ISSN 1464-8997

Geometry & Topology Monographs
Volume 2: Proceedings of the Kirbyfest
Pages 291–298

On regularly fibered complex surfaces

D Kotschick

Abstract We show that a compact complex surface which fibers smoothly over a curve of genus ≥ 2 with fibers of genus ≥ 2 fibers holomorphically. We deduce an improvement of a result in [16], and a characterisation of fibered surfaces with zero signature.

AMS Classification 14H15; 14J29, 20F34, 32L30, 57M50

Keywords Surface bundle, fibered surface, mapping class group

Dedicated to Robion C Kirby on the occasion of his 60th birthday.

1 Introduction

In this paper we begin a study of complex structures on the total spaces X of fiber bundles whose base B and fiber F are compact orientable two–manifolds. We shall assume throughout that the genera $g(B) = g$ and $g(F) = h$ are at least 2. For fixed g and h we have infinitely many homotopy types of orientable total spaces X, corresponding to conjugacy classes of representations of the fundamental group of B in the mapping class group of F. By the Thurston construction all the total spaces are symplectic, compare [16].

Suppose now that X admits some complex structure. The assumption g, $h \geq 2$ implies in particular that X is minimal and of general type. It has fixed topological Euler characteristic $c_2(X) = (2g-2)\cdot(2h-2)$ and therefore there are only finitely many possible values for $c_1^2(X)$. By the boundedness results of Moishezon and Gieseker for the moduli space of surfaces of general type, we conclude that among the infinitely many total spaces of surface bundles with fixed g and h there are at most finitely many which admit a complex structure. In this paper we take the first step or two towards characterising them.

In section 2 we shall prove a result which implies that every complex structure as above admits a regular holomorphic map f to B endowed with a suitable complex structure. In fact, f is in the same homotopy class as the bundle projection. (This then makes the above finiteness result a consequence of Parshin–Arakelov finiteness, without the need to appeal to Moishezon and Gieseker.)

In section 3 we give some applications of the fibration criterion. We characterise the total spaces of surface bundles admitting complex structures under the additional assumption that the signature vanishes, and we also sharpen a result from [16]. This result concerns the minimal genus of a holomorphic representative for a given second cohomology class in the classifying space of the mapping class group of F, and thus bears on the complex analytic version of Problem 2.18 in Kirby's problem list [12].

Acknowledgement I am grateful to F Catanese and to D Toledo for useful comments.

2 A fibration criterion

We shall say that a compact complex surface is *regularly fibered* if it admits an everywhere regular holomorphic map onto a smooth curve. It follows that the base curve and all the fibers are compact, and that the surface is a smooth fiber bundle, though not usually a complex analytic bundle, as the complex structure of the curves can vary. We shall assume throughout that the genera of the base and of the fiber are at least 2.

Nontrivial examples of regularly fibered surfaces were first exhibited by Kodaira [13] and by Atiyah [2], and later by many others, see eg [10, 7]. Sometimes the terms Kodaira surface and, more often, Kodaira fibration are used to denote these surfaces. That terminology gives rise to confusion, not only because the term Kodaira surface is more commonly used for certain non-Kähler complex surfaces of Kodaira dimension zero, but also because some authors seem to use Kodaira fibration to denote any regularly fibered surface (with base and fiber of genus at least 2) whereas others implicitly restrict the term to mean only the examples of Kodaira [13] (and maybe Kas [10]) constructed as branched covers of products.

Here is the fibration criterion saying that any surface satisfying the obvious necessary conditions is indeed regularly fibered.

Proposition 1 *Let S be a compact complex surface whose fundamental group fits into an extension*

$$1 \to \pi_1(F) \to \pi_1(S) \xrightarrow{\pi} \pi_1(B) \to 1 \ , \tag{1}$$

where F and B are closed oriented 2–manifolds with genera $g(F) = h \geq 2$ and $g(B) = g \geq 2$.

(1) *The topological Euler characteristic $e(S) \geq e(F) \cdot e(B) > 0$.*

(2) *The following conditions are equivalent:*

- $e(S) = e(F) \cdot e(B)$,
- *π is induced by a regular fibration of S over B endowed with a suitable complex structure,*
- *S is aspherical.*

Proof Corresponding to the extension (1) there is a fiber bundle $F \to X \to B$, with X a classifying space for $\pi_1(S)$. As X realises the smallest possible Euler characteristic among all orientable 4–manifolds with fundamental group $\pi_1(S)$, cf [14], we obtain $e(S) \geq e(X) = e(F) \cdot e(B)$, as claimed, where the last equality follows from the multiplicativity of the Euler characteristic in fiber bundles.

For the characterisation of the case of equality, notice that if S is regularly fibered, then it is aspherical by the homotopy exact sequence of the fibration. Further, if S is aspherical, then by the uniqueness of classifying spaces it is homotopy equivalent to X and therefore has the same Euler characteristic. Thus the crucial step is to show that the equality of Euler characteristics implies that π is induced by a regular holomorphic map.

First of all, as S is minimal with $c_2(S) = e(S) > 0$ and with $b_1(S) \geq 4$, it must be of general type. In particular, it is Kähler. By the theorem of Siu and Beauville, see Chapter 2 of [1] and also [8], there is a surjective holomorphic map with connected fibers $f\colon S \to C$, with C a compact complex curve, such that the map π in (1) factors through f_*. This implies that $ker(f_*)$ is a (finitely generated) subgroup of $ker(\pi) = \pi_1(F)$. Thus $ker(f_*)$ is the fundamental group of an orientable surface $\overline{F}$ which is a covering of F. If $\overline{F}$ were noncompact, $ker(f_*)$ would be a free group, contradicting the fact that $\pi_1(S)$ has cohomological dimension 4. Thus $\overline{F}$ is compact, with

$$g(\overline{F}) \geq g(F) = h \ . \tag{2}$$

On the other hand, denoting the generic fiber of f by F', we have that $ker(f_*) = \pi_1(\overline{F})$ is a quotient of $\pi_1(F')$ and so $g(\overline{F}) \leq g(F')$. Now by the theorem of Zeuthen–Segre a singular fiber makes a positive contribution to the Euler characteristic, so we have $e(S) \geq (2g(C) - 2) \cdot (2g(F') - 2)$, so that $e(S) = (2g(B) - 2) \cdot (2g(F) - 2)$ and $g(C) \geq g(B) = g$ imply

$$g(\overline{F}) \leq g(F') \leq g(F) = h \ . \tag{3}$$

Combining (2) and (3), we conclude that $g(\overline{F}) = g(F') = g(F) = h$ and therefore $g(C) = g(B) = g$. Thus C gives a complex structure on B and f is a holomorphic map inducing π. As we are in the case of equality for the Zeuthen–Segre inequality, f must be everywhere regular. □

In Chapter 2 of [1] the genus $g(M)$ of a compact Kähler manifold M was defined. This is the maximal genus of a compact curve C onto whose fundamental group $\pi_1(M)$ surjects. The Siu–Beauville theorem shows that if $\pi\colon \pi_1(M) \rightarrow \pi_1(C)$ is any surjective homomorphism with $g(M) = g(C)$, then π is induced by a holomorphic map with connected fibers. This conclusion does not necessarily hold if $g(M) > g(C)$, although surjective homomorphisms will exist in abundance. An interesting aspect of the proof of Proposition 1 is that it shows the homomorphism π in (1) is induced by a holomorphic map with connected fibers although the genus of S may very well be larger than $g(B) = g$: just take a trivial extension with $h = g(F) > g(B) = g$.

Remark 1 A version of the second part of Proposition 1 has been proved idependently by Hillman [5], but his proof is more complicated. He begins by using the work of Gromov and of Arapura–Bressler–Ramachandran on L^2–cohomology (see Chapter 4 of [1]) to produce a holomorphic map to a curve. In [9] an extension of the argument is proposed in the case where the kernel of π in (1) is not assumed to be a surface group, but can be any finitely presentable group. It turns out that this more general result can be deduced from Proposition 1 or from the result of [5] using standard arguments on the cohomology of Poincaré duality groups [6].

As an immediate consequence of Proposition 1 we have:

Corollary 2 *If S is any compact complex surface homotopy equivalent to a surface bundle X over a surface with base and fiber of genera at least 2, then S is regularly fibered and is diffeomorphic to X.*

This generalises results of Kas [10] and of Jost–Yau [8] who showed that deformations of Kodaira's examples [13] are regularly fibered. In those examples one obtains a description of a component of the moduli space in terms of moduli spaces of curves underlying the construction. In the general case, the corollary says that all components of the moduli space of complex structures on this particular manifold are made up of regularly fibered surfaces, but there is no direct description in terms of the moduli of curves.

3 Applications

The original motivation for studying regularly fibered surfaces was that they provide examples of smooth fibre bundles for which the signature is not multiplicative [2, 13], in this case that just means non-zero. In [16] we proved some bounds on the signatures of surface bundles over surfaces. We shall now slightly improve Theorem 3 of [16]:

Theorem 3 *Let X be a surface bundle over a surface, with the genera of the base and the fiber ≥ 2. If X admits a complex structure (not necessarily compatible with the orientation), or an Einstein metric, then*

$$3|\sigma(X)| < e(X) \ . \tag{4}$$

Proof Suppose X admits a complex structure. After possibly reversing the orientation, we may assume that the complex structure is compatible with the orientation.

The argument in [16] was as follows: X is a minimal surface of general type for which the underlying manifold endowed with the other, non-complex, orientation is symplectic and therefore has non-zero Seiberg–Witten invariants. Thus Theorem 1 of [15] gives

$$\sigma(X) \geq 0 \ .$$

This, together with the Miyaoka–Yau inequality

$$3\sigma(X) \leq e(X) \ ,$$

implies $3|\sigma(X)| \leq e(X)$.

We can now reach the same conclusion in a different way, and we can also show that the inequality must be strict, as claimed. By Corollary 2 the surface X is regularly fibered, so that the non-negativity of the signature follows from Arakelov's theorem. Moreover the Miyaoka–Yau inequality is strict for regularly fibered surfaces, as proved by Liu [18].

Suppose that X admits an Einstein metric. As it is also symplectic, it has non-zero Seiberg–Witten invariants and by the result of [17] satisfies $3\sigma(X) \leq e(X)$. The same argument for the manifold with the other orientation gives $-3\sigma(X) \leq e(X)$. It remains to exclude the case of equality.

Suppose that X admits an Einstein metric and that for a suitable choice of orientation $3\sigma(X) = e(X)$. Then Le Brun [17] showed that the Einstein metric must be Kähler–Einstein, so that X must be a complex surface which is a ball quotient. But then by the above argument for the complex case, we have a contradiction. □

We now return to the issue of characterising those surface bundles over surfaces which admit complex structures. Here is such a characterisation in the easiest case, when the signature of the total space vanishes.

Theorem 4 *Let X be the total space of a surface bundle over a surface, with the genera of the base B and of the fiber F at least 2. The following are equivalent:*

(1) *X admits a complex structure and has zero signature,*

(2) *the monodromy representation $\rho\colon \pi_1(B) \to \Gamma_h$ has finite image*[1].

Proof Suppose X admits a complex structure, then by Proposition 1 X is regularly fibered. If the signature vanishes, we are in the borderline case of Arakelov's theorem, which says that the signature is nonnegative, and is zero only if all the fibers are isomorphic, so the fibration is isotrivial. In this case we can pull back the fibration to a finite cover of B to obtain a product. This implies that the kernel of the monodromy representation has finite index in $\pi_1(B)$.

Conversely, assume that we have a bundle with finite monodromy. Then it must have zero signature. By the positive resolution of the Nielsen realisation problem [11] we can choose a complex structure on F and a lift of the monodromy representation to the diffeomorphism group of F so that the monodromy acts by complex analytic diffeomorphisms of F. Fixing an arbitrary complex structure on B, we obtain a complex structure on X by viewing it as $(F\times\tilde{B})/\pi_1(B)$, where $\pi_1(B)$ acts on $\tilde{B}$ by deck transformations and acts on F through the chosen lift of the monodromy representation to the diffeomorphism group. □

Remark 2 Under the conditions of the theorem X is finitely covered by a product, and is uniformised by the polydisk, compare Theorem 1 in [15]. Thus Theorem 4 is related to Catanese's characterisation [3] of complex surfaces finitely covered by products.

By the finiteness results for the complex case, Theorem 4 has the following immediate consequence:

Corollary 5 *For fixed $g = g(B)$ and $h = g(F)$, both ≥ 2, there are only finitely many conjugacy classes of representations $\rho\colon \pi_1(B) \to \Gamma_h$ with finite image.*

[1] Γ_h is the mapping class group of the fiber F, where $h = g(F)$.

This is a considerable strengthening of the following result of Harvey [4]:

Corollary 6 *The finite subgroups of the mapping class group Γ_h fall into finitely many conjugacy classes.*

Proof Every finitely generated subgroup of Γ_h is the monodromy group of a surface bundle of zero signature over some base B, where the genus of B can be taken to be the number of generators of the subgroup, see Proposition 4 of [16]. As the order of the finite subgroups of Γ_h is bounded[2], we have an a priori bound on $g = g(B)$ and can apply the previous corollary. $\square$

It is clear that the first corollary is much stronger than the second one, as there are usually many different monodromy representations with the same image, compare section 3 of [16].

Finally, note that every surface bundle with fibers of genus 2 has zero signature, and so is covered by Theorem 4. In the higher genus case there are always bundles of non-zero signature, and for these a characterisation of the monodromy representations arising from complex surfaces is not yet available. We shall return to this in a future paper. Here we just remark that this problem need not be the same as trying to decide which extensions of surface groups by surface groups are Kähler groups. If a surface bundle admits a complex structure, then its fundamental group is Kähler and in fact projective. However, it is possible that there are surface bundles which admit no complex structure but still have Kähler fundamental groups.

References

[1] **J Amorós**, **M Burger**, **K Corlette**, **D Kotschick**, **D Toledo**, *Fundamental groups of compact Kähler manifolds*, Amer. Math. Soc. Providence, Rhode Island (1996)

[2] **M F Atiyah**, *The signature of fibre bundles*, from: "Global Analysis", Papers in Honour of K Kodaira, Tokyo University Press (1969)

[3] **F Catanese**, *Fibred surfaces, varieties isogenous to a product and related moduli spaces*, preprint

[4] **W J Harvey**, *Branch loci in Teichmüller space*, Trans. Amer. Math. Soc. 153 (1971) 387–399

[2] by $84(h-1)$

[5] **J A Hillman**, *Complex surfaces which are fibre bundles*, Top. and Appl. (to appear)

[6] **J A Hillman**, personal communication

[7] **F Hirzebruch**, *The signature of ramified coverings*, from: "Global Analysis", Papers in Honour of K Kodaira, Tokyo University Press (1969)

[8] **J Jost**, **S-T Yau**, *Harmonic mappings and Kähler manifolds*, Math. Annalen, 262 (1983) 145–166

[9] **M Kapovich**, *On normal subgroups in the fundamental groups of complex surfaces*, preprint `mathGT/9808085`

[10] **A Kas**, *On deformations of a certain type of irregular algebraic surface*, Amer. Jour. Math. 90 (1968) 789–804

[11] **S Kerckhoff**, *The Nielsen realization problem*, Ann. of Math. 117 (1983) 235–265

[12] **R Kirby**, *Problems in low-dimensional topology*, from: "Geometric Topology" (W H Kazez, editor) Studies in Advanced Mathematics Vol. 2, Part 2, American Mathematical Society and International Press (1997)

[13] **K Kodaira**, *A certain type of irregular algebraic surfaces*, J. Analyse Math. 19 (1967) 207–215

[14] **D Kotschick**, *Four–manifold invariants of finitely presentable groups*, from: "Topology, Geometry and Field Theory" (K Fukaya et. al. editors) World Scientific (1994)

[15] **D Kotschick**, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. 29 (1997) 145–149

[16] **D Kotschick**, *Signatures, monopoles and mapping class groups*, Math. Research Letters, 5 (1998) 227–234

[17] **C Le Brun**, *Einstein metrics and Mostow rigidity*, Math. Research Letters, 2 (1995) 1–8

[18] **K Liu**, *Geometric height inequalities*, Math. Research Letters, 3 (1996) 693–702

Mathematisches Institut, Universität München
Theresienstr. 39, 80333 München, Germany

Email: `dieter@member.ams.org`

Received: 19 January 1999 Revised: 8 February 1999