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INCREASING UNIONS OF STEIN SPACES WITH SINGULARITIES

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Abstract. We show that if X is a Stein space and, if $\Omega \subset X$ is exhaustable by a sequence $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots$ of open Stein subsets of X, then Ω is Stein. This generalizes a well-known result of Behnke and Stein which is obtained for $X = \mathbb{C}^n$ and solves the union problem, one of the most classical questions in Complex Analytic Geometry. When X has dimension 2, we prove that the same result follows if we assume only that $\Omega \subset\subset X$ is a domain of holomorphy in a Stein normal space. It is known, however, that if X is an arbitrary complex space which is exhaustable by an increasing sequence of open Stein subsets $X_1 \subset X_2 \subset \cdots \subset X_n \subset \ldots$, it does not follow in general that X is holomorphically-convex or holomorphically-separate (even if X has no singularities). One can even obtain 2-dimensional complex manifolds on which all holomorphic functions are constant.

Key words: Stein spaces, q-complete spaces, q-convex functions, strictly plurisubharmonic functions.

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1. Introduction

Let X be a Stein space and $D \subset X$ an open subset which is the union of an increasing sequence of Stein open subsets of X.

Does it follow that D is necessarily Stein?

It is known from a classical theorem due to Behnke and Stein [1] that if $D_1 \subset D_2 \subset \cdots \subset D_n \subset \ldots$ is an increasing sequence of Stein open sets in $\mathbb{C}^n$, then their union $\bigcup_{j \geqslant 1} D_j$ is Stein.

In 1977, Markoe [2] proved the following:

Let X be a reduced complex space which the union of an increasing sequence $X_1 \subset X_2 \subset \cdots \subset X_n \subset \ldots$ of Stein domains.

Then X is Stein if and only if the 1^{th} cohomology group of X with values in the structure sheaf $\mathscr{O}_X$ vanishes ($H^1(X, \mathscr{O}_X) = 0$).

Similarly, it is known (see [3]) that in an arbitrary complex space X an increasing union of Stein spaces $(X_n)_{n \geqslant 0}$ is itself Stein if $H^1(X, \mathscr{O}_X)$ is separated.

It has been proved earlier by Fornaess in [4], [5] and [6] that, if an additional condition is not imposed on $H^1(X, \mathscr{O}_X)$, the space X is not necessarily holomorphically-convex or holomorphically-separate.

It was shown in [7] that if $(D_j)_{j\geqslant 1}$ is an increasing sequence of Stein domains in a normal Stein space X, then $D = \bigcup_{j\geqslant 1} D_j$ is a domain of holomorphy (i. e. for each $x \in \partial D$ there is $f \in O(D)$ which is not holomorphically extendable through x).

It was proved in [8] that if X is a complex space and $(D_j)_{j\geqslant 1}$ is an increasing sequence of Stein open subsets of X, then $D = \bigcup D_j$ is 2-complete. We recall that a complex space X is said to be q-complete if there exists an exhaustion function $\phi \in C^\infty(X, \mathbb{R})$ which is q-convex on the whole space X, that is every point $x \in X$ has an open neighborhood U isomorphic to a closed analytic set in a domain $D \subset \mathbb{C}^n$ such that the restriction $\phi|_U$ has an extension $\tilde{\phi} \in C^\infty(D)$ whose Levi form $L(\tilde{\phi}, z)$ has at most $q-1$ negative or zero eingenvalues at any point z of D.

Here we solve affirmatively the above problem in the general case. We show that if X is a Stein space and, if Ω is an increasing sequence of Stein open subsets of X, then there exists an increasing sequence $(\Omega'_\nu)_{\nu\geqslant 1}$ of open subsets of Ω such that $\Omega = \bigcup_{\nu\geqslant 1} \Omega'_\nu$ and there are continuous strictly psh functions $\psi''_\nu : \Omega'_\nu \to]0, +\infty[$ with the following properties

(a) $\psi''_j > 2^{\nu+2}$ on $\Omega'_{\nu+2} \backslash \Omega'_{\nu+1}$ for every $j \geqslant \nu + 1$.

(b) $(\psi''_\nu)_{\nu\geqslant 1}$ is stationary on every compact subset of Ω.

This implies that the function $\psi : \Omega \to \mathbb{R}$ defined by $\psi = \lim \psi''_\nu$ is a continuous strictly psh exhaustion function on Ω.

2. The Union Problem

In order to solve the problem in dimension 2, it is sufficient to show

Theorem 1. *Every domain of holomorphy D which is relatively compact in a 2-dimensional normal Stein space X is Stein.*

$\lhd$ By the theorem of Andreotti—Narasimhan [9] we have only to prove that D is locally Stein and, we may of course assume that X is connected.

Let $p \in \partial D \cap \mathrm{Sing}(X)$, and choose a connected Stein open neighborhood U of p with $U \cap \mathrm{Sing}(X) = \{p\}$ and such that U is biholomorphic to a closed analytic set in a domain M in some $\mathbb{C}^N$. Let E be a complex affine subspace of $\mathbb{C}^N$ of maximal dimension such that p is an isolated point of $E \cap U$.

By a coordinate transformation, one can obtain that $z_i(p) = 0$ for all $i \in \{1, 2, \cdots, N\}$ and we may assume that there is a connected Stein open neighborhood V of p in M such that $U \cap V \cap \{z_1(x) = z_2(x) = 0\} = \{p\}$.

We may suppose that $N \geqslant 4$ and, let $E_1 = V \cap \{z_2(x) = \cdots = z_{N-1}(x) = 0\}$, $E_2 = \{x \in E_1 : z_1(x) = 0\}$. Then $A = (U \cap V) \cup E_1$ is a Stein closed analytic set in V as the union of two closed analytic subsets of V.

Let $\xi : \tilde{A} \to A$ be a normalization of A. Then $\xi : \tilde{A} \backslash \xi^{-1}(p) \to A \backslash \{p\}$ is biholomorphic and, clearly $\xi^{-1}(A \cap E_2) = \{x \in \tilde{A} : z_1(\xi(x)) = \cdots = z_{N-1}(\xi(x)) = 0\}$ is everywhere 1-dimensional. It follows from a theorem of Simha [10] that $\tilde{A} \backslash \xi^{-1}(A \cap E_2)$ is Stein. Hence $A \backslash E_2 = \xi(\tilde{A} \backslash \xi^{-1}(A \cap E_2))$ itself is Stein.

Since $p \in E_2$ is the unique singular point of A, then $U \cap V \cap D$ is Stein, being a domain of holomorphy in the Stein manifold $A \backslash E_2$. $\rhd$

Let now X be a Stein space of dimension $n \geqslant 2$ and $\Omega \subset X$ an open subset which is the union of an increasing sequence $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \ldots$ of Stein open sets in X. Let $\phi_\nu : \Omega_\nu \to]0, +\infty[$ be a smooth strictly psh exhaustion function on Ω_ν, and let $(d_\nu)_{\nu\geqslant 1}$ be a sequence with $d_\nu < d_{\nu+1}$, and $\mathrm{Sup}\, d_\nu = +\infty$. One may assume that if $\Omega'_\nu = \{x \in \Omega_\nu : \phi_\nu(x) < d_\nu\}$, then $\Omega'_\nu \subset\subset \Omega'_{\nu+1}$.

Lemma 1. *There exist for each $\nu \geqslant 1$ an exhaustion function $\varphi_\nu \in C^\infty(\Omega_\nu)$ which is strictly psh in a neighborhood of $\overline{\Omega'}_\nu \setminus \Omega'_{\nu-1}$, a locally finite covering $(U_\nu)_{\nu\geqslant 1}$ of Ω by open sets $U_\nu \subset \Omega'_{\nu+1}$, and constants $c_\nu \in \mathbb{R}$, $\nu \geqslant 1$, with the following properties:*

(a) *For each $\nu \geqslant 1$ there exists a function $\psi_\nu : \Omega'_{\nu+1} \to]0,+\infty[$ such that $\psi_\nu|_{U_\nu}$ is strictly psh and $\psi_\nu = \psi_{\nu-1}$ on $\{x \in U_\nu : \varphi_{\nu+1}(x) < c_\nu\} \cap U_{\nu-1}$.*

(b) *For every index $\nu \geqslant 1$, there exists $\varepsilon_\nu > 0$ such that*

$$\Omega'_{\nu-1} \setminus \overline{\Omega'}_{\nu-2} \subset \big\{x \in U_\nu : \varphi_{\nu+1}(x) < c_\nu - \varepsilon_\nu\big\}$$

and

$$\big\{x \in U_\nu : \varphi_{\nu+1}(x) < c_\nu + \varepsilon_\nu\big\} \subset U_{\nu-1}.$$

$\lhd$ There exists a C^∞ exhaustion function $\varphi_{\nu+1}$ on $\Omega_{\nu+1}$ which is strictly plurisubharmonic in a neighborhood of $\overline{\Omega'}_{\nu+1} \setminus \Omega'_\nu$ such that, if $m_{\nu+1} = \min_{\overline{\Omega'}_{\nu+1}\setminus\Omega'_\nu} \varphi_{\nu+1}$ and $M_{\nu+1} = \max_{\overline{\Omega'}_{\nu-1}} \varphi_{\nu+1}$, then $m_{\nu+1} > M_{\nu+1}$.

In fact, we choose $\theta_\nu \in C_0^\infty(\Omega_{\nu+1})$ with compact support in $\Omega_{\nu+1}\setminus\overline{\Omega'}_{\nu-1}$ so that $0 \leqslant \theta_\nu \leqslant 1$ and $\theta_\nu(x) = 1$ when $x \in \overline{\Omega'}_{\nu+1} \setminus \Omega'_\nu$. Let ξ be a point of $\partial\Omega'_{\nu-1}$ such that $\phi_{\nu+1}(\xi) = \max_{\overline{\Omega'}_{\nu-1}} \phi_{\nu+1}$. Then it is clear that

$$\varphi_{\nu+1} = \phi_{\nu+1} + \phi_{\nu+1}(\xi)\theta_\nu$$

satisfies the requirements.

We now assume that $\Omega_0 = \varnothing$ and put

$$U_1 = \Omega'_2, \quad \text{and} \quad U_\nu = (\Omega'_{\nu+1} \setminus \overline{\Omega'}_{\nu-2}) \quad \text{for} \quad \nu \geqslant 2.$$

Then $(U_\nu)_{\nu\geqslant 1}$ is a locally finite covering of Ω. Moreover, if we set

$$c'_\nu = m_{\nu+1} = \operatorname{Inf}\Big\{\varphi_{\nu+1}(x),\ x \in \big(\overline{\Omega'}_{\nu+1} \setminus \Omega'_\nu\big)\Big\},$$

then

$$\big(\overline{\Omega'}_{\nu-1} \setminus \overline{\Omega'}_{\nu-2}\big) \subset \big\{x \in U_\nu : \varphi_{\nu+1}(x) < c'_\nu\big\} \subset \big(\Omega'_\nu \setminus \overline{\Omega'}_{\nu-2}\big) \subset U_{\nu-1}.$$

Furthermore, there exist $c_\nu > 0$ and $\varepsilon_\nu > 0$ such that $c_\nu + \varepsilon_\nu = c'_\nu$ and $\big(\overline{\Omega'}_{\nu-1} \setminus \overline{\Omega'}_{\nu-2}\big) \subset \big\{x \in U_\nu : \varphi_{\nu+1}(x) < c_\nu - \varepsilon_\nu\big\}$.

Moreover, if the function $\theta_\nu \in C_0^\infty\big(\Omega_{\nu+1}\setminus\Omega'_{\nu-1}\big)$ is chosen so that $\theta_\nu = 1$ on

$$(\Omega'_{\nu+1}\setminus\Omega'_\nu) \cup \Big\{x \in \overline{\Omega'}_\nu\setminus\Omega'_{\nu-1} : \operatorname{Inf}_{\Omega'_{\nu+1}\setminus\Omega'_\nu} \phi_{\nu+1} - \frac{\varepsilon_\nu}{2} \leqslant \phi_{\nu+1}(x) \leqslant \operatorname{Inf}_{\Omega'_{\nu+1}\setminus\Omega'_\nu} \phi_{\nu+1} + M_{\nu+1}\Big\},$$

then clearly we obtain $\big\{x \in U_\nu : c_\nu + \frac{\varepsilon_\nu}{2} \leqslant \varphi_{\nu+1}(x) \leqslant c_\nu + \varepsilon_\nu\big\} \subset \{\theta_\nu = 1\}$. Therefore with such a choice of θ_ν there exists for each ν a function $\psi_\nu : \Omega'_{\nu+1} \to]0,+\infty[$ such that $\psi_\nu|_{U_\nu}$ is strictly plurisubharmonic and, $\psi_\nu = \psi_{\nu-1}$ on $\big\{x \in U_\nu : \varphi_{\nu+1}(x) < c_\nu + \frac{\varepsilon_\nu}{2}\big\}$.

In fact, if $\nu = 1$, then it is obvious that $\psi_1 = \phi_2$ has the required properties for $\Omega_1 = \varnothing$, since $U_1 = \Omega'_2$ and $\big\{x \in U_1 : \varphi_2(x) < c_1 + \frac{\varepsilon_1}{2}\big\}$ is contained in Ω'_1.

We now assume that $\nu \geqslant 2$ and, that $\psi_1, \ldots, \psi_{\nu-1}$ have been constructed. Let $\chi_\nu(t) = a_\nu\big(t - c_\nu - \frac{\varepsilon_\nu}{2}\big)$ where a_ν is a positive constant, and consider the function $\psi_\nu : \Omega'_{\nu+1} \to]0, +\infty[$ defined by

$$\psi_\nu = \begin{cases} \psi_{\nu-1} \ \text{ on } \ \{\varphi_{\nu+1} \leqslant c_\nu - \varepsilon_\nu\}, \\ \max(\psi_{\nu-1}, \chi_\nu(\varphi_{\nu+1})) \ \text{ on } \ \{c_\nu - \varepsilon_\nu \leqslant \varphi_{\nu+1} \leqslant c_\nu + \varepsilon_\nu\}, \\ \chi_\nu(\phi_{\nu+1} + \phi_{\nu+1}(\xi)) \ \text{ on } \ \{\varphi_{\nu+1} \geqslant c_\nu + \varepsilon_\nu\}. \end{cases}$$

Since on $U'_\nu = \big\{x \in U_\nu : \varphi_{\nu+1}(x) < c_\nu + \frac{\varepsilon_\nu}{2}\big\} \subset U_{\nu-1}$ we have $\psi_{\nu-1} > 0 > \chi_\nu(\varphi_{\nu+1})$ and $\psi_{\nu-1}$ is strictly psh on $U_{\nu-1}$, then $\psi_\nu|_{U'_\nu} = \psi_{\nu-1}|_{U'_\nu}$ is strictly psh on U'_ν. On the other hand, the subset $\big\{c_\nu + \frac{\varepsilon_\nu}{2} \leqslant \varphi_{\nu+1} \leqslant c_\nu + \varepsilon_\nu\big\} \subset U_{\nu-1}$ is contained in $\{\theta_\nu = 1\}$, which implies that $\psi_\nu = \max(\psi_{\nu-1}, \chi_\nu(\phi_{\nu+1} + \phi_{\nu+1}(\xi)))$ on $\big\{c_\nu + \frac{\varepsilon_\nu}{2} \leqslant \varphi_{\nu+1} \leqslant c_\nu + \varepsilon_\nu\big\}$. Then clearly the function ψ_ν is well-defined and satisfies the required conditions, if a_ν is taken so that $a_\nu \frac{\varepsilon_\nu}{2} > \max_{\{\varphi_{\nu+1} = c_\nu + \varepsilon_\nu\} \cap \Omega'_\nu} \psi_{\nu-1}$. $\rhd$

Theorem 2. *If X is a Stein space and Ω an open subset of X which is an increasing union of Stein open sets in X, then Ω is Stein.*

$\lhd$ We shall prove that there exists for each $\nu \geqslant 1$ a continuous strictly psh function ψ''_ν in a neighborhood of $\overline{\Omega'_\nu}$ such that $\psi''_j > 2^{\nu+1}$ on $\Omega'_{\nu+2} \backslash \Omega'_{\nu+1}$ for every $j \geqslant \nu + 2$ and $(\psi''_\nu)_{\nu \geqslant 1}$ is stationary on every compact set in Ω.

In fact, let φ'_ν be the function defined by

$$\varphi'_\nu = \begin{cases} \psi_\nu \ \text{ on } \ \Omega'_{\nu+1} \backslash \overline{\Omega'}_{\nu-1}, \\ \psi_\mu \ \text{ on } \ \big\{x \in U_{\mu+1} : \varphi_{\mu+2}(x) < c_{\mu+1} - \varepsilon_{\mu+1}\big\} \ \text{ for } \ \mu \leqslant \nu. \end{cases}$$

Then, by Lemma 1, φ'_ν is a continuous strictly plurisubharmonic function on $\Omega'_{\nu+1}$.

Moreover, we have $\varphi'_\nu = \varphi'_{\nu-1}$ on $\{x \in U_{\mu+1} : \varphi_{\mu+2}(x) < c_{\mu+1} - \varepsilon_{\mu+1}\}$ for all $\mu \leqslant \nu - 1$.

Let now K be a compact set in Ω and $\nu \geqslant 2$ such that $K \subset \Omega'_{\nu-1}$. Since $\varphi'_\nu = \varphi'_{\nu-1}$ on $K \cap (\overline{\Omega'}_\mu \backslash \overline{\Omega'}_{\mu-1}) \subset \{x \in U_{\mu+1} : \varphi_{\mu+2}(x) < c_{\mu+1} - \varepsilon_{\mu+1}\}$ for all $\mu \leqslant \nu - 1$, then $\varphi'_\nu = \varphi'_{\nu-1}$ on K. This implies that the sequence $(\varphi'_\nu)_{\nu \geqslant 1}$ is stationary on every compact subset of Ω.

Let now $\nu \geqslant 1$ be an arbitrary natural number. Then there exists a smooth function $\psi'_\nu \in C^\infty(X)$ which is strictly plurisubharmonic in a neighborhood of $(X \backslash \Omega'_{\nu+1}) \cup \overline{\Omega'}_\nu$ such that $\psi'_\nu > 2^{\nu+2}$ in $\overline{\Omega'}_{\nu+2} \backslash \Omega'_{\nu+1}$ but $\psi'_\nu < 0$ in $\overline{\Omega'}_\nu$.

In fact, let $h \in C^\infty(X)$ be a strictly plurisubharmonic exhaustion function such that $h < 0$ in $\overline{\Omega'}_\nu$, and let $\chi_\nu \in C^\infty(X)$ be a smooth function with compact support in $\Omega'_{\nu+1}$ such that $\chi_\nu = 1$ in $\overline{\Omega'}_\nu$. Then it is clear that

$$h_\nu = h + b_\nu \chi_\nu,$$

where $b_\nu = \min_{x \in \overline{\Omega'}_{\nu+2} \backslash \Omega'_{\nu+1}} h(x)$, is a smooth exhaustion function on X which is strictly plurisubharmonic in a neighborhood of $(X \backslash \Omega'_{\nu+1}) \cup \overline{\Omega'}_\nu$ such that if $m'_\nu = \min_{y \in \overline{\Omega'}_{\nu+2} \backslash \Omega'_{\nu+1}} h_\nu(y)$ and $M'_\nu = \max_{y \in \overline{\Omega'}_\nu} h_\nu(y)$, then $m'_\nu > M'_\nu$.

Let $\varepsilon'_\nu > 0$ be such that $m'_\nu > M'_\nu + \varepsilon'_\nu$. Then we can choose a sufficiently big constant $C_\nu > 1$ so that

$$\psi'_\nu(x) = C_\nu\big(h_\nu(x) - M'_\nu - \varepsilon'_\nu\big)$$

is $> 2^{\nu+2}$ in $(\overline{\Omega'}_{\nu+2} \backslash \Omega'_{\nu+1})$, $\psi'_\nu < 0$ in $\overline{\Omega'}_\nu$, and strictly plurisubharmonic in a neighborhood of $(X \backslash \Omega'_{\nu+1}) \cup \overline{\Omega'}_\nu$.

If now we consider the following function defined in Lemma 1

$$\psi_\nu = \begin{cases} \psi_{\nu-1} & \text{on } \{\varphi_{\nu+1} \leqslant c_\nu - \varepsilon_\nu\}, \\ \max\big(\psi_{\nu-1}, \chi_\nu(\varphi_{\nu+1})\big) & \text{on } \big\{c_\nu - \varepsilon_\nu \leqslant \varphi_{\nu+1} \leqslant c_\nu + \varepsilon_\nu\big\}, \\ \chi_\nu\big(\phi_{\nu+1} + \phi_{\nu+1}(\xi)\big) & \text{on } \big\{\varphi_{\nu+1} \geqslant c_\nu + \varepsilon_\nu\big\} \end{cases}$$

and the fact that $c_\nu + \varepsilon_\nu = \operatorname{Inf}\big\{\varphi_{\nu+1}(x),\, x \in \big(\overline{\Omega'}_{\nu+1} \backslash \Omega'_\nu\big)\big\}$, we find that

$$\big(\Omega'_{\nu+1} \backslash \overline{\Omega'}_\nu\big) \subset \big\{x \in U_\nu :\, \varphi_{\nu+1}(x) \geqslant c_\nu + \varepsilon_\nu\big\}$$

and, on the set $\big(\Omega'_{\nu+1} \backslash \overline{\Omega'}_\nu\big)$ we have

$$\varphi'_\nu = \psi_\nu = \chi_\nu(\phi_{\nu+1} + \phi_{\nu+1}(\xi)) \geqslant a_\nu\left(\varphi_{\nu+1} - c_\nu - \frac{\varepsilon_\nu}{2}\right) \geqslant a_\nu \frac{\varepsilon_\nu}{2}.$$

We can therefore choose a_ν again big enough so that $a_\nu \frac{\varepsilon_\nu}{2} > \psi'_\nu$ on $\big(\overline{\Omega'}_{\nu+1} \backslash \Omega'_\nu\big)$. Moreover, by suitable choice of the constants a_μ we can also achieve that $\varphi'_\nu > \psi'_\mu$ on $\big(\Omega'_{\mu+1} \backslash \Omega'_\mu\big)$ for all $\mu < \nu$. In fact, since $\big(\Omega'_\mu \backslash \overline{\Omega'}_{\mu-1}\big) \subset \big\{x \in U_{\mu+1} :\, \varphi_{\mu+2}(x) < c_{\mu+1} - \varepsilon_{\mu+1}\big\}$, then, for every $2 \leqslant \mu \leqslant \nu$, $\varphi'_\nu = \psi_\mu$ on $\big(\Omega'_\mu \backslash \overline{\Omega'}_{\mu-1}\big)$. If we set $A_\mu = \big(\Omega'_\mu \backslash \overline{\Omega'}_{\mu-1}\big) \cap \big\{x \in U_\mu :\, \varphi_{\mu+1}(x) < c_\mu - \varepsilon_\mu\big\}$, then $\psi_\mu = \psi_{\mu-1}$ on A_μ. Since in addition $\big(\Omega'_\mu \backslash \overline{\Omega'}_{\mu-1}\big) \subset \big\{x \in U_{\mu-1} :\, \varphi_\mu(x) \geqslant c_{\mu-1} + \varepsilon_{\mu-1}\big\}$, then on the set A_μ we have $\varphi'_\nu = \psi_\mu = \psi_{\mu-1} \geqslant \chi_{\mu-1}(\varphi_\mu) \geqslant a_{\mu-1}\frac{\varepsilon_{\mu-1}}{2}$. Let now $x \in \big(\Omega'_\mu \backslash \Omega'_{\mu-1}\big)$. If $x \notin A_\mu$, since $x \in U_\mu$, then $\varphi_{\mu+1}(x) \geqslant c_\mu - \varepsilon_\mu$. Because $\big(\Omega'_\mu \backslash \overline{\Omega'}_{\mu-1}\big) \subset \big\{x \in U_{\mu-1} :\, \varphi_\mu(x) \geqslant c_{\mu-1} + \varepsilon_{\mu-1}\big\}$, we obtain, if $\varphi_{\mu+1}(x) \leqslant c_\mu + \varepsilon_\mu$,

$$\varphi'_\nu(x) = \psi_\mu(x) = \max\big(\psi_{\mu-1}(x), \chi_\mu(\varphi_{\mu+1}(x))\big) \geqslant \psi_{\mu-1}(x) = \chi_{\mu-1}(\varphi_\mu(x)) > a_{\mu-1}\frac{\varepsilon_{\mu-1}}{2}.$$

$$\text{Or} \quad \varphi'_\nu(x) = \psi_\mu(x) \geqslant \chi_\mu(\varphi_{\mu+1})(x) > a_\mu \frac{\varepsilon_\mu}{2}, \quad \text{if} \quad \varphi_{\mu+1}(x) \geqslant c_\mu + \varepsilon_\mu.$$

So we may of course take the constants a_μ sufficiently large so that $a_{\mu-1}\frac{\varepsilon_{\mu-1}}{2} > \psi'_{\mu-1}$ and $a_\mu \frac{\varepsilon_\mu}{2} > \psi'_{\mu-1}$ on $(\overline{\Omega'}_\mu \backslash \Omega'_{\mu-1})$ for all $\mu \leqslant \nu$. Since only finitely many conditions are required to get $\varphi'_\nu > \psi'_\mu$ on $(\Omega'_{\mu+1} \backslash \Omega'_\mu)$ for $\mu \leqslant \nu$, it follows that the function $\psi''_\nu : \Omega'_{\nu+1} \to \mathbb{R}$ given by $\psi''_\nu = \max(\varphi'_\nu, \psi'_\nu, \psi'_{\nu-1}, \ldots, \psi'_1)$ is obviously continuous and strictly plurisubharmonic in $\Omega'_{\nu+1}$. Also it is clear that for every $j \geqslant \nu + 1$, $\psi''_j \geqslant \psi'_\nu > 2^{\nu+2}$ on $(\Omega'_{\nu+2} \setminus \overline{\Omega'}_{\nu+1})$.

Let now $K \subset \Omega$ be a compact subset and $\nu \geqslant 2$ such that $K \subset \Omega'_{\nu-1}$. Since $\varphi'_\nu > 0 > \psi'_\nu$ on $\overline{\Omega'}_{\nu-1}$ and $\varphi'_\nu = \varphi'_{\nu-1}$ on K, then $\max(\varphi'_{\nu-1}, \psi'_{\nu-1}, \psi'_{\nu-2}, \cdots, \psi'_1) = \max(\varphi'_\nu, \psi'_\nu, \psi'_{\nu-1}, \cdots, \psi'_1)$ on K, which implies that the sequence $(\psi''_\nu)_{\nu \geqslant 1}$ is stationary on every compact subset of Ω.

This proves that the limit ψ'' of (ψ''_ν) is a continuous strictly plurisubharmonic exhaustion function on Ω, which shows that Ω is Stein. $\triangleright$

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ВОЗРАСТАЮЩЕЕ ОБЪЕДИНЕНИЕ ПРОСТРАНСТВ СТЕЙНА С СИНГУЛЯРНОСТЯМИ

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Аннотация. В статье показано, что если X — пространство Стейна, а множество $\Omega \subset X$ исчерпаемо последовательностью открытых множеств Стейна $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_n \subset \ldots$, содержащихся в X, то Ω — также множество Стейна. Этот факт обобщает хорошо известный результат Бенке и Стейна, полученный для $X = \mathbb{C}^n$, и решет проблему объединения — один из классических вопросов комплексной аналитической геометрии. В том случае, когда X двумерно, для справедливости полученного результата достаточно предположить, что $\Omega \subset\subset X$ — область голоморфности в нормальном пространстве Стейна. В то же время, известно, что произвольное комплексное пространство X, исчерпаемое возрастающей последовательностью открытых множеств Стейна $X_1 \subset X_2 \subset \cdots \subset X_n \subset \ldots$, не является, вообще говоря, голоморфно выпуклым или голоморфно отделимым (даже если X не имеет сингулярностей). Имеются даже двумерные комплексные многообразия, на которых все голоморфные функции постоянны.

Ключевые слова: пространство Стейна, q-полное пространство, q-выпуклая функция, строго плюрисубгармонические функции.

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