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1. Higher dimensional local fields

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We give here basic definitions related to n-dimensional local fields. For detailed exposition, see [P] in the equal characteristic case, [K1, §8] for the two-dimensional case and [MZ1], [MZ2] for the general case. Several properties of the topology on the multiplicative group are discussed in [F].

1.1. Main definitions

Suppose that we are given a surface S over a finite field of characteristic p, a curve $C \subset S$, and a point $x \in C$ such that both S and C are regular at x. Then one can attach to these data the quotient field of the completion $\widehat{(\widehat{\mathcal{O}}_{S,x})_C}$ of the localization at C of the completion $\widehat{\mathcal{O}}_{S,x}$ of the local ring $\mathcal{O}_{S,x}$ of S at x. This is a two-dimensional local field over a finite field, i.e., a complete discrete valuation field with local residue field. More generally, an n-dimensional local field F is a complete discrete valuation field with $(n-1)$-dimensional residue field. (Finite fields are considered as 0-dimensional local fields.)

Definition. A complete discrete valuation field K is said to have the structure of an n-dimensional local field if there is a chain of fields $K = K_n, K_{n-1}, \dots, K_1, K_0$ where K_{i+1} is a complete discrete valuation field with residue field K_i and K_0 is a finite field. The field $k_K = K_{n-1}$ (resp. K_0) is said to be the *first* (resp. the *last*) residue field of K.

Remark. Most of the properties of n-dimensional local fields do not change if one requires that the last residue K_0 is *perfect* rather than finite. To specify the exact meaning of the word, K can be referred to as an n-dimensional local field *over* a finite (resp. perfect) field. One can consider an n-dimensional local field over an arbitrary field K_0 as well. However, in this volume mostly the higher local fields over finite fields are considered.

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Examples. 1. $\mathbb{F}_q((X_1))\dots((X_n))$. 2. $k((X_1))\dots((X_{n-1}))$, k a finite extension of $\mathbb{Q}_p$.

3. For a complete discrete valuation field F let

$$K = F\{\{T\}\} = \left\{\sum_{-\infty}^{+\infty} a_i T^i : a_i \in F,\ \inf v_F(a_i) > -\infty,\ \lim_{i\to-\infty} v_F(a_i) = +\infty\right\}.$$

Define $v_K(\sum a_i T^i) = \min v_F(a_i)$. Then K is a complete discrete valuation field with residue field $k_F((t))$.

Hence for a local field k the fields

$$k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n)),\quad 0 \leqslant m \leqslant n-1$$

are n-dimensional local fields (they are called *standard fields*).

Remark. $K((X))\{\{Y\}\}$ is isomorphic to $K((Y))((X))$.

Definition. An n-tuple of elements $t_1,\dots,t_n \in K$ is called a *system of local parameters* of K, if t_n is a prime element of K_n, t_{n-1} is a unit in $\mathcal{O}_K$ but its residue in K_{n-1} is a prime element of K_{n-1}, and so on.

For example, for $K = k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n))$, a convenient system of local parameter is $T_1,\dots,T_m,\pi,T_{m+2},\dots,T_n$, where π is a prime element of k.

Consider the maximal m such that $\operatorname{char}(K_m) = p$; we have $0 \leqslant m \leqslant n$. Thus, there are $n+1$ types of n-dimensional local fields: fields of characteristic p and fields with $\operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$, $0 \leqslant m \leqslant n-1$. Thus, the mixed characteristic case is the case $m = n-1$.

Suppose that $\operatorname{char}(k_K) = p$, i.e., the above m equals either $n-1$ or n. Then the set of Teichmüller representatives $\mathcal{R}$ in $\mathcal{O}_K$ is a field isomorphic to K_0.

Classification Theorem. *Let K be an n-dimensional local field. Then*

(1) K *is isomorphic to* $\mathbb{F}_q((X_1))\dots((X_n))$ *if* $\operatorname{char}(K) = p$*;*

(2) K *is isomorphic to* $k((X_1))\dots((X_{n-1}))$*,* k *is a local field, if* $\operatorname{char}(K_1) = 0$*;*

(3) K *is a finite extension of a standard field* $k\{\{T_1\}\}\dots\{\{T_m\}\}((T_{m+2}))\dots((T_n))$ *and there is a finite extension of* K *which is a standard field if* $\operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$.

Proof. In the equal characteristic case the statements follow from the well known classification theorem for complete discrete valuation fields of equal characteristic. In the mixed characteristic case let k_0 be the fraction field of $W(\mathbb{F}_q)$ and let $T_1,\dots,T_{n-1},\pi$ be a system of local parameters of K. Put

$$K' = k_0\{\{T_1\}\}\dots\{\{T_{n-1}\}\}.$$

Then K' is an absolutely unramified complete discrete valuation field, and the (first) residue fields of K' and K coincide. Therefore, K can be viewed as a finite extension of K' by [FV, II.5.6].

Alternatively, let $t_1, \dots, t_{n-1}$ be any liftings of a system of local parameters of k_K. Using the canonical lifting $h_{t_1,\dots,t_{n-1}}$ defined below, one can construct an embedding $K' \hookrightarrow K$ which identifies T_i with t_i.

To prove the last assertion of the theorem, one can use *Epp's theorem on elimination of wild ramification* (see 17.1) which asserts that there is a finite extension l/k_0 such that $e\left(lK/lK'\right) = 1$. Then lK' is standard and lK is standard, so K is a subfield of lK. See [Z] or [KZ] for details and a stronger statement. □

Definition. The *lexicographic order* of $\mathbb{Z}^n$: $\mathbf{i} = (i_1, \dots, i_n) \leqslant \mathbf{j} = (j_1, \dots, j_n)$ if and only if

$$i_l \leqslant j_l,\ i_{l+1} = j_{l+1}, \dots, i_n = j_n \text{ for some } l \leqslant n\,.$$

Introduce $\mathbf{v} = (v_1, \dots, v_n)\colon K^* \to \mathbb{Z}^n$ as $v_n = v_{K_n}$, $v_{n-1}(\alpha) = v_{K_{n-1}}(\alpha_{n-1})$ where α_{n-1} is the residue of $\alpha t_n^{-v_n(\alpha)}$ in K_{n-1}, and so on. The map $\mathbf{v}$ is a valuation; this is a so called discrete valuation of rank n. Observe that for $n > 1$ the valuation $\mathbf{v}$ does depend on the choice of $t_2, \dots, t_n$. However, all the valuations obtained this way are in the same class of equivalent valuations.

Now we define several objects which do not depend on the choice of a system of local parameters.

Definition.
$O_K = \{\alpha \in K : \mathbf{v}(\alpha) \geqslant 0\}$, $M_K = \{\alpha \in K : \mathbf{v}(\alpha) > 0\}$, so $O_K/M_K \simeq K_0$. The group of *principal units* of K with respect to the valuation $\mathbf{v}$ is $V_K = 1 + M_K$.

Definition.

$$P\,(i_l, \dots, i_n) = P_K\,(i_l, \dots, i_n) = \{\alpha \in K : (v_l(\alpha), \dots, v_n(\alpha)) \geqslant (i_l, \dots, i_n)\}.$$

In particular, $O_K = P\big(\underbrace{0, \dots, 0}_{n}\big)$, $M_K = P\big(1, \underbrace{0, \dots, 0}_{n-1}\big)$, whereas $\mathcal{O}_K = P(0)$, $\mathcal{M}_K = P(1)$. Note that if $n > 1$, then

$$\cap_i M_K^i = P\big(1, \underbrace{0, \dots, 0}_{n-2}\big),$$

since $t_2 = t_1^{i-1}(t_2/t_1^{i-1})$.

Lemma. *The set of all non-zero ideals of O_K consists of all*

$$\{P\,(i_l, \dots, i_n) : (i_l, \dots, i_n) \geqslant (0, \dots, 0)\,, \quad 1 \leqslant l \leqslant n\}.$$

The ring O_K is not Noetherian for $n > 1$.

Proof. Let J be a non-zero ideal of O_K. Put $i_n = \min\{v_n(\alpha) : \alpha \in J\}$. If $J = P(i_n)$, then we are done. Otherwise, it is clear that

$$i_{n-1} := \inf\{v_{n-1}(\alpha) : \alpha \in J, v_n(\alpha) = i_n\} > -\infty.$$

If $i_n = 0$, then obviously $i_{n-1} \geqslant 0$. Continuing this way, we construct $(i_l, \dots, i_n) \geqslant (0, \dots, 0)$, where either $l = 1$ or

$$i_{l-1} = \inf\{v_{l-1}(\alpha) : \alpha \in J, v_n(\alpha) = i_n, \dots, v_l(\alpha) = i_l\} = -\infty.$$

In both cases it is clear that $J = P(i_l, \dots, i_n)$.

The second statement is immediate from $P(0,1) \subset P(-1,1) \subset P(-2,1) \dots$. □

For more on ideals in O_K see subsection 3.0 of Part II.

1.2. Extensions

Let L/K be a finite extension. If K is an n-dimensional local field, then so is L.

Definition. Let $t_1, \dots, t_n$ be a system of local parameters of K and let $t_1^{'}, \dots, t_n^{'}$ be a system of local parameters of L. Let $\mathbf{v}, \mathbf{v}'$ be the corresponding valuations. Put

$$E(L|K) := \left(v_j^{'}(t_i)\right)_{i,j} = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ \dots & e_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & e_n \end{pmatrix},$$

where $e_i = e_i(L|K) = e(L_i|K_i)$, $i = 1, \dots, n$. Then e_i do not depend on the choice of parameters, and $|L : K| = f(L|K) \prod_{i=1}^n e_i(L|K)$, where $f(L|K) = |L_0 : K_0|$.

The expression "unramified extension" can be used for extensions L/K with $e_n(L|K) = 1$ and L_{n-1}/K_{n-1} separable. It can be also used in a narrower sense, namely, for extensions L/K with $\prod_{i=1}^n e_i(L|K) = 1$. To avoid ambiguity, sometimes one speaks of a "semiramified extension" in the former case and a "purely unramified extension" in the latter case.

1.3. Topology on K

Consider an example of n-dimensional local field

$$K = k\,\{\{T_1\}\}\,\dots\{\{T_m\}\}\,((T_{m+2}))\dots((T_n)).$$

Expanding elements of k into power series in π with coefficients in $\mathcal{R}_k$, one can write elements of K as formal power series in n parameters. To make them convergent power series we should introduce a topology in K which takes into account topologies of the residue fields. We do not make K a topological field this way, since multiplication is only sequentially continuous in this topology. However, for class field theory sequential continuity seems to be more important than continuity.

1.3.1.

Definition.

(a) If F has a topology, consider the following topology on $K = F((X))$. For a sequence of neighbourhoods of zero $(U_i)_{i\in\mathbb{Z}}$ in F, $U_i = F$ for $i \gg 0$, denote $U_{\{U_i\}} = \left\{\sum a_i X^i : a_i \in U_i\right\}$. Then all $U_{\{U_i\}}$ constitute a base of open neighbourhoods of 0 in $F((X))$. In particular, a sequence $u^{(n)} = \sum a_i^{(n)} X^i$ tends to 0 if and only if there is an integer m such that $u^{(n)} \in X^m F[[X]]$ for all n and the sequences $a_i^{(n)}$ tend to 0 for every i.

Starting with the discrete topology on the last residue field, this construction is used to obtain a well-defined topology on an n-dimensional local field of characteristic p.

(b) Let K_n be of mixed characteristic. Choose a system of local parameters $t_1, \dots, t_n = \pi$ of K. The choice of $t_1, \dots, t_{n-1}$ determines a canonical lifting

$$h = h_{t_1,\dots,t_{n-1}} \colon K_{n-1} \to \mathcal{O}_K$$

(see below). Let $(U_i)_{i\in\mathbb{Z}}$ be a system of neighbourhoods of zero in K_{n-1}, $U_i = K_{n-1}$ for $i \gg 0$. Take the system of all $U_{\{U_i\}} = \left\{\sum h(a_i)\pi^i,\ a_i \in U_i\right\}$ as a base of open neighbourhoods of 0 in K. This topology is well defined.

(c) In the case $\operatorname{char}(K) = \operatorname{char}(K_{n-1}) = 0$ we apply constructions (a) and (b) to obtain a topology on K which depends on the choice of the coefficient subfield of K_{n-1} in $\mathcal{O}_K$.

The definition of the canonical lifting $h_{t_1,\dots,t_{n-1}}$ is rather complicated. In fact, it is worthwhile to define it for any $(n-1)$-tuple $(t_1, \dots, t_{n-1})$ such that $v_i(t_i) > 0$ and $v_j(t_i) = 0$ for $i < j \leqslant n$. We shall give an outline of this construction, and the details can be found in [MZ1, §1].

Let $F = K_0((\overline{t_1}))\dots((\overline{t_{n-1}})) \subset K_{n-1}$. By a lifting we mean a map $h\colon F \to \mathcal{O}_K$ such that the residue of $h(a)$ coincides with a for any $a \in F$.

Step 1. An auxiliary lifting $H_{t_1,\dots,t_{n-1}}$ is uniquely determined by the condition

$$H_{t_1,\dots,t_{n-1}}\left(\sum_{i_1=0}^{p-1}\cdots\sum_{i_{n-1}=0}^{p-1} \overline{t_1}^{\,i_1}\dots\overline{t_{n-1}}^{\,i_{n-1}} a^p_{i_1,\dots,i_{n-1}}\right)$$
$$= \sum_{i_1=0}^{p-1}\cdots\sum_{i_{n-1}=0}^{p-1} t_1^{i_1}\dots t_{n-1}^{i_{n-1}}(H_{t_1,\dots,t_{n-1}}(a_{i_1,\dots,i_{n-1}}))^p.$$

Step 2. Let k_0 be the fraction field of $W(K_0)$. Then $K' = k_0\{\{T_1\}\}\dots\{\{T_{n-1}\}\}$ is an n-dimensional local field with the residue field F. Comparing the lifting $H = H_{T_1,\dots,T_{n-1}}$ with the lifting h defined by

$$h\Big(\sum_{\mathbf{r}\in\mathbb{Z}^{n-1}} \theta_{\mathbf{r}}\overline{T_1}^{r_1}\dots\overline{T_{n-1}}^{r_{n-1}}\Big) = \sum_{\mathbf{r}\in\mathbb{Z}^{n-1}} [\theta_{\mathbf{r}}]T_1^{r_1}\dots T_{n-1}^{r_{n-1}},$$

we introduce the maps $\lambda_i\colon F \to F$ by the formula

$$h(a) = H(a) + pH(\lambda_1(a)) + p^2H(\lambda_2(a)) + \dots$$

Step 3. Introduce $h_{t_1,\dots,t_{n-1}}\colon F \to \mathcal{O}_K$ by the formula

$$h_{t_1,\dots,t_{n-1}}(a) = H_{t_1,\dots,t_{n-1}}(a) + pH_{t_1,\dots,t_{n-1}}(\lambda_1(a)) + p^2H_{t_1,\dots,t_{n-1}}(\lambda_2(a)) + \dots.$$

Remarks. 1. Observe that for a standard field $K = k\{\{T_1\}\}\dots\{\{T_{n-1}\}\}$, we have

$$h_{T_1,\dots,T_{n-1}}\colon \sum \theta_{\mathbf{i}}\overline{T_1}^{i_1}\dots\overline{T_{n-1}}^{i_{n-1}} \mapsto \sum [\theta_{\mathbf{i}}]T_1^{i_1}\dots T_{n-1}^{i_{n-1}},$$

where $\overline{T_j}$ is the residue of T_j in k_K, $j = 1, \dots, n-1$.

2. The idea of the above construction is to find a field $k_0\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$ isomorphic to K' inside K without a priori given topologies on K and K'. More precisely, let $t_1, \dots, t_{n-1}$ be as above. For $a = \sum_{-\infty}^{\infty} p^i h(a_i) \in K'$, let

$$f_{t_1,\dots,t_{n-1}}(a) = \sum_{-\infty}^{\infty} p^i h_{t_1,\dots,t_{n-1}}(a_i)$$

Then $f_{t_1,\dots,t_{n-1}}\colon K' \to K$ is an embedding of n-dimensional complete fields such that

$$f_{t_1,\dots,t_{n-1}}(T_j) = t_j, \qquad j = 1, \dots, n-1$$

(see [MZ1, Prop. 1.1]).

3. In the case of a standard mixed characteristic field the following alternative construction of the same topology is very useful.

Let $K = E\{\{X\}\}$, where E is an $(n-1)$-dimensional local field; assume that the topology of E is already defined. Let $\{V_i\}_{i\in\mathbb{Z}}$ be a sequence of neighbourhoods of zero in E such that

(i) there is $c \in \mathbb{Z}$ such that $P_E(c) \subset V_i$ for all $i \in \mathbb{Z}$;

(ii) for every $l \in \mathbb{Z}$ we have $P_E(l) \subset V_i$ for all sufficiently large i.

Put

$$\mathcal{V}_{\{V_i\}} = \Big\{\sum b_i X^i : b_i \in V_i\Big\}.$$

Then all the sets $\mathcal{V}_{\{V_i\}}$ form a base of neighbourhoods of 0 in K. (This is an easy but useful exercise in the 2-dimensional case; in general, see Lemma 1.6 in [MZ1]).

4. The formal construction of $h_{t_1,\dots,t_{n-1}}$ works also in case $\mathrm{char}(K) = p$, and one need not consider this case separately. However, if one is interested in equal

characteristic case only, all the treatment can be considerably simplified. (In fact, in this case $h_{t_1,\dots,t_{n-1}}$ is just the obvious embedding of $F \subset k_K$ into $\mathcal{O}_K = k_K[[t_n]]$.)

1.3.2. Properties.

(1) K is a topological group which is complete and separated.

(2) If $n > 1$, then every base of neighbourhoods of 0 is uncountable. In particular, there are maps which are sequentially continuous but not continuous.

(3) If $n > 1$, multiplication in K is not continuous. In fact, $UU = K$ for every open subgroup U, since $U \supset P(c)$ for some c and $U \not\subset P(s)$ for any s. However, multiplication is sequentially continuous:

$$\alpha_i \to \alpha, \quad 0 \neq \beta_i \to \beta \neq 0 \Longrightarrow \alpha_i \beta_i^{-1} \to \alpha\beta^{-1}.$$

(4) The map $K \to K$, $\alpha \mapsto c\alpha$ for $c \neq 0$ is a homeomorphism.

(5) For a finite extension L/K the topology of L = the topology of finite dimensional vector spaces over K (i.e., the product topology on $K^{|L:K|}$). Using this property one can redefine the topology first for "standard" fields

$$k\,\{\{T_1\}\}\,\dots\{\{T_m\}\}\,((T_{m+2}))\,\dots((T_n))$$

using the canonical lifting h, and then for arbitrary fields as the topology of finite dimensional vector spaces.

(6) For a finite extension L/K the topology of K = the topology induced from L. Therefore, one can use the Classification Theorem and define the topology on K as induced by that on L, where L is taken to be a standard n-dimensional local field.

Remark. In practical work with higher local fields, both (5) and (6) enables one to use the original definition of topology only in the simple case of a standard field.

1.3.3. About proofs. The outline of the proof of assertions in 1.3.1–1.3.2 is as follows. (Here we concentrate on the most complicated case $\operatorname{char}(K) = 0$, $\operatorname{char}(K_{n-1}) = p$; the case of $\operatorname{char}(K) = p$ is similar and easier, for details see [P]).

Step 1 (see [MZ1, §1]). Fix first $n-1$ local parameters (or, more generally, any elements $t_1, \dots, t_{n-1} \in K$ such that $v_i(t_i) > 0$ and $v_j(t_i) = 0$ for $j > i$).

Temporarily fix $\pi_i \in K$ ($i \in \mathbb{Z}$), $v_n(\pi_i) = i$, and $e_j \in P_K(0)$, $j = 1, \dots, d$, so that $\{\overline{e_j}\}_{j=1}^d$ is a basis of the F-linear space K_{n-1}. (Here F is as in 1.3.1, and $\bar\alpha$ denotes the residue of α in K_{n-1}.) Let $\{U_i\}_{i\in\mathbb{Z}}$ be a sequence of neighbourhoods of zero in F, $U_i = F$ for all sufficiently large i. Put

$$\mathcal{U}_{\{U_i\}} = \Big\{\sum_{i\geqslant i_0} \pi_i \cdot \sum_{j=1}^{d} e_j h_{t_1,\dots,t_{n-1}}(a_{ij}) : a_{ij} \in U_i, i_0 \in \mathbb{Z}\Big\}.$$

The collection of all such sets $\mathcal{U}_{\{U_i\}}$ is denoted by B_U.

Step 2 ([MZ1, Th. 1.1]). In parallel one proves that

– the set B_U has a cofinal subset which consists of subgroups of K; thus, B_U is a base of neighbourhoods of zero of a certain topological group $K_{t_1,\dots,t_{n-1}}$ with the underlying (additive) group K;

– $K_{t_1,\dots,t_{n-1}}$ does not depend on the choice of $\{\pi_i\}$ and $\{e_j\}$;

– property (4) in 1.3.2 is valid for $K_{t_1,\dots,t_{n-1}}$.

Step 3 ([MZ1, §2]). Some properties of $K_{t_1,\dots,t_{n-1}}$ are established, in particular, (1) in 1.3.2, the sequential continuity of multiplication.

Step 4 ([MZ1, §3]). The independence from the choice of $t_1, \dots, t_{n-1}$ is proved.

We give here a short proof of some statements in Step 3.

Observe that the topology of $K_{t_1,\dots,t_{n-1}}$ is essentially defined as a topology of a finite-dimensional vector space over a standard field $k_0\{\{t_1\}\}\dots\{\{t_{n-1}\}\}$. (It will be precisely so, if we take $\{\pi_i e_j : 0 \leqslant i \leqslant e-1, 1 \leqslant j \leqslant d\}$ as a basis of this vector space, where e is the absolute ramification index of K, and $\pi_{i+e} = p\pi_i$ for any i.) This enables one to reduce the statements to the case of a standard field K.

If K is standard, then either $K = E((X))$ or $K = E\{\{X\}\}$, where E is of smaller dimension. Looking at expansions in X, it is easy to construct a limit of any Cauchy sequence in K and to prove the uniqueness of it. (In the case $K = E\{\{X\}\}$ one should use the alternative construction of topology in Remark 3 in 1.3.1.) This proves (1) in 1.3.2.

To prove the sequential continuity of multiplication in the mixed characteristic case, let $\alpha_i \to 0$ and $\beta_i \to 0$, we shall show that $\alpha_i\beta_i \to 0$.

Since $\alpha_i \to 0, \beta_i \to 0$, one can easily see that there is $c \in \mathbb{Z}$ such that $v_n(\alpha_i) \geqslant c$, $v_n(\beta_i) \geqslant c$ for $i \geqslant 1$.

By the above remark, we may assume that K is standard, i.e., $K = E\{\{t\}\}$. Fix an open subgroup U in K; we have $P(d) \subset U$ for some integer d. One can assume that $U = \mathcal{V}_{\{V_i\}}$, V_i are open subgroups in E. Then there is m_0 such that $P_E(d-c) \subset V_m$ for $m > m_0$. Let

$$\alpha_i = \sum_{-\infty}^{\infty} a_i^{(r)} t^r, \quad \beta_i = \sum_{-\infty}^{\infty} b_i^{(l)} t^l, \qquad a_i^{(r)}, b_i^{(l)} \in E.$$

Notice that one can find an r_0 such that $a_i^{(r)} \in P_E(d-c)$ for $r < r_0$ and all i. Indeed, if this were not so, one could choose a sequence $r_1 > r_2 > \dots$ such that $a_{i_j}^{(r_j)} \notin P_E(d-c)$ for some i_j. It is easy to construct a neighbourhood of zero V'_{r_j} in E such that $P_E(d-c) \subset V'_{r_j}$, $a_{i_j}^{(r_j)} \notin V_{r_j}$. Now put $V'_r = E$ when r is distinct from any of r_j, and $U' = \mathcal{V}_{\{V'_r\}}$. Then $a_{i_j} \notin U'$, $j = 1, 2, \dots$ The set $\{i_j\}$ is obviously infinite, which contradicts the condition $\alpha_i \to 0$.

Similarly, $b_i^{(l)} \in P_E(d-c)$ for $l < l_0$ and all i. Therefore,

$$\alpha_i\beta_i \equiv \sum_{r=r_0}^{m_0} a_i^{(r)} t^r \cdot \sum_{l=l_0}^{m_0} b_i^{(l)} t^l \mod U,$$

and the condition $a_i^{(r)} b_i^{(l)} \to 0$ for all r and l immediately implies $\alpha_i \beta_i \to 0$.

1.3.4. Expansion into power series. Let $n = 2$. Then in characteristic p we have $\mathbb{F}_q((X))((Y)) = \{\sum \theta_{ij} X^j Y^i\}$, where θ_{ij} are elements of $\mathbb{F}_q$ such that for some i_0 we have $\theta_{ij} = 0$ for $i \leqslant i_0$ and for every i there is $j(i)$ such that $\theta_{ij} = 0$ for $j \leqslant j(i)$.

On the other hand, the definition of the topology implies that for every neighbourhood of zero U there exists i_0 and for every $i < i_0$ there exists $j(i)$ such that $\theta X^j Y^i \in U$ whenever either $i \geqslant i_0$ or $i < i_0, j \geqslant j(i)$.

So every formal power series has only finitely many terms $\theta X^j Y^i$ outside U. Therefore, it is in fact a *convergent* power series in the just defined topology.

Definition. $\Omega \subset \mathbb{Z}^n$ is called admissible if for every $1 \leqslant l \leqslant n$ and every $j_{l+1}, \dots, j_n$ there is $i = i(j_{l+1}, \dots, j_n) \in \mathbb{Z}$ such that

$$(i_1, \dots, i_n) \in \Omega,\ i_{l+1} = j_{l+1}, \dots, i_n = j_n \Rightarrow i_l \geqslant i.$$

Theorem. *Let $t_1, \dots, t_n$ be a system of local parameters of K. Let s be a section of the residue map $O_K \to O_K/M_K$ such that $s(0) = 0$. Let Ω be an admissible subset of $\mathbb{Z}^n$. Then the series*

$$\sum_{(i_1,\dots,i_n)\in\Omega} b_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n} \textit{ converges} \quad (b_{i_1,\dots,i_n} \in s(O_K/M_K))$$

and every element of K can be uniquely written this way.

Remark. In this statement it is essential that the last residue field is finite. In a more general setting, one should take a "good enough" section. For example, for $K = k\{\{T_1\}\} \dots \{\{T_m\}\}((T_{m+2})) \dots ((T_n))$, where k is a finite extension of the fraction field of $W(K_0)$ and K_0 is perfect of prime characteristic, one may take the Teichmüller section $K_0 \to K_{m+1} = k\{\{T_1\}\} \dots \{\{T_m\}\}$ composed with the obvious embedding $K_{m+1} \hookrightarrow K$.

Proof. We have

$$\sum_{(i_1,\dots,i_n)\in\Omega} b_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n} = \sum_{b\in s(O_K/M_K)} \Big(b \cdot \sum_{(i_1,\dots,i_n)\in\Omega_b} t_1^{i_1} \dots t_n^{i_n} \Big),$$

where $\Omega_b = \{(i_1, \dots, i_n) \in \Omega : b_{i_1,\dots,i_n} = b\}$. In view of the property (4), it is sufficient to show that the inner sums converge. Equivalently, one has to show that given a neighbourhood of zero U in K, for almost all $(i_1, \dots, i_n) \in \Omega$ we have $t_1^{i_1} \dots t_n^{i_n} \in U$. This follows easily by induction on n if we observe that $t_1^{i_1} \dots t_{n-1}^{i_{n-1}} = h_{t_1,\dots,t_{n-1}}(\overline{t_1}^{i_1} \dots \overline{t_{n-1}}^{i_{n-1}})$.

To prove the second statement, apply induction on n once again. Let $r = v_n(\alpha)$, where α is a given element of K. Then by the induction hypothesis

$$\overline{t_n^{-r}\alpha} = \sum_{(i_1,\dots,i_{n-1})\in\Omega_r} \overline{b_{i_1,\dots,i_n}}\,\big(\overline{t_1}\big)^{i_1}\dots\big(\overline{t_{n-1}}\big)^{i_{n-1}},$$

where $\Omega_r \subset \mathbb{Z}^{n-1}$ is a certain admissible set. Hence

$$\alpha = \sum_{(i_1,\dots,i_{n-1})\in\Omega_r} b_{i_1,\dots,i_n} t_1^{i_1}\dots t_{n-1}^{i_{n-1}} t_n^r + \alpha',$$

where $v_n(\alpha') > r$. Continuing this way, we obtain the desired expansion into a sum over the admissible set $\Omega = (\Omega_r \times \{r\}) \cup (\Omega_{r+1} \times \{r+1\}) \cup \dots$

The uniqueness follows from the continuity of the residue map $\mathcal{O}_K \to K_{n-1}$. $\square$

1.4. Topology on K^*

1.4.1. 2-dimensional case, $\operatorname{char}(k_K) = p$**.**

Let A be the last residue field K_0 if $\operatorname{char}(K) = p$, and let $A = W(K_0)$ if $\operatorname{char}(K) = 0$. Then A is canonically embedded into $\mathcal{O}_K$, and it is in fact the subring generated by the set $\mathcal{R}$.

For a 2-dimensional local field K with a system of local parameters t_2, t_1 define a base of neighbourhoods of 1 as the set of all $1 + t_2^i \mathcal{O}_K + t_1^j A[[t_1, t_2]]$, $i \geqslant 1$, $j \geqslant 1$. Then every element $\alpha \in K^*$ can be expanded as a convergent (with respect to the just defined topology) product

$$\alpha = t_2^{a_2} t_1^{a_1} \theta \prod (1 + \theta_{ij} t_2^i t_1^j)$$

with $\theta \in \mathcal{R}^*, \theta_{ij} \in \mathcal{R}, a_1, a_2 \in \mathbb{Z}$. The set $S = \{(j,i) : \theta_{ij} \neq 0\}$ is admissible.

1.4.2. In the general case, following Parshin's approach in characteristic p [P], we define the topology τ on K^* as follows.

Definition. If $\operatorname{char}(K_{n-1}) = p$, then define the topology τ on

$$K^* \simeq V_K \times \langle t_1 \rangle \times \dots \times \langle t_n \rangle \times \mathcal{R}^*$$

as the product of the induced from K topology on the group of principal units V_K and the discrete topology on $\langle t_1 \rangle \times \dots \times \langle t_n \rangle \times \mathcal{R}^*$.

If $\operatorname{char}(K) = \operatorname{char}(K_{m+1}) = 0$, $\operatorname{char}(K_m) = p$, where $m \leqslant n-2$, then we have a canonical exact sequence

$$1 \to 1 + P_K\big(1, \underbrace{0, \dots, 0}_{n-m-2}\big) \to O_K^* \to O_{K_{m+1}}^* \to 1.$$

Define the topology τ on $K^* \simeq O_K^* \times \langle t_1 \rangle \times \cdots \times \langle t_n \rangle$ as the product of the discrete topology on $\langle t_1 \rangle \times \cdots \times \langle t_n \rangle$ and the inverse image of the topology τ on $O_{K_{m+1}}^*$. Then the intersection of all neighbourhoods of 1 is equal to $1 + P_K\big(1, \underbrace{0, \ldots, 0}_{n-m-2}\big)$ which is a uniquely divisible group.

Remarks. 1. Observe that K_{m+1} is a mixed characteristic field and therefore its topology is well defined. Thus, the topology τ is well defined in all cases.

2. A base of neighbourhoods of 1 in V_K is formed by the sets

$$h(U_0) + h(U_1)t_n + \ldots + h(U_{c-1})t_n^{c-1} + P_K(c),$$

where $c \geqslant 1$, U_0 is a neighbourhood of 1 in V_{k_K}, $U_1, \ldots, U_{c-1}$ are neighbourhoods of zero in k_K, h is the canonical lifting associated with some local parameters, t_n is the last local parameter of K. In particular, in the two-dimensional case τ coincides with the topology of 1.4.1.

Properties.

(1) Each Cauchy sequence with respect to the topology τ converges in K^*.

(2) Multiplication in K^* is sequentially continuous.

(3) If $n \leqslant 2$, then the multiplicative group K^* is a topological group and it has a countable base of open subgroups. K^* is not a topological group with respect to τ if $m \geqslant 3$.

Proof. (1) and (2) follow immediately from the corresponding properties of the topology defined in subsection 1.3. In the 2-dimensional case (3) is obvious from the description given in 1.4.1. Next, let $m \geqslant 3$, and let U be an arbitrary neighbourhood of 1. We may assume that $n = m$ and $U \subset V_K$. From the definition of the topology on V_K we see that $U \supset 1 + h(U_1)t_n + h(U_2)t_n^2$, where U_1, U_2 are neighbourhoods of 0 in k_K, t_n a prime element in K, and h the canonical lifting corresponding to some choice of local parameters. Therefore,

$$\begin{aligned} UU + P(4) &\supset (1 + h(U_1)t_n)(1 + h(U_2)t_n^2) + P(4) \\ &= \{1 + h(a)t_n + h(b)t_n^2 + h(ab)t_n^3 : a \in U_1, b \in U_2\} + P(4). \end{aligned}$$

(Indeed, $h(a)h(b) - h(ab) \in P(1)$.) Since $U_1 U_2 = k_K$ (see property (3) in 1.3.2), it is clear that UU cannot lie in a neighbourhood of 1 in V_K of the form $1 + h(k_K)t_n + h(k_K)t_n^2 + h(U')t_n^3 + P(4)$, where $U' \neq k_K$ is a neighbourhood of 0 in k_K. Thus, K^* is not a topological group. $\square$

Remarks. 1. From the point of view of class field theory and the existence theorem one needs a stronger topology on K^* than the topology τ (in order to have more open subgroups). For example, for $n \geqslant 3$ each open subgroup A in K^* with respect to the topology τ possesses the property: $1 + t_n^2 \mathcal{O}_K \subset (1 + t_n^3 \mathcal{O}_K)A$.

A topology λ_* which is the sequential saturation of τ is introduced in subsection 6.2; it has the same set of convergence sequences as τ but more open subgroups. For example [F1], the subgroup in $1 + t_n \mathcal{O}_K$ topologically generated by $1 + \theta t_n^{i_n} \dots t_1^{i_1}$ with $(i_1, \dots, i_n) \neq (0, 0, \dots, 1, 2)$, $i_n \geqslant 1$ (i.e., the sequential closure of the subgroup generated by these elements) is open in λ_* and does not satisfy the above-mentioned property.

One can even introduce a topology on K^* which has the same set of convergence sequences as τ and with respect to which K^* is a topological group, see [F2].

2. For another approach to define open subgroups of K^* see the paper of K. Kato in this volume.

1.4.3. Expansion into convergent products. To simplify the following statements we assume here char $k_K = p$. Let B be a fixed set of representatives of non-zero elements of the last residue field in K.

Lemma. *Let $\{\alpha_i : i \in I\}$ be a subset of V_K such that*

$$\alpha_i = 1 + \sum_{\mathbf{r} \in \Omega_i} b_{\mathbf{r}}^{(i)} t_1^{r_1} \dots t_n^{r_n}, \tag{$*$}$$

where $b \in B$, and $\Omega_i \subset \mathbb{Z}_+^n$ are admissible sets satisfying the following two conditions:

(i) *$\Omega = \bigcup_{i \in I} \Omega_i$ is an admissible set;*

(ii) *$\bigcap_{j \in J} \Omega_j = \emptyset$, where J is any infinite subset of I.*

Then $\prod_{i \in I} \alpha_i$ converges.

Proof. Fix a neighbourhood of 1 in V_K; by definition it is of the form $(1 + U) \cap V_K$, where U is a neighbourhood of 0 in K. Consider various finite products of $b_{\mathbf{r}}^{(i)} t_1^{r_1} \dots t_n^{r_n}$ which occur in $(*)$. It is sufficient to show that almost all such products belong to U.

Any product under consideration has the form

$$\gamma = b_1^{k_1} \dots b_s^{k_s} t_1^{l_1} \dots t_n^{l_n} \tag{$**$}$$

with $l_n > 0$, where $B = \{b_1, \dots, b_s\}$. We prove by induction on j the following claim: for $0 \leqslant j \leqslant n$ and fixed $l_{j+1}, \dots, l_n$ the element γ almost always lies in U (in case $j = n$ we obtain the original claim). Let

$$\hat{\Omega} = \{\mathbf{r}_1 + \dots + \mathbf{r}_t : t \geqslant 1, \mathbf{r}_1, \dots, \mathbf{r}_t \in \Omega\}.$$

It is easy to see that $\hat{\Omega}$ is an admissible set and any element of $\hat{\Omega}$ can be written as a sum of elements of Ω in finitely many ways only. This fact and condition (ii) imply that any particular n-tuple $(l_1, \dots, l_n)$ can occur at the right hand side of $(**)$ only finitely many times. This proves the base of induction ($j = 0$).

For $j > 0$, we see that l_j is bounded from below since $(l_1, \dots, l_n) \in \hat{\Omega}$ and $l_{j+1}, \dots, l_n$ are fixed. On the other hand, $\gamma \in U$ for sufficiently large l_j and arbitrary $k_1, \dots, k_s, l_1, \dots, l_{j-1}$ in view of [MZ1, Prop. 1.4] applied to the neighbourhood of zero $t_{j+1}^{-l_{j+1}} \dots t_n^{-l_n} U$ in K. Therefore, we have to consider only a finite range of values $c \leqslant l_j \leqslant c'$. For any l_j in this range the induction hypothesis is applicable. $\square$

Theorem. *For any* $\mathbf{r} \in \mathbb{Z}_+^n$ *and any* $b \in B$ *fix an element*

$$a_{\mathbf{r},b} = \sum_{\mathbf{s}\in\Omega_{\mathbf{r},b}} b_{\mathbf{s}}^{\mathbf{r},b} t_1^{s_1} \dots t_n^{s_n},$$

such that $b_{\mathbf{r}}^{\mathbf{r},b} = b$, *and* $b_{\mathbf{s}}^{\mathbf{r},b} = 0$ *for* $\mathbf{s} < \mathbf{r}$. *Suppose that the admissible sets*

$$\{\Omega_{\mathbf{r},b} : \mathbf{r} \in \Omega_*, b \in B\}$$

satisfy conditions (i) *and* (ii) *of the Lemma for any given admissible set* Ω_*.

1. Every element $a \in K$ *can be uniquely expanded into a convergent series*

$$a = \sum_{\mathbf{r}\in\Omega_a} a_{\mathbf{r},b_{\mathbf{r}}},$$

where $b_{\mathbf{r}} \in B$, $\Omega_a \subset \mathbb{Z}_n$ *is an admissible set.*

2. Every element $\alpha \in K^*$ *can be uniquely expanded into a convergent product:*

$$\alpha = t_n^{a_n} \dots t_1^{a_1} b_0 \prod_{\mathbf{r}\in\Omega_\alpha} \left(1 + a_{\mathbf{r},b_{\mathbf{r}}}\right),$$

where $b_0 \in B$, $b_{\mathbf{r}} \in B$, $\Omega_\alpha \subset \mathbb{Z}_n^+$ *is an admissible set.*

Proof. The additive part of the theorem is [MZ2, Theorem 1]. The proof of it is parallel to that of Theorem 1.3.4.

To prove the multiplicative part, we apply induction on n. This reduces the statement to the case $\alpha \in 1 + P(1)$. Here one can construct an expansion and prove its uniqueness applying the additive part of the theorem to the residue of $t_n^{-v_n(\alpha-1)}(\alpha - 1)$ in k_K. The convergence of all series which appear in this process follows from the above Lemma. For details, see [MZ2, Theorem 2]. $\square$

Remarks. 1. Conditions (i) and (ii) in the Lemma are essential. Indeed, the infinite products $\prod_{i=1}^{\infty}(1 + t_1^i + t_1^{-i} t_2)$ and $\prod_{i=1}^{\infty}(1 + t_1^i + t_2)$ do not converge. This means that the statements of Theorems 2.1 and 2.2 in [MZ1] have to be corrected and conditions (i) and (ii) for elements $\varepsilon_{\mathbf{r},\theta}$ $(\mathbf{r} \in \Omega_*)$ should be added.

2. If the last residue field is not finite, the statements are still true if the system of representatives B is not too pathological. For example, the system of Teichmüller representatives is always suitable. The above proof works with the only ammendment: instead of Prop. 1.4 of [MZ1] we apply the definition of topology directly.

Corollary. *If* $\operatorname{char}(K_{n-1}) = p$*, then every element* $\alpha \in K^*$ *can be expanded into a convergent product:*

$$(***) \qquad \alpha = t_n^{a_n} \dots t_1^{a_1} \theta \prod (1 + \theta_{i_1,\dots,i_n} t_1^{i_1} \dots t_n^{i_n}), \quad \theta \in \mathcal{R}^*, \quad \theta_{i_1,\dots,i_n} \in \mathcal{R},$$

with $\{(i_1, \dots, i_n) : \theta_{i_1,\dots,i_n} \neq 0\}$ *being an admissible set. Any series* $(***)$ *converges.*

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