Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 1, 3–12

A Note on Convergence of Level Sets

F. Camilli

Abstract. Given a sequence of functions f_n converging in some topology to a function f, in general the 0-level set of f_n does not give a good approximation of the one of f. In this paper we show that, if we consider an appropriate perturbation of the 0-level set of f_n, we get a sequence of sets converging to the 0-level set of f, where the type of set convergence depends on the type of convergence of f_n to f.

Keywords: *Perturbed level sets, set convergence, capacity*

AMS subject classification: 28 A 12, 46 E 35

1. Introduction

In several fields (phase transition, free boundary problems, front propagation, etc.), a set of interest for the solution of the problem is represented by a level or a sublevel set of a function f. Let us suppose that by means of some approximation technique (f.e. discretization, regularization, rescaling of an order parameter) we get a sequence of functions converging in some topology to f. In general, no matter how strong is the convergence of f_n to f, the level sets of f_n do not give a good approximation of the ones of f.

Pursuing an idea used in Baiocchi and Pozzi [1], we show that appropriately perturbing the level sets of f_n (the same can be done for the sublevels or the superlevels), we get a sequence of sets defined by means of f_n converging to the level set of f. The type of set convergence is the convergence to zero of the measure of the symmetric difference between the level set of f_n and the correpsonding one of f, and the measure depends on the type of convergence of the sequence f_n.

We analyze the case of convergence in L^p and in $W^{1,p}$, but this technique could be useful in other situations.

The paper is organized as follows. In Section 2, we analyze the case of convergence in L^∞ and $W^{1,\infty}$ and the associated convergence of perturbed level sets in set-theoretical sense. In Section 3 we first consider the case of convergence in L^p, which gives the convergence in the sense of Lebesgue measure. Then we analyze the case of convergence in $W^{1,p}$ and the corresponding set convergence in the sense of capacity and Hausdorf measure.

F. Camilli: Univ. di Torino, Dip. di Matematica, Via Carlo Alberto 10, 10123 Torino (Italy)
New address: Univ. dell‘ Aquila, Dip. di Energetica, 67040 Roio Poggio (AQ), Italy
e-mail: camilli@ing.univaq.it

ISSN 0232-2064 / $ 2.50

2. The case $p = \infty$

In this section we will study (extending the result given in [1]) the case of the convergence in L^∞. We will see that the natural set convergence associated to the L^∞ convergence is the convergence in set-theoretical sense.

Definition 2.1. Given a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$, we set

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m \qquad \text{and} \qquad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m.$$

We say that $\{A_n\}_{n\in\mathbb{N}}$ converges to A in *set-theoretical sense* and write $A = \lim_{n\to\infty} A_n$ if

$$A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n.$$

We have the following result.

Proposition 2.1. *Let f_n and f be continuous functions on $\mathbb{R}^N$ such that*

$$\|f - f_n\|_{L^\infty(\mathbb{R}^N)} = \varepsilon_n \tag{2.1}$$

where $\varepsilon_n \to 0$ for $n \to \infty$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a sequence such that

$$\left.\begin{array}{ll} \delta_n > 0 & (n\in\mathbb{N}) \\ \delta_n \to 0 & (n\to\infty) \\ \dfrac{\varepsilon_n}{\delta_n} \to 0 & (n\to\infty). \end{array}\right\} \tag{2.2}$$

Set, for any $n \in \mathbb{N}$,

$$\begin{aligned} \Gamma &= \big\{x\in\mathbb{R}^N : f(x) = 0\big\} \\ \Gamma_n &= \big\{x\in\mathbb{R}^N : |f_n(x)| \le \delta_n\big\}. \end{aligned} \tag{2.3}$$

Then $\Gamma \subset \Gamma_n$, for n sufficiently large, and

$$\Gamma = \lim_{n\to\infty} \Gamma_n. \tag{2.4}$$

Proof. Let $\overline{n} \in \mathbb{N}$ be such that $\delta_n \ge \varepsilon_n$ for any $n \ge \overline{n}$ (recall that $\frac{\varepsilon_n}{\delta_n} \to 0$). If $x \in \Gamma$, then, for $n \ge \overline{n}$, we have from (2.1)

$$|f_n(x)| \le |f(x)| + \|f_n - f\|_{L^\infty(\mathbb{R}^N)} = \varepsilon_n \le \delta_n,$$

hence $x \in \Gamma_n$. Hence $\Gamma \subset \Gamma_n$ for $n \ge \overline{n}$ and therefore $\Gamma \subset \liminf_{n\to\infty}\Gamma_n$. Let us prove yet that $\limsup_{n\to\infty}\Gamma_n \subset \Gamma$. If $x \in \limsup_{n\to\infty}\Gamma_n$, then by definition there exists a subsequence $\{\Gamma_{n_k}\}_{k\ge 1}$ such that $x \in \Gamma_{n_k}$ for any $k \in \mathbb{N}$. It follows that $|f_{n_k}(x)| \le \delta_{n_k}$ for any $k \in \mathbb{N}$ and therefore $f(x) = \lim_{k\to\infty} f_{n_k}(x) = 0$ which yields $x \in \Gamma$ ∎

Remark 2.1 Observe that if Γ_n and Γ are contained in a compact set K, then the previous proposition gives the convergence to zero of the Hausdorff distance between Γ_n and Γ.

In the next proposition we show that improving the convergence of f_n to f, we get some additional information on the type of convergence of Γ_n to Γ.

Proposition 2.2. *Let $f, f_n \in C^1(\mathbb{R}^N)$ $(n \in \mathbb{N})$ be such that*

$$\|f - f_n\|_{W^{1,\infty}(\mathbb{R}^N)} = \varepsilon_n$$

where $\varepsilon_n \to 0$ for $n \to \infty$. Let δ_n and Γ and Γ_n be defined as in (2.2) – (2.3). *Set*

$$\Gamma^{reg} = \Big\{x \in \mathbb{R}^N : f(x) = 0 \text{ and } \nabla f(x) \neq 0\Big\}$$
$$\Gamma^{sing} = \Big\{x \in \mathbb{R}^N : f(x) = 0 \text{ and } \nabla f(x) = 0\Big\}$$

and

$$\Gamma_n^{reg} = \Big\{x \in \mathbb{R}^N : |f_n(x)| \leq \delta_n \text{ and } |\nabla f_n(x)| > \delta_n\Big\}$$
$$\Gamma_n^{sing} = \Big\{x \in \mathbb{R}^N : |f_n(x)| \leq \delta_n \text{ and } |\nabla f_n(x)| \leq \delta_n\Big\}.$$

Then

$$\Gamma^{reg} = \lim_{n\to\infty} \Gamma_n^{reg} \qquad \text{and} \qquad \Gamma^{sing} = \lim_{n\to\infty} \Gamma_n^{sing}.$$

Proof. Let $\overline{n} \in \mathbb{N}$ be such that $\delta_n \geq \varepsilon_n$ for $n \geq \overline{n}$. Then, for $n \geq \overline{n}$, $\Gamma \subset \Gamma_n$ and, if $x \in \Gamma^{sing}$, we have

$$|\nabla f_n(x)| \leq |\nabla f(x)| + \|\nabla f_n - \nabla f\|_{L^\infty(\mathbb{R}^N)} = \varepsilon_n \leq \delta_n.$$

Therefore $\Gamma^{sing} \subset \Gamma_n^{sing}$ for $n \geq \overline{n}$. If $x \in \limsup_{n\to\infty} \Gamma_n^{sing}$, then $x \in \Gamma_{n_k}^{sing}$ for a subsequence Γ_{n_k}. It follows that $|f_{n_k}(x)| \leq \delta_{n_k}$ and $|\nabla f_{n_k}(x)| \leq \delta_{n_k}$ for any $k \in \mathbb{N}$ and therefore

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) = 0 \qquad \text{and} \qquad \nabla f(x) = \lim_{k\to\infty} \nabla f_{n_k}(x) = 0.$$

Therefore $x \in \Gamma^{sing}$ and $\Gamma^{sing} = \lim_{n\to\infty} \Gamma_n^{sing}$. Since (2.4) holds, we get also $\Gamma^{reg} = \lim_{n\to\infty} \Gamma_n^{reg}$ ∎

We conclude this section giving an estimate of the Hausdorff distance between Γ and Γ_n in the case that Γ is regular.

Proposition 2.3. *Assume the same hypothesis as in Proposition* 2.1*, with δ_n and Γ, Γ_n defined as in* (2.2) – (2.3). *Moreover, assume that Γ is compact and that f is differentiable with $\nabla f \neq 0$ on Γ. Then there exists a constant $C > 0$ such that*

$$d_{\mathcal{H}}(\Gamma, \Gamma_n) \leq C(\varepsilon_n + \delta_n) \tag{2.5}$$

for n sufficiently large, where $d_{\mathcal{H}}$ denotes the Hausdorff distance.

Proof. By the assumptions on f and Γ, there exist $\eta_0 > 0$ and $C_0 > 0$ such that $|\nabla f(x)| \geq C_0$ on $\Gamma_{\eta_0} = \{x : d(x, \Gamma) \leq \eta_0\}$. For $\eta \leq \eta_0$, consider $y \in \partial(\Gamma_\eta) = \partial\{x : d(x, \Gamma) \leq \eta\}$ and let $x \in \Gamma$ be such that $d(y, \Gamma) = |y - x| = \eta$. Then

$$|(y - x) \cdot \nabla f(x)| = \eta |\nabla f(x)| \geq C_0 \eta.$$

Since $f(x) = 0$, if ω is a modulus of continuity of ∇f on Γ_{η_0}, then

$$|f(y)| \geq |(y - x) \cdot \nabla f(x)| - \omega(|y - x|)|y - x| \geq \eta(C_0 - \omega(\eta)). \tag{2.6}$$

For n sufficiently large in such a way that $C_0 - \omega(\delta_n + \varepsilon_n) \geq \frac{C_0}{2}$ and $2\frac{\delta_n + \varepsilon_n}{C_0} \leq \eta_0$, from (2.6) with $\eta = 2\frac{\delta_n + \varepsilon_n}{C_0}$ we get $|f(y)| \geq \delta_n + \varepsilon_n$ and therefore $|f_n(y)| \geq \delta_n$ on $\partial\Gamma_\eta$. It follows that $\Gamma_n \subset \Gamma_\eta$. Since $\Gamma \subset \Gamma_n$ for n sufficiently large, we finally get $d_{\mathcal{H}}(\Gamma, \Gamma_n) \leq d_{\mathcal{H}}(\Gamma, \Gamma_\eta) \leq \eta$ and therefore (2.5), with $C = \frac{2}{C_0}$ ∎

All the results of this section have an analogue in the case of sub- and superlevel sets of f_n and f.

3. The case $1 \leq p < \infty$

We first analyze the case of convergence in $L^p(\mathbb{R}^N)$. We prove that in this case an appropriate notion of set convergence is the convergence to 0 of the Lebesgue measure of $\Gamma\Delta\Gamma_n$. In the following, $\mathcal{L}^N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

Proposition 3.1. *Let $f_n, f \in L^p(\mathbb{R}^N)$ $(1 \leq p < \infty;\ n \in \mathbb{N})$ such that*

$$\|f - f_n\|_{L^p(\mathbb{R}^N)} = \varepsilon_n \tag{3.1}$$

where $\varepsilon_n \to 0$ for $n \to \infty$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a sequence such that

$$0 < \delta_n \quad (n \in \mathbb{N}) \qquad \text{and} \qquad \delta_n \to 0 \quad (n \to \infty). \tag{3.2}$$

Define, for any $n \in \mathbb{N}$,

$$\begin{aligned} \Gamma &= \{x \in \mathbb{R}^N : f(x) = 0\} \\ \Gamma_n &= \{x \in \mathbb{R}^N : |f_n(x)| \leq \delta_n\}. \end{aligned} \tag{3.3}$$

Then:

(i) *If $\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = 0$, we have*

$$\lim_{n\to\infty} \mathcal{L}^N(\Gamma\Delta\Gamma_n) = 0 \tag{3.4}$$

$$\mathcal{L}^N\Big(\Gamma \triangle \limsup_{n\to\infty} \Gamma_n\Big) = 0. \tag{3.5}$$

(ii) *If*

$$\sum_n \left(\frac{\varepsilon_n}{\delta_n}\right)^p < \infty, \tag{3.6}$$

we also have

$$\mathcal{L}^N\Big(\Gamma \triangle \liminf_{n\to\infty} \Gamma_n\Big) = 0. \tag{3.7}$$

Therefore $\Gamma = \lim_{n\to\infty} \Gamma_n$ up to a set of 0-Lebesgue measure.

Proof. We first observe that, since we are considering only the measure of Γ and Γ_n, we can assume that these sets are defined by means of any element in the class of equivalence of f and f_n. We have

$$\Gamma\Delta\Gamma_n = (\Gamma \setminus \Gamma_n) \cup (\Gamma_n \setminus \Gamma)$$

and

$$\begin{aligned} \Gamma \setminus \Gamma_n &= \Big\{x \in \mathbb{R}^N : f(x) = 0 \text{ and } |f_n(x)| > \delta_n\Big\} \\ \Gamma_n \setminus \Gamma &= \Big\{x \in \mathbb{R}^N : f(x) \neq 0 \text{ and } |f_n(x)| \leq \delta_n\Big\} \end{aligned}$$

(the previous and all the others inclusions in this proof are intended up to sets of null Lebesgue measure).

Since $\Gamma \subset \Gamma_n \subset \{x \in \mathbb{R}^N : |f(x) - f_n(x)| > \delta_n\}$, from the Cebycev inequality we get

$$\mathcal{L}^N(\Gamma \setminus \Gamma_n) \leq \frac{1}{\delta_n^p} \int_{\mathbb{R}^N} |f(x) - f_n(x)|^p dx = \left(\frac{\varepsilon_n}{\delta_n}\right)^p \tag{3.8}$$

and therefore

$$\lim_{n\to\infty} \mathcal{L}^N(\Gamma \setminus \Gamma_n) = 0. \tag{3.9}$$

Let us prove that

$$\mathcal{L}^N\Big(\limsup_{n\to\infty}(\Gamma_n \setminus \Gamma)\Big) = 0. \tag{3.10}$$

Set $\widetilde{\Gamma} = \limsup_{n\to\infty}(\Gamma_n \setminus \Gamma)$ and let $x \in \widetilde{\Gamma}$. Then there exists a subsequence $\{n_k\}_{k\geq 1}$ such that $|f_{n_k}(x)| \leq \delta_{n_k}$ for any $k \in \mathbb{N}$. It follows that, $\mathcal{L}^N$-a.e. on $\widetilde{\Gamma}$,

$$\liminf_{n\to\infty} f_n(x) = 0 \qquad \text{and} \qquad \liminf_{n\to\infty} |f_n(x) - f(x)|^p = |f(x)|^p.$$

Applying the Fatou Lemma we get

$$\int_{\widetilde{\Gamma}} |f(x)|^p dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} |f(x) - f_n(x)|^p dx = 0.$$

Since $|f(x)| > 0$ on $\widetilde{\Gamma}$, we get (3.10).

Since for any sequence $\{A_n\}_{n\geq 1}$ of measurable sets we have

$$\limsup_{n\to\infty} \mathcal{L}^N(A_n) \leq \mathcal{L}^N\Big(\limsup_{n\to\infty} A_n\Big)$$

it follows from (3.10) that $\lim_{n\to\infty} \mathcal{L}^N(\Gamma_n \setminus \Gamma) = 0$ and therefore, together with (3.9), also (3.4) holds. From (3.10) and

$$\mathcal{L}^N\Big(\Gamma \setminus \limsup_{n\to\infty} \Gamma_n\Big) = \mathcal{L}^N\Big(\liminf_{n\to\infty}(\Gamma \setminus \Gamma_n)\Big) \leq \lim_{n\to\infty} \mathcal{L}^N(\Gamma \setminus \Gamma_n) = 0$$

we get (3.5).

Let us prove now statement (ii). Estimate (3.8) gives

$$\mathcal{L}^N\Big(\bigcup_{m=n}^{\infty}(\Gamma \setminus \Gamma_m)\Big) \leq \sum_{m=n}^{\infty} \left(\frac{\varepsilon_m}{\delta_m}\right)^p,$$

and therefore, for any $n \in \mathbb{N}$,

$$\mathcal{L}^N\Big(\Gamma \setminus \liminf_{n\to\infty} \Gamma_n\Big) = \mathcal{L}^N\Big(\limsup_{n\to\infty}(\Gamma \setminus \Gamma_n)\Big) \leq \sum_{m=n}^{\infty} \left(\frac{\varepsilon_m}{\delta_m}\right)^p.$$

For (3.6) we get $\mathcal{L}^N\big(\Gamma\setminus\liminf_{n\to\infty} \Gamma_n\big) = 0$. Since (3.10) yields $\mathcal{L}^N\big(\liminf_{n\to\infty} \Gamma_n\setminus\Gamma\big) = 0$ we get (3.7) ∎

Remark 3.1. Since we have

$$\left|\mathcal{L}^N(\Gamma) - \mathcal{L}^N(\Gamma_n)\right| \le \mathcal{L}^N(\Gamma \triangle \Gamma_n)$$

then $\mathcal{L}^N(\Gamma\Delta\Gamma_n) \to 0$ implies that $\mathcal{L}^N(\Gamma_n) \to \mathcal{L}^N(\Gamma)$. The vice versa in general is not true. The result $\mathcal{L}^N(\Gamma\Delta\Gamma_n) \to 0$ gives a more complete information respect to the convergence of the measure of Γ_n to the measure of Γ. In fact, it shows that the measure of the part of Γ_n which does not approximate Γ tends to 0, while the measure of $\Gamma \setminus \Gamma_n$ can be estimated by means of (3.8).

If we know that $\|f_n - f\|_{W^{1,p}(\mathbb{R}^N)} = \varepsilon_n$, then we can prove a result similar to Proposition 2.2 for the convergence of regular and singular parts of Γ_n to Γ. In this case, a more accurate way of studying properties of sets defined through Sobolev functions is given by the notion of *capacity*. We will show that, in the case of convergence in $W^{1,p}(\mathbb{R}^N)$ $(1 \le p < N)$ we get convergence of Γ_n to Γ up to sets of 0 capacity. Let us recall the definition and some basic properties of the capacity we will need in the following (see [2 - 4] for more details).

Definition 3.1. *Let $1 \le p < N$ and set*

$$K^p = \left\{\varphi : \mathbb{R}^N \to \mathbb{R} \,\middle|\, 0 \le \varphi \in L^{p^*}(\mathbb{R}^N) \text{ with } \nabla\varphi \in L^p(\mathbb{R}^N, \mathbb{R}^N)\right\}$$

where $p^* = \frac{Np}{N-p}$. For $A \subset \mathbb{R}^N$, we define

$$\mathrm{Cap}_p(A) = \inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|\, dy \,\middle|\, \varphi \in K^p \text{ with } A \subset \{\varphi \ge 1\}^\circ\right\}.$$

It is possible to prove that Cap_p is an exterior measure on subsets of $\mathbb{R}^N$. For a function $\varphi \in L^1_{loc}(\mathbb{R}^N)$, the *precise representative* φ^* of φ is defined by

$$\varphi^*(x) = \begin{cases} \lim\limits_{r\to 0^+} ⨍_{B(x,r)} \varphi(y)\, dy & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

where $⨍_{B(x,r)} \varphi(y)\, dy = \int_{B(x,r)} \varphi(y)\, dy / \mathcal{L}^N(B(x,r))$. We have (see [2: Theorem 4.8.1]) the following

Theorem 3.1. *Let $\varphi \in W^{1,p}(\mathbb{R}^N)$ $(1 \le p < N)$. Then:*

(i) *There is a Borel set $E \subset \mathbb{R}^N$ such that $\mathrm{Cap}_p(E) = 0$ and*

$$\lim_{r\to 0^+} ⨍_{B(x,r)} \varphi(y)\, dy = \varphi^*(x) \qquad (x \in \mathbb{R}^N \setminus E).$$

(ii) *In addition,*

$$\lim_{r\to 0^+} ⨍_{B(x,r)} |\varphi(y) - \varphi^*(x)|^{p^*} dy = 0 \qquad (x \in \mathbb{R}^N \setminus E).$$

(iii) *The precise representative φ^* is quasi-continuous.*

Because of the previous theorem, any function in the space $W^{1,p}(\mathbb{R}^N)$ admits a quasi-continuous representative. We have the following convergence result for the perturbed level sets.

Proposition 3.2. *Let $f, f_n \in W^{1,p}(\mathbb{R}^N)$ $(1 \le p < N)$ be such that*

$$\|f - f_n\|_{W^{1,p}(\mathbb{R}^N)} = \varepsilon_n$$

where $\varepsilon_n \to 0$ for $n \to \infty$. Let δ_n and Γ, Γ_n be defined as in (3.2) – (3.3) by means of the precise representatives of f and f_n. Then:

(i) *If $\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = 0$, then*

$$\mathrm{Cap}_p\Big(\limsup_{n\to\infty} \Gamma_n \triangle \Gamma\Big) = 0. \tag{3.11}$$

(ii) *If*

$$\sum_n \Big(\frac{\varepsilon_n}{\delta_n}\Big)^p < \infty, \tag{3.12}$$

then we have also

$$\mathrm{Cap}_p\Big(\Gamma \triangle \liminf_{n\to\infty} \Gamma_n\Big) = 0 \tag{3.13}$$

and therefore $\Gamma = \lim_{n\to\infty} \Gamma_n$ up to a set of zero capacity.

Proof. Let us prove statement (i). Since the sets Γ and Γ_n are defined by means of the precise representatives of f and f_n, then they are well defined, i.e. up to sets of zero capacity. In the following all the relations involving Γ and Γ_n are intended to be satisfied Cap_p-a.e. We have

$$\Gamma \setminus \Gamma_n = \Big\{x \in \mathbb{R}^N \,:\, f(x) = 0 \text{ and } |f_n(x)| > \delta_n\Big\}.$$

Let us prove that, defining

$$B_n = \Big\{x \in \mathbb{R}^N \,\Big|\, \fint_{B(x,r)} |f_n - f|\, dy > \delta_n \text{ for some } r > 0\Big\}, \tag{3.14}$$

then

$$\mathrm{Cap}_p(\Gamma \setminus \Gamma_n) \le \mathrm{Cap}_p(B_n). \tag{3.15}$$

In fact, if $x \in \Gamma \setminus \Gamma_n$, then, up to a set of zero capacity, we have

$$\lim_{r\to 0} \fint_{B(x,r)} |f_n - f|\, dy = |f(x) - f_n(x)| > \delta_n.$$

Therefore there exists $r_0 > 0$ such that $\fint_{B(x,r)\ B(x,r_0)} |f_n - f|\, dy > \delta_n$ and so (3.15) holds. Recall that (see [2: Lemma 4.8.1]), if $\varphi \in K^p$, then there exists a constant C, depending only on N and p, such that for any $\eta > 0$

$$\mathrm{Cap}_p\bigg(\Big\{x \in \mathbb{R}^N \,\Big|\, \fint_{B(x,r)} \varphi(y)\, dy > \eta \text{ for some } r > 0\Big\}\bigg) \le \frac{C}{\eta^p} \int_{\mathbb{R}^N} |D\varphi|^p dy. \tag{3.16}$$

From (3.14) and (3.16) we get

$$\mathrm{Cap}_p(\Gamma \setminus \Gamma_n) \le \frac{C}{\delta_n^p} \int_{\mathbb{R}^N} |\nabla f - \nabla f_n|^p dy \le C\Big(\frac{\varepsilon_n}{\delta_n}\Big)^p \tag{3.17}$$

and therefore $\lim_{n\to\infty} \mathrm{Cap}_p(\Gamma \setminus \Gamma_n) = 0$. From the previous equality and the properties of the capacity, we get

$$\mathrm{Cap}_p\Big(\Gamma \setminus \limsup_{n\to\infty} \Gamma_n\Big) = \mathrm{Cap}_p\Big(\liminf_{n\to\infty}(\Gamma \setminus \Gamma_n)\Big) \le \lim_{n\to\infty} \mathrm{Cap}_p(\Gamma \setminus \Gamma_n) = 0. \tag{3.18}$$

Let A be the set

$$A = \left\{ x \in \mathbb{R}^N \,\middle|\, \limsup_{r\to 0^+} \frac{1}{r^{n-p}} \int_{B(x,r)} |\nabla f(y)|^p dy > 0 \right\}.$$

Then $\mathrm{Cap}_p(A) = 0$ (see [2: Theorem 2.4.3]) and from the Poincaré inequality we have

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - (f)_{x,r}|^{p^*} dy = 0 \qquad (x \in \mathbb{R}^N \setminus A) \tag{3.19}$$

where $(f)_{x,r} = \fint_{B(x,r)} f(y)\, dy$. From Theorem 3.1, for any $n \in \mathbb{N}$ there exists a Borel set E_n such that $\mathrm{Cap}_p(E_n) = 0$ and

$$\lim_{r\to 0^+} \fint_{B(x,r)} |f_n(y) - f_n(x)|^{p^*} dy = 0 \qquad (x \in \mathbb{R}^N \setminus E_n). \tag{3.20}$$

Set $\Delta_n = B_n \cup E_n \cup A$, where B_n has been defined in (3.14). If $x \in \Gamma_n \setminus \Delta_n$, then from Theorem 3.1, (3.14) and (3.19) - (3.20) we get

$$\begin{aligned} \limsup_{r\to 0} |(f)_{x,r}| &\le \limsup_{r\to 0} |(f)_{x,r} - f_n(x)| + \delta_n \\ &\le \limsup_{r\to 0} \Big\{ \fint_{B(x,r)} |f - (f)_{x,r}|\, dy \\ &\quad + \fint_{B(x,r)} |f - f_n|\, dy + \fint_{B(x,r)} |f_n - f_n(x)|\, dy \Big\} + \delta_n \\ &\le 2\delta_n. \end{aligned} \tag{3.21}$$

Moreover, inequality (3.16) gives

$$\mathrm{Cap}_p(\Delta_n) \le \mathrm{Cap}_p(B_n) + \mathrm{Cap}_p(E_n) + \mathrm{Cap}_p(A) \le C\Big(\frac{\varepsilon_n}{\delta_n}\Big)^p. \tag{3.22}$$

Set $\Delta = \liminf_{n\to\infty} \Delta_n$ and $\widetilde{\Gamma} = \limsup_{n\to\infty}(\Gamma_n \setminus \Gamma)$. From (3.21) - (3.22) it follows that if $x \in \widetilde{\Gamma} \setminus \Delta$, then $\lim_{r\to 0^+} (f)_{x,r} = 0$. Therefore from Theorem 3.1 we get $\widetilde{\Gamma} \setminus \Delta \subset \Gamma$ and, since $\mathrm{Cap}_p(\Delta) \le \liminf_{n\to\infty} \mathrm{Cap}_p(\Delta_n) = 0$, it follows also that

$$\mathrm{Cap}_p\Big(\limsup_{n\to\infty}(\Gamma_n \setminus \Gamma)\Big) = 0.$$

The previous equality and (3.18) imply (3.11).

Let us prove statement (ii). If $x \in \Gamma \setminus B_n$, then

$$\limsup_{r\to 0^+} \fint_{B(x,r)} |f_n|\, dy \le \limsup_{r\to 0^+} \left(\fint_{B(x,r)} |f|\, dy + \fint_{B(x,r)} |f - f_n|\, dy \right) \le \delta_n. \tag{3.23}$$

Thus (3.23) yields $\Gamma \setminus B_n \subset \Gamma_n$ for any n and therefore

$$\liminf_{n\to\infty} (\Gamma \setminus B_n) = \Gamma \setminus \limsup_{n\to\infty} B_n \subset \liminf_{n\to\infty} \Gamma_n.$$

Set $B = \limsup_{n\to\infty} B_n$. Then, for any $n \in \mathbb{N}$,

$$\mathrm{Cap}_p(B) \le \sum_{m=n}^{\infty} \mathrm{Cap}_p(B_m) \le \sum_{m=n}^{\infty} \left(\frac{\varepsilon_m}{\delta_m}\right)^p$$

and, for hypothesis (3.12) , we get $\mathrm{Cap}_p(B) = 0$ and $(\Gamma \setminus B) \subset \liminf_{n\to\infty} \Gamma_n$. From statement (i) we get (3.13) ∎

Remark 3.2. For the capacity we do not have an analogy of property (3.4). While, as we have proved, $\lim_{n\to\infty} \mathrm{Cap}_p(\Gamma \setminus \Gamma_n) = 0$ in general, it is *not* true that $\lim_{n\to\infty} \mathrm{Cap}_p(\Gamma_n \setminus \Gamma) = 0$ as it can been easy seen taking $f_n \equiv f$.

Taking into account the relation between capacity and Hausdorff measure (see [2, 3]), from the previous proposition we get the following result about convergence in the sense of the Hausdorff measure.

Corollary 3.1. *Under the same hypothesis of Proposition* 3.2*, we have the following:*

(i) *If* $\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = 0$*, then for any* $\sigma > 0$

$$\mathcal{H}^{N-p+\sigma}\left(\limsup_{n\to\infty} \Gamma_n \triangle \Gamma \right) = 0. \tag{3.24}$$

(ii) *If* $\sum_n \left(\frac{\varepsilon_n}{\delta_n}\right)^p < \infty$*, then we also have, for any* $\sigma > 0$*,*

$$\mathcal{H}^{N-p+\sigma}\left(\Gamma \triangle \liminf_{n\to\infty} \Gamma_n \right) = 0. \tag{3.25}$$

If $p = 1$*, then* (3.24) – (3.25) *hold also for* $\sigma = 0$*.*

If $p > N$, since $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ with continuous immersion, we can apply the results of Section 2 to the continuous representatives of f and f_n. Therefore, from the convergence of f_n to f we get the convergence in the set theoretical sense of Γ_n to Γ.

Acknowledgement. The author wishes to thank Prof. C.Baiocchi of the University of Roma “La Sapienza” for suggesting the problem.

References

[1] Baiocchi, C. and G. A. Pozzi: *Error estimates and free-boundary convergence for a finite difference discretization of a parabolic variational inequality.* RAIRO Anal. Numér. 11 (1977), 315 – 353.

[2] Evans, L. C. and R. F. Gariepy: *Measure Theory and Fine p Properties of Functions.* Boca Raton: CRC Press 1992.

[3] Federer, H. and W. Ziemer: *The Lebesgue set of a function whose distribution derivatives are p-th power summable.* Indiana Univ. Math. J. 22 (1972), 139 – 158.

[4] Ziemer, W.: *Weakly Differentiable Functions.* New York: Springer-Verlag 1989.

Received 22.03.1998