sin x &= \sin x_0 \cos h|\underline{x}| + \omega(x)\sin h|\underline{x}| \cos x_0
\end{aligned}$$

follow. The right-hand sides of $\sin x$ *and* $\cos x$ *can be used by definition in the case of* $|\underline{x}| = 0$.

Proof. For the proof we refer to our book [8: pp. 53 - 55] ∎

Definition 3.6. *Let x be a paravector in $C\ell_{0,n}$ with $x \neq x_0 < 0$. Then* $\log x$ *is defined by*

$$\log x = \ln|x| + \omega(x)\arccos\frac{x_0}{|x|} \qquad \big(|\underline{x}| \neq 0 \text{ or } |\underline{x}| = 0, x_0 > 0\big).$$

Theorem 3.7. *Let x be a paravector in $C\ell_{0,n}$ with $x \neq x_0 < 0$. Then we have:*

(i) $e^{\log x} = x$ *and* $\log e^x = x$.

(ii) $\log 1 = 0$ *and* $\log e_i = \frac{\pi}{2}$ $(i = 1, ..., n)$.

(iii) $1 - \frac{1}{|x|} - \arctan\frac{|\underline{x}|}{|x_0|} \leq \log|x| \leq |x| - 1 + \arctan\frac{|\underline{x}|}{|x_0|}$.

(iv) $\log(xy) = \log x + \log y$ *if* $xy = yx$.

Proof. The proof follows immediately from Definition 3.6 ∎

Finally, it is also possible to introduce a general power function.

Definition 3.8. Let α be a real number. The general power function x^α is defined by $x^\alpha = e^{\alpha \log x}$.

The following example will confirm this definition.

Example 3.9. Let $x = \underline{x}$ and $\alpha = \frac{1}{n}$. We obtain after a straightforward calculation

$$\underline{x}^{\frac{1}{n}} = \sqrt[n]{|\underline{x}|}\left[\cos\left(\frac{\pi}{2n} + \frac{2ki}{n}\right) + \omega(x)\sin\left(\frac{\pi}{2n} + \frac{2k\pi}{n}\right)\right]$$

for $k = 0, 1, ..., n-1$.

4. On the Cauchy-Fueter operator

In order to prepare differentiability properties of the elementary functions introduced above we need a suitable decomposition of the so-called Cauchy-Fueter operator

$$\partial = \partial_0 + D \qquad \text{where } \partial_0 = \frac{\partial}{\partial x_0} \text{ and } D = \sum_{i=1}^{n} e_i \partial_i,\ \partial_i = \frac{\partial}{\partial x_i}.$$

With the denotations

$$L = \sum_{i=1}^{n} e_i L_i(x) \text{ with } L_i(x) = |\underline{x}|\partial_i - x_i \ell_\omega \qquad \text{and} \qquad \ell_\omega = \sum_{i=1}^{n} \omega_i \partial_i$$

we will obtain the decomposition

$$\partial = \frac{1}{2}\left(\partial_0 - \frac{1}{|\underline{x}|}L + \omega \ell_\omega\right).$$

For a better understanding it is useful to deduce properties of the decomposition operators L and ℓ_ω.

Proposition 4.1. *Let $f \in C^1(\mathbb{R}^n)$ be a paravector-valued function. Then:*

(i) $l_\omega \underline{x} = \omega$.

(ii) $\ell_\omega \omega = 0$.

(iii) $\ell_\omega f = \frac{d}{d|\underline{x}|} f$.

(iv) $|\underline{x}|\partial_j \omega_k = \delta_{jk} - \omega_k \omega_j = L_j \omega_k$.

(v) $\sum_{i=1}^{n} \omega_i L_i = \mathrm{Sc}\, \omega L = 0$.

Proof. These relations are simple consequences from the definition of the operators L_i and ℓ_ω ∎

Proposition 4.2.

(i) *Let* $f \in C^1(\mathbb{R})$ *and* $f = f(|\underline{x}|)$ *a paravector-valued function. Then* $Lf = 0$.

(ii) *Let* $\varphi \in C^1(\mathbb{R})$ *and* $\varphi = \varphi(\omega)$ *a scalar-valued function. Then*

$$L\varphi = |\underline{x}| \left[\sum_{k=1}^{n} \partial_{\omega_k} \varphi e_k - \sum_{k=1}^{n} \omega_k \partial_{\omega_k} \varphi \right] = |\underline{x}| \left[\operatorname{grad}_\omega \varphi - \omega (\operatorname{grad}_\omega \varphi \cdot \omega) \right].$$

(iii) $\ell_\omega \varphi(\omega) = 0$ *is valid.*

Proof. Statement (i): For $j = 1, ..., n$ we obtain by definition $L_j(|\underline{x}|) = |\underline{x}|\partial_j |\underline{x}| - x_j \ell_\omega |\underline{x}| = 0$. Furthermore, we have

$$L_j f(|\underline{x}|) = |\underline{x}|\partial_j f - x_j \sum_{i=1}^{n} \omega_i \partial_i f = \ell_\omega f(|\underline{x}|\partial_j |\underline{x}| - x_j \ell_\omega |\underline{x}|) = \ell_\omega f L_j |\underline{x}| = 0.$$

Statement (ii): Let $\varphi = \varphi(\omega)$. Then

$$L\varphi = |\underline{x}| \operatorname{grad}_\omega \varphi - \omega \left[(\operatorname{grad}_\omega \varphi) \cdot \omega \right]$$

where $\operatorname{grad}_\omega \varphi = \sum_{i=1}^{n} e_i \partial_{\omega_i} = \varphi$. Indeed, we have

$$(L\varphi)(\omega) = \sum_{i=1}^{n} e_i L_i \varphi(\omega) = |\underline{x}| \sum_{i=1}^{n} [e_i \partial_i - \omega_i \ell_\omega] \varphi(\omega).$$

Setting now $\varphi^*(x) := \varphi(\omega)$,

$$(\partial_j \varphi^*)(x) = \sum_{j=1}^{n} \partial_{\omega_k} \varphi \partial_j \omega_k = \sum_{k=1}^{n} \partial_{\omega_k} \varphi (\delta_{jk} - \omega_j \omega_k).$$

Hence,

$$\begin{aligned} \sum_{j=1}^{n} e_j \partial_j \varphi^*(x) &= \sum_{j=1}^{n} \sum_{k=1}^{n} e_j \partial_{\omega_k} \varphi \partial_j \omega_k \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} e_j \partial_{\omega_k} \varphi \delta_{jk} - \sum_{k=1}^{n} \sum_{j=1}^{n} e_j \omega_j \partial_{\omega_k} \varphi \omega_k \\ &= \operatorname{grad}_\omega \varphi - \omega \sum_{k=1}^{n} \partial_{\omega_k} \varphi \omega_k \\ &= \operatorname{grad}_\omega \varphi - \omega (\operatorname{grad}_\omega \varphi \cdot \omega). \end{aligned}$$

Statement(iii): We have

$$\ell_\omega \varphi(\omega) = \sum_{k=1}^{n} \partial_k \varphi \ell_\omega \omega_k = \operatorname{grad}_\omega \varphi \cdot \ell_\omega \omega = 0.$$

Thus the proposition is proved ∎

Remark. Following [3] we get for L the vector representation

$$L = \sum_{j=1}^{n} e_j (v^j \cdot D)$$

where $v^j = |\underline{x}|e_j - \omega_j \underline{x}$, which can be easily seen by the help of Proposition 3.1 ∎

Proposition 4.4. *ωL is the spherical Dirac operator.*

Proof. Because of Proposition 4.1/(iv) we find

$$\begin{aligned}
\omega L &= \sum_{j \neq k} e_j e_k \omega_j L_k \\
&= \sum_{j<k} e_j e_k (\omega_j L_k - \omega_k L_j) \\
&= \sum_{j<k} e_j e_k \Big[(\omega_j |\underline{x}| \partial_k - \omega_j x_k \ell_\omega) - \omega_k |\underline{x}| \partial_j - x_j \omega_k \ell_\omega\Big] \\
&= \sum_{j<k} e_j e_k (x_j \partial_k - x_k \partial_j)
\end{aligned}$$

and the statement is proved ∎

Theorem 4.5. *We have the equality*

$$(\omega L + L\omega)u = (1-n)u$$

for the anti-commutator. Special cases:

(i) *If $u \equiv u_0(|\underline{x}|)$, where u_0 a real-valued function, then $L\omega u_0 = (1-n)u_0$.*

(ii) *If $u = \omega$, then $(\omega L)\omega = (n-1)\omega$.*

Proof. We have

$$L\omega = \sum_{j,k=1}^{n} e_j e_k L_j \omega_k + \sum_{j,k=1}^{n} e_j e_k \omega_k L_j.$$

Using Proposition 4.1/(iii)-(iv) we get

$$L\omega = \sum_{j,k=1}^{n} (\delta_{jk} - \omega_k \omega_j) e_j e_k - \omega L = -n + 1 - \omega L$$

and the statement is proved ∎

Proposition 4.6. *The Cauchy-Fueter operator permits the decomposition*

$$\partial = \frac{1}{2}(\partial_0 + \omega \ell_\omega) - \frac{1}{2}\frac{1}{|\underline{x}|} L.$$

Proof. It is easy to see that

$$\omega L = \omega \sum_{j=1}^{n} e_j \left[|\underline{x}| \partial_j - x_j \ell_\omega\right] = \underline{x} D + |\underline{x}| \ell_\omega.$$

From this we obtain the assumption by a straightforward calculation ∎

Corollary 4.7. *Let* $\xi = \underline{\xi}(|\underline{x}|)$*. Then:*

(i) $L(\xi\omega) = (n-3)\xi + 2(\xi,\omega)\omega$.

(ii) $D(\xi\omega) = \frac{1}{|\underline{x}|}L(\xi\omega) + \frac{d}{d|\underline{x}|}\xi$.

Proof. Corollary 4.6, the generalized Leibniz rule [8: p. 40] and Proposition 4.4 deliver

$$D(\xi\omega) = (D\xi)\omega - \xi D\omega - 2\sum_{j=1}^{n}\xi_j\partial_j\omega = \omega\xi'\omega - 2\sum_{j=1}^{n}\sum_{k=1}^{n}e_k\xi(\partial_j\omega_k) + \frac{n-1}{|\underline{x}|}\xi.$$

Furthermore, it follows

$$\begin{aligned} D(\xi\omega) &= \omega\Big(\frac{d}{d|\underline{x}|}\xi\Big)\omega + \frac{(n-1)}{|\underline{x}|}\xi - \frac{2}{|\underline{x}|}\sum_{j=1}^{n}\sum_{k=1}^{n}e_k\xi_j[\delta_{jk} - \omega_j\omega_k] \\ &= \omega\Big(\frac{d}{d|\underline{x}|}\xi\Big)\omega + \frac{n-1}{|\underline{x}|}\xi - \frac{2}{|\underline{x}|}\xi + \frac{2}{|\underline{x}|}(\xi,\omega)\omega \\ &= \omega\Big(\frac{d}{d|\underline{x}|}\xi\Big)\omega + \frac{1}{|\underline{x}|}(n-3)\xi + \frac{2(\xi,\omega)}{|\underline{x}|}\omega. \end{aligned}$$

Obviously, we get from $\ell_\omega(\xi\omega) = \big(\frac{d}{d|\underline{x}|}\xi\big)\omega$ and Corollary 4.6

$$L(\xi\omega) = -|\underline{x}|\Big(D(\xi\omega) - \omega\Big(\frac{d}{d|\underline{x}|}\xi\Big)\omega\Big) = (n-3)\xi + 2(\xi,\omega)\omega$$

and the proof is finished ∎

5. Fueter's mapping

The elementary functions introduced above are not Clifford regular. Above all this is caused by the occurence of the operator L and his action

$$L(u_0\omega) = (1-n)u_0$$

(cf. Theorem 4.5). Such a cross-mapping which acts from the vector part to the real part is disturbing the structure of this simple type of elementary functions. In order to maintain such differentiability properties it seems to be useful to introduce the "reduced" operator $\partial_{ra} = \frac{1}{2}(\partial_0 - \omega\ell_\omega)$. Analogously to the complex case we abbreviate $\partial_{ra}u =: u'$.

Definition 5.1. Let

$$f = f_0 + \omega(x)f_1, \qquad f_i : \mathbb{R}^n \oplus \mathbb{R}^1 \to \mathbb{R}^1,\ f_i = f_i(x_0, |\underline{x}|)\ \ (i = 1,2), h = h_0\omega(x)|\underline{h}|.$$

Such a paravector-valued function f is called *radially differentiable* or *radially Clifford regular* if

$$\lim_{h\to 0}\frac{[f(x+h) - f(x)]\overline{h}}{|h|^2} = A_f(x)$$

exists.

Corollary 5.2. *For radially differentiable functions $A_f(x) = f'(x)$ holds.*

Corollary 5.3. *The paravector-valued function $f = f_0 + \omega f_1$ is radially differentiable if and only if $\overline{\partial}_{ra} f = 0$*

Corollary 5.4. *All above defined elementary functions are radially Clifford regular. Hence it follows:*

(i) $(e^x)' = e^x$.

(ii) $(\sin x)' = \cos x$ *and* $(\cos x)' = -\sin x$.

(iii) $(\sinh x)' = \cosh x$ *and* $(\cosh x)' = \sinh x$.

(iv) $(\log x)' = \frac{1}{x}$.

(v) $x^\alpha = \alpha x^{\alpha-1} \quad (\alpha \in \mathbb{R})$.

Proof. The proof follows straightforward by using the elementary relations $\ell_\omega \omega = 0, \ell_\omega |x| = \frac{|x|}{|x|}$ and $\varepsilon = \ell_\omega \underline{x} = \omega$ ∎

Now we will demonstrate how to transform these radially differentiable elementary functions to Clifford regular functions. The key-idea goes back to R. Fueter and was later generalized by M. Sce [19], F. Sommen [20, 21] and T. Qian [18]. Let $u = u(x_0, |\underline{x}|)$ and $v = v(x_0, |\underline{x}|)$ real-valued functions and $h = u + \omega v$. We will denote by τ_n the mapping

$$q = \tau_n(h) = \kappa_n \Delta^{\frac{(n-1)}{2}} h,$$

where Δ denotes the Laplacian and κ_n a normalization factor. For even n the operator $\Delta^{\frac{n-1}{2}}$ has to be considered as Fourier multiplier operator induced by the symbol $(2\pi i|\xi|)^{n-1}$.

Theorem 5.5 (Qian [18]). *Let $h = u + \omega v$ be radially differentiable in $\mathbb{R}^{n+1}$ and $n = 2k + 1$. Then for any $k \in \mathbb{N}$ we find*

$$\tau_n(h) = \frac{1}{2^k k} \Delta^k h = (k-1)! \left[\left(\frac{1}{|\underline{x}|} \ell_\omega \right)^k u + \omega \left(\ell_\omega \frac{1}{|\underline{x}|} \right)^k v \right]. \tag{5.2}$$

Corollary 5.6. *Let $k = 1$. A radially quaternionic regular function $h = u + \omega v$ fulfils the partial differential equation*

$$|\underline{x}|^2 \Delta h - 2|\underline{x}| \ell_\omega h + 2\mathrm{Vec}\, h = 0.$$

Remark. A componentwise consideration of equation (5.2) leads for the scalar part to the Laplace-Beltrami equation in the hyperbolic metric and for the vector components to the Laplace-Beltrami equation to the eigenvalue -2.

Now we will give a lot of examples which will emphasize how effectively Fueter's mapping τ_n is working. For abbreviation we write

$$b_k(x) = \frac{\Gamma\left(\frac{1}{2} - k\right)}{\Gamma\left(2k+1\right))} \left(\frac{2}{|\underline{x}|} \right)^{\frac{k-1}{2}} \frac{1}{k\, 2^k}.$$

Example 5.7 (Exponential function). Let $n = 2k+1$ and $h(x) = e^x$. We obtain by applying Fueter' s mapping

$$\mathrm{EXP}_k x := b_k(x)\left[J_{k-\frac{1}{2}}(|\underline{x}|) + \omega J_{k+\frac{1}{2}}(|\underline{x}|)\right] e^{x_0}$$

where

$$J_\lambda(|\underline{x}|) = \frac{\left(\frac{|\underline{x}|}{2}\right)^\lambda}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\lambda\right)} \int_{-1}^{1} e^{i|\underline{x}|t}(1-t^2)^{\lambda-\frac{1}{2}}\,dt \qquad (\lambda \in \mathbb{R})$$

are Bessel functions of first kind. Indeed, we obtain from [10: p. 740] the equalities

$$J_{k+\frac{1}{2}}(|\underline{x}|) = (-1)^{k+1}\sqrt{\frac{2}{\pi}}|\underline{x}|^{k+\frac{1}{2}}\left(\frac{1}{|\underline{x}|}\ell_\omega\right)^{k+1}\cos|\underline{x}| = (-1)^k\sqrt{\frac{2}{\pi}}\left(\ell_\omega\frac{1}{|\underline{x}|}\right)^k \sin|\underline{x}|.$$

A straightforward calculation leads to the above defind exponential function $\mathrm{EXP}_k x$. In the special case for $k = 1$ we have

$$b_1(x) = -\frac{\Gamma\left(-\frac{1}{2}\right)}{2!}\sqrt{\frac{2}{|\underline{x}|}} = \sqrt{\frac{2\pi}{|\underline{x}|}}$$

and

$$J_{\frac{1}{2}}(|\underline{x}|) = \sqrt{\frac{2}{\pi|\underline{x}|}}\sin|\underline{x}|$$
$$J_{\frac{3}{2}}(|\underline{x}|) = \sqrt{\frac{2}{\pi|\underline{x}|}}\left(\frac{\sin|\underline{x}|}{|\underline{x}|} - \cos|\underline{x}|\right).$$

Hence,

$$\mathrm{EXP}_1 x = e^{x_0}\left[\frac{\sin|\underline{x}|}{|\underline{x}|} + \omega\left(\frac{\sin|\underline{x}| - |\underline{x}|\cos|\underline{x}|}{|\underline{x}|^2}\right)\right] = e^{x_0}\left(\mathrm{sinc}\,|\underline{x}| - \omega\mathrm{sinc}'|\underline{x}|\right)$$

where $\mathrm{sinc}\,|\underline{x}| = \frac{\sin|\underline{x}|}{|\underline{x}|}$ and $\mathrm{sinc}'|\underline{x}| := \frac{d}{d|\underline{x}|}(\mathrm{sinc}\,|\underline{x}|)$.

In [21] F. Sommen obtained by using of a similarity principle for Vekua systems a class of hypercomplex exponential functions which can be seen as generalization of Fueter type mappings.

Example 5.8 (Hyperbolic functions). Let $n = 2k+1$. Further, let $h(x)$ the radially hyperbolic sine function. We get

$$\mathrm{SINH}\, x := \tau_n(\sinh x) = b_k(x)\left[\sinh x_0 J_{k-\frac{1}{2}}(|\underline{x}|) + \omega\cosh x_0 J_{k+\frac{1}{2}}(|\underline{x}|)\right],$$

and in the special case $k = 1$ we have

$$\mathrm{SINH}\, x := \sinh x_0 \mathrm{sinc}\,|\underline{x}| - \omega\cos x_0 \mathrm{sinc}'|\underline{x}|.$$

Let $h(x)$ the radially cosine function. We obtain

$$\mathrm{COSH}\, x := \tau_n(\cosh x) = b_k(x)\Big[\cosh x_0 J_{k-\frac{1}{2}}(|\underline{x}|) + \omega \sinh x_0 J_{k+\frac{1}{2}}(|\underline{x}|)\Big],$$

and in the special case $k = 1$ we have

$$\mathrm{COSH}\, x := \cosh x_0 \operatorname{sinc} |\underline{x}| - \omega \sinh x_0 \operatorname{sinc}' |\underline{x}|.$$

Example 5.9 (Trigonometric functions). Let $n = 2k + 1$. It is easy to see that

$$y = \omega(x)x = -|\underline{x}| + \frac{\underline{x} x_0}{|\underline{x}|}.$$

Hence, $|\underline{y}| = |x_0|$ and $\omega(y) = \omega(x)\frac{x_0}{|x_0|}$. It follows that

$$\mathrm{EXP}_k y = b_k(x_0)\left[J_{k-\frac{1}{2}}(|\underline{x}|_0) + \omega(x)\frac{x_0}{|x_0|} J_{k+\frac{1}{2}}(|x_0|)\right] e^{-|x_0|}.$$

We can now define a Clifford regular SIN-function by

$$\begin{aligned}\mathrm{SIN}\, x :=& \frac{1}{2}\left(\mathrm{EXP}_k y + \mathrm{EXP}_k(-y)\right)\omega(y)\\ =& b_k(x)\left[\cosh |\underline{x}| J_{k+\frac{1}{2}}(|x_0|) + \omega(x)\sinh |\underline{x}| J_{k-\frac{1}{2}}(|x_0|)\frac{x_0}{|x_0|}\right].\end{aligned}$$

Analogously we can also define

$$\begin{aligned}\mathrm{COS}\, x :=& \frac{1}{2}\left[\mathrm{EXP}_k y + \mathrm{EXP}_k(-y)\right]\\ =& b_k(x_0)\left[\cosh |\underline{x}| J_{k-\frac{1}{2}}(|x_0|) - \omega(x)\sinh |\underline{x}| J_{k+\frac{1}{2}}(|x_0|)\frac{x_0}{|x_0|}\right].\end{aligned}$$

In the special case $k = 1$ we have

$$\begin{aligned}\mathrm{SIN}\, x :=& -\cosh |\underline{x}| \operatorname{sinc}' |x_0| + \omega(x)\sinh |\underline{x}| \operatorname{sinc} x_0 \frac{x_0}{|x_0|}\\ \mathrm{COS}\, x :=& \cosh |\underline{x}| \operatorname{sinc} |x_0| + \omega(x)\sinh |\underline{x}| \operatorname{sinc}' |x_0| \frac{x_0}{|x_0|}.\end{aligned}$$

Now we will give an application of this conception of elementary functions.

Theorem 5.10. *Let $n = 3$. The exponential function $\mathrm{EXP}_3 x$ fulfils the property*

$$\overline{\partial}\mathrm{EXP}_3(\lambda x) = \lambda\, \mathrm{EXP}_3 x \qquad (\lambda \in \mathbb{C}).$$

Proof. Using the decomposition $D = |\underline{x}|^{-1}L + \omega \ell_\omega$ we obtain

$$\begin{aligned}&(|\underline{x}|^{-1}L + \omega\ell_\omega)\left[\frac{\sin\lambda|\underline{x}|}{\lambda|\underline{x}|} - \omega(x)\ell_\omega\left(\frac{\sin\lambda|\underline{x}|}{\lambda|\underline{x}|}\right)\right]\\ &\quad = \frac{1}{\lambda}\left[|\underline{x}|^{-1}L\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right) + \omega\ell_\omega\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)\right]\\ &\quad\quad - \left[\omega(x)\ell_\omega\omega(x)\ell_\omega\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right) - |\underline{x}|^{-1}L\left(\omega(x)\ell_\omega\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)\right].\end{aligned}$$

From Proposition 4.2/(i) immediately

$$L\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)=0$$

and

$$\frac{1}{\lambda|\underline{x}|}L\left(\omega\ell_\omega\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)=-\frac{2}{\lambda|\underline{x}|}\ell_\omega\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)=-\frac{2\sin\lambda|\underline{x}|}{\lambda|\underline{x}|^3}+\frac{\cos\lambda|\underline{x}|}{|\underline{x}|^2}$$

follows. It remains to consider

$$\frac{1}{\lambda}\omega\ell_\omega\left(-\omega\ell_\omega\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)=\frac{1}{\lambda}\ell_\omega^2\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right).$$

A straightforward computation delivers

$$\frac{1}{\lambda}\ell_\omega^2\left(\frac{\sin\lambda|\underline{x}|}{|\underline{x}|}\right)=\frac{\lambda\sin\lambda|\underline{x}|}{|\underline{x}|}-2\frac{\cos\lambda|\underline{x}|}{|\underline{x}|^2}+\frac{2\sin\lambda|\underline{x}|}{\lambda|\underline{x}|^3}.$$

Finally, we have $D\mathrm{EXP}_3(\lambda x)=-\lambda\mathrm{EXP}_3(\lambda x)$ and so $\overline{\partial}\mathrm{EXP}_3(\lambda x)=\lambda\mathrm{EXP}_3(\lambda x)$ ∎

From the definition of the exponential function it follows now

Corollary 5.11. *We have* $|\mathrm{EXP}_3x|^2>0$ *and* $\lim_{\underline{x}\to 0}\mathrm{EXP}_3x=e^{x_0}$.

We consider now the so-called $\overline{\partial}$-problem and obtain the following result.

Corollary 5.12. *Let* $L_n=a_n\overline{\partial}^n+\ldots+a_1\overline{\partial}+a_0\quad(a_k\in\mathbb{R})$. *Further, let* λ_k *be the roots of the algebraic equation* $a_n\lambda^n+\ldots+a_1\lambda+a_0=0$. *Then*

$$u_k=\mathrm{EXP}_3\lambda_k x\qquad(k=1,\ldots,n)$$

belong to $\ker L_n$, *e.g. we have constructed a set of solutions of the* n*-th order linear partial differential equation* $L_nu=0$ *relatively to the operator* $\overline{\partial}$.

Proof. We have to make the ansatz $u=\mathrm{EXP}_3\lambda x$. The result follows by using Theorem 5.10 ∎

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