Vladikavkaz Mathematical Journal
2020, Volume 22, Issue 4, P. 68–86

УДК 519.642
DOI 10.46698/n8076-2608-1378-r

NEW NUMERICAL METHOD FOR SOLVING NONLINEAR STOCHASTIC INTEGRAL EQUATIONS

R. Zeghdane[1]

[1] Department of Mathematics, Faculty of Mathematics and Informatics,
University of Bordj-Bou-Arreridj,
El-Anasser 34030, Bordj-Bou-Arreridj, Algeria

E-mail: rebihae@yahoo.fr

Abstract. The purpose of this paper is to propose the Chebyshev cardinal functions for solving Volterra stochastic integral equations. The method is based on expanding the required approximate solution as the element of Chebyshev cardinal functions. Though the way, a new operational matrix of integration is derived for the mentioned basis functions. More precisely, the unknown solution is expanded in terms of the Chebyshev cardinal functions including undetermined coefficients. By substituting the mentioned expansion in the original problem, the operational matrix reducing the stochastic integral equation to system of algebraic equations. The convergence and error analysis of the etablished method are investigated in Sobolev space. The method is numerically evaluated by solving test problems caught from the literature by which the computational efficiency of the method is demonstrated. From the computational point of view, the solution obtained by this method is in excellent agreement with those obtained by other works and it is efficient to use for different problems.

Key words: Chebyshev cardinal functions, stochastic operational matrix, Brownian motion, Itô integral, collocation method, numerical solution.

Mathematical Subject Classification (2010): 45G10, 65R20.

For citation: *Zeghdane, R.* New Numerical Method for Solving Nonlinear Stochastic Integral Equations, *Vladikavkaz Math. J.*, 2020, vol. 22, no. 4, pp. 68–86. DOI: 10.46698/n8076-2608-1378-r.

1. Introduction

In recent years, the cardinal functions have been finding an important role in numerical analysis [1]. Both mathematicians and physicists have devoted considerable effort to find robust and stable analytical and numerical methods for solving stochastic differential equations, Adomian method [2], implicit Taylor methods [3, 4] and recently the operational matrices ofintegration for orthogonal polynomials Legendre wavelets, Chebyshev polynomials, etc. [5–20]. Several analytical and numerical methods have been proposed for solving various types of stochastic problems with the classical Brownian motion [10, 12, 14, 21–23]. Noting that finding the exact solutions for most of these equations is hard, therefore, we have to apply approximate numerical methods to obtain numerical solutions. This motivates our interest to propose an efficient and accurate computational method for solving stochastic integral equations. In [24] M. H. Heydari & al. used Chebyshev cardinal wavelets and their application in solving nonlinear stochastic differential equations with fractional Brownian motion. M. H. Heydari obtained a new method based on the Chebyshev cardinal functions for variable-order fractional optimal control problems [25]. An effective direct method to determine the numerical solution of Volterra–Fredholm integro-differential equations based on Chebyshev cardinal functions and deterministic operational matrices was also found in [26]. The aim of this paper is to use cardinal Chebyshev functions to solve nonlinear stochastic integral equations:

$$X(t) = X_0 + \int_0^t k_1(t,s)\big[X(s)\big]^p\,ds + \int_0^t k_2(t,s)\big[X(s)\big]^q\,dB(s), \tag{1}$$

under the initial condition $X(0) = X_0$, where $X(t)$ is an unknown process, the function $k_1(t,s)$, $k_2(t,s)$ are defined on the square $0 \leqslant t,\ s \leqslant 1$, X_0 is a random variable, $B(s)$ is a Brownian motion and $p, q \in \mathbb{N}$. After, we apply cardinal Chebyshev functions to SDE in the following general form

$$X(t) = X_0 + \int_0^t a(s, X(s))\,ds + \int_0^t b(s, X(s))\,dB(s), \tag{2}$$

where $a(s, X(s,\omega))$, $b(s, X(s,\omega))$ for $s, t \in [0,1]$ are known stochastic processes defined on the same filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ with natural filtration $\mathscr{F}_t$, X_0 is the known random variable with $E|X_0|^2 < +\infty$ and $X(t)$ is unknown stochastic process which should be computed. The second integral in (2) is the Itô integral. Furthermore, all Lebesgue's and Itô integrals in (1) and (2) are well defined. The organization of this paper is as follows. In the second section, we give some preliminaries of stochastic calculus. We introduce Chebyshev cardinal functions and operational matrix of integration in Section 3. In Sections 4 and 5 we describe the numerical procedure of the numerical solution of the proposed problem. Convergence analysis of the method will be investigated in Section 6. To show the effectness of the numerical technique, some numerical examples are illustrated in Section 7. Finally, a brief conclusion is drawn on Section 8.

2. Preliminaries

Definition 1. Let $\mathscr{V} = \mathscr{V}(S,T)$ be the class of functions $g(t,\omega) : [0,\infty) \longrightarrow \mathbb{R}$ such that:
1) the function $g(t,\omega)$ be $\mathscr{B} \times \mathscr{F}$ measurable, where $\mathscr{B}$ is the Borel σ-algebra of $\mathbb{R}^+$;
2) the function $g(t,\omega)$ is $\mathscr{F}_t$-adapted (measurable);
3) $E\big[\int_S^T g^2(t,\omega)dt\big] < \infty$.

Lemma 1 (Itô isometry). *For each $X(t,\omega) \in \mathscr{V}(S,T)$, we have*

$$E\bigg(\int_0^t X(s,\omega)\,dB(s)\bigg)^2 = E\bigg(\int_0^t X^2(s,\omega)\,ds\bigg).$$

Lemma 2 (the Gronwall inequality). *Let α, $\beta : [t_0, T] \longrightarrow \mathbb{R}$ be integrable with*

$$0 \leqslant \alpha(t) \leqslant \beta(t) + L\int_{t_0}^t \alpha(s)\,ds \tag{3}$$

for $t \in [t_0, T]$, where $L > 0$. Then

$$0 \leqslant \alpha(t) \leqslant \beta(t) + L\int_{t_0}^t e^{L(t-s)}\beta(s)\,ds, \quad t \in [t_0, T]. \tag{4}$$

3. Chebyshev Cardinal Functions

In this section, to construct the so called Chebyshev cardinal functions for the set of orthogonal Chebyshev polynomials $T_N(x)$, we will use the Taylor expansion of $T_{N+1}(x)$ in neighborhood the j-th root of $T_{N+1}(x)$, which gives

$$T_{n+1}(x) \simeq T_{N+1}(x_j) + T_{N+1,x}(x - x_j) + o(x - x_j)^2.$$

Since the first term in the right hand side vanishes, we can define the cardinal function of degree N in $[-1,1]$ as follows [1, 27]

$$C_j(x) = \frac{T_{N+1}(x)}{T_{N+1,x}(x_j)(x - x_j)}, \quad x \in [-1,1], \tag{5}$$

where the subscript x denotes x differentiation and x_j are the zeros of $T_{N+1}(x)$ defined by

$$x_j = \cos\left(\frac{(2j-1)\pi}{2N+2}\right), \quad j = 1, \ldots, N+1, \tag{6}$$

with the kronecker property

$$C_j(x_i) = \delta_{ji} = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

3.1. Function approximation. To obtain cardinal functions in the interval $[0,1]$, we change the variable $t = \frac{x+1}{2}$, then the shifted Chebyshev cardinal functions are defined on the interval $[0,1]$ as follows:

$$C_i^*(t) = C_i(2t-1), \quad i = 1, \ldots, N+1. \tag{7}$$

Remark 1. The shifted Chebyshev cardinal functions are orthogonal with respect to the weight function $w^*(t) = w(2t-1)$ on $[0,1]$, where $w(t) = \frac{1}{\sqrt{1-t^2}}$ and we have

$$\left\langle C_i^*(t), C_j^*(t) \right\rangle = \int_0^1 C_i^*(t) C_j^*(t) w^*(t)\, dt = \frac{\pi}{2(N+1)}\, \delta_{ij}. \tag{8}$$

Theorem 1. *Any function $g(t)$ mean square integrable on $[0,1]$ can expanded by element of shifted cardinal Chebyshev function as follow*

$$g(t) = \sum_{j=1}^{N+1} u_j C_i^*(t) = U^T \Phi_N(t), \tag{9}$$

where

$$u_j = g(t_j), \quad t_j = \frac{x_j + 1}{2}, \quad j = 1, \ldots, N+1,$$

are the shifted points of x_j,

$$U = \left(u_1, u_2, \ldots, u_{N+1}\right)^T, \quad \Phi_N(t) = \left(C_1^*, C_2^*, \ldots, C_{N+1}^*\right)^T.$$

◁ If $g(t) = \sum_{j=1}^{N+1} u_j C_j^*(t)$, then

$$g(t_i) = \sum_{i=1}^{N+1} u_j C_j^*(t_i) = \sum_{i=1}^{N+1} u_j \delta_{ji}.$$

Then $u_i = g(t_i)$. ▷

Theorem 2. *Any function $g(t,s)$ mean square integrable on $[0,1]\times[0,1]$ can be approximated by cardinal functions as follow*

$$g(t,s) = \sum_{j=1}^{N+1}\sum_{i=1}^{N+1} g(t_i,s_j)C_i^*(t)C_j^*(t) = \Phi_N(t)^T K_1 \Phi_N(s), \tag{10}$$

where $K_{1,(i,j)} = g(t_i,s_j)$ and t_j, s_j are the corresponding shifted points of x_j.

◁ The proof proceeds in a similar way as the proof of Theorem 1. ▷

3.2. Deterministic and stochastic operational matrices. Let

$$\Phi_N(t) = \left(C_1^*, C_2^*, \dots, C_{N+1}^*\right)^T.$$

Lemma 3. *We have*

$$\int_0^t \Phi_N(s)\,ds = P^{-1}Q\Phi_N(t), \tag{11}$$

where the $(N+1)\times(N+1)$ matrix P is called the transform matrix (or Vandermonde's matrix) and is given by

$$P = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_{N+1} \\ t_1^2 & t_2^2 & \dots & t_{N+1}^2 \\ \dots & \dots & \dots & \dots \\ t_1^{N-1} & t_2^{N-1} & \dots & t_{N+1}^{N-1} \\ t_1^N & t_2^N & \dots & t_{N+1}^N \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} t_1 & t_2 & \dots & t_{N+1} \\ \frac{t_1^2}{2} & \frac{t_2^2}{2} & \dots & \frac{t_{N+1}^2}{2} \\ \dots & \dots & \dots & \dots \\ \frac{t_1^{N-1}}{N-1} & \frac{t_2^{N-1}}{N-1} & \dots & \frac{t_{N+1}^{N-1}}{N-1} \\ \frac{t_1^N}{N} & \frac{t_2^N}{N} & \dots & \frac{t_{N+1}^N}{N} \\ \frac{t_1^{N+1}}{N+1} & \frac{t_2^{N+1}}{N+1} & \dots & \frac{t_{N+1}^{N+1}}{N+1} \end{pmatrix}.$$

◁ Let $\psi_i(t) = t^{i-1}$ for $i = 1,\dots N+1$, by expanding $\psi_i(t)$ in $(N+1)$ terms of the shifted Chebyshev cardinal functions, we obtain

$$\psi_i(t) = \sum_{j=1}^{N+1} \psi_i(t_j)C_j^*(t), \quad i = 1,2,\dots,N+1.$$

Then

$$\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \dots \\ \psi_{N+1}(t) \end{pmatrix} = P \begin{pmatrix} C_1^*(t) \\ C_2^*(t) \\ \dots \\ C_{N+1}^*(t) \end{pmatrix} = P\Phi_N(t).$$

Since the matrix P is invertible, $\Phi_N(t) = P^{-1}\Psi_N(t)$, where

$$\Psi_N(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \dots \\ \psi_{N+1}(t) \end{pmatrix}.$$

Hence

$$\int_0^t \Phi_N(s)\,ds = \int_0^t P^{-1}\Psi_N(s)\,ds = P^{-1}\int_0^t \Psi_N(s)\,ds = P^{-1}\begin{pmatrix} t \\ \frac{t^2}{2} \\ \dots \\ \frac{t^{N+1}}{N+1} \end{pmatrix}.$$

Now, let $g_i(t) = \frac{t^i}{i}$, $i = 1, 2, \dots, N+1$, we have $g_i(t) = \sum_{j=1}^{N+1} g_i(t_j)C_j^*(t) = Q\Phi_N(t)$. Then

$$\int_0^t \Phi_N(s)\,ds = P^{-1}Q\Phi_N(t). \quad \triangleright$$

Lemma 4. *Assume* $\Phi_N(t) = \left(C_1^*, C_2^*, \dots, C_{N+1}^*\right)^T$ *and* $U = (u_1, u_2, \dots, u_{N+1})^T$. *Then*

$$\Phi_N(t)\Phi_N^T(t)U = \widetilde{U}\Phi_N(t), \tag{12}$$

where $\widetilde{U} = \mathrm{diag}[u_1, u_2, \dots, u_{N+1}]$.

$\triangleleft$ We have

$$\Phi_N(t)\Phi_N^T(t)U \simeq \begin{pmatrix} C_1^*(t)C_1^*(t) & C_1^*(t)C_2^*(t) & \dots & C_1^*(t)C_{N+1}^*(t) \\ C_2^*(t)C_1^*(t) & C_2^*(t)C_2^*(t) & \dots & C_2^*(t)C_{N+1}^*(t) \\ \dots & \dots & \dots & \dots \\ C_{N+1}^*(t)C_1^*(t) & C_{N+1}^*(t)C_2^*(t) & \dots & C_{N+1}^*(t)C_{N+1}^*(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{N+1} \end{pmatrix}$$

and expanding $C_i(t)C_j(t)$, $i, j = 1, 2, \dots, N+1$, by the elements of Chebyshev cardinal functions, we get

$$C_i(t)C_j(t) \simeq \sum_{k=1}^{N+1} C_i(t_k)C_j(t_k)C_k(t) \simeq \sum_{k=1}^{N+1} \delta_{ik}\delta_{jk}C_k(t).$$

From this we conclude

$$\Phi_N(t)\Phi_N^T(t)U \simeq \begin{pmatrix} C_1^*(t) & 0 & \dots & 0 \\ 0 & C_2^*(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C_{N+1}^*(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{N+1} \end{pmatrix} = \widetilde{U}\Phi_N(t). \quad \triangleright$$

Lemma 5 [26]**.** *If we consider* $X(t) \simeq U^T\Phi_N(t)$, *then for every* $p \in \mathbb{N}$, *we have*

$$\left[X(t)\right]^p \simeq U_p^T\Phi_N(t) \simeq U^T\left(\widetilde{U}\right)^{p-1}\Phi_N(t), \tag{13}$$

or

$$\left[X(t)\right]^p \simeq \left[u_1^p, u_2^p, \dots, u_{N+1}^p\right]\Phi_N(t), \tag{14}$$

where $\widetilde{U} = \mathrm{diag}(u_1, u_2, \dots, u_{N+1})$.

3.3. Stochastic operational matrices of integration. In this subsection, we give stochastic operational matrix of integration with respect to Brownian motion we have

$$\int_0^t \Phi_N(s)\,dB(s) = \int_0^t P^{-1}\Psi_N(s)\,dB(s) = P^{-1}\int_0^t \Psi_N(s)\,dB(s)$$
$$= P^{-1}\left[\int_0^t dB(s),\ \int_0^t s\,dB(s),\ \ldots,\ \int_0^t s^N\,dB(s)\right]^T$$

we apply Itô formula, we get

$$\begin{pmatrix} \int_0^t dB(s) \\ \int_0^t s\,dB(s) \\ \int_0^t s^2\,dB(s) \\ \cdots \\ \int_0^t s^N\,dB(s) \end{pmatrix} = B(t)\Psi_N(t) - \begin{pmatrix} 0 \\ \int_0^t B(s)\,ds \\ 2\int_0^t sB(s)\,ds \\ \cdots \\ N\int_0^t s^{N-1}B(s)\,ds \end{pmatrix} = A_N(t) = (a_i)_{i=0,\ldots,N},$$

where

$$a_i = t^iB(t) - i\int_0^t s^{i-1}B(s)\,ds, \quad i = 0,\ldots,N.$$

For the integral $\int_0^t s^{i-1}B(s)\,ds$, we can use Simpson rule as follow

$$\int_0^t s^{i-1}B(s)\,ds \simeq \frac{t}{6}\left(0^{i-1}B(0) + 4\left(\frac{t}{2}\right)^{i-1}B\left(\frac{t}{2}\right) + t^{i-1}B(t)\right), \quad i = 1,2,\ldots,$$

so

$$a_i = t^iB(t) - i\,\frac{t}{6}\left(4\left(\frac{t}{2}\right)^{i-1}B\left(\frac{t}{2}\right) + t^{i-1}B(t)\right) = \left(\left(1-\frac{i}{6}\right)B(t) - \frac{i}{3\cdot 2^{i-2}}B\left(\frac{t}{2}\right)\right)t^i,$$
$$i = 1,2,\ldots,$$

$$a_i = B(t) \quad \text{for} \quad i = 0.$$

Also we approximate $B(t)$ and $B\left(\frac{t}{2}\right)$ for $0 \leqslant t \leqslant 1$ by $B(0.5)$ and $B(0.25)$, then we obtain

$$P^{-1}A_N(t)$$
$$= P^{-1}\begin{pmatrix} B(0.5) & 0 & 0 & \ldots & 0 \\ 0 & \frac{5}{6}B(0.5) - \frac{2}{3}B(0.25) & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & (1-\frac{N}{6})B(0.5) - \frac{N}{3\cdot 2^{N-2}}B(0.25) \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ \ldots \\ t^N \end{pmatrix}.$$

Then

$$P^{-1}A_N(t) = P^{-1}A_s\Psi_N(t) = P^{-1}A_sP\Phi_N(t) = P_s\Phi_N(t), \tag{15}$$

where

$$A_s = \begin{pmatrix} B(0.5) & 0 & 0 & \dots & 0 \\ 0 & \frac{5}{6}B(0.5) - \frac{2}{3}B(0.25) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \left(1-\frac{N}{6}\right)B(0.5) - \frac{N}{3\cdot 2^{N-2}}B(0.25) \end{pmatrix}$$

and $P_s = P^{-1}A_sP$ is $(N+1)\times(N+1)$ stochastic operational matrix. Finally,

$$\int_0^t \Phi_N(t)\,dB(t) \simeq P_s\Phi_N(t). \tag{16}$$

4. Numerical Method for Solving Stochastic Integral Equation (1)

In section, we describe numerical technique for solving stochastic integral equation (1), first we approximate the functions $k_1(t,s)$, $k_2(t,s)$ and $X(t)$ by elements of the basis C_i^*, $i = 1, 2, \dots, N+1$, as follow

$$X(t) \simeq U^T\Phi_N(t), \quad k_1(t,s) \simeq \Phi_N(t)^TK_1\Phi_N(s), \quad k_2(t,s) \simeq \Phi_N^T(t)K_2\Phi_N(s). \tag{17}$$

Then, we approximate the integrals $\int_0^t k_1(t,s)[X(s)]^p ds$ and $\int_0^t k_2(t,s)[X(s)]^q dB(s)$, we obtain

$$\begin{aligned}\int_0^t k_1(t,s)[X(s)]^p ds &\simeq \int_0^t \Phi_N(t)^TK_1\Phi_N(s)\Phi_N(s)^TU_p ds \simeq \Phi_N(t)^TK_1\int_0^t \Phi_N(s)\Phi_N(s)^TU_p ds \\ &\simeq \Phi_N(t)^TK_1\widetilde{U}_p\left[\int_0^t \Phi_N(s)\,ds\right] \simeq \Phi_N(t)^TK_1\widetilde{U}_pP^{-1}Q\Phi_N(t),\end{aligned} \tag{18}$$

where $\widetilde{U}_p = \mathrm{diag}(u_1^p, u_2^p, \dots, u_{N+1}^p)$ and U_p are the coefficients of $X^p(t)$ in the basis $\Phi_N(t)$. Let U_q be the coefficients of $X^q(t)$ in the basis $\Phi_N(t)$. Then we have

$$\begin{aligned}\int_0^t k_2(t,s)\left[X(s)\right]^q dB(s) &\simeq \int_0^t \Phi_N(t)^TK_2\Phi_N(s)\Phi_N(s)^TU_q\,dB(s) \\ \simeq \Phi_N(t)^TK_2\int_0^t \Phi_N(s)\Phi_N(s)^TU_q\,dB(s) &\simeq \Phi_N(t)^TK_2\widetilde{U}_q\left[\int_0^t \Phi_N(s)\,dB(s)\right] \\ &\simeq \Phi_N(t)^TK_2\widetilde{U}_qP_s\Phi_N(t).\end{aligned} \tag{19}$$

We replace equations (17), (18) and (19) in equation (1), we get

$$U^T\Phi_N(t) - X_0 - \Phi_N(t)^TK_1\widetilde{U}_pP^{-1}Q\Phi_N(t) - \Phi_N(t)^TK_2\widetilde{U}_qP_s\Phi_N(t) = 0. \tag{20}$$

To solve equation (20), we have three methods.

1. First, by collacting equation (20) in $(N+1)$ points t_j, $j = 1, 2, \ldots, N+1$, shifted points of x_j, we obtain

$$U^T\Phi_N(t_j) - X_0 - \Phi_N(t_j)^T K_1 \widetilde{U}_p P^{-1} Q \Phi_N(t_j) - \Phi_N(t_j)^T K_2 \widetilde{U}_q P_s \Phi_N(t_j) = 0, \quad j = 1, 2, \ldots, N+1. \tag{21}$$

We have $\Phi_N(t_j) = e_j^N$, where e_j^N denotes the column of ordre j of identity matrix I of order $N+1$. Then we obtain a nonlinear system included $N+1$ unknowns $(u_1, u_2, \ldots, u_{N+1})^T$ and $N+1$ equations, Newton method can be used to obtain accurate solution of nonlinear systems.

2. Here, we approximate $\Phi_N(t_j)^T K_1 \widetilde{U}_p P^{-1} Q \Phi_N(t_j)$ and $\Phi_N(t_j)^T K_2 \widetilde{U}_q P_s \Phi_N(t_j)$ as follow

Lemma 6. *We have*

$$\Phi_N(t)^T K_1 \widetilde{U}_p P^{-1} Q \Phi_N(t) \simeq M_1 \Phi_N(t), \tag{22}$$

$$\Phi_N(t)^T K_2 \widetilde{U}_q P_s \Phi_N(t) \simeq M_2 \Phi_N(t), \tag{23}$$

where M_1 and M_2 are $(N+1)$ row vectors including elements equal to the diagonal entries of $K_1 \widetilde{U}_p P^{-1} Q$ and $K_2 \widetilde{U}_q P_s$ respectibely.

$\lhd$ It is easy to proof identity (22) and (23). $\rhd$

We replace (22) and (23) in equation (20), we get

$$U^T\Phi_N(t) - X_0 - M_1\Phi_N(t) - M_2\Phi_N(t) = 0. \tag{24}$$

Hence

$$\left[U^T - A_0 - M_1 - M_2\right]\Phi_N(t) = 0, \tag{25}$$

where A_0 is $(N+1)$ row vector including elements equal to X_0. The obtained system (25) is a nonlinear system with $N+1$ unknowns $(u_1, u_2, \ldots, u_{N+1})^T$.

3. We can use orthogonality condition.

5. Solving Stochastic Integral Equation (2)

We approximate equation (2) as follows:

$$z_1(t) = a(t, X(t)), \quad z_2(t) = b(t, X(t)), \quad t \in [0, 1]. \tag{26}$$

By using equation (2) and (26), we have

$$\begin{cases} z_1(t) = a\left(t,\ X_0 + \int\limits_0^t z_1(s)\, ds + \int\limits_0^t z_2(s) dB(s)\right), \\ z_2(t) = b\left(t,\ X_0 + \int\limits_0^t z_1(s)\, ds + \int\limits_0^t z_2(s) dB(s)\right). \end{cases} \tag{27}$$

By expanding $z_1(t)$ and $z_2(t)$ by elements of cardinal functions, we get

$$z_1(t) = U_1^T\Phi_N(t), \quad z_2(t) = U_2^T\Phi_N(t). \tag{28}$$

By substituting equation (28) in (27), we obtain

$$\begin{cases} z_1(t) = a\Big(t,\ X_0 + \int_0^t U_1^T \Phi_N(s)\,ds + \int_0^t U_2^T \Phi_N(s)\,dB(s)\Big), \\ z_2(t) = b\Big(t,\ X_0 + \int_0^t U_1^T \Phi_N(s)\,ds + \int_0^t U_2^T \Phi_N(s)\,dB(s)\Big), \end{cases} \tag{29}$$

which is equivalent to

$$\begin{cases} z_1(t) = a\Big(t,\ X_0 + U_1^T \int_0^t \Phi_N(s)\,ds + U_2^T \int_0^t \Phi_N(s)\,dB(s)\Big), \\ z_2(t) = b\Big(t,\ X_0 + U_1^T \int_0^t \Phi_N(s)\,ds + U_2^T \int_0^t \Phi_N(s)\,dB(s)\Big). \end{cases} \tag{30}$$

By using equation (11) and (16), we get

$$\begin{cases} U_1^T \Phi_N(t) = a\big(t,\ X_0 + U_1^T P^{-1} Q \Phi_N(t) + U_2^T P_s \Phi_N(t)\big), \\ U_2^T \Phi_N(t) = b\big(t,\ X_0 + U_1^T P^{-1} Q \Phi_N(t) + U_2^T P_s \Phi_N(t)\big). \end{cases} \tag{31}$$

We collocate (29) at shifted points t_j, $j = 1, 2, \dots N+1$, and we arrive at

$$\begin{cases} U_1^T e_j^N = a\big(t_j,\ X_0 + U_1^T P^{-1} Q e_j^N + U_2^T P_s e_j^N\big), \\ U_2^T e_j^N = b\big(t_j,\ X_0 + U_1^T P^{-1} Q e_j^N + U_2^T P_s e_j^N\big), \end{cases} \tag{32}$$

where e_j^N denotes the column of ordre j of identity matrix I of order $N+1$. The system (32) can be solved for the unknown U_1 and U_2 with Matlab software packages or by the Newton's iterative method. By determining U_1 and U_2, we can determine the approximate solution of $X(t)$ as follow

$$X_N(x) = X_0 + U_1^T P^{-1} Q \Phi_N(t) + U_2^T P_s \Phi_N(t). \tag{33}$$

6. Convergence Analysis

In this section, we investigate the convergence and error analysis of the proposed method in the Sobolev space.

Definition 2 [28]. The Sobolev space $H_w^m(a,b)$ is defined as follow:

$$H_w^m(a,b) = \big\{u \in L_w^2(a,b),\ u^{(j)}(t) \in L_w^2(a,b),\ j = 0, 1, \dots, m\big\}, \tag{34}$$

where w be a weight function and $m \geqslant 0$ be an integer.

Remark 2. The Sobolev space $H_w^m(a,b)$ is endowed with the following weighted inner product

$$\big\langle u(t), v(t)\big\rangle_{m,w} = \sum_{i=1}^{m} \int_a^b u^{(j)} v^{(j)} w(t)\,dt. \tag{35}$$

The space $H_w^m(a,b)$ is a Hilbert space with the following norm

$$\big\|u(t)\big\|_{H_w^m(a,b)} = \left(\sum_{i=1}^{m} \big\|u^{(j)}\big\|_{L_w^2(a,b)}\right)^{\frac{1}{2}}. \tag{36}$$

Lemma 7 [28]. *Let*

$$u \in H_w^m(-1,1), \quad w(t) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad I_N u = \sum_{j=1}^{N+1} u_j C_j(t)$$

be the Chebyshev interpolant of $u(t)$ *Then, the truncated error* $u - I_N u$ *satisfies*

$$\|u - I_N u\|_{L_w^2(-1,1)} \leqslant \widehat{C}_m N^{-m} \Bigg(\sum_{j=\min(m,N)}^{m} \|u^{(j)}\|_{L_w^2(-1,1)} \Bigg)^{\frac{1}{2}}, \tag{37}$$

where $\widehat{C}_m$ *is a positive constant independent of* N *and dependent on* m. *Moreover, in the maximum norm, it yields*

$$\|u - I_N u\|_{L_w^\infty(-1,1)} \leqslant \widehat{C}_m N^{\frac{1}{2}-m} \Bigg(\sum_{j=\min(m,N)}^{m} \|u^{(j)}\|_{L_w^2(-1,1)} \Bigg)^{\frac{1}{2}}, \tag{38}$$

where $\widehat{C}_m$ *is a positive constant independent of* N *and dependent on* m, *and* $\|u\|_{L_w^\infty(-1,1)} = \sup_{-1 \leqslant t \leqslant 1} |u(t)|$.

Theorem 3. *Let*

$$u \in H_{w^*}^m(0,1), \;\; w^*(t) = w(2t-1) \quad \text{and} \quad I_N^* u = \sum_{j=1}^{N+1} u_j C_j^*(t), \;\; u_j = u(t_j)$$

be the Chebyshev interpolant of $u(t)$. *Then, the truncated error* $u - I_N^* u$ *satisfies*

$$\|u - I_N^* u\|_{L_{w^*}^2(0,1)} \leqslant \widehat{C}_m N^{-m} \Bigg(\sum_{j=\min(m,N)}^{m} \left(\frac{1}{2}\right)^{2j} \|u^{(j)}\|_{L_{w^*}^2(0,1)} \Bigg)^{\frac{1}{2}}, \tag{39}$$

where $\widehat{C}_m$ *is a positive constant independent of* N *and dependent on* m. *Moreover, in the maximum norm, it yields*

$$\|u - I_N u\|_{L^\infty(0,1)} \leqslant \widehat{C}_m N^{\frac{1}{2}-m} \sqrt{2} \Bigg(\sum_{j=\min(m,N)}^{m} \left(\frac{1}{2}\right)^{2j} \|u^{(j)}\|_{L_{w^*}^2(0,1)} \Bigg)^{\frac{1}{2}}, \tag{40}$$

where $\widehat{C}_m$ *is a positive constant independent of* N *and dependent on* m, *and* $\|u\|_{L^\infty(0,1)} = \sup_{0 \leqslant t \leqslant 1} |u(t)|$.

$\lhd$ The proof proceeds in a same manner as the one of Theorem (5.4) in [24]. $\rhd$

Theorem 4. *Suppose* $X(t) \in H_w^m(0,1)$ *and* $X_N(x)$ *be the exact and approximate solutions of equation* (2), *respectively, furthermore, we suppose that*

(H1) $|a(t, X_1(t)) - a(t, X_2(t))| + |b(t, X_1(t)) - b(t, X_2(t))| \leqslant L|X_1 - X_2|$ (*Lipschitz condition*),

(H2) $|a(t, X(t))| + |b(t, X(t))| \leqslant L(1 + |X|)$ (*Linear growth condition*), *where* $t \in [0,1]$, $X_1, X_2 \in \mathbb{R}$ *and* L_i *are positive constants for* $i = 1, 2$.

(H3) $E|X_0|^2 < \infty$.

Then $X_n(t)$ *converges to* $X(t)$ *in* L^2.

$\lhd$ Let $e_N(t) = X(t) - X_N(t)$ be an error function of approximate solution $X_N(t)$ to the exact solution $X(t)$,

$$X(t) - X_N(t) = \int_a^t \big(z_1(s) - \bar{z}_1(s)\big)\,ds + \int_a^t \big(z_2(s) - \bar{z}_2(s)\big)\,dB(s), \tag{41}$$

where $z_i(t)$, $i = 1, 2$, are given by $z_1(t) = a(t, X(t))$, $z_2(t) = b(t, X(t))$, also $\bar{z}_i(t)$, $i = 1, 2$, is approximated form of $z_i(t)$ by schifted cardinal Chebyshev function

$$\bar{z}_1(t) = \mathrm{app}_N(a(t, X_N(t)), \quad \bar{z}_2(t) = \mathrm{app}_N(b(t, X_N(t))$$

$$z_1^N(t) = a(t, X_N(t)), \quad z_2^N(t) = b(t, X_N t)),$$

$$e_N(t) = \int_0^t \big(z_1(s) - \bar{z}_1(s)\big)\,ds + \int_0^t \big(z_2(s) - \bar{z}_2(s)\big)\,dB(s),$$

$$E\left|e_N(t)\right|^2 = E\left(\left|\int_0^t \big(z_1(s) - \bar{z}_1(s)\big)\,ds + \int_0^t \big(z_2(s) - \bar{z}_2(s)\big)\,dB(s)\right|\right)^2.$$

Using the inequality $(a+b)^2 \leqslant 2(a^2 + b^2)$, we get

$$E\left|e_N(t)\right|^2 \leqslant 2E\left|\int_0^t \big(z_1(s) - \bar{z}_1(s)\big)\,ds\right|^2 + 2E\left|\int_0^t \big(z_2(s) - \bar{z}_2(s)\big)\,dB(s)\right|^2$$

by the Schwartz inequality and Itô isometry, we get

$$E\left|e_N(t)\right|^2 \leqslant 2E\left(\int_0^t \left|z_1(s) - \bar{z}_1(s)\right|^2 ds\right) + 2E\left(\int_0^t \left|z_2(s) - \bar{z}_2(s)\right|^2 ds\right),$$

consequently,

$$2E\left(\int_0^t \left|z_1(s) - \bar{z}_1(s)\right|^2 ds\right) \leqslant 4E\left(\int_0^t \left|z_1(s) - z_1^N(s)\right|^2 ds\right) + 4E\left(\int_0^t \left|z_1^N(s) - \bar{z}_1(s)\right|^2 ds\right),$$

$$2E\left(\int_0^t \left|z_2(s) - \bar{z}_2(s)\right|^2 ds\right) \leqslant 4E\left(\int_0^t \left|z_2(s) - z_2^N(s)\right|^2 ds\right) + 4E\left(\int_0^t \left|z_2^N(s) - \bar{z}_2(s)\right|^2 ds\right).$$

By using Theorem 3, there exists $\alpha_j(m, N)$, $j = 1, 2$, such that

$$E\left\|z_j^N(s) - \bar{z}_j(s)\right\|^2 \leqslant \big(\alpha_j(m, N)\big)^2, \quad j = 1, 2,$$

where

$$\alpha_i(m, N) = \widehat{C}_m N^{-m}\left(\sum_{j=\min(m,N)}^{m} \left(\frac{1}{2}\right)^{2j} \left\|\big(z_i^N\big)^{(j)}\right\|_{L^2_{w^*}(0,1)}\right)^{\frac{1}{2}}, \quad i = 1, 2.$$

Then

$$E\left|e_n(t)\right|^2 \leqslant 4\big(\alpha_1(m,N)+\alpha_2(m,N)\big)^2+4\left(\int_0^t E\left|z_1(s)-z_1^n(s)\right|^2 ds+\int_0^t E\left|z_2(s)-z_2^n(s)\right|^2 ds\right).$$

By using Lipschitz condition, we get

$$E\left|e_n(t)\right|^2 \leqslant 4\big(\alpha_1(m,N)+\alpha_2(m,N)\big)^2+8L\int_0^t E\left|e_n(s)\right|^2 ds, \tag{42}$$

hence by Gronwall inequality we obtain $E|e_N(t)|^2 \longrightarrow 0$, as $N \longrightarrow \infty$. ▷

Remark 3. We can see that if m is sufficiently large than the error in Lemma (7) is sufficiently small.

7. Numerical Examples

To demonstrate the accuracy and effectiveness of the method proposed herein, we have applied it to several examples. These examples are solved in different references, so the numerical results obtained here can be compared with those of other numerical methods. In order to analyze the error of the method we introduce the absolute error, with M simulations

$$e_N(t)=\big|X(t)-X_N(t)\big|.$$

Example 1. Consider the deterministic Volterra integral equation of the kind as follows [29]:

$$-\frac{1}{15}\big(-8\exp(2t)+6\sin(t)+3\cos(t)+5\exp(-t)\big)-\int_0^t\big(\exp(s-t)+\sin(t-s)X(s)\big)\,ds,$$

where the exact solution is $X(t)=\exp(2t)$. The numerical results are summarized in Table 1.

Table 1. The absolute errors obtained by the proposed method with different values of N for Example 1

t	$N=4$	$N=10$	$N=15$
0	8.1011 E−3	6.4010 E−9	8.3377 E−14
0.2	5.3252 E−3	6.6734 E−9	2.5157 E−13
0.4	9.8748 E−3	1.0813 E−9	3.5527 E−15
0.6	1.0258 E−3	8.5579 E−9	2.0872 E−14
0.8	1.5953 E−2	4.6935 E−9	1.4264 E−12
1	7.2225 E−2	1.1335 E−7	3.9968 E−13

Example 2. Consider the deterministic Volterra integral equation of the second kind as follows:

$$X(t)=\cos(t)-\int_0^t(t-s)\cos(t-s)X(s)\,ds,$$

where the exact solution is $X(t)=\frac{1}{3}(2\cos\sqrt{3t}+1)$. The numerical results are shown in Fig. 1–2.

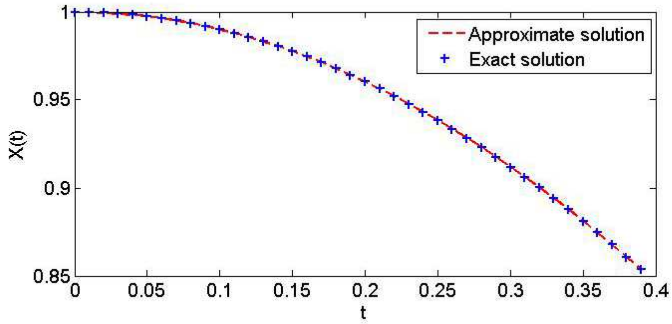

Fig. 1. The graphs of exact and approximate solutions for $N = 4$ for Example 2.

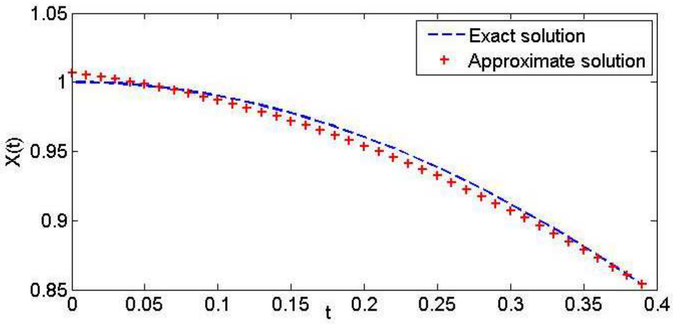

Fig. 2. The graphs of exact and approximate solutions for $N = 2$ for Example 2.

The proposed method, can be also applied to nonlinear deterministic Fredhoml integral equations.

Example 3. Consider the Fredholm integral equation of the second kind [30]

$$X(t) = \exp\left(2t + \left(\frac{1}{3}\right)\right) + \int_0^1 \exp\left(2t - \left(\frac{5}{3}\right)s\right) X(s)\, ds, \tag{43}$$

with the exact solution $X(t) = \exp(2t)$. The computational results are compared with that obtained in [30] and are illustrated in Table 2.

Table 2. The absolute errors obtained by the proposed method with different values of N for Example 3

t	$N = 5$	$N = 6$	$N = 10$	$m = 64$ [30]	$m = 128$ [30]
0	1.0589 E−4	7.6683 E−6	6.2495 E−11	5.6999 E−5	4.0000 E−5
0.2	8.3927 E−5	7.8574 E−6	4.6068 E−11	1.2000 E−4	1.9999 E−5
0.4	4.1799 E−5	8.3148 E−6	5.3258 E−11	9.9992 E−5	3.0000 E−5
0.6	4.4243 E−5	8.7391 E−6	5.5025 E−11	4.5999 E−4	4.9999 E−5
0.8	9.9533 E−5	9.1235 E−6	5.0805 E−11	7.5999 E−4	2.9999 $E-5$
1	1.4078 E−4	9.8403 E−6	7.3655 E−11	3.5000 E−4	4.9999 E−5

Example 4. Consider the deterministic Riccati differential equation

$$u'(t) + u^2(t) - 1 = 0, \quad u(0) = 0. \tag{44}$$

The exact solution is given by $u(t) = \frac{\exp(2t)-1}{\exp(2t)+1}$. The numerical results of this example are given in Table 3, and are compared with the results obtained in [31].

Table 3. The absolute errors obtained by the proposed method with two values of N for Example 4

t	$N=6$ (Present method)	$N=12$ (Present method)	$m=12$ [31]
0.1	1.2775 E−6	1.6259 E−11	1.11 E−10
0.2	2.6439 E−6	2.5123 E−12	2.04 E−10
0.3	1.3688 E−7	2.3986 E−11	2.10 E−12
0.4	2.8560 E−6	2.2805 E−11	2.23 E−10
0.5	7.3035 E−7	1.3141 E−11	4.03 E−10
0.6	2.39994 E−6	2.3181 E−13	1.79 E−10
0.7	9.8334 E−7	1.1980 E−11	8.59 E−11
0.8	2.5757 E−6	1.4748 E−11	2.70 E−10
0.9	2.8394 E−7	5.0553 E−12	1.89 E−10
1.0	2.6817 E−6	2.3652 E−11	2.66 E−11

Example 5. Let us consider the problem

$$X(t) = X_0 + \int_0^t a^2 \cos(X(s)) \sin^3(X(s))\, ds - a \int_0^t \sin^2(X(s))\, dB(s), \quad t \in [0,1]. \tag{45}$$

The exact solution is $X(t) = \operatorname{arccot}(aB(s) + \cot(X_0))$. The computed errors for $N = 5$, $a = 1/8$ and $X_0 = \pi/32$, $X_0 = 0.1$, $X_0 = 0.01$, $X_0 = 1$ are summarized in Table 4.

Table 4. The absolute errors obtained by the proposed method with different values of X_0 for Example 5

t	$X0=0.01$	$X0=\pi/32$	$X0=0.1$	$X0=1$
0	8.2145 E−6	4.0132 E−4	8.3099 E−4	6.2593 E−2
0.1	7.7400 E−6	6.8875 E−4	7.8514 E−4	5.9772 E−2
0.2	1.0725 E−6	8.6983 E−4	1.1750 E−4	1.1500 E−2
0.3	4.6979 E−7	4.2429 E−4	4.6663 E−4	3.2472 E−2
0.4	4.2535 E−6	9.1225 E−5	3.0996 E−5	1.2364 E−3
0.5	7.6467 E−6	1.3240 E−4	7.8170 E−4	6.1292 E−2
0.6	3.0515 E−6	1.2116 E−4	3.2278 E−4	2.8464 E−2
0.7	6.1677 E−6	3.2922 E−4	6.1092 E−4	4.1968 E−2
0.8	1.5208 E−6	6.1442 E−4	1.3615 E−4	4.9793 E−3
0.9	3.2037 E−6	9.8149 E−4	3.0564 E−4	1.7478 E−2

Example 6 (Stochastic Lotka–Volterra model). Lotka–Volterra model also known as the predator-prey equations, in deterministic subclasses, are well-known and have been an active area of research concerning ecological population modeling [32]. The logistic model is often represented as follow:

$$\begin{cases} dX(t) = X(t)\big(b_1 - a_{11}X(t) - a_{12}Y(t)\big)dt + \sigma_1 X(t)dB_1(t), \\ dY(t) = Y(t)\big(b_2 - a_{21}X(t) - a_{22}Y(t)\big)dt + \sigma_2 Y(t)dB_1(t), \end{cases}$$

with initial conditions $X(0) = X_0$, $Y(0) = Y_0$, where a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, σ_1 and σ_2 are parameters. The application of the proposed method, gives the corresponding nonlinear system

$$\begin{cases} U^T = X_0^T + b_1 U^T P^{-1} Q - a_{11} \widetilde{U}_2^T P^{-1} Q - a_{12} U^T \widetilde{V} P^{-1} Q + \sigma_1 U^T P_s^{B_1}, \\ V^T = Y_0^T + b_2 V^T P^{-1} Q - a_{21} V^T \widetilde{U} P^{-1} Q - a_{22} \widetilde{V}_2^T P^{-1} Q + \sigma_2 V^T P_s^{B_2}, \end{cases}$$

where

$$X(t) = U^T Q_N(t), \quad Y(t) = V^T Q_N(t), \quad X^2(t) = \widetilde{U}_2^T Q_N(t), \quad Y^2(t) = \widetilde{V}_2^T Q_N(t),$$
$$\widetilde{V} = \operatorname{diag}\big[v_1, v_2, \dots, v_{N+1}\big], \quad \widetilde{U} = \operatorname{diag}\big[u_1, u_2, \dots, u_{N+1}\big],$$
$$\widetilde{V}_2 = \big(v_1^2, v_2^2, \dots v_{N+1}^2\big)^T, \quad \widetilde{U}_2 = \big(u_1^2, u_2^2, \dots u_{N+1}^2\big)^T,$$

with $U = (u_1, u_2, \dots, u_{N+1})$, $V = (v_1, v_2, \dots, v_{N+1})$. In this example, we take $X(0) = 0.5$, $Y(0) = 1$ and $b_1 = 20$, $B_2 = -30$, $a_{11} = a_{22} = 0$, $a_{12} = a_{21} = 25$ and $\sigma_1 = \sigma_2 = 1$. We take $M = 80$ simulations for $N = 8$ and $M = 30$ for $N = 5$, we compute the means of $X(t)$ and $Y(t)$. The numerical results are shown in Fig. 3–4.

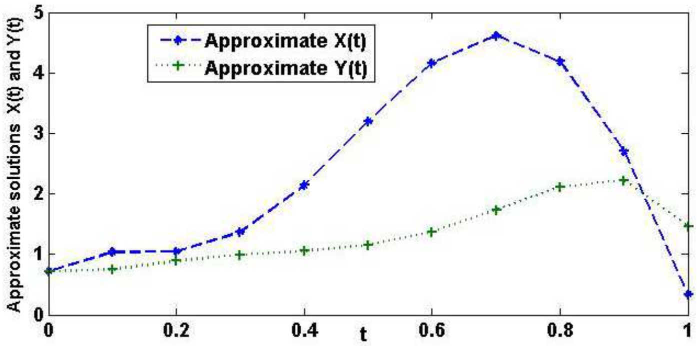

Fig. 3. Approximate solutions for $M = 80$ and $N = 8$ for Example 6.

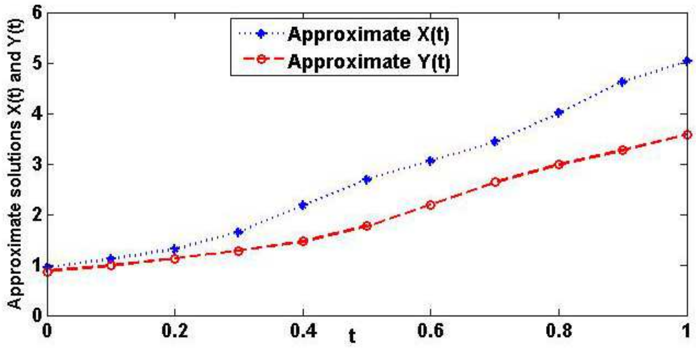

Fig. 4. Approximate solutions for $M = 30$ and $N = 5$ for Example 6.

Example 7. Consider the following nonlinear stochastic Itô integral equation

$$X(t) = 1 + \int_0^t X(t)\Big(\frac{1}{32} - X^2(t)\Big)\,dt + \int_0^t 0.25X(t)\,dB(t), \quad t \in [0,1], \tag{46}$$

with the exact solution

$$X(t) = \frac{\exp(0.25B(t))}{\sqrt{1 + 2\int_0^t \exp(0.5B(s))\,ds}}, \tag{47}$$

where $X(t)$ is a stochastic process defined on the probability space $(\Omega, \mathscr{F}, P)$. The numerical results with $M = 150$ simulations are shown in Table 5 and are compared with the results obtained in [10].

Table 5. The absolute errors obtained by the proposed method with different values of N for Example 7

t	$N = 4$	$N = 8$	$N = 10$	$N = 4$ [10]	$N = 8$ [10]	$N = 10$ [10]
0	1.6360 E−3	4.0901 E−2	1.4337 E−1	8.17 E−2	2.76 E−2	9.29 E−2
0.1	3.4591 E−2	7.6948 E−2	8.2714 E−2	5.29 E−2	2.51 E−2	6.31 E−2
0.2	1.1814 E−1	6.9798 E−2	1.3960 E−3	2.89 E−2	2.59 E−2	3.86 E−2
0.3	9.468 E−2	2.4183 E−2	1.7747 E−2	6.7 E−3	3.06 E−2	1.65 E−2
0.4	7.5338 E−2	9.7591 E−3	8.4280 E−3	1.59 E−2	3.84 E−2	4.3 E−3
0.5	7.7120 E−2	6.4695 E−3	5.9106 E−2	4.12 E−2	4.87 E−2	2.41 E−2
0.6	6.1632 E−2	9.0339 E−3	1.2380 E−2	7.25 E−2	6.08 E−2	4.31 E−2
0.7	5.4542 E−2	1.0809 E−1	4.8138 E−2	1.141 E−1	7.42 E−2	6.12 E−2
0.8	6.9447 E−2	6.1975 E−2	4.8274 E−2	1.714 E−1	8.89 E−2	7.87 E−2
0.9	6.6438 E−2	2.3192 E−2	8.8528 E−2	2.512 E−1	1.055 E−1	9.55 E−2

Example 8 (the basic Black–Scholes model). Consider the following linear stochastic equation

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \quad X(0) = X_0, \quad t \in [0,1], \tag{48}$$

where the exact solution is given by

$$X(t) = \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)\right).$$

The results obtained for $\lambda = -10$, $\mu = 1$, $N = 5$ and $M = 100$ simulations of this example are given in Table 6 and in Fig. 5–6.

Table 6. Computed errors for Example 8

t	$X_0 = 0.001$	$X_0 = 0.01$	$X_0 = 1$
0	6.0012 E−5	2.1938 E−3	1.2309 E−1
0.1	7.1472 E−4	2.5322 E−3	9.5855 E−2
0.2	8.3066 E−4	1.5726 E−3	2.8627 E−2
0.3	7.0929 E−4	6.0209 E−4	4.0362 E−2
0.4	4.8256 E−4	4.1796 E−4	1.8396 E−2
0.5	2.3794 E−4	3.8518 E−4	2.7327 E−2
0.6	5.5484 E−5	4.4513 E−4	3.7373 E−2
0.7	2.6947 E−5	5.3577 E−4	2.9311 E−2
0.8	1.8354 E−6	5.6066 E−4	5.1787 E−3
0.9	8.6807 E−5	5.4154 E−4	2.0294 E−2

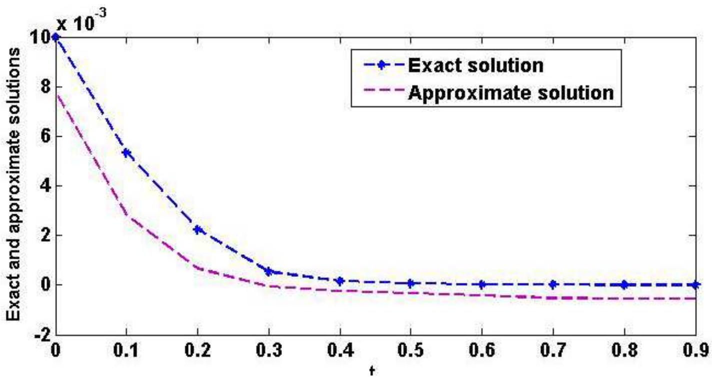

Fig. 5. Exact and approximate solutions for $X_0 = 0.01$ for Example 8.

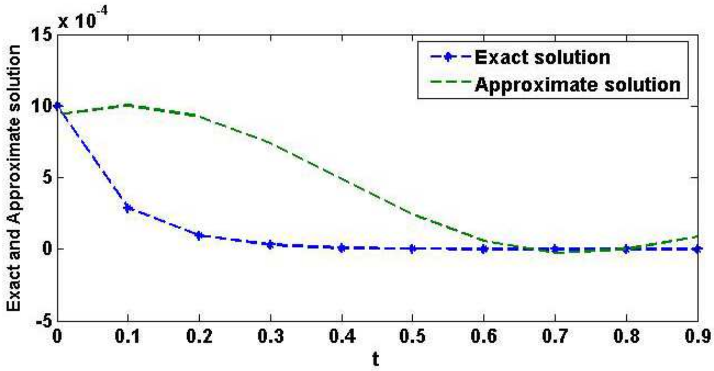

Fig. 6. Exact and approximate solutions for $X_0 = 0.001$ for Example 8.

8. Conclusion

Some stochastic differential equations can be written as stochastic Volterra integral equations. There are many stochastic integral equations which can not be solved analytically. In recent decade, many researcher are trying to develop the numerical methods for solving stochastic integral equations. In this paper, we introduced the cardinal Chebyshev functions, then the deterministic and stochastic operational matrices of these orthogonal functions have been obtained. These matrices can be also used to solve linear and nonlinear differential equations. These cardinal functions was used and applied for solving linear and nonlinear Volterra integral equations. The convergence and error analysis of the proposed method were investigated. Finally, several examples were included to demonstrate the applicability of the presented approach, the method of Chebyshev cardinal functions proposed in this paper can be further expanded to solve systems of stochastic integro-and integral equations for futur studies.

Acknowledgments. The author wish to express their gratitude to the Editor and referees for their helpful comments, suggestions and careful reading of the paper which have helped to improve the quality of the paper.

References

1. *Boyd, J. P. Chebyshev and Fourier Spectral Methods*, New York, Dover Publications, Inc., 2000.
2. *Wazwaz, A.-M. A First Course in Integral Equations*, New Jersey, World Scientific, 1997. DOI: 10.1142/9570.
3. *Kloeden, P. E and Platen, E. Numerical Solution of Stochastic Differential Equations*, Berlin, Springer, 1992.
4. *Milstein, G. N. Numerical Integration of Stochastic Differential Equations*, Mathematics and its Application, vol. 313, Dordrecht, Kluwer, 1995. DOI: 10.1007/978-94-015-8455-5.
5. *Asgari, M., Hashemizadeh, E., Khodabin, M. and Maleknedjad, K.* Numerical Solution of Nonlinear Stochastic Integral Equation by Stochastic Operational Matrix Based on Bernstein Polynomials, *Bull. Math. Soc. Sci. Math. Roumanie*, 2014, vol. 57 (105), no. 1, pp. 3–12.
6. *Mirazee, F. and Samadyar, N.* Application of Hat Basis Functions for Solving Two-Dimensional Stochastic Fractional Integral Equations, *Journal of Computational and Applied Mathematics*, 2018, vol. 37, pp. 4899–4916. DOI: 10.1007/s40314-018-0608-4.
7. *Mohammadi, F.* A Wavalet-Based Computational Method for Solving Stochastic Itô–Volterra Integral Equations, *Journal of Computational Physics*, 2015, vol. 298, pp. 254–265. DOI: 10.1016/J.JCP.2015.05.051.
8. *Mahmoudi, Y.* Wavelet Galerkin Method for Numerical Solution of Nonlinear Integral Equation, *Applied Mathematics and Computation*, 2005, vol. 167, no. 2, pp. 1119–1129. DOI: 10.1016/j.amc.2004.08.004.

9. *Maleknejad, K., Mollapourasl, R. and Alizadeh, M.* Numerical Solution of Volterra Type Integral Equation of the First Kind with Wavelet Basis, *Applied Mathematics and Computation*, 2007, vol. 194, no. 2, pp. 400–405. DOI: 10.1016/j.amc.2007.04.031.
10. *Mirzaee, F., Samadyar, N. and Hoseini, S. F.* Euler Polynomial Solutions of Nonlinear Stochastic Itô–Volterra Integral Equations, *Journal of Computational and Applied Mathematics*, 2018, vol. 330, pp. 574–585. DOI: 10.1016/j.cam.2017.09.005.
11. *Mirazee, F., Alipour, S. and Samadyar, N.* Numerical Solution Based on Hybrid of Block-Pulse and Parabolic Function for Solving a System of Nonlinear Stochastic Itô–Volterra Integral Equations of Fractional Order, *Journal of Computational and Applied Mathematics*, 2019, vol. 349, pp. 157–171. DOI: 10.1016/j.cam.2018.09.040.
12. *Mirazee, F. and Samadyar, N.* On the Numerical Solution of Stochastic Quadratic Integral Equations via Operational Matrix Method, *Mathematical Methods in the Applied Sciences*, 2018, vol. 41, no. 12, pp. 4465–4479. DOI: 10.1002/mma.4907.
13. *Mirazee, F. and Alipour, S.* Approximation Solution of Nonlinear Quadratic Integral Equations of Fractional Order via Piecewise Linear Functions, *Journal of Computational and Applied Mathematics*, 2018, vol. 331, pp. 217–227. DOI: 10.1016/j.cam.2017.09.038.
14. *Mirazee, F. and Hamzeh, A.* A Computational Method for Solving Nonlinear Stochastic Volterra Integral Equations, *Journal of Computational and Applied Mathematics*, 2016, vol. 306, pp. 166-0178. DOI: 10.1016/j.cam.2016.04.012.
15. *Mirazee, F. and Samadyar, N.* Numerical Solutions Based on Two-Dimensional Orthonormal Bernstein Polynomials for Solving Some Classes of Two-Dimensional Nonlinear Integral Equations of Fractional Order, *Applied Mathematics and Computation*, 2019, vol. 344, pp. 191–203. DOI: 10.1016/j.amc.2018.10.020.
16. *Mirazee, F. and Samadyar, N.* Using Radial Basis Functions to Solve Two Dimensional Linear Stochastic Integral Equations on Non-Rectangular Domains, *Engineering Analysis with Boundary Elements*, 2018, vol. 92, pp. 180–195. DOI: 10.1016/j.enganabound.2017.12.017.
17. *Mirazee, F. and Samadyar, N.* Numerical Solution of Nonlinear Stochastic Itô–Volterra Integral Equations Driven by Fractional Brownian Motion, *Mathematical Methods in the Applied Sciences*, 2018, vol. 41, no. 4, pp. 1410–1423. DOI: 10.1002/mma.4671.
18. *Shang, X. and Han, D.* Numerical Solution of Fredholm Integral Equations of the First Kind by Using Linear Legendre Multi-Wavelets, *Applied Mathematics and Computation*, 2007, vol. 191, no. 2, pp. 440–444. DOI: 10.1016/j.amc.2007.02.108.
19. *Yousefi, S. and Razzaghi, M.* Legendre Wavelets Method for the Nonlinear Volterra–Fredholm Integral Equations, *Mathematics and Computers in Simulation*, 2005, vol. 70, no. 1, pp. 1–8. DOI: 10.1016/j.matcom. 2005. 02. 035.
20. *Yousefi, S. A., Lotfi, A. and Dehghan, M.* He's Variational Iteration Method for Solving Nonlinear Mixed Volterra–Fredholm Integral Equations, *Computers and Mathematics with Applications*, 2009, vol. 58, no. 11–12, pp. 2172–2176. DOI: 10.1016/j.camwa.2009.03.083.
21. *Maleknejad, K., Khodabin, M. and Rostami, M.* Numerical Method for Solving m-Dimensional Stochastic Itô Volterra Integral Equations by Stochastic Operational Matrix Based on Block-Pulse Functions, *Comput.Math. Appl*, 2012, vol. 63, no. 1, pp. 133–143. DOI: 10.1016/j.camwa.2011.10.079.
22. *Mao, X.* Approximate Solutions for a Class of Stochastic Evolution Equations with Variable Delays. II, *Numerical Functional Analysis and Optimization*, 1994, vol. 15, no. 1–2, pp. 65–76. DOI: 10.1080/01630569408816550.
23. *Mirzaee, F. and Hadadiyan, E.* A Collocation Technique for Solving Nonlinear Stochastic Itô–Volterra Integral Equation, *Applied Mathematics and Computation*, 2014, vo. 247, pp. 1011–1020. DOI: 10.1016/j.amc.2014.09.047.
24. *Heydari, M. H., Mahmoudi, M. R., Shakiba, A. and Avazzadeh, Z.* Chebyshev Cardinal Wavelets and Their Application in Solving Nonlinear Stochastic Differential Equations with Fractional Brownian Motion, *Communications in Nonlinear Science and Numerical Simulation*, 2018, vol. 64, pp. 98–121. DOI: 10.1016/j.cnsns.2018.04.018.
25. *Heydari, M. H.* A New Direct Method Based on the Chebyshev Cardinal Functions for Variable-Order Fractional Optimal Control Problems, *Journal of the Franklin Institute*, 2018, vol. 355, no. 12, pp. 4970–4995. DOI: 10.1016/j.jfranklin.2018.05.025.
26. *Heydari, M., Avazzadeh, Z. and Loghmani, G. B.* Chebyshev Cardinal Functions for Solving Volterra–Fredholm Integro-Differential Equations Using Operational Matrices, *Iranian Journal of Science and Technology, A1*, 2012, vol. 36, no. 1, pp. 13–24. DOI: 10.22099/IJSTS.2012.2050.
27. *Funaro, D. Polynomial Approximation of Differential Equations*, New York, Springer-Verlag, 1992.

28. *Canuto, C., Hussaini, M., Quarteroni, A. and Zang, T. Spectral Methods in Fluid Dynamics*, Berlin, Springer, 1988.
29. *Blyth, W. F., May, R. L. and Widyaningsih, P.* Volterra Integral Equations Solved in Fredholm form Using Walsh Functions, *ANZIAM Journal* 2004, vol. 45 (E), pp. 269–282. DOI: 10.21914/anziamj.v45i0.887.
30. *Reihani, M. H. and Abadi, Z.* Rationalized Haar Functions Method for Solving Fredholm and Volterra Integral Equations, *Journal of Computational and Applied Mathematics*, 2007, vol. 200, pp. 12–20. DOI: 10.1016/j.cam.2005.12.026.
31. *Parand, K. and Delkhosh, M.* Operational Matrix to Solve Nonlinear Riccati Differential Equations of Arbitrary Order, *St. Pertersburg Polytechnical University Journal: Physics and Mathematics*, 2007, vol. 3, no. 3, pp. 242–254. DOI: 10.1016/j.spjpm.2017.08.001.
32. *Arató, M.* A Famous Nonlinear Stochastic Equation (Lotka–Volterra Model with Diffusion), *Mathematical and Computer Modelling*, 2003, vol. 38, no. 7–9, pp. 709-726. DOI: 10.1016/S0895-7177(03)90056-2.

Received October 12, 2020

Rebiha Zeghdane
Department of Mathematics, Faculty of Mathematics
and Informatics, University of Bordj-Bou-Arreridj,
El-Anasser 34030, Bordj-Bou-Arreridj, Algeria,
Associate Professor
E-mail: rebihae@yahoo.fr

Владикавказский математический журнал
2020, Том 22, Выпуск 4, С. 68–86

НОВЫЙ ЧИСЛЕННЫЙ МЕТОД РЕШЕНИЯ НЕЛИНЕЙНЫХ СТОХАСТИЧЕСКИХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ

Ребиха Зегдане[1]

[1] Университет Бордж-Бу-Арреридж,
Алжир, 34030, Бордж-Бу-Арреридж, Эль-Анасе
E-mail: rebihae@yahoo.fr,

Аннотация. Цель статьи — применить кардинальные функции Чебышева к численному решению стохастических интегральных уравнений Вольтерра. Метод основан на разложении искомого приближенного решения по кардинальным функциями Чебышева. Для упомянутых базисных функций выводится новая операционная матрица интегрирования. Точнее, искомое решение разлагается в терминах кардинальных функций Чебышева с неизвестными коэффициентами. Подставляя указанное разложение в исходную задачу, операционная матрица сводит стохастическое интегральное уравнение к системе алгебраических уравнений. Исследованы сходимость и оценка погрешности в пространстве C Соболева. Метод подвергнут численной оценке путем решения тестовых задач, взятых из литературы, с помощью которых демонстрируется вычислительная эффективность метода. С вычислительной точки зрения решение, полученное этим методом, отлично согласуется с решениями, полученными в других работах, и его эффективно использовать при решении различных задач.

Ключевые слова: кардинальные функции Чебышева, стохастическая операциональная матрица, броуновское движение, интеграл Ито, метод коллокации, численное решение.

Mathematical Subject Classification (2010): 45G10, 65R20.

Образец цитирования: *Zeghdane, R.* New Numerical Method for Solving Nonlinear Stochastic Integral Equations // Владикавк. мат. журн.—2020.—Т. 22, № 4.—С. 68–86 (in English). DOI: 10.46698/n8076-2608-1378-r.