ISSN 1464-8997 **259**

Geometry & Topology Monographs
Volume 2: Proceedings of the Kirbyfest
Pages 259–290

Classification of unknotting tunnels for two bridge knots

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Abstract In this paper, we show that any unknotting tunnel for a two bridge knot is isotopic to either one of known ones. This together with Morimoto–Sakuma's result gives the complete classification of unknotting tunnels for two bridge knots up to isotopies and homeomorphisms.

AMS Classification 57M25; 57M05

Keywords Two bridge knots, unknotting tunnel

1 Introduction

Let K be a knot in the 3–sphere S^3. The *exterior* of K is the closure of the complement of a regular neighborhood of K, and is denoted by $E(K)$. A *tunnel* for K is an embedded arc τ in S^3 such that $\tau \cap K = \partial\tau$. Then we denote $\tau \cap E(K)$ by $\hat{\tau}$, where we regard τ as obtained from $\hat{\tau}$ by a radial extension. Let τ_1, τ_2 be tunnels for K. We say that τ_1 and τ_2 are *homeomorphic* if there is a self homeomorphism f of $E(K)$ such that $f(\hat{\tau}_1) = \hat{\tau}_2$. We say that τ_1 and τ_2 are *isotopic* if $\hat{\tau}_1$ is ambient isotopic to $\hat{\tau}_2$ in $E(K)$.

We say that a tunnel τ for K is *unknotting* if $S^3 - \mathrm{Int}\, N(K \cup \tau; S^3)$ is a genus two handlebody. We note that the unknotting tunnels for K is essentially the genus 2 Heegaard splittings of $E(K)$; if τ is an unknotting tunnel, then we can obtain a genus 2 Heegaard splitting (C_1, C_2), where C_1 is a regular neighborhood of $\partial E(K) \cup \hat{\tau}$ in $E(K)$, and $C_2 = cl(E(K) - C_1)$, and every genus 2 Heegaard splitting of $E(K)$ is obtained in this manner. Moreover, such Heegaard splittings are isotopic (homeomorphic resp.) if and only if the corresponding unknotting tunnels are isotopic (homeomorphic resp.). We say that a knot K is a 2–bridge knot if K admits a (genus zero) 2–bridge position, that is, there exists a genus zero Heegaard splitting $B_1 \cup_P B_2$ of S^3 such that $K \cap B_i$ is a system of 2–string trivial arcs in B_i $(i = 1, 2)$. It is known that

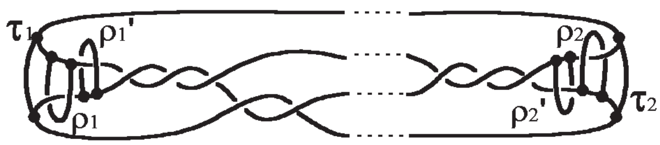

Figure 1.1

each 2{bridge knot admits six unknotting tunnels as depicted in Figure 1.1 or Figure 3.1 (see [17], or [8]).

Then the purpose of this paper is to prove:

Theorem 1.1 *Every unknotting tunnel for a non-trivial 2{bridge knots is isotopic to one of the above six unknotting tunnels.*

We note that the isotopy, and homeomorphism classes of the above tunnels are completely classi ed by Morimoto{Sakuma [12] and Y.Uchida [18], and that it is known that the unknotting tunnels for a trivial knot are mutually isotopic (see, for example [15]). Hence these results together with the above theorem give the complete classi cation of isotopy, and homeomorphism classes of unknotting tunnels for two{bridge knots.

2 Preliminaries

Throughout this paper, we work in the di erentiable category. For a submanifold H of a manifold K, $N(H;K)$ denotes a regular neighborhood of H in K. Let N be a manifold embedded in a manifold M with $\dim N = \dim M$. Then $\mathrm{Fr}_M N$ denotes the frontier of N in M. For the de nitions of standard terms in 3{dimensional topology, we refer to [6].

Let M be a compact 3{manifold, γ a union of mutually disjoint arcs or simple closed curves properly embedded in M, F a surface embedded in M, which is in general position with respect to γ, and ‘(F) a simple closed curve with ‘ \ γ = ; .

De nition 2.1 A surface D in M is a *γ{disk,* if D is a disk intersecting γ in at most one transverse point.

De nition 2.2 We say that ‘ is *γ{inessential* if ‘ bounds a γ{disk in F. We say that ‘ is *γ{essential* if it is not γ{inessential.

Definition 2.3 We say that a disk D is a γ–*compressing disk* for F if; D is a γ–disk, and $D \cap F = \partial D$, and ∂D is a γ–essential simple closed curve in F. The surface F is γ–*compressible* if it admits a γ–compressing disk, and it is γ–*incompressible* if it is not γ–compressible.

Definition 2.4 Let F_1, F_2 be surfaces embedded in M such that $\partial F_1 = \partial F_2$, or $\partial F_1 \cap \partial F_2 = \emptyset$. We say that F_1 and F_2 are γ–*parallel*, if there is a submanifold N in M such that $(N, N \cap \gamma)$ is homeomorphic to $(F_1 \times I, P \times I)$ as a pair, where P is a nion of points in $\mathrm{Int}(F_1)$, and F_1 (F_2 resp.) is the closure of the component of $\partial(F_1 \times I) - (\partial F_1 \times \{1/2\})$ containing $F_1 \times \{0\}$ ($F_1 \times \{1\}$ resp.) if $\partial F_1 = \partial F_2$, or F_1 (F_2 resp.) is the surface corresponding to $F_1 \times \{0\}$ ($F_1 \times \{1\}$ resp.) if $\partial F_1 \cap \partial F_2 = \emptyset$.

The submanifold N is called a γ–*parallelism* between F_1 and F_2.

We say that F is γ–*boundary parallel* if there is a subsurface F' in ∂M such that F and F' are γ–parallel.

Definition 2.5 We say that F is γ–*essential* if F is γ–incompressible, and not γ–boundary parallel.

Let α be an arc properly embedded in F with $\alpha \cap \gamma = \emptyset$.

Definition 2.6 We say that α is γ–*inessential* if there is a subarc β of ∂F such that $\partial\beta = \partial\alpha$, and $\alpha \cup \beta$ bounds a disk D in F such that $D \cap \gamma = \emptyset$, and α is γ–*essential* if it is not γ–inessential.

Definition 2.7 We say that F is γ–*boundary compressible* if there is a disk Δ in M such that $\Delta \cap F = \partial\Delta \cap F = \alpha$ is an γ–essential arc in F, and $\Delta \cap \partial M = \partial\Delta \cap \partial M = cl(\partial\Delta - \alpha)$.

Definition 2.8 Let F_1, F_2 be mutually disjoint surfaces in M which are in general position with respect to γ. We say that F_1 and F_2 are γ–*isotopic* if there is an ambient isotopy φ_t $(0 \le t \le 1)$ of M such that; $\varphi_0 = id_M$; $\varphi_1(F_1) = F_2$, and; $\varphi_t(\gamma) = \gamma$ for each t.

Genus g n–bridge position

Let $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a system of mutually disjoint arcs properly embedded in M.

Definition 2.9 We say that τ is a *system of n–string trivial arcs* if there exists a system of mutually disjoint disks $\{D_1, \ldots, D_n\}$ in M such that, for each i ($i = 1, \ldots, n$), we have (1) $D_i \cap \tau = \partial D_i \cap \gamma_i = \gamma_i$, and (2) $D_i \cap \partial M$ is an arc, say α_i, such that $\alpha_i = cl(\partial D_i - \gamma_i)$.

Example 2.10 Let τ be a system of 2–string trivial arcs in a 3–ball B. The pair (B, τ) is often refered as *2–string trivial tangle*, or a *rational tangle*.

Let K be a link in a closed 3–manifold M. Let $M = A \cup_P B$ be a genus g Heegaard splitting. Then the next definition is borrowed from [3].

Definition 2.11 We say that K is in a *(genus g) n–bridge position* (with respect to the Heegaard splitting $A \cup_P B$) if $K \cap A$ ($K \cap B$ resp.) is a system of n–string trivial arcs in A (B resp.).

In this paper, we abbreviate a genus 0 n–bridge position to an n–bridge position. A knot K is called an *n–bridge knot* if it admits an n–bridge position. It is known that the 2–bridge positions of a 2–bridge knot K are unique up to K–isotopy (see [13],[16], or Section 7 of [10]).

Definition 2.12 We say that a genus g bridge position of K with respect to $A \cup_P B$ is *weakly K–reducible* if there exist K–compressing disks D_A, D_B for P in A, B respectively such that $\partial D_A \cap \partial D_B = \emptyset$. The genus g bridge position of K with respect to $A \cup_P B$ is *strongly K–irreducible* if it is not weakly K–reducible.

Remark It is known that the 2–bridge positions of a 2–bridge knot are strongly K–irreducible (see Proposition 7.5 of [10]).

For a 2–bridge knot K we can obtain four genus one 1–bridge positions of K as follows.

Let $A \cup_P B$ be the Heegaard splitting which gives the 2–bridge position, and a_1, a_2, b_1, b_2 the closures of the components of $K - P$, where $a_1 \cup a_2$ ($b_1 \cup b_2$ resp.) is contained in A (B resp.). Let $T_1 = A \cup N(b_1; B)$, $\tau_1 = a_1 \cup b_1 \cup a_2$, $T_2 = cl(B - N(b_1; B))$, and $\tau_2 = b_2$. Then each T_i is a solid torus and it is easy to see that τ_i is a trivial arc in T_i ($i = 1, 2$). Hence, $T_1 \cup T_2$ gives genus one 1–bridge position of K. Moreover, by using a_1, a_2, b_2 for b_1, we can obtain other three genus one 1–bridge positions of K.

Let K be a knot with a genus one 1–bridge position with respect to $T_1 \cup T_2$. Let $_1$, $_2$ be tunnels for K embedded in T_1, T_2 respectively as in Figure 2.1. It is easy to see that $_1$, $_2$ are unknotting tunnels, and we call them the *unknotting tunnels associated to the genus one 1–bridge position*. In Section 8 of [10], it is shown that every genus one 1–bridge position for a non-trivial 2–bridge knot is obtained as above. Hence, by definition (see also Figure 3.1), it is easy to see:

Proposition 2.13 *Let $_1$, $_2$ be unknotting tunnels associated to a genus one 1–bridge position of a 2–bridge knot K. Then one of $_1$, $_2$ is isotopic to $_1$ or $_2$, and the other is isotopic to either $_1$, $'_1$, $_2$ or $'_2$.*

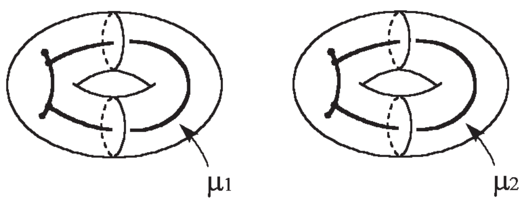

Figure 2.1

Let be an unknotting tunnel for K. Let $V_1 = N(K \cup ; S^3)$, $V_2 = cl(S^3 - V_1)$. Note that $V_1 \cup_Q V_2$ is a genus two Heegaard splitting of S^3.

Definition 2.14 We say that the Heegaard splitting $V_1 \cup V_2$ is *weakly K–reducible* if there exist K–compressing disks D_1, D_2 properly embedded in V_1, V_2 respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. The splitting is *strongly K–irreducible* if it is not weakly K–reducible.

Proposition 2.15 *If $(V_1; V_2)$ is weakly K–reducible, then either K is a trivial knot or K admits a genus one 1–bridge position, where is isotopic to one of the unknotting tunnels associated to the 1–bridge position.*

Proof Let D_1 (V_1), D_2 (V_2) be a pair of K–compressing disks which gives weak K–irreducibility.

Claim 1 We may suppose that D_1 (D_2 resp.) is non-separating in V_1 (V_2 resp.).

Proof of Claim 1 Suppose that D_2 is separating in V_2. Then D_2 cuts V_2 into two solid tori, say T_1, T_2. By exchanging the suffix, if necessary, we may suppose that ∂D_1 ∂T_1. Then take a meridian disk D'_2 in T_2 such that

$\partial D_2' \subset \partial V_2$. We may regard D_2' as a (non-separating essential) disk in V_2, and we have $\partial D_1 \cap \partial D_2' = \emptyset$. By regarding D_2' as D_2, we see that we may suppose that D_2 is non-separating in V_2.

Suppose that D_1 is separating in V_1. Since K does not intersect D_1 in one point, we have $D_1 \cap K = \emptyset$. The disk D_1 cuts V_1 into two solid tori U_1, U_2, where K is a core circle of U_1. If $\partial D_2 \subset \partial U_1$, then the above argument works to show that there exists a non-separating meridian disk for V_1 giving weak K–reducibility together with D_2. If $\partial D_2 \subset \partial U_2$, then we take a meridian disk D_1' for U_1 such that $\partial D_1' \subset V_1$, and D_1' intersects K transversely in one point. We may regard D_1' a (non-separating essential) K–disk in V_1, and we have $\partial D_1' \cap \partial D_2 = \emptyset$. By regarding D_1' as D_1, we see that we may suppose that D_1, D_2 are non-separating in V_1, V_2 respectively.

Now we have the following two cases.

Case 1 $D_1 \cap K = \emptyset$.

Let T be the solid torus obtained from V_1 by cutting along D_1. Since ∂D_2 is non-separating in ∂V_2 and S^3 does not contain non-separating 2–sphere, we see that ∂D_2 is an essential simple closed curve in ∂T. Since S^3 does not contain non-separating 2–sphere or punctured lens spaces, ∂D_2 is a longitude of T, and, hence, there is an annulus A in T such that $\partial A = K \cup \partial D_2$. Then $A \cup D_2$ gives a disk bounding K, and this shows that K is a trivial knot.

Case 2 $D_1 \cap K \neq \emptyset$.

Let $N = N(D_1; V_1)$, $T_1 = \mathrm{cl}(V_1 - N)$, $a_1 = K \cap T_1$, and $a_2 = K \cap N$. Note that a_2 is a core with respect to a natural 1–handle structure on N. It is easy to see that a_1 is a trivial arc in T_1. Let $T_2 = V_2 \cup N$. We regard a_2 as an arc properly embedded in T_2.

Claim 2 T_2 is a solid torus and a_2 is a trivial arc in T_2.

Proof of Claim 2 Let T' be the solid torus obtained from V_2 by cutting along D_2 and $B' = T' \cup N$. By the arguments in Case 1, we see that ∂D_1 is a longitude of T'. Hence B' is a 3–ball and a_2 is a trivial arc in B'. Since V_2 is obtained from B' by identifying two disks in $\partial B'$ corresponding to the copies of D_2, we see that T_2 is a solid torus, and a_2 is a trivial arc in T_2.

Hence we see that $T_1 \cup T_2$ gives a genus one 1–bridge position of K. By the construction of T_1, we see that   is isotopic to an unknotting tunnel associated to $T_1 \cup T_2$. □

3 Comparing 2-bridge position and an unknotting tunnel

In [14], Rubinstein-Scharlemann introduced a powerful machinery called *graphic* for studying positions of two Heegaard surfaces of a 3-manifold. Successively, Dr. Osamu Saeki and the author introduced an orbifold version of their setting, and showed that the results similar to Rubinstein-Scharlemann's hold in this setting [10]. In this section, we quickly review the arguments and apply it to compare decomposing 2-spheres giving 2-bridge positions, and genus 2 Heegaard splittings obtained from an unknotting tunnel for a 2-bridge knot.

Let K be a 2-bridge knot, that is, there exists a genus zero Heegaard splitting $B_1 \cup_P B_2$ of S^3 such that $K \cap B_i$ is a 2-strings trivial arcs in B_i $(i = 1, 2)$. Then the unknotting tunnels τ_1, τ_2 are contained in B_1, B_2 respectively as in Figure 3.1.

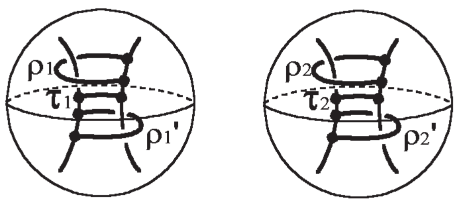

Figure 3.1

There is a diffeomorphism $f\colon P \times (0, 1) \to S^3 - (\tau_1 \cup \tau_2)$ such that $f(P \times \{1/2\})$ is the decomposing 2-sphere P, and that $f((p_1 \cup p_2 \cup p_3 \cup p_4) \times (0, 1)) = K \cap (S^3 - (\tau_1 \cup \tau_2))$ for some $p_1, p_2, p_3, p_4 \in P$.

Let τ be an unknotting tunnel for K. Let $\Gamma_1 = K \cup \tau$, $V_1 = N(\Gamma_1; S^3)$, $V_2 = c\ell(S^3 - V_1)$, and Γ_2 a spine of V_2 such that each vertex has valency 3. Note that $V_1 \cup_Q V_2$ is a genus two Heegaard splitting of S^3. Then there is a diffeomorphism $g\colon Q \times (0, 1) \to S^3 - (\Gamma_1 \cup \Gamma_2)$.

Let $P_s = f(P \times \{s\})$, and $Q_t = g(Q \times \{t\})$. Then for a fixed small constant $\varepsilon > 0$, we may suppose that $P_s \cap Q_t$ looks as one of the following, where $s \in (0, \varepsilon)$ or $(1 - \varepsilon, 1)$, and $t \in (0, \varepsilon)$.

(1) $P_s \cap Q_t$ consists of two transverse simple closed curves ℓ_1, ℓ_2 which are K-essential in P_s, and inessential in Q_t.

(2) $P_s \cap Q_t$ consists of a simple closed curve ℓ and a figure 8 σ such that; ℓ is K-essential in P_s, and inessential in Q_t, and σ is arising from a saddle tangency.

(3) $P_s \cap Q_t$ consists of three transverse simple closed curves ℓ_1, ℓ_2, and m such that; ℓ_1 and ℓ_2 bound pairwise disjoint K-disks in P_s each of which contains a puncture from K, ℓ_1 and ℓ_2 are parallel in Q_t, and; m is K-essential in P_s and inessential in Q_t,

(4) $P_s \cap Q_t$ consists of two transverse simple closed curves ℓ_1, ℓ_2, and a figure 8, γ such that; ℓ_1 and ℓ_2 bound pairwise disjoint K-disks in P_s each of which contains a puncture from K, ℓ_1 and ℓ_2 are parallel in Q_t, and; γ is arising from a saddle tangency.

(5) $P_s \cap Q_t$ consists of four transverse simple closed curves ℓ_1, ℓ_2, ℓ_3, and ℓ_4 such that ℓ_1, ℓ_2, ℓ_3, ℓ_4 bound mutually disjoint K-disks in P_s each containing a puncture from K, and ℓ_1 and ℓ_2 (ℓ_3 and ℓ_4 resp.) are pairwise parallel in Q_t.

Moreover, for a fixed $\varepsilon_1 \in (0, \varepsilon)$, if we move s from 0 to ε, then the intersection $P_s \cap Q_{\varepsilon_1}$ ($P_{1-s} \cap Q_{\varepsilon_1}$ resp.) is changed as (1) $\to$ (2) $\to$ (3) $\to$ (4) $\to$ (5).

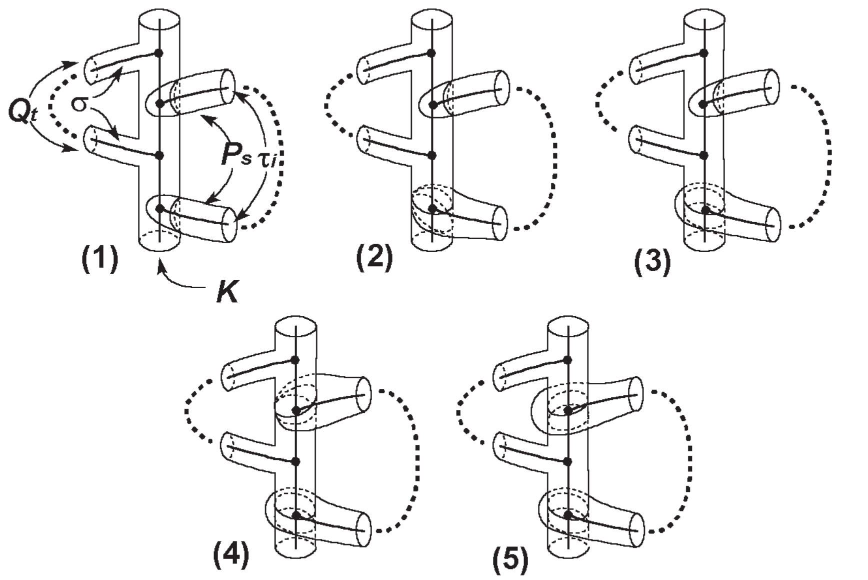

Figure 3.2

Then, by the arguments in Section 4 of [10], we see that by an arbitrarily small deformation of $f|_{(\varepsilon, 1-\varepsilon)}$, and $g|_{(\varepsilon, 1)}$ which does not alter $f|_{(0,\varepsilon] \cup [1-\varepsilon, 1)}$, and $g|_{(0,\varepsilon]}$, we may suppose that the maps are pairwise generic, that is:

There is a stratification of $\mathrm{Int}(I \times I)$ which consists of four parts below.

Regions Region is a component of the subset of $\mathrm{Int}(I \times I)$ consisting of values (s,t) such that P_s and Q_t intersect transversely, and this is an open set.

Edges Edge is a component of the subset consisting of values (s,t) such that P_s and Q_t intersect transversely except for one non-degenerate tangent point. The tangent point is either a “center” or a “saddle”. Edge is a 1–dimensional subset of $\mathrm{Int}(I \times I)$.

Crossing vertices Crossing vertex is a component of the subset consisting of points (s,t) such that P_s and Q_t intersect transversely except for two non-degenerate tangent points. Crossing vertex is an isolated point in $\mathrm{Int}(I \times I)$. In a neighborhood of a crossing vertex, four edges are coming in, where one can regard the crossing vertex as the intersection of two edges.

Birth-death vertices Birth-death vertex is a component of the subset consisting of points (s,t) such that P_s and Q_t intersect transversely except for a single degenerate tangent point. In particular, there is a parametrization (;) of $I \times I$ such that $P_s = \{(x,y,z) \mid z = 0\}$, and $Q_t = \{(x,y,z) \mid z = x^2 + \ + \ y + y^3\}$. Birth-death vertex is an isolated point in $\mathrm{Int}(I \times I)$, and in a neighborhood of a birth-death vertex, two edges are coming in, with one from center tangency, the other from saddle tangency.

Let Γ be the union of edges and vertices above. By the above, Γ is a 1–complex in $\mathrm{Int}(I \times I)$. Then we note that as in Section 3 of [14], Γ naturally extends to $\partial(I \times I)$. Here we note that, by the configurations (1)–(5) above, Γ looks as in Figure 3.3 near the bottom corners of $I \times I$. We note that the arguments in Section 6 of [10] which uses labels on the regions hold without changing proofs in this setting. Hence the argument in the proof of Proposition 5.9 of [14] which uses a simplicial map to a certain complex (called K in [14]) works in our setting, and this shows (note that $B_1 \cup_P B_2$ is always strongly K–irreducible (Remark of Definition 2.12)).

Proposition 3.1 *Suppose that $V_1 \cup_Q V_2$ is strongly K–irreducible, and K is not a trivial knot in S^3. Then there is an unlabelled region in $I \times I - \Gamma$.*

And we also have (see Corollary 6.22 of [10]):

Corollary 3.2 *Suppose that $V_1 \cup_Q V_2$ is strongly K–irreducible and K is not a trivial knot in S^3. Then, by applying K–isotopy, we may suppose that P and Q intersect in non-empty collection of simple closed curves which are K–essential in P, and essential in Q.*

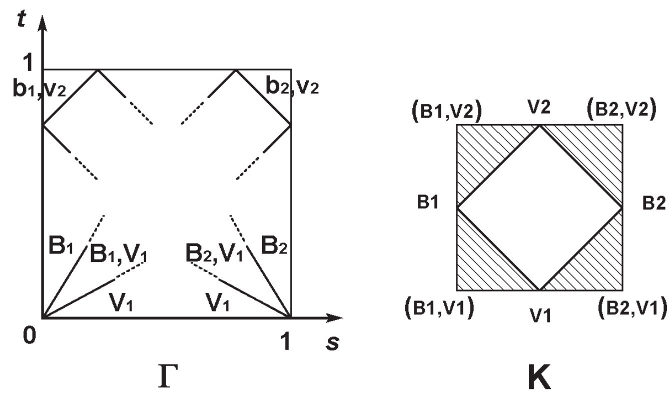

Figure 3.3

4 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. For the statements and the proofs of Lemmas B-1, C-1, C-2, C-3, D-2, D-3, D-4 which are used in this section, see Appendix of this paper. Let K be a non-trivial 2–bridge knot and $_1$, $_2$, $_1$, $'_1$, $_2$, $'_2$, , $B_1 \cup_P B_2$, $V_1 \cup_Q V_2$ be as in the previous section.

Proposition 4.1 *Suppose that $P \cap Q$ consists of non-empty collection of transverse simple closed curves which are K–essential in P and essential in Q. Then either*

(1) *is isotopic to either $_1$, or $_2$,*

(2) *$V_1 \cup_Q V_2$ is weakly K–reducible, or*

(3) *there is an essential annulus in $E(K)$.*

We note that the closures of $P - Q$ consist of two disks with each intersecting K in two points, and annuli. Since the disks are contained in V_1, $P \cap Q$ consists of even number of components. The proof of Theorem 4.1 is carried out by the induction on the number of the components. As the rst step of the induction, we show:

Lemma 4.2 *Suppose that $P \cap Q$ consists of two simple closed curves which are K–essential in P and essential in Q. Then we have the conclusion of Proposition 4.1.*

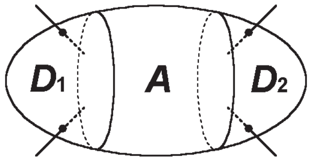

Figure 4.1

Proof Let D_1, A, D_2 be the closures of the components of $P - (P \cap Q)$ such that D_1, D_2 are disks, and A is an annulus.

We divide the proof into several cases.

Case 1 Either D_1 or D_2, say D_1, is separating in V_1.

We first show:

Claim 1 The annulus A is boundary parallel in V_2.

Proof Since D_1 is separating in V_1, the component of ∂A corresponding to ∂D_1 is separating in ∂V_2. Hence, by Lemma C-2, we see that A is compressible or boundary parallel in V_2. Suppose that A is compressible in V_2. Since S^3 does not contain non-separating 2–sphere, we see that D_2 is also separating in V_1, and, hence, D_1 and D_2 are pairwise parallel in V_1. Let A' be the annulus in Q such that $\partial A' = \partial A$. By exchanging suffix, if necessary, we may suppose that A' is properly embedded in B_1. Since each component of $K \cap B_1$ is an unknotted arc, we see that A' is an unknotted annulus in B_1, and this implies that A and A' are parallel in B_1, and, hence, in V_2 ie, A is boundary parallel. This completes the proof of Claim 1.

By Claim 1, we may suppose, by isotopy, that $B_1 \subset V_1$, and $\partial B_1 = D_1 \cup A' \cup D_2$, where A' is an annulus contained in ∂V_1 $(= Q)$.

Claim 2 Both D_1 and D_2 are K–incompressible in V_1.

Proof Assume, without loss of generality, that there is a K–compressing disk E_1 for D_1. Note that since $K \cap D_1$ consists of two points, ∂E_1 and ∂D_1 are parallel in $D_1 - K$. Let A_1 be the annulus in D_1 bounded by $\partial E_1 \cup \partial D_1$. Let D_1' be the disk in D_1 bounded by ∂E_1. Then we have the following two cases.

Case (a) $N(\partial E_1; E_1)$ is contained in B_1.

We consider the 2–sphere $D_1' \cup E_1$ in V_1. Let B_1' be the 3–ball in V_1 bounded by $D_1' \cup E_1$. Since K does not contain a local knot in V_1, we see that $K \cap B_1'$ is an unknotted arc properly embedded in B_1'. Hence there is an ambient isotopy of S^3 which moves $K \cap B_1'$ to an arc in D_1 joining $\partial(K \cap B_1')$, and which does not move $cl(K - B_1')$. On the other hand, $cl(K - B_1')$ is a component of the strings of the trivial tangle $(B_2, K \cap B_2)$. This shows that K is a trivial knot, a contradiction.

Case (b) $N(\partial E_1; E_1)$ is contained in B_2.

In this case, we first consider the disk $A' \cup A_1 \cup E_1$. By a slight deformation of $A' \cup A_1 \cup E_1$, we obtain a K–compressing disk E_2 for D_2 such that $N(\partial E_2; E_2)$ is contained in B_1. Then, by the argument as in Case (a), we see that K is a trivial knot, a contradiction.

This completes the proof of Claim 2.

Now we have the following two subcases.

Case 1.1 D_1 and D_2 are not K–parallel in V_1.

In this case, by Lemma D-4, we see that $\partial N((K \cup \tau_1); V_1)$ is isotopic to ∂V_1 in $S^3 - K$. This shows that τ is isotopic to τ_1.

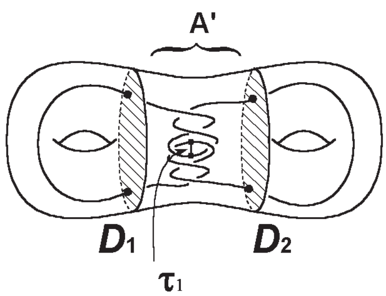

Figure 4.2

Case 1.2 D_1 and D_2 are K–parallel in V_1.

Let Q_1, Q_2 be the closures of the components of $Q - A'$ such that $\partial Q_i = \partial D_i$ $(i = 1, 2)$. Then Q_i is a torus with one hole properly embedded in B_2. By Lemma D-2, we may suppose, by exchanging suffix if necessary, that there is

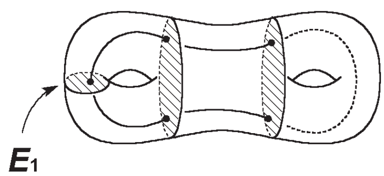

Figure 4.3

a K–compressing disk E_1 for Q_1 such that $E_1 \subset V_1$, and $E_1 \cap K$ consists of a point. We consider the genus one surface Q_2 properly embedded in B_2. By Lemma B-1, we see that Q_2 is K–compressible in B_2. Let E_2 be the K–compressing disk for Q_2. Now we have the following subsubcases.

Case 1.2.1 $N(\partial E_2; E_2)$ is contained in V_1.

By the K–incompressibility of D_2 (Claim 2), we see that $E_2 \cap K \neq \emptyset$; ie, $E_2 \cap K$ consists of a point. Then $E_1 \cup E_2$ cuts $(V_1; K)$ into a 2–string trivial tangle which is K–isotopic to $(B_1; K \cap B_1)$. Hence τ is isotopic to τ_1.

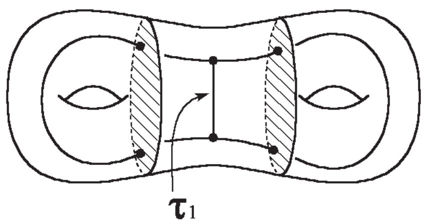

Figure 4.4

Case 1.2.2 $N(\partial E_2; E_2)$ is contained in V_2.

In this case, we first show:

Claim 1 $E_2 \cap Q_1 \neq \emptyset$.

Proof Suppose that $E_2 \cap Q_1 = \emptyset$. Then, by compressing Q_2 along E_2, we obtain a disk D' properly embedded in B_2 such that $\partial D' = \partial Q_2$, and D' separates the components of $B_2 \cap K$. Let $B_{2;1}$, $B_{2;2}$ be the closures of the components of $B_2 - D'$ such that $D_1 \subset B_{2;1}$, $D_2 \subset B_{2;2}$. Then we can isotope

$K \cap B_{2,i}$ rel ∂ in $B_{2,i}$ to an arc in D_i without moving $K \cap B_1$. Since D_1 and D_2 are K–parallel in V_1, this shows that K is a trivial knot, a contradiction.

Let $V_{1,2}$ be the closure of the component of $V_1 - D_1$ such that $\mathrm{Fr}_{B_2} V_{1,2} = Q_1$. Note that $V_{1,2}$ is a solid torus in B_2 with $V_{1,2} \cap P = \partial V_{1,2} \cap P = D_1$. By regarding $V_{1,2}$ as a very thin solid torus, we may suppose that $\mathrm{Int} E_2 \cap V_1$ consists of a disk $E_{2,1}$ intersecting K in one point. Then $E_2 \cap V_2$ is an annulus $A_{2,1}$ $(= c\ell(E_2 - E_{2,1}))$.

Claim 2 $A_{2,1}$ is incompressible in V_2.

Proof Assume that $A_{2,1}$ is compressible in V_2. Then, by compressing $A_{2,1}$, we obtain a disk E_2' in V_2 such that $\partial E_2' = \partial E_{2,1}$. Since $E_{2,1}$ intersects K in one point, $E_{2,1}$ is a non-separating disk in V_1. Hence, we see that $E_2' \cup E_{2,1}$ is a non-separating 2–sphere in S^3, a contradiction.

Then, by Lemma C-1, there is an essential disk D_2' in V_2 such that $D_2' \cap (E_2 \cap V_2) = \emptyset$, and, hence, $E_{2,1} \cap D_2' = \emptyset$. This shows that $V_1 \cup_Q V_2$ is weakly K–reducible.

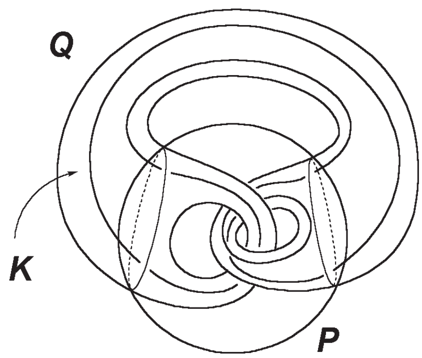

Figure 4.5

Case 2 Both D_1 and D_2 are non-separating in V_1.

In this case, we first show:

Claim 1 A is boundary parallel in V_2.

Proof Assume that A is not boundary parallel. Since S^3 does not contain non-separating 2–sphere, we see that A is incompressible in V_2. Hence, by

Lemma C-1, we see that there is an essential disk D for V_2 such that $D \cap A = \emptyset$, and that D cuts V_2 into two solid tori T_1, T_2, where $A \subset T_1$. Moreover, since S^3 does not contain a punctured lens space, we see that each component of ∂A represents a generator of the fundamental group of the solid torus T_1. However this contradicts Lemma C-3.

By Claim 1, we may suppose, by isotopy, that $B_1 \subset V_1$, and $\partial B_1 = D_1 \cup A' \cup D_2$, where A' is an annulus contained in $\partial V_1 (= Q)$.

Then we have the following subcases.

Case 2.1 Both D_1 and D_2 are K–incompressible in V_1.

This case is divided into the following two subsubcases.

Case 2.1.1 D_1 and D_2 are not K–parallel in V_1.

In this case, by Lemma D-4, we see that the given unknotting tunnel τ is isotopic to τ_1.

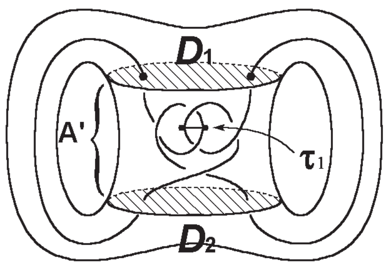

Figure 4.6

Case 2.1.2 D_1 and D_2 are K–parallel in V_1.

By Lemma D-3, there is a K–boundary compressing disk Δ for D_1 or D_2, say D_1, such that $\Delta \cap D_2 = \emptyset$. Let Q_1 be the closure of the component of $Q - (\partial D_1 \cup \partial D_2)$ which is a torus with two holes. Let $T_1 = Q_1 \cup D_1$. Then Δ is a compressing disk for T_1. Let D' be the disk obtained by compressing T_1 along Δ, and D'_2 a disk obtained by pushing IntD' slightly into Int$(V_1 \cap B_2)$. We may regard D'_2 is properly embedded in B_2. Suppose that D'_2 is K–compressible in B_2. Then we can show that K is a trivial knot by using the argument as in the proof of Claim 1 of Case 1.2.2. Hence D_2' is K–incompressible in B_2. Hence, by Lemma B-1 (3), either D_2' and D_2 are K–parallel or $D_2' \cup D_2$ bounds a 2–string trivial tangle in V_1, which is not a K–parallelism between D_2 and D_2'.

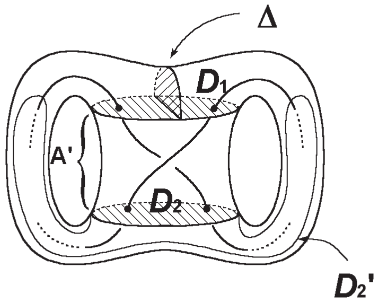

Figure 4.7

In the former case, we immediately see that the given unknotting tunnel is isotopic to $_1$. In the latter case, we have:

Claim 1 Suppose that $D_2' \cup D_2$ bounds a 2–string trivial tangle in V_1 which is not a K–parallelism between D_2 and D_2'. Then is isotopic to $_2$.

Proof By Lemma B-1 (2), we see that D_2' and $D_1 \cup A'$ bounds a K–parallelism in B_2. Hence, by isotopy, we can move P to the position such that B_2 V_1, and $\partial B_2 = D_2 \cup D_2'$. Then, by applying the argument of Case 2.1.1 with regarding D_2, D_2' as D_1, D_2 respectively, we see that is isotopic to $_2$.

Case 2.2 Either D_1 or D_2 is K–compressible in V_1.

Let E be a compressing disk for D_1 or D_2, say D_1, in V_1. Then ∂E and ∂D_1 are parallel in $D_1 - K$, and let A be the annulus in D_1 bounded by $\partial E \cup \partial D_1$. Let D be a disk properly embedded in V_1 which is obtained by moving Int($A \cup E$) slightly so that $D \cap (D_1 \cup D_2) = \partial D = \partial D_1$.

Claim 1 D B_1.

Proof Assume that D B_2. Then we may regard $A' \cup D$ is a K–compressing disk for D_2 in V_1. Then, by using the arguments in Case (a) of the proof of Claim 2 of Case 1, we can show that K is a trivial knot, a contradiction.

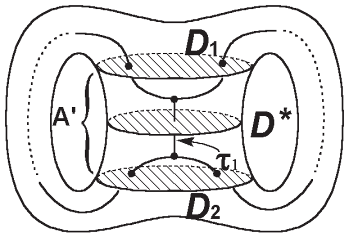

Figure 4.8

By Claim 1, τ_1 looks as in Figure 4.8.

Assertion Either "$K \cup \tau_1$ is a spine of V_1" or "there is an essential annulus in $E(K)$".

Proof of Assertion Let U_1 be a sufficiently small regular neighborhood of $K \cup \tau_1$, and $U_2 = cl(S^3 - U_1)$. Note that U_2 is a handlebody, because τ_1 is an unknotting tunnel for K. Let E_2 be a non-separating essential disk properly embedded U_2.

We may suppose that $D^* \cap U_1$ consists of a disk intersecting τ_1 in one point.

We suppose that $\#\{E_2 \cap D^*\}$ is minimal among all non-separating essential disks for U_2.

Claim 1 No component of $E_2 \cap D^*$ is a simple closed curve, an arc joining points in ∂U_2, or an arc joining points in ∂V_1.

Proof This can be proved by using standard innermost disk, outermost arc, and outermost circle arguments. The idea can be seen in the following figures.

Claim 2 $E_2 \cap D^* \neq \emptyset$.

Proof Assume that $E_2 \cap D^* = \emptyset$. Let T be the solid torus obtained by cutting U_1 along $D^* \cap U_1$. Note that T is a regular neighborhood of K. Since E_2 is non-separating in U_2, and S^3 does not contain a non-separating 2–sphere, ∂E_2 is an essential simple closed curve in ∂T, and ∂E_2 is not contractible in T. This shows that K bounds a disk which is an extension of E_2. Hence K is a trivial knot, a contradiction.

Hence $E_2 \cap D^*$ consists of a number of arcs joining points in ∂U_1 to points in ∂V_1. Here, by using cut and paste arguments, we remove the components of $E_2 \cap \partial V_1$ which are inessential in ∂V_1.

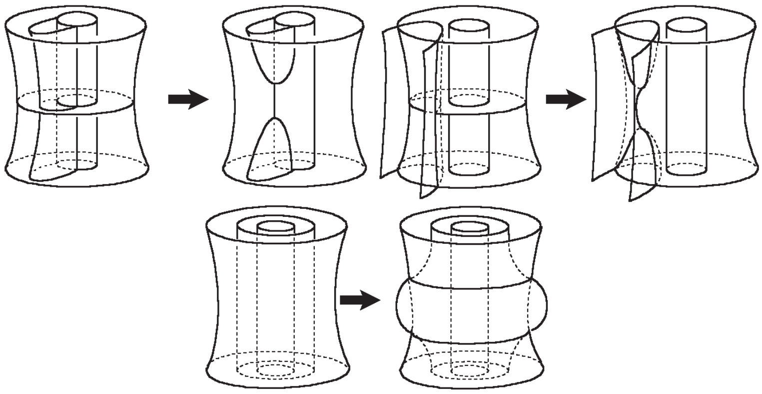

Figure 4.9

Claim 3 The components of $E_2 \cap \partial V_1$ are not nested in E_2.

Proof Let ℓ be a component of $E_2 \cap \partial V_1$ which is innermost in E_2, and G the disk in E_2 bounded by ℓ.

Subclaim 1 G is contained in V_2.

Proof Assume that G is contained in V_1. Since $G \cap (K \cup {}_1) = \emptyset$, this implies that ${}_1$ is contained in a regular neighborhood of K, contradicting the fact that ${}_1$ is an unknotting tunnel.

Subclaim 2 $\partial G \cap \partial D \neq \emptyset$.

Proof Assume that $\partial G \cap \partial D = \emptyset$. Then we can show that there is a non-separating disk G properly embedded in V_2 such that $\partial G \cap \partial D = \emptyset$ by using the argument as in the Proof of Claim 1 of the proof of Proposition 2.15. Then by using the argument as in the proof of Claim 2 above, we can show that K is a trivial knot, a contradiction.

Hence there exists a component of $E_2 \cap D$ connecting ℓ and ∂U_1. This means that ℓ is not surrounded by another component of $E_2 \cap \partial V_1$, and this gives the conclusion of Claim 3.

Claim 4 For each component ℓ of $E_2 \cap \partial V_1$, $\ell \cap D$ consists of more than one component.

Proof Assume that $\ell \cap D_\gamma$ consists of a point. Let G be the disk in E_2 bounded by ℓ. Then ∂D_γ and ∂G intersects in one point, and this shows that $\hat{\gamma}_1$ is a trivial arc in $E(K)$, a contradiction.

Let $E^2 = E_2 \cap V_1$. We call the boundary component of ∂E^2 corresponding to ∂E_2 the *outer boundary*. Other boundary components of E^2 (: the components of $E^2 \cap \partial V_1$) are called *inner boundary components*. Let V_1' be the solid torus obtained by cutting V_1 along D_γ. Let ℓ be an inner boundary component which is "outermost" with respect to the intersection $E^2 \cap D_\gamma$, that is:

Let A_ℓ be the union of the components of $E^2 \cap D_\gamma$ intersecting ℓ. Then except for at most one component, each component of $E^2 - A_\ell$ does not intersect other inner boundary components.

Let G be the disk in E_2 bounded by ℓ. Let $a_1, \ldots, a_n$ be the components of $E^2 \cap D_\gamma$, which are located on E^2 in this order, where $a_i \cup a_{i+1}$ $(i = 1, \ldots, n-1)$ cobounds a square Δ_i in E^2. Let $\Delta_i' = \Delta_i \cap V_1'$.

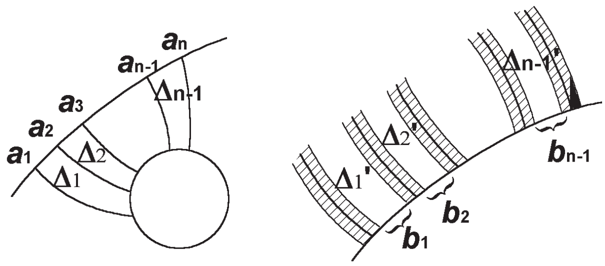

Figure 4.10

Let R' be the image of ∂V_1 in V_1'. Note that R' is a torus with two holes. Let $b_i = \Delta_i' \cap R$. Then by the minimality condition, we see that each b_i is an essential arc properly embedded in R'.

Claim 5 If $b_1, \ldots, b_{n-1}$ are mutually parallel in R', then there is an essential annulus in $E(K)$.

Proof Note that $\ell \cap R'$ consists of n components, that is, $b_1, \ldots, b_{n-1}$ above, and another component, say b_0.

Subclaim 1 b_0 is not parallel to b_i $(i = 1, \ldots, n-1)$ in R'.

Proof Assume that $b_0, b_1, \ldots, b_{n-1}$ are mutually parallel in R'. Then we can take a simple closed curve m in ∂V_1 such that m intersects ∂D^* transversely in one point, and $m \cap R'$ is ambient isotopic to b_i in R'. Let T^* be a regular neighborhood of $D^* \cup m$ in V_1 such that $\partial G \subset T^*$. Note that T^* is a solid torus, and ∂G wraps around ∂T^* longitudally n times. This show that the 3–sphere contains a lens space with fundamental group a cyclic group of order n, a contradiction.

By Subclaim 1, we see that we can take simple closed curves m_0, m_1 in ∂V_1 such that $m_0 \cap m_1 = \emptyset$, m_i $(i = 0, 1)$ intersects ∂D^* transversely in one point, $m_0 \cap R'$ is ambient isotopic to b_0 in R', and $m_1 \cap R'$ is ambient isotopic to b_i $(i = 1, \ldots, n-1)$ in R'.

Let W^* be a regular neighborhood of $D^* \cup m_0 \cup m_1$ in V_1 such that $\partial G \subset \partial W^*$, and $A^* = \mathrm{Fr}_{V_1} W^*$. Then W^* is a genus two handlebody, and A^* is an annulus in ∂W^*. Note that $cl(V_1 - W^*)$ is a regular neighborhood of K. Then we denote by $E'(K)$ the closure of the exterior of this regular neighborhood of K. Note that A^* is embedded in $\partial E'(K)$. Then attach $N(G; V_2)$ to W^* along $\partial G = \ell$. It is directly observed (see Figure 4.11) that we obtain a solid torus, say T^*, such that A^* wraps around ∂T^* longitudally n–times. Then, let $A^{*\prime} = cl(\partial T^* - A^*)$. Note that $A^{*\prime}$ is an annulus properly embedded in $E'(K)$.

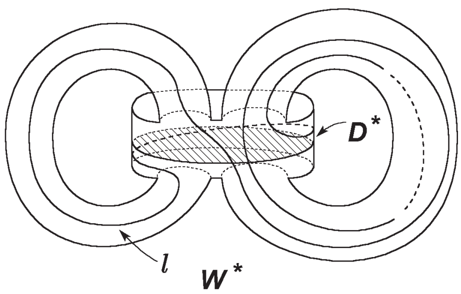

Figure 4.11

Assume that $A^{*\prime}$ is compressible in $E'(K)$. Then the compressing disk is not contained in T^* since $A^{*\prime}$ is incompressible in T^*. Hence T^* together with a regular neighborhood of this compressing disk produces a punctured lens space with fundamental group a cyclic group of order n in S^3, a contradiction. Hence $A^{*\prime}$ is incompressible in $E'(K)$. Then assume that $A^{*\prime}$ is boundary parallel, and let R be the corresponding parallelism. Since $n \geq 2$, R is not T^*. Hence $E'(K) = T^* \cup R$, and this shows that $E'(K)$ is a solid torus, which implies that K is a trivial knot, a contradiction. Hence A' is an essential annulus in $E'(K)$, and this completes the proof of Claim 5.

Suppose that $b_1, \ldots, b_{n-1}$ contains at least two proper isotopy classes in R'. We suppose that b_i, b_j $(i \neq j)$ belong to mutually different isotopy classes. Let r_1, r_2 be the components of $\partial R'$. Since ∂G and ∂D intersects transversely, we easily see that we may suppose that $b_i \cap r_1 \neq \emptyset$, and $b_j \cap r_2 \neq \emptyset$.

Let T be the solid torus obtained by cutting V_1 along D, and $T^2 = cl(T - N(K; T))$ (, hence, T^2 is homeomorphic to (torus) $\times [0, 1]$). Here we may regard that U_1 is obtained from $U_1 \cap T$ by adding a 1–handle h^1 corresponding to $N(D \cap U_1; U_1)$, where $\tau_1 \cap h^1$ is a core of h^1. Let τ', τ'' be the components of the image of τ_1 in T^2, where we may regard that $U_1 \cap T$ is obtained from $N(K; T)$ by adding $N(\tau' \cup \tau''; T^2)$.

Claim 6 $\tau' \cup \tau''$ is "vertical" in T^2 ie, $\tau' \cup \tau''$ is ambient isotopic to the union of arcs of the form $(p_1 \cup p_2) \times [0, 1]$, where p_1, p_2 are points in (torus).

Proof By extending Δ_i' (Δ_j' resp.) to the cores of $N(\tau' \cup \tau''; T^2)$, we obtain either an annulus which contains τ' or τ'' (if ∂b_i (∂b_j resp.) is contained in r_1 or r_2), or a rectangle two edges of which are τ' and τ'' (if ∂b_i (∂b_j resp.) joins r_1 and r_2) in T^2.

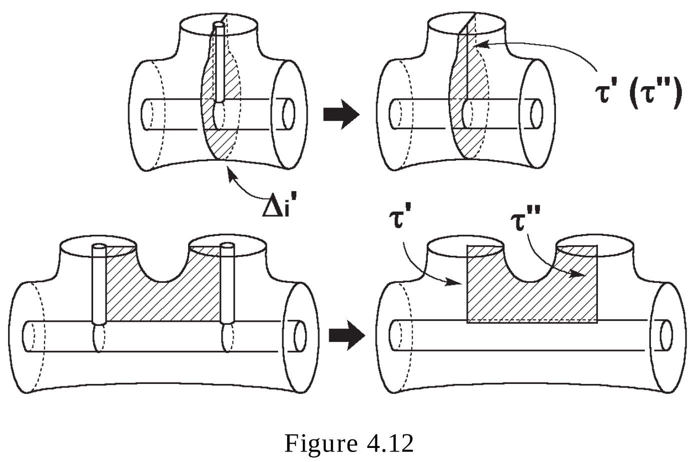

Figure 4.12

Then we have the following three cases.

Case 1 Both b_i and b_j join r_1 and r_2.

In this case, we obtain an annulus A by taking the union of the rectangles from $_i$ and $_j$. Since b_i and b_j are not ambient isotopic in R', A is incompressible in T^2. We note that every incompressible annulus in (torus) $\times [0,1]$ with one boundary component contained in (torus) $\times \{0\}$, the other in (torus) $\times \{1\}$ is "vertical"(for a proof of this, see, for example, [4]). Hence A is vertical, and this shows that $' \cup ''$ is vertical.

Case 2 Either b_i or b_j, say b_i, join r_1 and r_2, and ∂b_j is contained in r_1 or r_2.

In this case, we see that $'$ or $''$ is vertical by the existence of the annulus from $_j$. Then the existance of the rectangle from $_i$ shows that $'$ and $''$ are parallel, and this implies that $' \cup ''$ is vertical.

Case 3 ∂b_i is contained in r_1, and ∂b_j is contained in r_2.

In this case we see that $' \cup ''$ is vertical by the existence of the vertical annuli from $_i$ and $_j$.

By Claims 5, and 6, we see that $K \cup {}_1$ is a spine of V_1 or there is an essential annulus in $E(K)$, and this completes the proof of Assertion. □

Assertion shows that is isotopic to $_1$ or there is an essential annulus in $E(K)$, and this together with the conclusions of Cases 1, and 2.1 shows that we have the conclusions of Lemma 4.2 for all cases.

This completes the proof of Lemma 4.2. □

Lemma 4.3 *Suppose that $P \cap Q$ consists of more than two components. Then we can deform Q by an ambient isotopy in $E(K)$ to reduce $\#\{P \cap Q\}$ still with non-empty intersection each component of which is K-essential in P, and essential in Q.*

Proof Let $2n = \#\{P \cap Q\}$, and $D_1, A_1, A_2, \ldots, A_{2n-1}, D_2$ the closures of the components of $P - (P \cap Q)$ such that D_1, D_2 are disks and that they are located on P successively in this order.

Claim 1 Suppose that there is an annulus component A of $Q \cap B_i$ ($i = 1$ or 2) such that A is K-compressible in B_i. Then the K-compressing disk is disjoint from K.

Proof Let D be the K–compressing disk for A. Assume that $D \cap K \neq \emptyset$; ie, $D \cap K$ consists of a point. Then, by compressing A along D, we obtain two disks each of which intersects K in one point. But this is impossible, since each component of ∂A separates ∂B_1 into two disks each intersecting K in two points.

Claim 2 Suppose that there is an annulus component A_1^Q in $Q \cap B_1$, and an annulus component A_2^Q in $Q \cap B_2$. Then either A_1^Q or A_2^Q is K–incompressible in B_1 or B_2.

Proof We first suppose that A_2^Q is K–compressible in B_2. Then, by Claim 1, the K–compressing disk is disjoint from K. Hence, by compressing A_2^Q along the disk, we obtain two disks in B_2 which are K–essential in B_2 and disjoint from K. Let D_2 be one of the disks. Assume, moreover, that A_1^Q is also K–compressible. Then, by using the same argument, we obtain a K–essential disk D_1 in B_1 such that $D_1 \cap K = \emptyset$. Note that ∂D_1 and ∂D_2 are parallel in $P - K$. This implies that K is a two-component trivial link, a contradiction.

Claim 3 If $2n > 6$, then we have the conclusion of Lemma 4.3.

Proof Note that there are at most three mutually non-parallel, disjoint essential simple closed curves on Q. Hence if $2n > 6$, then there are three components, say ℓ_1, ℓ_2, ℓ_3, of $P \cap Q$ which are mutually parallel on Q. We may suppose that ℓ_1, ℓ_2, ℓ_3 are located on Q successively in this order. Let A_1 (A_2 resp.) be the annulus on Q bounded by $\ell_1 \cup \ell_2$ ($\ell_2 \cup \ell_3$ resp.). Without loss of generality, we may suppose that A_1 (A_2 resp.) is properly embedded in B_1 (B_2 resp.). Since K is connected, we may suppose, by exchanging suffix if necessary, that each component of ∂A_1 separates the boundary points of each component of $K \cap B_1$ on P. Since each component of $K \cap B_1$ is an unknotted arc, we see that A_1 is an unknotted annulus. Hence there is an annulus A_1' in P such that $\partial A_1' = \partial A_1$ and A_1' and A_1 are pairwise (K–)parallel in B_1. Let N be the parallelism between A_1' and A_1.

If $\mathrm{Int}(N) \cap Q \neq \emptyset$, then we can push the components of $\mathrm{Int}(N) \cap Q$ out of B_1 along the parallelism N, still with at least two components of intersection $\ell_1 \cup \ell_2$. If $\mathrm{Int}(N) \cap Q = \emptyset$, then we can push A_1 out of B_1 along this parallelism to reduce $\#\{P \cap Q\}$ by two.

According to Claim 3 and its proof, we suppose that $2n = 4$ or 6, and no three components of $P \cap Q$ are mutually parallel in Q. Note that the intersection numbers of any simple closed curves on Q with $P \cap Q$ are even, because P is a separating surface. This shows that $P \cap Q$ consists of two (in case when $n = 2$) or three (in case when $n = 3$) parallel classes in Q each of which consists of two components. Hence, each component of $Q \cap B_i$ ($i = 1$ or 2, say 1) is an annulus. If a component of $Q \cap B_1$ is K–incompressible in B_1, then, by the argument in the proof of Claim 3, we have the conclusion of Lemma 4.3. Hence, in the rest of the proof, we suppose that each component of $Q \cap B_1$ is a K–compressible annulus in B_1.

Let N_1 be the closure of the component of $B_1 - (Q \cap B_1)$ such that $(K \cap B_1)$ N_1. Note that N_1 is a 3–ball such that $\mathrm{Fr}_{B_1} N_1$ consists of some components of $Q \cap B_1$. Then, by the assumptions, we see that $\mathrm{Fr}_{B_1} N_1$ consists of either one, two, or three annuli.

Claim 4 If $\mathrm{Fr}_{B_1} N_1$ consists of an annulus, then there is a component of $Q \cap B_1$ which is K–boundary parallel in B_1, and, hence, we have the conclusion of Lemma 4.3.

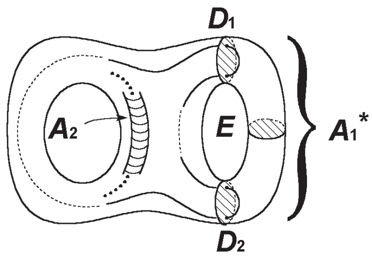

Figure 4.13

Proof Let $A_1 = \mathrm{Fr}_{B_1} N_1$. Since A_1 is compressible, there is a K–compressing disk E for A_1 in B_1. Note that $E \cap K = \emptyset$ (Claim 1). We may regard that E is properly embedded in V_1 and E is parallel to D_1 and D_2 in V_1. Since K is connected, we see that E is non-separating in V_1. By cutting V_1 along E, we obtain a solid torus T_1 such that K is a core circle of T_1. Recall that $D_1, A_1, A_2, \ldots, A_{2n-1}, D_2$ are the closures of the components of $P - Q$. Note that A_2 is properly embedded in $T_1 - K$. Since the 3–sphere does not contain a non-separating 2–sphere, we see that A_2 in incompressible in T_1. Since every incompressible surface in (torus) I is either vertical or boundary parallel annulus (see [4]), A_2 is boundary parallel in T_1. Let N be the parallelism for A_2, and $A_2 = N \cap \partial T_1$. Since K is connected, and K intersects D_1 and D_2, we see that A_2 is disjoint from the images of D_1 and D_2 in T_1. Hence we see that A_2 is disjoint from the images of E in ∂T_1. This shows that the parallelism N survives in V_1, and, hence, we have the conclusion of Lemma 4.3 by the argument as in the proof of Claim 3.

Claim 5 If $\mathrm{Fr}_{B_1} N_1$ consists of two annuli A_1, A_2, then there is a component of $Q \cap B_1$ which is (K–)boundary parallel in B_1, and, hence, we have the conclusion of Lemma 4.3.

Proof By exchanging suffix, if necessary, we may suppose that the annulus A_i is incident to D_i ($i = 1, 2$). If $n = 2$, then we have $\partial A_1 = \partial A_1$. If $n = 3$, then, by reversing the order of $A_1, \ldots, A_5$, and changing the suffix of A_i if necessary, we may suppose that $\partial A_1 = \partial A_1$. Then let N be the 3–manifold in B_1 such that $\partial N = A_1 \cup A_1$. Note that N is embedded in V_2 and $\mathrm{Fr}_{V_2} N = A_1$.

Subclaim Either D_1 or D_2, say D_1, is non-separating in V_1.

Proof Assume that both D_1 and D_2 are separating in V_1. Then D_1 and D_2 are parallel in V_1, but this contradicts the fact that N and K are connected.

Since D_1 is a non-separating disk in V_1, and S^3 does not contain a non-separating 2–sphere, we see that A_1 is incompressible in V_2. Then, since S^3 does not contain a punctured lens space with non-trivial fundamental group, we see that A_1 is boundary parallel in V_2 by Lemma C-3 (see the proof of Claim 1 in Case 2 of the proof of Lemma 4.2). Hence N is a parallelism between A_1 and A_1, and this shows that A_1 is K–boundary parallel in B_1 along this parallelism to give the conclusion of Lemma 4.3.

Claim 6 $\mathrm{Fr}_{B_1} N_1$ does not consist of three components.

Proof Assume that $\mathrm{Fr}_{B_1} N_1$ consists of three annuli A_1, A_2, and A_3, where $\partial A_1 = \partial A_1$, $\partial A_2 = \partial A_3$, and $\partial A_3 = \partial A_5$. Since A_1, A_2, A_3 are K–compressible in B_1, there are mutually disjoint K–compressing disks D_1, D_2, D_3 for A_1, A_2, A_3 respectively. We may regard that D_1, D_2, D_3 are properly embedded in V_1. Note that ∂D_1, ∂D_2, ∂D_3 are not mutually parallel in ∂V_1. Hence we see that $D_1 \cup D_2 \cup D_3$ cuts V_1 into two components X_1, X_2 such that one component of $K \cap B_1$ is contained in X_1, and the other component is contained in X_2 (see Figure 4.14). But this contradicts the fact that K is connected.

Claims 3, 4, 5, and 6 complete the proof of Lemma 4.3. □

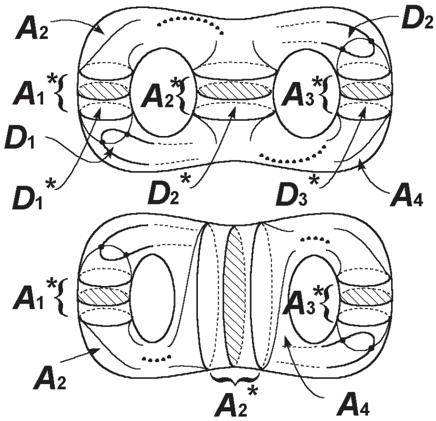

Figure 4.14

Proof of Proposition 4.1 By Lemma 4.3, we may suppose that $P \setminus Q$ consists of two transverse simple closed curves which are K{essential in P, and essential in Q. Then, by Lemma 4.2, we have the conclusion of Proposition 4.1.

Proof of Theorem 1.1 Let be an unknotting tunnel for a non-trivial 2{ bridge knot K, and $(V_1; V_2)$ a genus 2 Heegaard splitting of S^3 obtained from K [as above. If $(V_1; V_2)$ is weakly K{reducible, then by Propositions 2.13, and 2.15, we see that is isotopic to $_1$, $_2$, $_1$, $^0_1$, $_2$, or $^0_2$. If $(V_1; V_2)$ is strongly K{irreducible, then by Corollary 3.2, and Proposition 4.1, we see that is isotopic to $_1$ or $_2$, or $E(K)$ contains an essential annulus. If $E(K)$ contains an essential annulus, then K is a $(2; p)${torus knot. Then, by [1], it is known that every unknotting tunnel for K is isotopic to one of $_1$ or $_1$ (and that $_1$ and $_2$ are pairwise isotopic, and $_1$, $^0_1$, $_2$, $^0_2$ are mutually isotopic). Hence we have the conclusion of Theorem 1.1.

This completes the proof of Theorem 1.1.

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Received: 28 December 1998 Revised: 30 May 1999

Appendix A

Let γ be the union of mutually disjoint arcs and simple closed curves properly embedded in a 3–manifold N such that N admits a 2–fold branched covering space $p\colon \tilde{N} \to N$ along γ.

Let F be a surface properly embedded in N, which is in general position with respect to γ. Then, by using $\mathbb{Z}_2$–equivariant loop theorem [7], we see that:

Lemma A-1 F is γ–incompressible if and only if $\tilde{F}$ $(= p^{-1}(F))$ is incompressible.

Moreover, by using $\mathbb{Z}_2$–equivariant cut and paste argument as in [6, Proof of 10.3], we see that:

Lemma A-2 γ–incompressible surface F is γ–boundary compressible if and only if $\tilde{F}$ is boundary compressible.

By using $\mathbb{Z}_2$–Smith conjecture ([19], [11]) together with the $\mathbb{Z}_2$–equivariant cut and paste argument and the irreducibility of H, we have:

Lemma A-3 γ–incompressible surface F is γ–boundary parallel if and only if $\tilde{F}$ is boundary parallel. In particular, if $\tilde{N}$ is irreducible, and F is a disk intersecting γ in one point, and ∂F bounds a disk D in ∂H such that D intersects γ in one point, then F is γ–boundary parallel (in fact, F and D are γ–parallel).

Appendix B

For the proof of the following two lemmas, we refer Appendix B, and Appendix C of [10].

Let $(B; \ )$ be a 2–string trivial tangle.

Lemma B-1 Let F be a –incompressible surface in B. Then either:

(1) F is a disk disjoint from , and F separates the components of . Particularly, in this case, F is –essential,

(2) F is a –boundary parallel disk intersecting in at most one point,

(3) F is a –boundary parallel disk intersecting in two points and F separates $(B; \ )$ into the parallelism and a rational tangle, or

(4) F is a –boundary parallel annulus such that $F \cap \ = \emptyset$.

Let be a 1–string trivial arc in a solid torus T.

Lemma B-2 Let D be an –essential disk in T such that $D \cap$ consists of two points. Then there exists an –compressing disk D' for ∂T such that $D' \cap D = \emptyset$ and $D' \cap$ consists of one point. Moreover, by cutting $(T;\ )$ along D', we obtain a 2–string trivial tangle $(B;\ )$ such that D is a –incompressible disk in $(B;\ )$ (hence, D is –boundary parallel).

Appendix C

Let H be a genus 2 handlebody, and A an essential annulus properly embedded in H.

Lemma C-1 There exists an essential disk D in H such that $A \cap D = \emptyset$. Moreover the disk D can be taken as a separating disk, or a non-separating disk according as A is separating or non-separating.

Proof There exists boundary compressing disk for A. Apply a boundary compression on A along to obtain a disk D'. By moving D' by a tiny isotopy, we obtain a desired disk D. For a detail, see, for example, [9]. □

Lemma C-2 Each component of ∂A is non-separating in ∂H. And A is separating in H if and only if the components of ∂A are pairwise parallel in ∂H.

Proof Let D be as in Lemma C-1. By the proof of Lemma C-1, we see that A is isotopic to an annulus obtained from D by adding a band. By isotopy, we may suppose that $A \cap D = \emptyset$. Let T be the closure of the component of $H - N(D;H)$ such that $A \quad T$. Then T is a solid torus, and A is incompressible in T. Hence each component of ∂A is non-separating in ∂T. This implies that each component of ∂A is non-separating in ∂H. Let D_1, D_2 be the copies of D in ∂T. Note that ∂A separates ∂T into two annuli, say A_1, A_2. If D is separating in H, then $D_1 \cup D_2$ is contained in one of A_1 or A_2, say A_1. Then the components of ∂A are mutually parallel in ∂H through the annulus A_2. If D is non-separating in H, then, by exchanging the su x if necessary, we may suppose that D_1 is contained in A_1, and D_2 is contained in A_2. This shows that the components of ∂A are not parallel in ∂H. □

Lemma C-3 Let D be as in Lemma C-1. Suppose that A is separating in H. Then each component of ∂A does not represent a generator of the fundamental group of the solid torus obtained from H by cutting along D, which contains A.

Proof Let T be the solid torus obtained from H by cutting along D such that $A \quad T$. Then $(T;A)$ is homeomorphic to $(A \quad I; A \quad \{1/2\})$ as pairs. This shows that the closure of a component of $T - A$ gives a parallelism between A and a subsurface of ∂H. □

Appendix D

Let K be a knot in a genus two handlebody H with an essential disk E such that E cuts H into a solid torus, where K is a core circle of T. Note that there exists a two-fold branched cover $p\colon \tilde{H} \to H$ of H along K, where $\tilde{H}$ is a genus three handlebody.

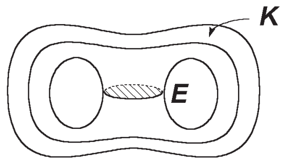

Figure D-1

Lemma D-1 Let D be a K-essential disk in H such that $D \cap K$ consists of two points. Then there exists a K-boundary compressing disk for D.

Proof Let $\tilde{D}$ be the lift of D in $\tilde{H}$. Then, by Lemmas A-1 and A-3, we see that $\tilde{D}$ is an essential annulus in a genus three handlebody $\tilde{H}$. Then $\tilde{D}$ is boundary compressible in $\tilde{H}$. Hence, by Lemma A-2, we see that D is K-boundary compressible in H. □

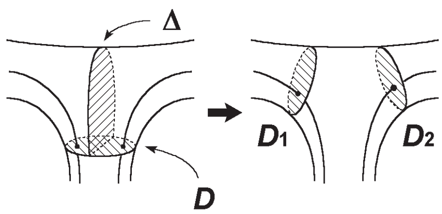

Figure D-2

By Lemma D-1, we obtain, by boundary compressing D along , two K-compressing disks, say D_1 and D_2, for ∂T such that $D_i \cap K$ consists of a point $(i = 1, 2)$.

Lemma D-2 Let D, D_1, D_2 be as above. Suppose, moreover, that D is separating in H. Then D_1 and D_2 are K-parallel in H, and, by cutting (H, K) along D_i ($i = 1$ or 2, say 1), we obtain a 1-string trivial arc in a solid torus, say $(T,\ )$. Moreover, D_2 is -boundary parallel in T.

Proof We note that D separates H into two solid tori T_1, T_2, where D_1, D_2 are properly embedded in T_1. Since each D_i intersects K in one point, D_i is an essential disk of T_1, and this shows that D_1 and D_2 are parallel in T_1, and in H. Then, $\mathbb{Z}_2$-Smith conjecture shows that they are actually K-parallel. Then, by using $\mathbb{Z}_2$-equivariant loop theorem, we see that we obtain a 1-string trivial tangle in a solid torus $(T,\ )$, by cutting (H, K) along D_1. Since D_1 and D_2 are K-parallel in H, we see that D_2 is -boundary parallel in T. □

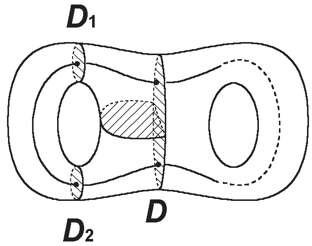

Figure D-3

Lemma D-3 Let D be as in Lemma D-1. Suppose, moreover, that D is non-separating in H. Then $D_1 \cup D_2$ is non-separating in H, and, by cutting $(H;K)$ along $D_1 \cup D_2$, we obtain a 2-string trivial tangle, say $(B; \ )$. Moreover, D is -boundary parallel in B.

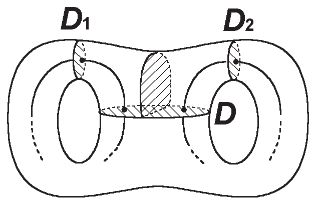

Figure D-4

Proof Let T be the solid torus obtained from H by cutting along D. We may suppose that D_1 and D_2 are properly embedded in T. Since each D_i intersects K in one point, we see that D_i is an essential disk in T. By the construction of D_1, D_2, we see that $D_1 \cup D_2$ separates the copies of D in T. This shows that $D_1 \cup D_2$ is non-separating in H. Then, by cutting $(H;K)$ along $D_1 \cup D_2$, we obtain a 2-string tangle in a 3-ball, say $(B; \ )$. Since $\tilde{H}$ is a genus three handlebody, we see that the 2-fold covering space of B branched along is a solid torus. Hence, $(B; \ )$ is a 2-string trivial tangle. By Lemma B-1 (3), we see that D is -boundary parallel in B. □

Lemma D-4 Let D, D' be pairwise disjoint, pairwise parallel, non K-parallel, K-essential disks in H such that $D \cap K$, and $D' \cap K$ consist of two points. Then there are two K-compressing disks D^1, D^2 for ∂H such that $D^1 \cup D^2$ is non-separating in H and is disjoint from $D \cup D'$, and, by cutting $(H;K)$ along $D^1 \cup D^2$, we obtain a 2-string trivial tangle, say $(B; \ )$. Moreover D, D' are -boundary parallel in $(B; \ )$, and, hence, $D \cup D'$ cobounds a 2-string trivial tangle in $(H;K)$.

Proof Let be a K-boundary compressing disk for $D \cup D'$. Without loss of generality, we may suppose that $\cap D \neq \emptyset$. We divide the proof into the following two cases.

Case 1 D and D' are non-separating in H.

Let D^1, D^2 be the disks obtained from D and as in Lemma D-3. Then, by the proof of Lemma D-3, it is easy to see that $D^1 \cup D^2$ satisfies the conclusion of Lemma D-4.

Case 2 D and D' are separating in H.

Let D^1 be the disk corresponding to D_1 or D_2 in Lemma D-2, and $(T, \ )$ the 1-string trivial arc in a solid torus T obtained from (H, K) by cutting along D^1. Then, by Lemma B-2, we see that there exists an -compressing disk D^2 for ∂T such that D^2 cuts $(T, \ )$ into a 2-string trivial tangle. Here we may suppose that D^2 is disjoint from the images of D^1 in ∂T, and, hence, we may regard that D^2 is properly embedded in H. Then $D^1 \cup D^2$ satisfies the conclusion of Lemma D-4.

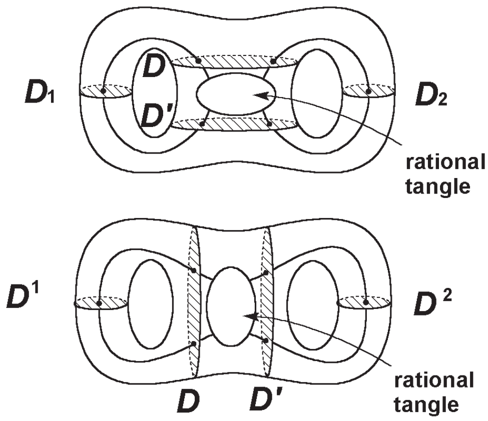

Figure D-5