Владикавказский математический журнал
2018, Том 20, Выпуск 1, С. 30–37

УДК **517.98**

THE UNIQUENESS OF THE SYMMETRIC STRUCTURE IN IDEALS OF COMPACT OPERATORS

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This paper is dedicated to the memory of Professor
Inomjon Gulomjonovich Ganiev

Let H be a separable infinite-dimensional complex Hilbert space, let $\mathscr{L}(H)$ be the C^*-algebra of bounded linear operators acting in H, and let $\mathscr{K}(H)$ be the two-sided ideal of compact linear operators in $\mathscr{L}(H)$. Let $(E, \|\cdot\|_E)$ be a symmetric sequence space, and let $\mathscr{C}_E := \{x \in \mathscr{K}(\mathscr{H}) : \{s_n(x)\}_{n=1}^{\infty} \in E\}$ be the proper two-sided ideal in $\mathscr{L}(H)$, where $\{s_n(x)\}_{n=1}^{\infty}$ are the singular values of a compact operator x. It is known that $\mathscr{C}_E$ is a Banach symmetric ideal with respect to the norm $\|x\|_{\mathscr{C}_E} = \|\{s_n(x)\}_{n=1}^{\infty}\|_E$.
A symmetric ideal $\mathscr{C}_E$ is said to have a unique symmetric structure if $\mathscr{C}_E = \mathscr{C}_F$, that is $E = F$, modulo norm equivalence, whenever $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ is isomorphic to another symmetric ideal $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$. At the Kent international conference on Banach space theory and its applications (Kent, Ohio, August 1979), A. Pelczynsky posted the following problem:
(P) Does every symmetric ideal have a unique symmetric structure?
This problem has positive solution for Schatten ideals $\mathscr{C}_p$, $1 \leqslant p < \infty$ (J. Arazy and J. Lindenstrauss, 1975). For arbitrary symmetric ideals problem (P) has not yet been solved. We consider a version of problem (P) replacing an isomorphism $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ by a positive linear surjective isometry. We show that if F is a strongly symmetric sequence space, then every positive linear surjective isometry $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ is of the form $U(x) = u^* x u$, $x \in \mathscr{C}_E$, where $u \in \mathscr{L}(H)$ is a unitary or antiunitary operator. Using this description of positive linear surjective isometries, it is established that existence of such an isometry $U : \mathscr{C}_E \to \mathscr{C}_F$ implies that $(E, \|\cdot\|_E) = (F, \|\cdot\|_F)$.

DOI: 10.23671/VNC.2018.1.11394.

Mathematical Subject Classification (2010): 46L52, 46B04.

Key words: symmetric ideal of compact operators, uniqueness of a symmetric structure, positive isometry.

1. Introduction

Let $(c_0, \|\cdot\|_\infty)$ be a Banach lattice of all sequences $a = \{\xi_n\}_{n=1}^{\infty}$ of real numbers convergent to zero, where $\|a\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|$ ($\mathbb{N}$ is the set of natural numbers). If $a = \{\xi_n\}_{n=1}^{\infty} \in c_0$, then a non-increasing rearrangement $a^* = \{\xi_n^*\}$ of a sequence a is a sequence $\{|\xi_n|\}$ in decreasing order.

A non-zero linear subspace $E \subset c_0$ with a Banach norm $\|\cdot\|_E$ is called *symmetric sequence space*, if the conditions $b \in E$, $a \in c_0$, $a^* \leqslant b^*$, imply that $a \in E$ and $\|a\|_E \leqslant \|b\|_E$.

Let H be a complex separable infinite-dimensional Hilbert space and let $\mathscr{L}(H)$ be an C^*-algebra of all bounded linear operators acting in H. We denote by $\mathscr{K}(H)$ (respectively, $\mathscr{F}(H)$) the two-sided ideal in $\mathscr{L}(H)$ of all compact (respectively, finite rank) linear operators. It is well known that $\mathscr{F}(\mathscr{H}) \subset \mathscr{I} \subset \mathscr{K}(\mathscr{H})$ for any proper two-sided ideal $\mathscr{I}$ in $\mathscr{L}(H)$ (see, for example, [10, Proposition 2.1]).

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then the set $\mathscr{C}_E := \{x \in \mathscr{K}(\mathscr{H}) : \{s_n(x)\}_{n=1}^{\infty} \in E\}$ is a proper two-sided ideal in $\mathscr{L}(H)$, where $\{s_n(x)\}_{n=1}^{\infty}$ are the singular values of x (i.e. the eigenvalues of $(x^*x)^{1/2}$ in decreasing order) [10, Theorem 2.5]. In addition, $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ is a Banach space with respect to the norm $\|x\|_{\mathscr{C}_E} = \|\{s_n(x)\}_{n=1}^{\infty}\|_E$ [14, Ch. 3, § 3.5]. In this case we say that $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ is a *symmetric ideal* (cf. [12, Ch. III]).

It is said that $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ to have *a unique symmetric structure*, if whenever $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ is isomorphic to another symmetric ideal $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$, then necessarily, $\mathscr{C}_E = \mathscr{C}_F$, i.e. $E = F$, with equivalent norms.

In Kent Conference (International Conference on Banach Space Theory and its Applications, Kent, Ohio, August 1979), A. Pelczynsky raised the following problem:

(P): Does every symmetric ideal have a unique symmetric structure?

In [3] it is proved that symmetric ideals $\mathscr{C}_p = \mathscr{C}_{l_p}$, $1 \leqslant p < \infty$, have unique symmetric structure. In addition, J. Arazy proved (see [4, Corollary 5.9]) the following

Theorem 1. *If a symmetric sequence space E does not contain a subspace isomorphic to c_0 and a space E does not contain a complemented subspace isomorphic to l_2, then $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ has a unique symmetric structure.*

Using the Theorem 1 it is easy to prove that for the Lorentz ideals the problem (P) is solved positively for $q \neq 2$ (see Section 2 below). If $q = 2$, then answer is also positive (O. Sadovskaya and F. Sukochev (unpublished)). At the same time, for arbitrary ideals the problem (P) has not yet been solved.

In this paper we consider the version of problem (P) (we call the problem (P^+)) in the case when isomorphism $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ is replaced by positive linear bijective isometry. We solve the problem (P^+) in the class of strongly symmetric ideals of compact operators.

2. Preliminaries

Let $p, q \in [1; \infty)$, and let

$$l_{p,q} = \left\{ a = \{\xi_n\}_{n=1}^{\infty} \in c_0 : \ \|a\|_{p,q} = \left(\sum_{n=1}^{\infty} (\xi_n^*)^q \left(n^{\frac{q}{p}} - (n-1)^{\frac{q}{p}} \right) \right)^{\frac{1}{q}} < \infty \right\}$$

be the Lorentz sequence space. It is known that $(l_{p,q}, \|\cdot\|_{p,q})$ is a symmetric sequence space $(1 \leqslant q \leqslant p < \infty)$, in addition,

$$l_{p,p} = l_p = \left\{ \{\xi_n\}_{n=1}^{\infty} \subset c_0, \ \|\{\xi_n\}\|_{p,p} = \|\{\xi_n\}\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

If $1 < p < q < \infty$, then $\|\cdot\|_{p,q}$ is a complete quasinorm on the vector lattice $l_{p,q}$; moreover, on $l_{p,q}$ there exists a norm $\|\cdot\|_{(p,q)}$, which equivalent to the quasinorm $\|\cdot\|_{p,q}$, such that $(l_{p,q}, \|\cdot\|_{(p,q)})$ is a symmetric sequence space [5, Ch. 4, § 4].

The following important property of the Lorentz sequence space is proved in the paper [7].

Theorem 2. *If $1 < p < \infty$, $1 \leqslant q < \infty$, then every infinite-dimensional closed subspace in $l_{p,q}$ has a closed subspace that is isomorphic to l_q.*

Consider the Lorentz symmetric ideal in $\mathscr{L}(H)$ defined by the equality

$$C_{p,q} = \{x \in \mathscr{K}(H) : \{s_n(x)\}_{n=1}^{\infty} \in l_{p,q}\},$$

equipped with the norm $\|x\|_{p,q} = \|\{s_n(x)\}\|_{p,q}$, $1 \leqslant q \leqslant p < \infty$ (respectively, $\|x\|_{p,q} = \|\{s_n(x)\}\|_{(p,q)}$, if $1 < p < q < \infty$).

Using Theorems 1 and 2, we can give a positive solution of the problem (P) for symmetric ideal $(C_{p,q}, \|\cdot\|_{p,q})$, $q \neq 2$.

Theorem 3. *Let $1 < p, q < \infty$, $q \neq 2$. If $(C_{p,q}, \|\cdot\|_{p,q})$ is isomorphic to symmetric ideal $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$, then $C_{p,q} = \mathscr{C}_E$, i. e. $E = l_{p,q}$, with equivalent norms.*

$\lhd$ It is known that the Lorentz sequences space $(l_{p,q}, \|\cdot\|_{p,q})$ (respectively, $(l_{p,q}, \|\cdot\|_{(p,q)})$) is reflexive if $1 < p, q < \infty$. And the conjugate space $(l_{p,q})^*$ is isomorphic to $l_{r,s}$, where $1 < r, s < \infty$, $\frac{1}{p} + \frac{1}{r} = 1$, $\frac{1}{q} + \frac{1}{s} = 1$ (see, for example, [5, Ch. 4, § 4, Theorem 4.7]). By Theorems 5.6, 5.11 [11] we have that the Lorentz symmetric ideal $(C_{p,q}, \|\cdot\|_{p,q})$, $1 < p, q < \infty$, is reflexive too, in addition, the conjugate space $(C_{p,q})^*$ is isomorphic to $C_{r,s}$, where $1 < r, s < \infty$, and $\frac{1}{p} + \frac{1}{r} = 1$, $\frac{1}{q} + \frac{1}{s} = 1$. Consequently, $(C_{p,q}, \|\cdot\|_{p,q})$, $1 < p, q < \infty$, does not contain a subspace isomorphic to c_0. Using now Theorems 1, 2, and the inequality $q \neq 2$, we have a positive solution of the problem (P). $\rhd$

Let $S_{p,q}$ denote the closed subspace of $C_{p,q}$ consisting of all block diagonal matrices $x = \operatorname{diag}\{x_k\}_{k=1}^{\infty}$ with x_k a $k \times k$-matrix for all k. By Corollary 5.8 [4] in the case when $S_{p,q}$ is not isomorphic to $C_{p,q}$ the Banach symmetric ideal $C_{p,q}$ has a unique symmetric structure. In an unpublished paper of O. Sadovskaya and F. Sukochev it is proved that $S_{p,q}$ it is not embedded in Banach spaces $C_{p,q}$, in particular, $S_{p,q}$ is not isomorphic to $C_{p,q}$ for all $1 < p, q < \infty$. Therefore, in the case $q = 2$, the Theorem 3 is true too.

Since the Banach spaces $(l_{p,q}, \|\cdot\|_{p,q})$ and $(l_{r,s}, \|\cdot\|_{r,s})$ are isomorphic if and only if $p = r$, $q = s$ (see [10]), Theorem 3 implies the following isomorphic classification of Banach symmetric ideal $(C_{p,q}, \|\cdot\|_{p,q})$.

Corollary 1. *Let $1 < p, q, r, s < \infty$. The Banach spaces $(C_{p,q}, \|\cdot\|_{p,q})$ and $(C_{r,s}, \|\cdot\|_{r,s})$ are isomorphic if and only if $p = r$, $q = s$.*

It should be noted that the Banach space $C_{2,2} = C_2$ is a separable Hilbert space and it is isomorphic to $l_2 = l_{2,2}$, in particular, C_2 has local unconditional structure. For all other variants of the values of the parameters p, q the Lorentz symmetric ideal $(C_{p,q}, \|\cdot\|_{p,q})$ has not local unconditional structure [13]. Since a Banach lattice has a locally unconditional structure, it follows that spaces $l_{p,q}$ and $C_{p,q}$ are not isomorphic, if $1 \leqslant p, q < \infty$, $p \neq 1$ and $q \neq 2$, or $p \neq 2$.

Define in $\mathscr{K}(H)$ (respectively, in c_0) the Hardy–Littlewood–Polya partial order $x \prec\prec y$ (respectively, $\{\xi\}_{n=1}^{\infty} \prec\prec \{\eta\}_{n=1}^{\infty}$), if

$$\sum_{n=1}^{k} s_n(x) \leqslant \sum_{n=1}^{k} s_n(y) \quad \left(\text{respectively, } \sum_{n=1}^{k} \xi_n^* \leqslant \sum_{n=1}^{k} \eta_n^*\right) \quad \forall\, k \in \mathbb{N}.$$

It is clear that $x \prec\prec y \Leftrightarrow \{s_n(x)\}_{n=1}^{\infty} \prec\prec \{s_n(y)\}_{n=1}^{\infty}$.

The Hardy–Littlewood–Polya partial order has the following important property

Proposition 1 [9, Proposition 2.1]. *If $x, y \in \mathscr{K}(H)$, $x = x^*$, $y \geqslant 0$, and $-y \leqslant x \leqslant y$, then $x \prec\prec y$.*

A symmetric ideal (a symmetric sequence space) $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ (respectively, $(E, \|\cdot\|_E)$) is called *a strongly symmetric ideal* (respectively, *a strongly symmetric sequences space*) if the condition $x \prec\prec y$, $x, y \in \mathscr{C}_E$ (respectively, $a \prec\prec b$, $a, b \in E$) implies that $\|x\|_{\mathscr{C}_E} \leqslant \|y\|_{\mathscr{C}_E}$ (respectively, $\|a\|_E \leqslant \|b\|_E$). It is clear that a symmetric ideal $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ is a strongly symmetric ideal if and only if a symmetric sequence space $(E, \|\cdot\|_E)$) is a strongly symmetric sequence space.

The Proposition 1 implies the following.

Corollary 2. *Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ be a strongly symmetric ideal. If $x, y \in \mathscr{C}_E$, $y \geqslant 0$, $x^* = x$ and $-y \leqslant x \leqslant y$, then $\|x\|_{\mathscr{C}_E} \leqslant \|y\|_{\mathscr{C}_E}$.*

3. Description of positive isometries of symmetric ideals

In this section we give a description of positive linear bijective isometry $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$, when $\mathscr{C}_F$ is a strongly symmetric ideal.

The following Proposition establishes positivity of the inverse mapping of positive surjective isometry.

Proposition 2. *Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ be a symmetric ideal and let $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ be a strongly symmetric ideal. Let $U\colon \mathscr{C}_E \to \mathscr{C}_F$ be a positive linear surjective isometry. Then an isometry U^{-1} is also positive.*

$\lhd$ Let $x \in \mathscr{C}_E$ and $U(x) \geqslant 0$. Since U is a positive surjective mapping it follows that $U(y^*) = U(y)^*$ for all $y \in \mathscr{C}_E$. Hence $x^* = x$. Let x_+ and x_- be a positive and negative parts of an operator x. It is clear that $x_+, x_- \in \mathscr{C}_E$. If $x_+ = 0$, then $U(x) \leqslant 0$. Consequently, $U(x) = 0$, which implies $x = 0$.

Let now $x_+ \neq 0$. Set $y_1 = U(x_+)$ and $y_2 = U(x_-)$. We have that

$$y_1 \geqslant 0,\ y_2 \geqslant 0 \quad \text{and} \quad y = y_1 - y_2 = U(x) \geqslant 0.$$

In addition,

$$y_1 + y_2 = U(|x|) \quad \text{and} \quad \|U(|x|)\|_{\mathscr{C}_F} = \|\,|x|\,\|_{\mathscr{C}_E} = \|x\|_{\mathscr{C}_E}.$$

Using mathematical induction, we show that

$$\|x_+ + kx_-\|_{\mathscr{C}_E} = \|y_1 + ky_2\|_{\mathscr{C}_F} \leqslant \|x\|_{\mathscr{C}_E} \tag{1}$$

for all $k \in \mathbb{N}$. If $k = 1$, then the inequality (1) is obvious. Suppose that it is true for $k = n$. Then

$$-(y_1 + ny_2) \leqslant (y_1 - y_2) - ny_2 = y_1 - (n+1)y_2 \leqslant y_1 + ny_2.$$

By Proposition 1 we have that $y_1 - (n+1)y_2 \prec\prec y_1 + ny_2$. Since $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ is a strongly symmetric ideal it follows that (see Corollary 2)

$$\|y_1 - (n+1)y_2\|_{\mathscr{C}_F} \leqslant \|y_1 + ny_2\|_{\mathscr{C}_F} \leqslant \|x\|_{\mathscr{C}_E}.$$

Thus

$$\begin{aligned}\|y_1 + (n+1)y_2\|_{\mathscr{C}_F} &= \|x_+ + (n+1)x_-\|_{\mathscr{C}_E} = \|\,|x_+ - (n+1)x_-|\,\|_{\mathscr{C}_E}\\ &= \|x_+ - (n+1)x_-\|_{\mathscr{C}_E} = \|y_1 - (n+1)y_2\|_{\mathscr{C}_F} \leqslant \|x\|_{\mathscr{C}_E}.\end{aligned}$$

Therefore the inequality (1) holds for all $k \in \mathbb{N}$. Since

$$k\|x_-\|_{\mathscr{C}_E} \leqslant \|x_+ + kx_-\|_{\mathscr{C}_E} \leqslant \|x\|_{\mathscr{C}_E} \quad \text{for all} \quad k \in \mathbb{N},$$

it follows that $\|x_-\|_{\mathscr{C}_E} = 0$, that is $x \geqslant 0$. $\rhd$

Remark 1. The proof of Proposition 2 is analogous to the proof of Theorem 1 in [1], where the positivity of the inverse mapping for isometries of Banach lattices is established.

Theorem 4. *Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ be a symmetric ideal and let $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ be a strongly symmetric ideal. Let $U\colon \mathscr{C}_E \to \mathscr{C}_F$ be a positive linear surjective isometry. Then $U(p)$*

(respectively, $U^{-1}(e)$) is an one-dimensional projection for any one-dimensional projection $p \in \mathscr{F}(H)$ (respectively, $e \in \mathscr{F}(H)$).

$\lhd$ Suppose that $U(p) = y$ is not a rank 1 operator. Since $y \geqslant 0$, $y \in \mathscr{C}_F$ it follows that there exist pairwise orthogonal one-dimensional projections $q_1, q_2 \in \mathscr{F}(H)$ and positive numbers λ_1, λ_2 such that $0 < \lambda_1 q_1 + \lambda_2 q_2 \leqslant y$. By Proposition 2 we have that

$$0 < U^{-1}(\lambda_1 q_1 + \lambda_2 q_2) \leqslant U^{-1}(y) = p.$$

If $U^{-1}(q_i) = x_i$, then $0 < \lambda_i x_i \leqslant p$, $i = 1, 2$. Since p is an one-dimensional projection, it follows that $\lambda_i x_i = \gamma_i p$ for some $\gamma_i > 0$. Consequently, $q_i = U(x_i) = U(\frac{\lambda_i}{\gamma_i} p) = \frac{\lambda_i}{\gamma_i} y$, $i = 1, 2$, which is impossible, because $q_1 q_2 = 0$.

Therefore, $U(p) = \lambda q$ for some one-dimensional projection q and positive number λ.

Now using the inequalities

$$1 = \|p\|_{\mathscr{C}_E} = \|U(p)\|_{\mathscr{C}_F} = \lambda \|q\|_{\mathscr{C}_F} = \lambda,$$

we have that $\lambda = 1$. Consequently, $U(p) = q$.

Similarly $U^{-1}(e)$ is an one-dimensional projection for any one-dimensional projection $e \in \mathscr{F}(H)$. $\rhd$

Corollary 3. *Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$, $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ and $U\colon \mathscr{C}_E \to \mathscr{C}_F$ be the same as in Theorem 4. Then $U(\mathscr{F}(H)) = \mathscr{F}(H)$.*

A linear bijective mapping $\varphi\colon \mathscr{L}(H) \to \mathscr{L}(H)$ is called an *Jordan isomorphism*, if $\varphi(x^2) = (\varphi(x))^2$ and $\varphi(x^*) = (\varphi(x))^*$ for all $x \in \mathscr{L}(H)$. If $\varphi\colon \mathscr{L}(H) \to \mathscr{L}(H)$ is an Jordan isomorphism, then there exists an unitary or an antiunitary operator $u \in \mathscr{L}(H)$ such that $\varphi(x) = u^* x u$ for all $x \in \mathscr{L}(H)$ (see, for example, [6, Ch. 3, § 3.2.1]).

Let H be a k-dimensional complex Hilbert space. In this case $\mathscr{L}(H) = \mathscr{K}(H)$. If $(E, \|\cdot\|_E) \subset c_0$ is a symmetric sequence space, then the set

$$\mathscr{C}_E(k) := \{x \in \mathscr{K}(H) : \{s_1(x), \dots, s_k(x), 0, \dots\} \in E\} = \mathscr{L}(H)$$

is a k-dimensional simmetric space with respect to the norm

$$\|x\|_{\mathscr{C}_E(k)} = \|\{s_1(x), \dots, s_k(x), 0, \dots\}\|_E.$$

Using the description of all positive linear surjective isometries of strongly symmetric spaces $E(M, \tau)$ in the case a finite von Neumann algebra M and a finite trace τ [8, Theorem 3.1], we have the following

Theorem 5. *Let $(E, \|\cdot\|_E) \subset c_0$ be a symmetric sequence space with a strongly symmetric norm. Let $U\colon \big(\mathscr{C}_E(k), \|\cdot\|_{\mathscr{C}_E(k)}\big) \to \big(\mathscr{C}_E(k), \|\cdot\|_{\mathscr{C}_E(k)}\big)$ be a positive linear surjective isometry. Then there exists an Jordan isomorphism $\varphi\colon \mathscr{L}(H) \to \mathscr{L}(H)$ such that $U(x) = \varphi(x)$ for all $x \in \mathscr{C}_E(k) = \mathscr{L}(H)$. In particular, $U(x)U(y) = U(y)U(x)$ if and only if $xy = yx$.*

The following Theorem gives a description of positive linear bijective isometries $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$, when $\mathscr{C}_F$ is a strongly symmetric ideal.

Theorem 6 (cf. [2, 16])**.** *Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ be a symmetric ideal and $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ be a strongly symmetric ideal. Let $U\colon \mathscr{C}_E \to \mathscr{C}_F$ be a positive linear surjective isometry. Then there exists an unitary or antiunitary operator $u \in \mathscr{L}(H)$ such that $U(x) = u^* x u$ for all $x \in \mathscr{C}_E$.*

$\lhd$ By Proposition 2 we have that an inverse isometry U^{-1} is also a positive map. Let $p, e, q, f \in \mathscr{F}(H)$ be an one-dimensional projections such that $U(p) = q$, $U(e) = f$ (see Theorem 4). If $p \cdot e = 0$, then by Theorem 5 we have that $q \cdot f = 0$.

Let $\{p_n\}_{n=1}^k \subset \mathscr{F}(H)$ be a pairwise orthogonal one-dimensional projections and $x = \sum_{n=1}^k \lambda_n p_n \in \mathscr{F}(H)$, $\lambda_n \in \mathbb{R}$, $n = 1, \dots, k$. Since $U(p_n) \cdot U(p_m) = 0$, $n \neq m$, $n, m = 1, \dots, k$, it follows that

$$U(x^2) = U\left(\sum_{n=1}^k \lambda_n^2 p_n\right) = \sum_{n=1}^k \lambda_n^2 U(p_n) = U(x)^2$$

and

$$tr(U(x)) = \sum_{n=1}^k \lambda_n tr(U(p_n)) = \sum_{n=1}^k \lambda_n = tr(x).$$

Therefore $U(x^2) = U(x)^2$ and $tr(U(x)) = tr(x)$ for all $x^* = x \in \mathscr{F}(H)$. In addition, U is a bijection of the set $\mathscr{P}(H)$ of all one-dimensional projections.

If $p, e, q, f \in \mathscr{P}(H)$ and $U(p) = q$, $U(e) = f$, then

$$\begin{aligned} 2tr(pe) = tr(pe) + tr(ep) = tr\left((p+e)^2 - p - e\right) tr\left(U\left((p+e)^2\right)\right) - 2 \\ = tr\left(U\left((p+e)^2\right)\right) - 2 = tr\left((q+f)^2\right) - 2 = 2tr(qf). \end{aligned}$$

Consequently, $tr(pe) = tr(U(p)U(e))$ for all $p, e \in \mathscr{P}(H)$. Now using Theorem 3.2.8 [6, Ch. 3, § 3.2] we get that there exists an unitary or antiunitary operator u such that $U(p) = u^* p u$ for all $p \in \mathscr{P}(H)$. Thus $U(x) = u^* x u$ for all $x \in \mathscr{F}(H)$.

Let $0 \leqslant x \in \mathscr{C}_E$ and $0 \leqslant x_n \in \mathscr{F}(H)$ be such a sequence that $x_n \uparrow x$. Since $U\colon \mathscr{C}_E \to \mathscr{C}_E$ is an order isomorphism (see Proposition 2) it follows that $u^* x_n u = U(x_n) \uparrow U(x)$. Consequently, $U(x) = u^* x u$ for all $x \in \mathscr{C}_E$. $\rhd$

4. Pelchinsky problem with respect positive isometries

Consider now the following version of problem (P):

(P^+): Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ and $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ are symmetric ideals and let there exists a positive isometry $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$. Is it true that then $(E, \|\cdot\|_E) = (F, \|\cdot\|_F)$?

Below we give a solution of the problem (P^+) for the class of strongly symmetric ideals of compact operators.

Theorem 7. *Let $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ be a symmetric ideal and let $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ be a strongly symmetric ideal. Let $U\colon \mathscr{C}_E \to \mathscr{C}_F$ be a positive linear surjective isometry. Then $(E, \|\cdot\|_E) = (F, \|\cdot\|_F)$.*

$\lhd$ By Theorem 6 there exists an unitary or an antiunitary operator $u \in \mathscr{L}(H)$ such that $U(x) = u^* x u$ for all $x \in \mathscr{C}_E$. Fix an orthonormal basis $\{\psi_n\}_{n=1}^\infty$ in a separable Hilbert space H. Let $p_n \in \mathscr{P}(H)$, $p_n(\psi_n) = \psi_n$, $n \in \mathbb{N}$. Consider real subspace $G_E = \{x = \sum_{n=1}^\infty \xi_n p_n : \xi_n \in \mathbb{R}, x \in \mathscr{C}_E\}$ in the space $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$. It is clear that $\{\xi_n\}_{n=1}^\infty \in E$ and $\|x\|_{\mathscr{C}_E} = \|\{\xi_n\}\|_E$ for all $x = \sum_{n=1}^\infty \xi_n p_n \in G_E$. Consequently, the correspondence $G_E \ni x \leftrightarrow \{\xi_n\} \in E$ identifies the Banach spaces $(G_E, \|\cdot\|_{\mathscr{C}_E})$ and $(E, \|\cdot\|_E)$.

Since u is an unitary or antiunitary operator it follows that $\upsilon_n = u^* \psi_n u$, $n \in \mathbb{N}$, is an orthonormal basis in a Hilbert space H. Let $q_n \in \mathscr{P}(H)$, $q_n(\upsilon_n) = \upsilon_n$, $n \in \mathbb{N}$. Set $G_F = \left\{x = \sum_{n=1}^\infty \eta_n q_n : \eta_n \in \mathbb{R}, x \in \mathscr{C}_F\right\}$. It is clear that the correspondence $G_F \ni x \leftrightarrow \{\eta_n\} \in F$ identifies the Banach spaces $(G_F, \|\cdot\|_{\mathscr{C}_F})$ and $(F, \|\cdot\|_F)$. Since $G_F = u^* G_E u$ we get that $(E, \|\cdot\|_E) = (F, \|\cdot\|_F)$. $\rhd$

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Received 29 November, 2017

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ЕДИНСТВЕННОСТЬ СИММЕТРИЧНОЙ СТРУКТУРЫ В ИДЕАЛАХ КОМПАКТНЫХ ОПЕРАТОРОВ

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Пусть H — сепарабельное бесконечномерное комплексное гильбертово пространство, $\mathscr{L}(H)$ — C^*-алгебра ограниченных линейных операторов, действующих в H, $\mathscr{K}(H)$ — двусторонний идеал в $\mathscr{L}(H)$ всех компактных операторов. Пусть $(E, \|\cdot\|_E)$ — симметричное пространство последовательностей, $\mathscr{C}_E := \{x \in \mathscr{K}(\mathscr{H}) : \{s_n(x)\}_{n=1}^{\infty} \in E\}$ — собственный двусторонний идеал в $\mathscr{L}(H)$, порожденный $(E, \|\cdot\|_E)$, где $\{s_n(x)\}_{n=1}^{\infty}$ сингулярные числа компактного оператора x. Известно, что $\mathscr{C}_E$ — банахов симметричный идеал относительно нормы $\|x\|_{\mathscr{C}_E} = \|\{s_n(x)\}_{n=1}^{\infty}\|_E$.

Говорят, что симметричный идеал $\mathscr{C}_E$ имеет единственную симметричную структуру, если наличие изоморфизма из $(\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E})$ на другой симметричный идеал $(\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ обязательно влечет равенство $\mathscr{C}_E = \mathscr{C}_F$, т. е. $E = F$, с точностью до эквивалентных норм. На международной конференции по теории банаховых пространств и их приложений (Kent, Ohio, August 1979), А. Пельчинский поставил следующую проблему:
(P): Каждый ли симметричный идеал имеет единственную симметричную структуру?
Эта проблема получила положительное решение в работе J. Arazy и J. Lindenstrauss (1975) для идеалов Шаттена $\mathscr{C}_p$, $1 \leqslant p < \infty$. В случае произвольных симметричных идеалов проблема (P) до сих пор не решена. Мы рассматриваем вариант проблемы (P), заменяя наличие изоморфизма $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ на существование положительной линейной сюръективной изометрии. Мы показываем, что в случае строго симметричного пространства последовательностей F, каждая положительная линейная сюръективная изометрия $U : (\mathscr{C}_E, \|\cdot\|_{\mathscr{C}_E}) \to (\mathscr{C}_F, \|\cdot\|_{\mathscr{C}_F})$ имеет следующий вид: $U(x) = u^*xu$ для всех $x \in \mathscr{C}_E$, где $u \in \mathscr{L}(H)$ есть унитарный или антиунитарный оператор. Используя это описание положительных линейных сюръективных изометрий, доказывается, что наличие такой изометрии $U : \mathscr{C}_E \to \mathscr{C}_F$ влечет равенство $(E, \|\cdot\|_E) = (F, \|\cdot\|_F)$.

Ключевые слова: симметричный идеал компактных операторов, единственность симметричной структуры, положительная изометрия.