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ISOMORPHISM BETWEEN THE ALGEBRA OF MEASURABLE FUNCTIONS
AND ITS SUBALGEBRA OF APPROXIMATELY DIFFERENTIABLE FUNCTIONS[#]

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Abstract. The present paper is devoted to study of certain classes of homogeneous regular subalgebras of the algebra of all complex-valued measurable functions on the unit interval. It is known that the transcendence degree of a commutative unital regular algebra is one of the important invariants of such algebras together with Boolean algebra of its idempotents. It is also known that if (Ω, Σ, μ) is a Maharam homogeneous measure space, then two homogeneous unital regular subalgebras of $S(\Omega)$ are isomorphic if and only if their Boolean algebras of idempotents are isomorphic and transcendence degrees of these algebras coincide. Let $S(0, 1)$ be the algebra of all (classes of equivalence) measurable complex-valued functions and let $AD^{(n)}(0, 1)$ ($n \in \mathbb{N} \cup \{\infty\}$) be the algebra of all (classes of equivalence of) almost everywhere n -times approximately differentiable functions on $[0, 1]$. We prove that $AD^{(n)}(0, 1)$ is a regular, integrally closed, ρ -closed, c -homogeneous subalgebra in $S(0, 1)$ for all $n \in \mathbb{N} \cup \{\infty\}$, where c is the continuum. Further we show that the algebras $S(0, 1)$ and $AD^{(n)}(0, 1)$ are isomorphic for all $n \in \mathbb{N} \cup \{\infty\}$. As an application of these results we obtain that the dimension of the linear space of all derivations on $S(0, 1)$ and the order of the group of all band preserving automorphisms of $S(0, 1)$ coincide and are equal to 2^c . Finally, we show that the Lie algebra $\text{Der } S(0, 1)$ of all derivations on $S(0, 1)$ contains a subalgebra isomorphic to the infinite dimensional Witt algebra.

Keywords: regular algebra, algebra of measurable functions, isomorphism, band preserving isomorphism.

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1. Introduction

In his pioneering papers [1–3] J. von Neumann built the theory on the correspondence between complemented orthomodular lattices and regular rings and proved that if given two $*$ -regular rings with orders $n \geq 3$ (which means that it contains a ring of matrices of order n),

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then any lattice isomorphism of their lattices of projections can be uniquely extended as a ring isomorphism of the whole ring. One of important classes of $*$ -regular rings are the $*$ -algebra of operators affiliated with a finite von Neumann algebra. Let M be a finite von Neumann algebra and let $S(M)$ be the $*$ -algebra of all measurable operators affiliated with M . In particular, in the case of type II_1 von Neumann algebras the above mentioned result of J. von Neumann means that any ring isomorphism of $S(M)$ completely determined by its action on the lattice of projections.

In [4] (see also [5]) it was proved that for the type II_1 von Neumann algebras M and N any ring isomorphism from $S(M)$ onto $S(N)$ is a real algebra isomorphism which is continuous in the locally measure topology. Moreover, there exist a real $*$ -isomorphism $\Psi : S(M) \rightarrow S(N)$ and an invertible element $a \in S(N)$ such that $\Phi(x) = a\Psi(x)a^{-1}$ for all $x \in S(M)$. Earlier M. Mori [6] characterized lattice isomorphisms between projection lattices $P(M)$ and $P(N)$ of von Neumann algebras M and N , respectively, by means of ring isomorphisms between the algebras of locally measurable operators $LS(M)$ and $LS(N)$ when M and N are von Neumann algebras of type I_∞ or III .

At the same time, the structure of isomorphisms on regular algebras in the abelian case is completely different from the non commutative one. It is well-known that if a von Neumann algebra M is abelian, then it is $*$ -isomorphic to the algebra $L_\infty(\Omega)$ of all (classes of equivalence of) essentially bounded measurable complex-valued functions on a measure space (Ω, Σ, μ) and therefore, $S(M) \cong S(\Omega)$ is the algebra of all measurable complex functions on Ω . In 2006 A. G. Kusraev [7] proved that $S(\Omega)$ admits discontinuous in the measure topology automorphisms which identically act on the Boolean algebra $\nabla(S(\Omega))$ of all idempotents of $S(\Omega)$ if and only if $\nabla(S(\Omega))$ is non atomic. In [8] we have introduced a new notion of transcendence degree of a commutative unital regular algebra. The transcendence degree of a commutative unital regular algebra is one of the important invariants of such algebras together with Boolean algebra of its idempotents. Namely, we have proved that if (Ω, Σ, μ) is a Maharam homogeneous measure space, then two homogeneous unital regular subalgebras of $S(\Omega)$ are isomorphic if and only if their Boolean algebras of idempotents are isomorphic and their transcendence degrees coincide. In the present paper we shall study some certain classes homogeneous subalgebras of the algebra of all measurable functions on the unit interval.

The paper is organised as follows.

In Section 2 we give some preliminaries from the theory of regular algebras and formulate open problems concerning the structure of homogeneous commutative regular algebras.

In Section 3 we shall consider the algebra $AD^{(n)}(0,1)$ of all classes of complex-valued measurable functions which consists of an almost everywhere n -times approximately differentiable functions on $[0,1]$. We prove that $AD^{(n)}(0,1)$ is a regular, integrally closed, ρ -closed, c -homogeneous subalgebra in $S(0,1)$ for all $n \in \mathbb{N} \cup \{\infty\}$, where c is the continuum (Theorem 3.1). Further we show that the algebras $S(0,1)$ and $AD^{(n)}(0,1)$ are isomorphic for all $n \in \mathbb{N} \cup \{\infty\}$ (Corollary 3.1).

In Section 4 we prove that the dimension of the linear space of all derivations on $S(0,1)$ and the order of the group of all band preserving automorphisms of $S(0,1)$ both equal 2^c (see Theorem 4.1 and 4.2).

2. Regular Algebras

In the present section we give some preliminaries from the theory of regular algebras and formulate open problems concerning the structure of homogeneous commutative regular algebras. More detailed info concerning regular algebras can be found in [3, 9, 10].

Recall that an algebra \mathcal{A} is said to be regular (in the sense of von Neumann) if, for any $a \in \mathcal{A}$ there is an element $x \in \mathcal{A}$ such that $axa = a$.

Let \mathcal{A} be a commutative unital regular algebra with a unity $\mathbf{1}$ over a field \mathbb{F} and let $\nabla = \nabla(\mathcal{A})$ be the set of all idempotents \mathcal{A} . On ∇ a partial order is defined as follows: $e \leq f$ if and only if $ef = e$. Then ∇ becomes a Boolean algebra with the greatest element $\mathbf{1}$, where $\mathbf{1} - e$ is the complement of an element e . For $a \in \mathcal{A}$, let $i(a)$ be the unique solution of the equations $axa = a$ and $xax = x$ [11, Theorem 1.17]. The element $s(a) = ai(a) \in \nabla$ is called the support of a [4, 11].

Let μ be a finite strictly positive countable-additive measure on ∇ . Define the function $\rho : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R}$, by setting [12], [13, p. 898]:

$$\rho(x, y) = \mu(s(x - y)), \quad x, y \in \mathcal{A}. \quad (1)$$

Then \mathcal{A} is a topological ring in the metric topology given by ρ (see [12, Proposition 2.6]).

Below we shall assume that \mathcal{A} is a commutative unital regular algebra over an algebraically closed field \mathbb{F} of characteristic zero, μ is a finite strictly positive countable-additive measure on ∇ and \mathcal{A} is complete with respect to the metric ρ defined as in (1).

Let \mathcal{B} be a regular subalgebra of \mathcal{A} and let $\mathcal{B}[x]$ denote the algebra of all polynomials with coefficients from subalgebra \mathcal{B} . An element $a \in \mathcal{A}$ is said to be integral with respect to \mathcal{B} , if there exists a unitary polynomial $p \in \mathcal{B}[x]$, such that $p(a) = 0$. The integral closure of \mathcal{B} is the set $\mathcal{B}^{(i)}$ of all elements that are integral with respect to \mathcal{B} . A subalgebra \mathcal{B} is said to be integrally closed, if $\mathcal{B}^{(i)} = \mathcal{B}$ (see [12]).

Recall that $x, y \in \mathcal{A}$ differ at $e \in \nabla$ provided that from $fx = fy$, it follows that $ef = 0$ for all $f \in \nabla$. Clearly, this condition is equivalent to $e \leq s(x - y)$. A subset \mathcal{S} of \mathcal{A} is said to be locally linearly independent, if for an arbitrary nonzero $e \in \nabla$ and each family of elements $x_1, \dots, x_n \in \mathcal{S}$ that differ pairwise at e and $\lambda_1, \dots, \lambda_n \in \mathbb{F}$, the condition $e(\lambda_1 x_1 + \dots + \lambda_n x_n) = 0$ implies that $\lambda_1 = \dots = \lambda_n = 0$. For $\mathcal{S} \subset \mathcal{A}$ denote by $\langle \mathcal{S} \rangle$ the set of elements of the form $x_1^{n_1} \dots x_k^{n_k}$, where $x_1, \dots, x_k \in \mathcal{S}$ and $k, n_1, \dots, n_k \in \mathbb{N}$. A set $\mathcal{S} \subset \mathcal{A}$ is called locally algebraically independent if $\langle \mathcal{S} \rangle$ is locally linearly independent. A locally algebraically independent subset \mathcal{S} is said to be maximal if it is not a proper subset of a locally algebraically independent set. By the Zorn's Lemma, every locally algebraically independent set is a subset of a maximal locally algebraically independent subset [14].

Let \mathcal{S} be a non empty subset in \mathcal{A} . Denote by $\mathcal{A}(\mathcal{S})$ the least regular, integrally closed, ρ -closed subalgebra of \mathcal{A} generated by \mathcal{S} and ∇ . A subset \mathcal{S} in \mathcal{A} is called faithful, if $s(x) = \mathbf{1}$ for all $x \in \mathcal{S}$. A commutative unital regular algebra \mathcal{A} is called homogeneous, if there is a faithful locally algebraically independent subset \mathcal{M} such that $\mathcal{A}(\mathcal{M}) = \mathcal{A}$. Note that the cardinality of any algebraic independent system \mathcal{M} in \mathcal{A} is not greater than the dimension of the algebra \mathcal{A} , that is, $|\mathcal{M}| \leq \dim \mathcal{A}$. The largest cardinal number γ such that there exists a faithful maximal locally algebraically independent subset \mathcal{M} in \mathcal{A} with the cardinality γ and $\mathcal{A}(\mathcal{M}) = \mathcal{A}$ is called the transcendence degree of \mathcal{A} and is denoted as $\text{trdeg}(\mathcal{A}) = \gamma$. In this case we say that \mathcal{A} is γ -homogeneous. In other words, \mathcal{A} is γ -homogeneous, if there is a faithful locally algebraically independent subset \mathcal{M} with the cardinality γ such that $\mathcal{A}(\mathcal{M}) = \mathcal{A}$ and any faithful system in \mathcal{A} of cardinality bigger than γ is algebraically dependent.

Recall that a weight of the Boolean algebra ∇ is the least cardinality of sets which generate ∇ , i.e.,

$$\tau(\nabla) = \inf \{ |X| : X \text{ generates } \nabla \}.$$

For $0 \neq e \in \nabla$ set $\nabla_e = \{x \in \nabla : x \leq e\}$. A Boolean algebra ∇ is said to be homogeneous, if $\tau(\nabla_e) = \tau(\nabla)$ for any non zero $e \in \nabla$ [13, 15, 16].

Let (Ω, Σ, μ) be a measure space with a finite strictly positive countable-additive measure μ and let $S(\Omega) = S(\Omega, \Sigma, \mu)$ be the algebra of all (classes of) \mathbb{F} -valued measurable functions on (Ω, Σ, μ) . It is clear that $S(\Omega)$ is a commutative unital regular algebra. A rank metric ρ on $S(\Omega)$ defined as in (1):

$$\rho(x, y) = \mu(\{\omega \in \Omega : x(\omega) \neq y(\omega)\}), \quad x, y \in S(\Omega).$$

Recall that a measure space (Ω, Σ, μ) is called Maharam homogeneous if the Boolean algebra ∇ of the algebra $S(\Omega)$ is homogeneous [13, Page 908]. An automorphism $\phi : \nabla \rightarrow \nabla$ is called measure-preserving, if $\mu(e) = \mu(\phi(e))$ for all $e \in \nabla$. Every measure-preserving automorphism of ∇ can be extended to an automorphism of $S(\Omega)$, which we also denote by ϕ [13, p. 907].

Let $G = \{\phi\}$ be a group of automorphisms of ∇ . The group G is called ergodic, if for every $0 \neq e \in \nabla$ the following equality holds: $\bigvee_{\phi \in G} \phi(e) = \mathbf{1}$.

In [8, Proposition 4.4] the following criteria of homogeneity of regular subalgebras was obtained.

Proposition 2.1. *Let (Ω, Σ, μ) be a Maharam homogeneous measure space with a finite strictly positive countable-additive measure μ and let \mathcal{A} be an integrally closed and ρ -closed regular subalgebra in $S(\Omega)$. Suppose that G is a measure-preserving ergodic group of automorphisms on ∇ such that $\phi(\mathcal{A}) = \mathcal{A}$ for all $\phi \in G$. Then \mathcal{A} is homogeneous.*

From the above it follows that if (Ω, Σ, μ) is a Maharam homogeneous measure space, then $S(\Omega)$ is a homogeneous commutative regular algebra. In particular, $S(0, 1)$ is a homogeneous commutative regular algebra (see [8]).

At the end of this Section we formulate open problems concerning homogeneous commutative regular algebras.

PROBLEM 2.1: Is any commutative regular algebra \mathcal{A} a direct sum of homogeneous commutative regular algebras?

PROBLEM 2.2: Does cardinal number of any maximal faithful locally algebraically independent subset of a homogeneous commutative regular algebra \mathcal{A} coincide with $\text{trdeg}(\mathcal{A})$?

PROBLEM 2.3: Let (Ω, Σ, μ) be a Maharam homogeneous measure space with a finite strictly positive countable-additive measure μ . Find a connection between the transcendence degree of the algebra $S(\Omega)$ and the weight of the Boolean algebra $\nabla(S(\Omega))$.

In the next Section we shall give a solution of the last problem in the case of separable Maharam homogeneous measure space.

3. Algebra of Approximately Differentiable Functions

Let $E \subset \mathbb{R}$ be a Lebesgue measurable set, $f : E \rightarrow \mathbb{C}$ be a measurable function. Let $t_0 \in \mathbb{R}$ be a point in which E has density 1. Recall that a number ℓ is called the approximate limit of f at t_0 if the set $\{t \in E : |f(t) - \ell| < \varepsilon\}$ has density one at t_0 for each $\varepsilon > 0$, and ℓ is denoted by $\text{ap} - \lim_{t \rightarrow t_0} f(t)$. If the approximate limit

$$f'_{\text{ap}}(t_0) := \text{ap} - \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

exists and it is finite, then it is called approximate derivative of the function f at t_0 and the function f is called approximately differentiable at t_0 [17]. Further, for the sake of convenience the approximate derivative f'_{ap} we denote as \dot{f} .

Let $AD(0, 1)$ be the set of all classes of complex-valued measurable functions which consists of an almost everywhere approximately differentiable functions on $[0, 1]$.

Note that by [18, Proposition 4.7 and Corollary 4.11] the algebra $AD(0, 1)$ is a regular, integrally closed and ρ -closed proper subalgebra in $S(0, 1)$. Moreover,

$$\nabla(S(0, 1)) = \nabla(AD(0, 1)). \quad (2)$$

Further, the algebra $AD(0, 1)$ is homogeneous, and the algebras $S(0, 1)$ and $AD(0, 1)$ are isomorphic [8, Theorem 4.8].

First we shall give a solution of Problem 2.3 in the case of separable Maharam homogeneous measure space. In particular, we obtain the following connection between the transcendence degree of the algebra $S(0, 1)$ and the weight of the Boolean algebra $\nabla(S(0, 1))$:

$$\text{trdeg } S(0, 1) = c = 2^{\aleph_0} = 2^{\tau(\nabla(S(0, 1)))}.$$

Proposition 3.1. *The transcendence degrees of the algebra of complex-valued measurable functions and the algebra of approximately differentiable functions on the unit interval coincide and are equal to continuum, i. e.*

$$\text{trdeg } S(0, 1) = \text{trdeg } AD(0, 1) = c,$$

where c is the continuum.

◁ Let $\{d_i : i \in I\}$ be a Hamel basis for the field of real numbers over the field of rational numbers. For each $i \in I$ define a function $x_i \in S(0, 1)$ as follows

$$x_i(t) = t^{|d_i|}, \quad t \in [0, 1]. \quad (3)$$

Then $\{x_i : i \in I\}$ is a faithful locally algebraically independent subset of $S(0, 1)$ of continuum cardinality [19]. Since $\{x_i : i \in I\} \subset AD(0, 1)$, it follows that $\text{trdeg } AD(0, 1) \geq c$.

Note that each $x \in C[0, 1]$ is uniquely determined by its restriction on $\mathbb{Q} \cap [0, 1]$. Hence,

$$|C[0, 1]| \leq |\mathbb{C}^{\mathbb{Q} \cap [0, 1]}| = c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = c. \quad (4)$$

Note that by Lusin's theorem, it follows that

$$S(0, 1) = \overline{C[0, 1]}^\rho. \quad (5)$$

Further, from (4) and (5), we can conclude that

$$|S(0, 1)| = c. \quad (6)$$

Indeed, let us embed the space $S(0, 1)$ into the space $C[0, 1]^{\aleph_0}$ as follows: for any $x \in S(0, 1)$ take a sequence $\{x_n\} \subset C[0, 1]$ such that $x_n \xrightarrow{\rho} x$ and consider the mapping

$$x \in S(0, 1) \longmapsto \{x_n\} \in C[0, 1]^{\aleph_0}.$$

Note that we identify $C[0, 1]$ with classes of measurable functions that contain continuous functions. It is clear that this mapping is injective. Therefore

$$c \leq |S(0, 1)| \leq |C[0, 1]^{\aleph_0}| = c^{\aleph_0} = c.$$

So,

$$c \leq \text{trdeg } AD(0, 1) \leq \text{trdeg } S(0, 1) \leq |S(0, 1)| \leq c,$$

and therefore $\text{trdeg } S(0, 1) = \text{trdeg } AD(0, 1) = c$. The proof is completed. \triangleright

REMARK 3.1. In the previous Proposition we have show that $S(0, 1)$ is c -homogeneous. In the proof of Proposition 3.1 we have constructed continual faithful algebraically independent system in $S(0, 1)$, and by Zorn's Lemma this system can be completed to a maximal faithful algebraically independent system. But in general case, to point a concrete maximal continual faithful maximal algebraically independent system in $S(0, 1)$ is a non trivial problem.

Let $AD^{(n)}(0, 1)$ be the set of all classes of complex-valued measurable functions which consists of an almost everywhere n -times approximately differentiable functions on $[0, 1]$ and let

$$AD^\infty(0, 1) = \bigcap_{n \geq 1} AD^{(n)}(0, 1).$$

Theorem 3.1. *The algebra $AD^{(n)}(0, 1)$ is a regular, integrally closed, ρ -closed and c -homogeneous subalgebra in $S(0, 1)$ for all $n \in \mathbb{N} \cup \{\infty\}$.*

\triangleleft First we shall prove for natural n .

The proof is by induction on n . For $n = 1$ all properties except homogeneity have shown in [8], and c -homogeneity already proved in Proposition 3.1. Assume that we have proved for $n - 1$.

Let us first show that $AD^{(n)}(0, 1)$ is regular. Let f be a almost everywhere n -times approximately differentiable function on $[0, 1]$, that is, $[f] \in AD(0, 1)$. Then the function g on $[0, 1]$ defined as

$$g(t) = \begin{cases} \frac{1}{f(t)}, & \text{if } f(t) \neq 0, \\ 0, & \text{if } f(t) = 0 \end{cases}$$

is also n -times approximately differentiable almost everywhere in $[0, 1]$. Hence, $[g] \in AD(0, 1)$ and $f^2g = f$. This means that $AD(0, 1)$ is regular.

Let $x \in S(0, 1)$ be an integral element with respect $AD^{(n)}(0, 1)$. By the induction assumption $AD^{(n-1)}(0, 1)$ is integrally closed and therefore $x \in AD^{(n-1)}(0, 1)$. By [12, Proposition 3.3], there exist a partition $\{e_1, \dots, e_k\}$ of $s(x)$ and natural numbers $n_1 < n_2 < \dots < n_k$ such that for every $i = 1, \dots, k$ the minimal degree of a unitary polynomial g_i over $AD^{(n-1)}(0, 1)$ for which $g_i(e_i x) = 0$ is n_i . Let $i \in \{1, \dots, k\}$ be a fixed index and let

$$(e_i x)^{n_i} + a_{n_i-1}(e_i x)^{n_i-1} + \dots + a_1(e_i x) + a_0 = 0,$$

where $a_0, \dots, a_{n_i-1} \in e_i AD^{(n)}(0, 1)$. Let $i \in \{1, \dots, k\}$ be a fixed index. Taking the approximate derivative in the last equality we obtain that

$$\dot{x}p + q = 0,$$

where

$$p = n_i(e_i x)^{n_i-1} + \sum_{j=1}^{n_i-1} a_j j (e_i x)^{j-1} \quad \text{and} \quad q = \sum_{j=0}^{n_i-1} \dot{a}_j (e_i x)^j.$$

Note that $p, q \in AD^{(n-1)}(0, 1)$, and therefore $e_i \dot{x} = -qi(p)$ belongs to $AD^{(n-1)}(0, 1)$. Thus $e_i x \in AD^{(n)}(0, 1)$ for all $i = 1, \dots, k$, hence $x \in AD^{(n)}(0, 1)$.

Let us show that $AD^{(n)}(0, 1)$ is ρ -closed. Take $\{x_k\} \subset AD^{(n)}(0, 1)$ such that $x_k \xrightarrow{\rho} x \in S(0, 1)$. By the induction assumption $AD^{(n-1)}(0, 1)$ is ρ -closed and therefore $x \in$

$AD^{(n-1)}(0, 1)$. By [18, Proposition 4.12], the approximate differentiation operator $\partial_{AD} : AD(0, 1) \rightarrow S(0, 1)$ is non-expansive, that is,

$$\rho(\partial_{AD}(a), \partial_{AD}(b)) \leq \rho(a, b) \quad (7)$$

for all $a, b \in AD(0, 1)$. Using (7) we have that $\partial_{AD}(x_k) \xrightarrow{\rho} \partial_{AD}(x)$. Since $AD^{(n-1)}(0, 1)$ is ρ -closed and $\{\partial_{AD}(x_k)\} \subset AD^{(n-1)}(0, 1)$, it follows that $\partial_{AD}(x) \in AD^{(n-1)}(0, 1)$. Thus $x \in AD^{(n)}(0, 1)$, and therefore $AD^{(n)}(0, 1)$ is ρ -closed.

Since $AD^{(n)}(0, 1)$ is invariant under rational translation modulo 1, by Proposition 2.1, it follows that it is homogeneous.

Let $\{x_i : i \in I\}$ be a locally algebraically independent subset in $S(0, 1)$ as in (3). Since $\{x_i : i \in I\}$ is a subset of $AD^{(n)}(0, 1)$, it follows that

$$c \leq \text{trdeg } AD^{(n)}(0, 1) \leq \text{trdeg } S(0, 1) = c,$$

and therefore $\text{trdeg } AD^{(n)}(0, 1) = c$.

Now we shall consider the case $n = \infty$.

Let $x \in AD^\infty(0, 1)$ be a non zero element. Then its partial inverse $i(x) \in AD^{(n)}(0, 1)$ for all $n \in \mathbb{N}$. Thus $i(x) \in AD^\infty(0, 1)$ and $i(x)x^2 = x$. Hence $AD^\infty(0, 1)$ is regular. Further, since $AD^{(n)}(0, 1)$ is ρ -closed for all n , it follows that $AD^\infty(0, 1)$ is also ρ -closed.

Let $x \in S(0, 1)$ be an integral element with respect to $AD^\infty(0, 1)$. Then x is an integral element with respect to $AD^{(n)}(0, 1)$ for all $n \geq 1$. Since $AD^{(n)}(0, 1)$ is integrally closed, it follows that $x \in AD^{(n)}(0, 1)$ for all $n \geq 1$. Thus $x \in AD^\infty(0, 1)$.

Again using invariancy of $AD^\infty(0, 1)$ under rational translation modulo 1, we obtain that it is homogeneous. Further, let $x_i(t) = t^{|d_i|}$ as in (3). Since $\{x_i\}_{i \in I} \subset AD^\infty(0, 1)$, it follows that $\text{trdeg } AD^\infty(0, 1) = c$. The proof is completed. \triangleright

Corollary 3.1. *The algebras $S(0, 1)$ and $AD^{(n)}(0, 1)$ are isomorphic for all $n \in \mathbb{N} \cup \{\infty\}$.*

\triangleleft By (2), the Boolean algebras $\nabla(S(0, 1))$ and $\nabla(AD(0, 1))$ coincide. Since $\dot{e} = 0$ for all $e \in \nabla(S(0, 1))$, it follows that $\nabla(S(0, 1)) \equiv \nabla(AD^{(n)}(0, 1))$ for all $n \in \mathbb{N} \cup \{\infty\}$. By Theorem 3.1, we have that $\text{trdeg } S(0, 1) = \text{trdeg } AD^{(n)}(0, 1) = c$ for all $n \in \mathbb{N} \cup \{\infty\}$. Therefore by [8, Theorem 4.6], we obtain that the algebras $S(0, 1)$ and $AD^{(n)}(0, 1)$ are isomorphic. The proof is completed. \triangleright

We have mentioned that by Lusin's theorem, it follows that $S(0, 1) = \overline{C[0, 1]}^\rho$ (see (5)). On the other hand, by Whitney's theorem [20, Theorem 1] (see also [17, Theorem 3.1.16]), any approximately differentiable function on $[0; 1]$ is continuously differentiable outside of a closed subset of arbitrarily small measure, it follows that

$$AD(0, 1) = \overline{C^{(1)}[0, 1]}^\rho.$$

At the same time for $n \geq 2$ we have the following result.

Proposition 3.2. *Let $n \geq 2$. Then*

$$\overline{C^{(n)}[0, 1]}^\rho \subsetneq AD^{(n)}(0, 1),$$

where $C^{(n)}[0, 1]$ is the algebra of all n -times continuously differentiable complex-valued functions on $[0, 1]$.

\triangleleft By [21] for any $\varepsilon \in (0, 1)$ there are a closed subset $A \subset [0, 1]$ and a monotone non negative function $f \in C^{(1)}[0, 1]$ such that

- (a) $m(A) = 1 - \varepsilon$;

(b) $\dot{f}|_A \equiv 0$;

(c) $\{t \in A : f(t) = g(t)\}$ is a finite set for any $g \in C^{(2)}[0, 1]$.

Consider the function $h = \chi_A f$. Then $\dot{h} = 0$, hence $h \in AD^{(n)}[0, 1]$ for all $n \geq 1$. From properties (a) and (c) for any $g \in C^{(n)}[0, 1]$ ($n \geq 2$) we have that

$$\rho(h, g) = \mu(\{t \in [0, 1] : h(t) \neq g(t)\}) \geq \mu(\{t \in A : f(t) \neq g(t)\}) = 1 - \varepsilon.$$

Thus $h \notin \overline{C^{(n)}[0, 1]^p}$ for all $n \geq 2$. The proof is completed. \triangleright

Similar to the proof of Theorem 3.1 we obtain the following result.

Theorem 3.2. *The algebra $\overline{C^{(n)}[0, 1]^p}$ is a regular, integrally closed, ρ -closed and c -homogeneous subalgebra in $S(0, 1)$ for all $n \in \mathbb{N} \cup \{\infty\}$.*

4. The Space of All Derivations and Group of Band Preserving Automorphisms of Algebras of Measurable Functions

4.1. The dimension of the space of all derivations on algebras of measurable functions. Recall that a linear mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$.

Let $\partial_{AD} : AD(0, 1) \rightarrow S(0, 1)$ be the approximate differentiation operator. It is clear that ∂_{AD} maps $AD^\infty(0, 1)$ into itself. Let $\alpha : S(0, 1) \rightarrow AD^\infty(0, 1)$ be an isomorphism (see Corollary 3.1). Setting

$$\partial_{AD}^{(\alpha)}(x) = \alpha^{-1}(\partial_{AD}(\alpha(x))), \quad x \in S(0, 1),$$

we obtain a non zero derivation on $S(0, 1)$.

Recall that the existence of non zero derivations on $S(0, 1)$ were obtained independently by A. F. Ber, F. A. Sukochev, V. I. Chilin [12] and A. G. Kusraev [7]. In [19, Proposition 6] it was shown that the dimension of the linear space of all derivations on the algebra $S(0, 1)$ is uncountable.

For the proof of the next result we need the following Erdős–Kaplansky Theorem [22, p. 258]. Let V be an infinite dimensional linear space over a field \mathbb{F} and let V^* be its dual space. Then

$$\dim V^* = |\mathbb{F}|^{\dim V}. \quad (8)$$

Proposition 3.1 allows us to determine the dimension of all derivations on $S(0, 1)$. Namely, we shall prove the following result.

Teopema 4.1. *Let $\text{Der } S(0, 1)$ be the linear space of all derivations of $S(0, 1)$. Then*

$$\dim \text{Der } S(0, 1) = 2^c.$$

\triangleleft Let \mathcal{M} be a maximal locally algebraically independent subset in $S(0, 1)$. Denote by $\mathbb{F}(\mathcal{M}, S(0, 1))$ the set of all mappings $f : \mathcal{M} \rightarrow S(0, 1)$ such that

$$s(f(x)) \leq s(x) \quad (9)$$

for all $x \in \mathcal{M}$. Then the mapping

$$\delta \in \text{Der } S(0, 1) \mapsto \delta|_{\mathcal{M}} \in \mathbb{F}(\mathcal{M}, S(0, 1)) \quad (10)$$

defines a linear isomorphism between $\text{Der } S(0, 1)$ and $\mathbb{F}(\mathcal{M}, S(0, 1))$ (see [19, Theorem 1]).

By Proposition 3.1 the algebra $S(0, 1)$ is c -homogeneous and therefore we can choose a faithful maximal locally algebraically independent subset \mathcal{M} of $S(0, 1)$ of the continuum cardinality. In this case, since $s(x) = \mathbf{1}$ for all $x \in \mathcal{M}$, it follows that the linear space $\mathbb{F}(\mathcal{M}, S(0, 1))$ coincide with the linear space $S(0, 1)^{\mathcal{M}}$ of all mappings from \mathcal{M} into $S(0, 1)$. Further, using (6) we obtain that

$$\dim \text{Der } S(0, 1) = \dim S(0, 1)^{\mathcal{M}} \leq |S(0, 1)|^{\text{trdeg } S(0, 1)} = c^c = (2^{\aleph_0})^c = 2^{\aleph_0 \cdot c} = 2^c.$$

It is clear that the mapping

$$f \in (\text{span } \mathcal{M})^* \longmapsto \tilde{f} \in S(0, 1)^{\mathcal{M}}$$

defined as follows

$$\tilde{f}(x) = f(x)\mathbf{1}, \quad x \in \mathcal{M}$$

is injective. Therefore $\dim S(0, 1)^{\mathcal{M}} \geq \dim (\text{span } \mathcal{M})^*$. Finally,

$$\dim \text{Der } S(0, 1) = \dim S(0, 1)^{\mathcal{M}} \geq \dim (\text{span } \mathcal{M})^* \stackrel{(8)}{=} |\mathbb{C}|^{|\mathcal{M}|} = c^c = 2^c.$$

The proof is completed. \triangleright

EXAMPLE 1. It is well-known that for an arbitrary algebra \mathcal{A} the linear space $\text{Der}(\mathcal{A})$ equipped with the following Lie bracket

$$[D_1, D_2] = D_1 D_2 - D_2 D_1, \quad D_1, D_2 \in \text{Der}(\mathcal{A})$$

becomes a Lie algebra.

We shall show that the Lie algebra $\text{Der } S(0, 1)$ contains a subalgebra isomorphic to the infinite dimensional Witt algebra.

Let \mathcal{M} be a faithful maximal locally algebraically independent subset of $S(0, 1)$ and let $\xi \in \mathcal{M}$ be a fixed element. For any $n \in \mathbb{Z}$ set

$$\delta_n(\zeta) = \begin{cases} \xi^{n+1}, & \text{if } \zeta = \xi, \\ 0, & \text{if } \zeta \neq \xi. \end{cases}$$

Since $s(\delta_n(\zeta)) \leq s(\zeta)$ for all $\zeta \in \mathcal{M}$, it follows that $\delta_n \in \mathbb{F}(\mathcal{M}, S(0, 1))$ for all $n \in \mathbb{Z}$ (see (9)). By (10), for each $n \in \mathbb{Z}$ there is a unique derivation $d_n \in \text{Der } S(0, 1)$ such that $d_n|_{\mathcal{M}} = \delta_n$. Since d_n is a derivation, using Leibniz rule, we obtain that $d_n(\xi^k) = k\xi^{n+k}$ for all $n, k \in \mathbb{Z}$. Taking into account the latter observation, we have that

$$\begin{aligned} [d_n, d_m](\xi) &= d_n(d_m(\xi)) - d_m(d_n(\xi)) = d_n(\xi^{m+1}) - d_m(\xi^{n+1}) \\ &= (m+1)\xi^{n+m+1} - (n+1)\xi^{n+m+1} \\ &= (m-n)\xi^{n+m+1} = (m-n)d_{n+m}(\xi) \end{aligned}$$

and

$$[d_n, d_m](\zeta) = 0, \quad \zeta \neq \xi$$

for all $n, m \in \mathbb{Z}$. Thus $[d_n, d_m]|_{\mathcal{M}} = (m-n)d_{n+m}|_{\mathcal{M}}$, and hence according to (10),

$$[d_n, d_m] = (m-n)d_{n+m} \tag{11}$$

for all $n, m \in \mathbb{Z}$. The relation (11) shows that the subspace $W = \text{span}\{d_n : n \in \mathbb{Z}\}$ is a Lie subalgebra of $\text{Der } S(0, 1)$ which is isomorphic to the infinite dimensional Witt algebra.

It should be noted that Witt algebras compose one of the four classes of Cartan type Lie algebras originally introduced in 1909 by E. Cartan [23] when he studied infinite dimensional simple Lie algebras over complex numbers.

Based on latter note we may formulate the following problem.

PROBLEM 4.1: Any finite or countable dimensional complex simple Lie algebra can be embedded into the Lie algebra $\text{Der } S(0, 1)$.

REMARK 4.1. Let us consider on $S(0, 1)$ a metric ρ_μ defined as [24]

$$\rho_\mu(x, y) = \inf \left\{ \varepsilon > 0 : \mu(\{t \in [0, 1] : |x(t) - y(t)| > \varepsilon\}) \leq \varepsilon \right\}.$$

Note that ρ_μ -convergence in $S(0, 1)$ coincide with the convergence in measure. Since the subspace of all simple functions $\mathbb{F}(\nabla)$ is ρ_μ -dense in $S(0, 1)$ and any derivation of $S(0, 1)$ vanishes on $\mathbb{F}(\nabla)$, it follows that any ρ_μ -continuous derivation on $S(0, 1)$ is identically zero. So, any non zero derivation of $S(0, 1)$ is ρ_μ -discontinuous.

Note that this phenomenon is a purely commutative effect. In the case of type II_1 von Neumann algebra M all derivations of the regular algebra $S(M)$ are inner, in particular, is continuous in the measure topology [25, 26].

4.2. The order of the group of all band preserving automorphisms of algebras of measurable functions. Recall that two elements $x, y \in \mathcal{A}$ are said to be orthogonal (notation $x \perp y$), if $s(x)s(y) = 0$. A linear mapping $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is said to be band preserving, if [7, 27]

$$x \perp y \implies \Phi(x) \perp y.$$

Since $x \perp \mathbf{1} - s(x)$, we see that Φ is band preserving if and only if $s(\Phi(x)) \leq s(x)$ for all $x \in \mathcal{A}$.

Let $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ be an automorphism. Then Φ is band preserving if and only if Φ acts identically on $\mathbb{F}_c(\nabla)$ [8, Proposition 3.3].

Assume that \mathcal{A} is homogeneous and \mathcal{M} is a faithful maximal locally algebraically independent subset of \mathcal{A} . Denote by $G(\mathcal{M}, \mathcal{A})$ the set of all mappings $g : \mathcal{M} \rightarrow \mathcal{A}$ such that $g(\mathcal{M}) = \{g(x) : x \in \mathcal{M}\}$ is also a faithful maximal locally algebraically independent subset of \mathcal{A} .

A permutation of \mathcal{M} is a bijection from \mathcal{M} onto \mathcal{M} . The group of all permutations of \mathcal{M} is denoted by $\text{Sym}(\mathcal{M})$. Let $\mathbb{F}_c(\nabla) = \overline{\text{span} \nabla^\rho}$. Then $\mathbb{F}_c(\nabla)$ is a regular subalgebra in \mathcal{A} (see [12]). Let $\mathbb{F}_c(\nabla)^\bullet$ be the group of all invertible elements from $\mathbb{F}_c(\nabla)$.

Denote by $\text{Aut}_\nabla(\mathcal{A})$ the group of all band preserving automorphisms of \mathcal{A} . We may view to the group $\text{Aut}_\nabla(\mathcal{A})$ as a Galois group of the ring extension \mathcal{A} over $\mathbb{F}_c(\nabla)$. There is an injective mapping

$$g \in G(\mathcal{M}, \mathcal{A}) \rightarrow \Phi_g \in \text{Aut}_\nabla(\mathcal{A}),$$

in particular, the group $\text{Aut}_\nabla(\mathcal{A})$ contains a subgroup isomorphic to the group $\text{Sym}(\mathcal{M}) \times (\mathbb{F}_c(\nabla)^\bullet)^{|\mathcal{M}|}$ (see [8, Theorem 4.2]).

Similarly, as in the proof of Theorem 4.1 we have that

$$|\text{Aut}_\nabla S(0, 1)| \leq |S(0, 1)^{S(0, 1)}| = c^c = 2^c.$$

On other hand, taking into that $\text{Aut}_\nabla S(0, 1)$ contains a subgroup isomorphic to $(\mathbb{F}_c(\nabla)^\bullet)^{|\mathcal{M}|}$, we obtain that

$$|\text{Aut}_\nabla S(0, 1)| \geq |\mathbb{F}_c(\nabla)^\bullet|^{|\mathcal{M}|} \geq c^{|\mathcal{M}|} = c^c = 2^c.$$

So, we have proved the following result.

Theorem 4.2. *The order of the group of all band-preserving automorphisms of the algebra of all complex-valued measurable functions on the unit interval is equal 2^c , i. e.*

$$|\text{Aut}_{\nabla} S(0, 1)| = 2^c.$$

REMARK 4.2. We note that A. E. Gutman, A. G. Kusraev, S. S. Kutateladze in [14, Comment 5.3.7], concerning the notion of algebraic independence in regular rings wrote: *This notion, presenting the external interpretation of the internal notion of algebraic independence (or transcendence), seems to turn out useful in studying the descents of fields or general regular rings.* As we can see, our results confirm the above prediction.

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ИЗОМОРФИЗМ МЕЖДУ АЛГЕБРОЙ ИЗМЕРИМЫХ ФУНКЦИЙ И ЕЕ ПОДАЛГЕБРОЙ АСИМПТОТИЧЕСКИ ДИФФЕРЕНЦИРУЕМЫХ ФУНКЦИЙ

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Аннотация. Настоящая работа посвящена изучению некоторых классов однородных регулярных подалгебр алгебры всех комплекснозначных измеримых функций на единичном интервале. Известно, что степень трансцендентности унитарной коммутативной регулярной алгебры является одним

из важных инвариантов таких алгебр наряду с булевой алгеброй ее идемпотентов. Также известно, что если (Ω, Σ, μ) — однородное пространство с мерой Магарам, то две однородные унитарные регулярные подалгебры в $S(\Omega)$ изоморфны тогда и только тогда, когда их булевы алгебры идемпотентов изоморфны, и степени трансцендентности этих алгебр совпадают. Пусть $S(0, 1)$ — алгебра всех (классов эквивалентности) измеримых комплекснозначных функций, и пусть $AD^{(n)}(0, 1)$ ($n \in \mathbb{N} \cup \{\infty\}$) — алгебра всех (классов эквивалентности) почти всюду n -раз асимптотически дифференцируемых функции на $[0, 1]$. В работе доказано, что $AD^{(n)}(0, 1)$ является регулярной, цело-замкнутой, ρ -замкнутой, c -однородной подалгеброй в $S(0, 1)$ для всех $n \in \mathbb{N} \cup \{\infty\}$, где c — континуум. Далее мы покажем, что алгебры $S(0, 1)$ и $AD^{(n)}(0, 1)$ изоморфны для всех $n \in \mathbb{N} \cup \{\infty\}$. В качестве приложения этих результатов установлено, что размерность линейного пространства всех дифференцирований на $S(0, 1)$ и порядок группы всех сохраняющих полосу автоморфизмов алгебры $S(0, 1)$ совпадают и равны 2^c . Мы также покажем, что алгебра Ли $\text{Der } S(0, 1)$, всех дифференцирований алгебры $S(0, 1)$, содержит подалгебру, изоморфную бесконечномерной алгебре Витта.

Ключевые слова: регулярная алгебра, алгебра измеримых функций, изоморфизм, сохраняющие полосу изоморфизм.

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