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# Normalizers of tori

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**Abstract**

We determine the groups which can appear as the normalizer of a maximal torus in a connected 2–compact group. The technique depends on using ideas of Tits to give a novel description of the normalizer of the torus in a connected compact Lie group, and then showing that this description can be extended to the 2–compact case.

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# 1 Introduction

Suppose that $G$ is a connected compact Lie group and that $T \subset G$ is a maximal torus, or in other words a maximal connected abelian subgroup. The normalizer $NT$ of $T$ lies in a short exact sequence

$$1 \to T \to NT \to W \to 1 \tag{1.1}$$

in which $W$ is a finite group called the *Weyl group* of $G$. In this paper we use ideas of Tits [34] to give a particularly simple description of the groups which appear as such an $NT$ (see Proposition 1.6). This leads to an analogous determination of the groups which appear as the normalizer $N\breve{T}$ of a maximal 2–discrete torus $\breve{T}$ in a connected 2–compact group (see Section 1.16).

In the connected compact Lie group case, $NT$ determines $G$ up to isomorphism; see [8] for semisimple groups, and [30] or [28] in general. In listing the possible $NT$s we are thus giving an alternative approach to the classification of connected compact Lie groups. In contrast, it is not known that the normalizer of a maximal 2–discrete torus in a connected 2–compact group $X$ determines $X$ up to equivalence. However, this seems likely to be true [27, 23, 2], and we hope that the results of this paper will eventually contribute to a classification of connected 2–compact groups.

**1.2 Compact Lie groups** A glance at (1.1) reveals that $NT$ is determined up to isomorphism by three ingredients:

(1) the torus $T$,

(2) the finite group $W$, with its conjugation action on $T$, and

(3) an extension class $k \in H^2(W;T)$.

Let $\mathrm{T}(1)$ denote the circle $\mathbb{R}/\mathbb{Z}$. The torus $T$ is a product $\mathrm{T}(1)^r$ of circles, and so is determined by the number $r$, which is the *rank* of $T$ (or of $G$). The natural map $\mathrm{Aut}(T) \to \mathrm{Aut}(\pi_1 T) \cong \mathrm{GL}(r,\mathbb{Z})$ is an isomorphism, and the resulting conjugation homomorphism $W \to GL(r,\mathbb{Z})$ embeds $W$ as a finite subgroup of $\mathrm{GL}(r,\mathbb{Z})$ generated by reflections. Here a *reflection* is a matrix which is conjugate in $\mathrm{GL}(r,\mathbb{Q})$ to the diagonal matrix $\mathrm{diag}(-1,1,\ldots,1)$. This takes care of ingredients (1) and (2). Ingredient (3) is more problematical. The most direct way to approach (3) would be to list all of the elements of $H^2(W;T)$ and then point to ones which correspond to $NT$ extensions, but it would be a daunting task to make all of these cohomology calculations (one for each finite group acting on a torus by reflections) and then name the resulting elements in a way which would make it convenient to single out the ones giving an $NT$.

We take another approach, which is based on [34]. Suppose that $T$ is a torus and $W$ is a finite group of automorphisms of $T$ generated by reflections. A *marking* for $(T, W)$ is defined to be a collection $\{h_\sigma\}$ of elements of $T$, one for each reflection $\sigma$ in $W$, which satisfy some simple conditions (see Definition 2.12). The triple $(T, W, \{h_\sigma\})$ is called a *marked reflection torus*. The following proposition is less deep than it may seem; it is derived from a simple algebraic correspondence between marked reflection tori and classical root systems (see Section 2).

**1.3 Proposition** (See Remark 5.6) *Suppose that $G$ is a connected compact Lie group with maximal torus $T$ and Weyl group $W$. Then $G$ determines a natural marking $\{h_\sigma\}$ for the pair $(T, W)$, and the assignment $G \mapsto (T, W, \{h_\sigma\})$ gives a bijection between isomorphism classes of connected compact Lie groups and isomorphism classes of marked reflection tori.*

To each such marked reflection torus we associate a group $\nu(T, W, \{h_\sigma\})$, called the *normalizer extension of $W$ by $T$* (see Definition 3.5), which lies in a short exact sequence

$$1 \to T \to \nu(T, W, \{h_\sigma\}) \to W \to 1. \tag{1.4}$$

The following result is essentially due to Tits [34].

**1.5 Proposition** (See Theorem 5.7) *Suppose that $G$ is a connected compact Lie group and that $(T, W, \{h_\sigma\})$ is the corresponding marked reflection torus. Then the normalizer of $T$ in $G$ is isomorphic to $\nu(T, W, \{h_\sigma\})$.*

This leads to our characterization of $NT$s.

**1.6 Proposition** *Suppose that $N$ is an extension of a finite group $W$ by a torus $T$. Then $N$ is isomorphic to the normalizer of a torus in a connected compact Lie group if and only if*

- *the conjugation action of $W$ on $T$ expresses $W$ as a group of automorphisms of $T$ generated by reflections, and*
- *there exists a marking $\{h_\sigma\}$ of $(T, W)$ such that $N$ is isomorphic to $\nu(T, W, \{h_\sigma\})$.*

In fact the marking $\{h_\sigma\}$ in Proposition 1.6 is unique, if it exists (see Remark 5.6).

**1.7 2–compact groups** A 2–*compact group* is by definition a pair $(X, BX)$ of spaces together with an equivalence $X \sim \Omega BX$, where $H_*(X;\mathbb{Z}/2)$ is finite and $BX$ is 2–complete. Usually we refer to $X$ itself as the 2–compact group and leave $BX$ understood. Any such $X$ has a maximal 2–*discrete torus* $\breve{T}$ (see Section 1.16), which is an ordinary discrete group of the form $(\mathbb{Z}/2^\infty)^r$ for some $r > 0$. There is also an associated normalizer $N\breve{T}$ which lies in a short exact sequence

$$1 \to \breve{T} \to N\breve{T} \to W \to 1$$

in which $W$ is a finite group called the *Weyl group* of $X$. The automorphism group of $\breve{T}$ is isomorphic to $\mathrm{GL}(r,\mathbb{Z}_2)$, and again the conjugation action of $W$ on $\breve{T}$ embeds $W$ as a subgroup of $\mathrm{Aut}(\breve{T})$ generated by reflections. In this context a *reflection* is an element of $\mathrm{GL}(r,\mathbb{Z}_2)$ which is conjugate in $\mathrm{GL}(r,\mathbb{Q}_2)$ to $\mathrm{diag}(-1,1,\ldots,1)$. As above, we define a *marking* for $(\breve{T}, W)$ to be a suitable (see Definition 6.8) collection of elements $\{h_\sigma\}$ in $\breve{T}$, and we associate to each such marking a group $\nu(\breve{T}, W, \{h_\sigma\})$, called the *normalizer extension of $W$ by $\breve{T}$* (see Definition 6.15), which lies in a short exact sequence

$$1 \to \breve{T} \to \nu(\breve{T}, W, \{h_\sigma\}) \to W \to 1. \tag{1.8}$$

The structure $(\breve{T}, W, \{h_\sigma\})$ goes by the name of *marked 2–discrete reflection torus*. Our main results are the following ones.

**1.9 Proposition** (See Lemma 9.9) *Suppose that $X$ is a connected 2–compact group with maximal 2–discrete torus $\breve{T}$ and Weyl group $W$. Then $X$ determines a natural marking $\{h_\sigma\}$ for $(\breve{T}, W)$, and thus a marked 2–discrete reflection torus $(\breve{T}, W, \{h_\sigma\})$.*

**1.10 Proposition** (See Proposition 9.12) *Suppose that $X$ is a connected 2–compact group and that $(\breve{T}, W, \{h_\sigma\})$ is the corresponding (see Proposition 1.9) marked 2–discrete reflection torus. Then the normalizer of $\breve{T}$ in $X$ is isomorphic to $\nu(\breve{T}, W, \{h_\sigma\})$.*

It turns out that there are not very many marked 2–discrete reflection tori to work with. If $(T, W, \{h_\sigma\})$ is a marked reflection torus, there is an associated marked 2–discrete reflection torus $(T^\delta, W, \{h_\sigma\})$, where

$$T^\delta = \{x \in T \mid 2^k x = 0 \text{ for } k \gg 0\}. \tag{1.11}$$

Say that a 2–discrete marked reflection torus is *of Coxeter type* if it is derived in this way from an ordinary marked reflection torus. Let $\mathcal{T}_\Delta$ denote the marked 2–discrete reflection torus derived from the exceptional 2–compact group $\mathrm{DI}(4)$ [10, 24]. There is an evident notion of cartesian product for marked 2–discrete reflection tori (see Section 7).

**1.12 Proposition** (See Propositions 6.10 and 7.4) *Any marked 2–discrete reflection torus can be written as a product $\mathcal{T}_1 \times \mathcal{T}_2$, where $\mathcal{T}_1$ is of Coxeter type and $\mathcal{T}_2$ is a product of copies of $\mathcal{T}_\Delta$.*

It follows from Proposition 1.12 that any marked 2–discrete reflection torus arises from some 2–compact group: those of Coxeter type from 2–completions of connected compact Lie groups (Section 8 and Lemma 9.17), and the others from products of copies of DI(4). This can be used to give a characterization parallel to Proposition 1.6 of normalizers of maximal 2–discrete tori in connected 2–compact groups.

**1.13 Remark** Proposition 1.12 is in line with the conjecture that any connected 2–compact group $X$ can be written as $X_1 \times X_2$, where $X_1$ is the 2–completion of a connected compact Lie group and $X_2$ is a product of copies of DI(4) (see Proposition 9.13). Taken together, Propositions 1.10 and 1.12 imply that the conjecture is true as far as normalizers of maximal 2–discrete tori are concerned.

**1.14 Remark** If $p$ is an odd prime, the normalizer of a maximal $p$–discrete torus in a connected $p$–compact group is always the semidirect product of the torus and the Weyl group [1]. This explains why we concentrate on $p = 2$; for odd $p$ the extension we study is trivial. The techniques of this paper work to some extent for odd primes and can be used to show that the extension is trivial when the Weyl group is a Coxeter group; most of the remaining cases can be handled with the general methods of [15].

**1.15 Organization of the paper** Section 2 describes various ways of interpreting a root system, one of which is in terms of a *marked reflection torus*. Section 3 shows how a marked reflection torus $(T, W, \{h_\sigma\})$ gives rise to a *normalizer extension* $\nu(W)$ of $W$ by $T$; the normalizer extension is defined with the help of a *reflection extension* $\rho(W)$ of $W$. In Section 4 we find generators and relations for $\rho(W)$ by identifying this group with an extension $\tau(W)$ of $W$ constructed by Tits. Section 5 exploits these generators and relations to show that the normalizer $NT$ of a maximal torus in a connected compact Lie group $G$ is isomorphic to the normalizer extension obtained from the marked reflection torus (equivalently root system) associated to $G$. The arguments are due to Tits [34], although we put them in a different context and present them in a way that can be generalized to 2–compact groups. Section 6 generalizes the material from Sections 2 and 3 to a 2–primary setting, and Section 7 goes on to give a classification of marked 2–discrete reflection tori. Section 8 gathers together some technical information about 2–completions of connected compact Lie groups. Finally, Section 9 proves the main results about the normalizer of a maximal 2–discrete torus in a connected 2–compact group; the approach depends in part on a technical lemma, which is proved in Section 10.

**1.16 Notation and terminology** By a *maximal 2–discrete torus* $\breve{T}$ in a 2–compact group $X$, we mean the discrete approximation [11, 6.4] to a maximal 2–compact torus $\widehat{T}$ in $X$ [11, 8.9] [12, 2.15]. The *normalizer* of $\breve{T}$ is the discrete approximation in the sense of [12, 3.12] to the normalizer of $\widehat{T}$ [11, 9.8] (Aside 9.1). We take for granted the basic properties of 2–compact groups (see for instance the survey articles [20, 22, 26]), although we sometimes recall the definitions to help orient the reader. Since we only work with $p$–compact groups for $p = 2$, we sometimes abbreviate 2*–complete* to *complete* and 2*–discrete* to *discrete*.

Given an abelian group $A$ and an involution $\sigma$ on $A$, we write $A^-(\sigma)$ for the kernel of $1+\sigma$ on $A$, and $A^+(\sigma)$ the kernel of $1-\sigma$.

The following equations give the relationship between our notions of *root system* (Definition 2.2) and *geometric root system* (see Section 2.23) and similar notions of Tits, Serre, and Bourbaki.

$$\begin{aligned}
\textit{root system} &= \text{système de racines [34]}\\
&= \text{root data [32]}\\
&= \text{diagramme radiciel réduit [5, §4, Sect. 8]}\\
\textit{geometric root system} &= \text{système de racines [31] [4]}
\end{aligned}$$

In a nutshell, for us *root systems* are integral objects, while *geometric root systems* are real objects. It is a classical result [4] that geometric root systems classify connected compact semisimple Lie groups up to isogeny; equivalently, they classify simply connected compact Lie groups (or center–free connected compact Lie groups) up to isomorphism. A root system (in our sense) is an integral analog of a geometric root system, an analog which has the additional feature of allowing for the existence of central tori; root systems classify arbitrary connected compact Lie groups up to isomorphism. The existence of a natural correspondence between root systems and connected compact Lie groups is documented in [5, §4, Sect. 9]. This correspondence can be derived fairly easily from the classical correspondence between simple geometric root systems and simply connected simple compact Lie groups. However, there is an indirect path to the correspondence that goes through the intermediate category of connected reductive linear algebraic groups over $\mathbb{C}$ (see Section 2.23).

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## 2 Root systems, lattices and tori

In this section we show that three different structures are equivalent: *root systems*, *marked reflection lattices*, and *marked reflection tori*. The conclusion is that a root system amounts to a reflection lattice $L$ together with a bit of extra marking data for each reflection; this marking data is particularly easy to specify in terms of the torus $\mathrm{T}(1) \otimes L$.

**2.1 Root systems** The following definition is adapted from [34, 4.1]. A *lattice* $L$ (over $\mathbb{Z}$) is a free abelian group of finite rank.

**2.2 Definition** A *root system* in a lattice $L$ consists of a finite subset $R$ of $L$ (the set of *roots*) and for each $r \in R$ a homomorphism $n_r\colon L \to \mathbb{Z}$ (the *coroot* associated to $r$) such that following axioms are satisfied:

(R1) taken together, $R$ and $\bigcap_r \ker(n_r)$ span $\mathbb{Q} \otimes L$;

(R2) for each $r \in R$, $n_r(r) = -2$;

(R3) if $r \in R$, $k \in \mathbb{Z}$, and $kr \in R$, then $k = \pm 1$;

(R4) if $r, t \in R$, then $t + n_r(t) r \in R$.

**2.3 Remark** This structure is called a *système de racines* in [34] and a *diagramme radiciel rèduit* in [5]. It is not the conventional *système de racines* of Serre [31, Section 5] or Bourbaki [4, VI], which we will call a *geometric root system*. Basically, a root system in our sense is an integral form of a geometric root system; see Section 2.23 below.

**2.4 Marked Reflection Lattices** A *reflection* on a lattice $L$ is an automorphism $\sigma$ such that $\sigma$ is conjugate in $\mathrm{Aut}(\mathbb{Q} \otimes L) \cong \mathrm{GL}(r, \mathbb{Q})$ to a diagonal matrix $\mathrm{diag}(-1, 1, \ldots, 1)$.

**2.5 Definition** Suppose that $\sigma$ is a reflection on a lattice $L$. A *strict marking* for $\sigma$ is a pair $(b, \beta)$, where $b \in L$ and $\beta\colon L \to \mathbb{Z}$ is a homomorphism such that for any $x \in L$

$$\sigma(x) = x + \beta(x) b. \tag{2.6}$$

Two strict markings $(b, \beta)$ and $(b', \beta')$ are *equivalent* if $(b, \beta) = \pm(b', \beta')$. A *marking* for $\sigma$ is an equivalence class of strict markings.

Suppose that $L$ is a lattice, $\sigma$ is a reflection on $L$ with marking $\pm(b,\beta)$, and $w$ is an automorphism of $L$. We let $w\cdot(b,\beta)$ denote the marking for $w\sigma w^{-1}$ given by $(w(b),\beta\circ w^{-1})$.

**2.7 Definition** A *reflection lattice* is a lattice $L$ together with a finite subgroup $W$ of $\mathrm{Aut}(L)$ which is generated by the reflections it contains. A *marked reflection lattice* is a reflection lattice $(L,W)$ together with markings $\pm(b_\sigma,\beta_\sigma)$, one for each reflection $\sigma$ in $W$, such that for $w\in W$,

$$w\cdot(b_\sigma,\beta_\sigma)=\pm(b_{w\sigma w^{-1}},\beta_{w\sigma w^{-1}}).$$

**2.8 Marked reflection tori** Let $\mathrm{T}(n)$ denote the $n$–torus $(\mathbb{R}/\mathbb{Z})^n$. A *torus* is a compact Lie group isomorphic to $\mathrm{T}(n)$ for some $n$. Any torus $T$ gives a lattice $\pi_1T$; conversely, a lattice $L$ gives a torus $\mathrm{T}(1)\otimes L$. These two constructions are inverse to one another up to natural isomorphism, and induce an equivalence between the category of tori (with continuous homomorphisms) and the category of lattices.

**2.9 Remark** We write the group operation in a torus additively. If $T$ is a torus, the lattice $\pi_1T$ can also be described as the group $\mathrm{Hom}(\mathrm{T}(1),T)$ of continuous homomorphisms from the 1–torus to $T$ (these are *cocharacters* of $T$).

**2.10 Definition** An automorphism $\sigma$ of $T$ is said to be a *reflection* (respectively, *trivial mod* 2) if the induced automorphism of $\pi_1T$ is a reflection (respectively, trivial mod 2). An element $x$ of $T$ is said to be *strongly $\sigma$–negative* if $x$ lies in the identity component $T_0^-(\sigma)$ of $T^-(\sigma)$ (see Section 1.16), or, in other words, if $x$ lies in the identity component of the group

$$T^-(\sigma)=\{x\in T\mid \sigma(x)=-x\}.$$

**2.11 Remark** A reflection $\sigma$ on $T$ is trivial mod 2 if and only if $\sigma$ acts as the identity on ${}_2T=\{x\in T\mid 2x=0\}$. If $T=\mathrm{T}(1)\otimes L$, then $x\in T$ is strongly $\sigma$–negative if and only if $x\in \mathrm{T}(1)\otimes L^-(\sigma)$.

**2.12 Definition** Suppose that $\sigma$ is a reflection on a torus $T$. A *marking* for $\sigma$ is an element $h\in T$ such that

(1) $h$ is strongly $\sigma$–negative (Remark 2.11),

(2) $2h=0$, and

(3) $h\neq 0$ if $\sigma$ is nontrivial mod 2.

**2.13 Definition** A *reflection torus* is a torus $T$ together with a finite subgroup $W$ of $\mathrm{Aut}(T)$ which is generated by the reflections it contains. A *marked reflection torus* is a reflection torus $(T, W)$ together with markings $h_\sigma \in T$, one for each reflection $\sigma$ in $W$, such that for $w \in W$,

$$h_{w\sigma w^{-1}} = w(h_\sigma).$$

**2.14 Remark** If $(L, W)$ is a reflection lattice, or $(T, W)$ is a reflection torus, then $W$ is a classical Weyl group [19, 2.9] and hence a Coxeter group [19, Section 1]. Let $S = \{s_1, \dots, s_\ell\}$ be a set of simple reflections in $W$ [19, 1.3], and $m_{i,j}$ the order of $s_i s_j$, so that $\{m_{i,j}\}$ is the Coxeter matrix associated to $S$. By [19, 1.9], $W$ is generated by $S$ subject only to the relations $(s_i s_j)^{m_{i,j}} = 1$.

**2.15 Equivalence between the three structures** The goal of this section is to prove the following two statements.

**2.16 Proposition** *If $L$ is a lattice, then there is a natural bijection between marked reflection structures on $L$ and root systems in $L$.*

**2.17 Proposition** *If $L$ is a lattice with associated torus $T = \mathrm{T}(1) \otimes L$, then there is a natural bijection between marked reflection structures on $L$ and marked reflection structures on $T$.*

**2.18 Remark** It is not hard to enumerate the possible markings for a reflection $\sigma$ on $L$. Write $\ker(1+\sigma)$ and $\mathrm{im}(1-\sigma)$ for the kernel and image of the indicated endomorphisms of $L$; since $\sigma$ is a reflection, both of these groups are infinite cyclic. If $b \in L$ is an element such that $\mathrm{im}(1-\sigma) \subset \langle b \rangle$, then for every $x \in L$ there is a unique integer $\beta(x)$ such that $\sigma(x) = x + \beta(x)b$; it is easy to check that $\beta\colon L \to \mathbb{Z}$ is a homomorphism, and so $(b, \beta)$ is a strict marking. In particular, strict markings for $\sigma$ correspond bijectively to such elements $b$, and markings to such elements $b$ taken up to sign. Any such $b$ lies in $\ker(1+\sigma)$. It is easy to see that there are inclusions

$$2\ker(1+\sigma) \subset \mathrm{im}(1-\sigma) \subset \ker(1+\sigma).$$

where the left hand inclusion is an equality if and only if $\sigma$ reduces to the identity mod 2. Let $b_0$ be a generator of $\ker(1+\sigma)$. Then $\sigma$ has exactly two markings (determined by $\pm b_0$ and $\pm 2b_0$) if $\sigma$ is trivial mod 2, and one marking (determined by $\pm b_0$) otherwise.

**Proof of Proposition 2.16** Suppose to begin with that $(L, W)$ is a marked reflection lattice. For each reflection $\sigma \in W$, let $\pm(b_\sigma, \beta_\sigma)$ be the corresponding marking, so that

$$\sigma(x) = x + \beta_\sigma(x) b_\sigma. \tag{2.19}$$

The set of roots in the corresponding root system is given by

$$R = \{\pm b_\sigma \mid \sigma \in W \text{ a reflection}\}.$$

For each root $r \in R$, the coroot $n_r$ is $\beta_\sigma$ if $r = b_\sigma$ and $-\beta_\sigma$ if $r = -b_\sigma$. It remains to check axioms (R1)-(R4). For (R1), note that $\bigcap_r \ker(n_r)$ is the fixed point set $L^W$ of the action of $W$ on $L$, and observe that for any $x \in L$ there is an expression

$$(\#W)x = \sum_{w \in W} w(x) + \sum_{w \in W} (x - w(x)),$$

with the first summand on the right in $L^W$ and the second in the span of $R$. Condition (R2) comes from combining (2.19) with the equality $\sigma^2(x) = x$. For each $w \in W$ and reflection $\sigma \in W$, $w(b_\sigma) = \pm b_{w\sigma w^{-1}}$ (Definition 2.7); applied to the special case in which $w$ is another reflection, this gives (R4). For (R3) we take the following argument from [31, V.1]. Suppose that $\sigma$ and $\tau$ are reflections in $W$ such that both $b_\sigma$ and $b_\tau$ are multiples of a single element $b \in L$; we must show that $\sigma = \tau$. Let $w = \sigma\tau$. Then $w(b) = b$, and $w$ acts as the identity on $\mathbb{Q} \otimes (L/\langle b \rangle)$; in particular, all of the eigenvalues of $w$ are equal to 1. But $w \in W$ has finite order, and so $w$ must be the identity map.

On the other hand, suppose that $R$ is a root system in $L$. For each root $r$ let $\sigma_r \colon L \to L$ be given by

$$\sigma_r(x) = x + n_r(x) r.$$

It follows from (R2) that $\sigma_r$ is a reflection on $L$, with a marking $\pm(r, n_r)$; take $W$ to be the group generated the transformations $\sigma_r$. By (R4) each reflection $\sigma_r$ preserves the finite set $R$ of roots, and so $W$ also preserves $R$. To see that $W$ is finite, it is enough to check that any element of $W$ which acts trivially on $R$ is the identity, but this follows from (R1) and the fact that the subspace $\bigcap_r \ker(n_r)$ is pointwise fixed by each reflection $\sigma_r$ and hence pointwise fixed by $W$. Note that by the above eigenvalue argument, $w\sigma_r w^{-1} = \sigma_{w(r)}$: both transformations send $w(r)$ to $-w(r)$ and act as the identity on $\mathbb{Q} \otimes (L/\langle w(r) \rangle)$. Checking that the markings are preserved under conjugation thus amounts to showing that if $\sigma_r = \sigma_t$ then $r = \pm t$ (note that as in Remark 2.18 a marking $\pm(r, n_r)$ is determined by its first component). But since $r$ and $t$ then determine markings of the same reflection, it follows from Remark 2.18 that $r = \pm t$, $r = \pm 2t$, or $t = \pm 2r$, and the last two possibilities are ruled out by (R3). The one remaining issue is that there might be reflections in $W$ which do not appear in the set $\{\sigma_r \mid r \in R\}$; these reflections have not been marked in the above discussion. But by [19, 1.14] there are no such additional reflections. □

Proposition 2.17 is a consequence of the following lemma, which is proved by examining the discussion in Remark 2.18.

**2.20 Lemma** *Let $\sigma$ be a reflection on the lattice $L$, treated also as a reflection on $T = \mathrm{T}(1) \otimes L = (\mathbb{R} \otimes L)/L$. Then sending $b \in L$ to the residue class of $b/2$ in $T$ gives a bijection between markings $\pm(b, \beta)$ of $\sigma$ and elements $h \in T$ satisfying the conditions of Definition 2.12.*

**2.21 Duality** A reflection $\sigma$ on $L$ induces a reflection on the dual lattice $L^{\#} = \mathrm{Hom}(L, \mathbb{Z})$; sending $\pm(b, \beta)$ to $\pm(\beta, b)$ gives a bijection between markings of $\sigma$ and markings of the dual reflection. It follows that if $(L, W)$ is a marked reflection lattice with markings $\pm(b_\sigma, \beta_\sigma)$, then $(L^{\#}, W)$ is a marked reflection lattice with markings $\pm(\beta_\sigma, b_\sigma)$. Similarly, interchanging roots and coroots give a bijection between root systems in $L$ and root systems in $L^{\#}$.

**2.22 Counting root systems** As in Remark 2.18, the number of markings of a reflection lattice $(L, W)$ is $2^k$, where $k$ is the number of conjugacy classes of reflections in $W$ which are trivial mod $2$. By Proposition 2.16, this is also the number of distinct root systems with reflection lattice $(L, W)$.

**2.23 Another notion of root system** We point out the relationship between our notion of root system (which is Tits') and that of Serre and Bourbaki. Recall that what we term a *geometric root system* (Remark 2.3) is a pair $(V, R)$ such that $V$ is a finite-dimensional real vector space, $R$ is a finite subset of $V$, and the following conditions are satisfied [31, Section V].

- $R$ generates $V$ as a vector space, and does not contain $0$.
- For each $r \in R$ there exists a reflection $s_r$ on $V$ which carries $R$ to itself and has $s_r(r) = -r$.
- For each $r, t \in R$, $s_r(t) - t$ is an integral multiple of $r$.

The geometric root system is said to be *reduced* if for each $r \in R$, $r$ and $-r$ are the only roots proportional to $r$.

The following proposition is immediate. We will say that a root system (see Definition 2.2) is *semisimple* if $\bigcap_r \ker(n_r) = \{0\}$.

**2.24 Proposition** *If $(L, R, \{n_r\})$ is a semisimple root system, then $(\mathbb{R}\otimes L, R)$ is a a reduced geometric root system.*

Conversely, if $(V, R)$ is a reduced geometric root system, then for each $r \in R$ define a homomorphism $H_r\colon V \to \mathbb{R}$ by the formula $s_r(x) = x + H_r(x)r$. Let $L_{\max}$ be the lattice in $V$ given by $\{x \mid \forall r, H_r(x) \in \mathbb{Z}\}$, and let $L_{\min}$ be the lattice in $L$ spanned by $R$. Note that if $L \subset V$ is a lattice between $L_{\min}$ and $L_{\max}$, then $H_r(L) \subset \mathbb{Z}$.

**2.25 Proposition** *Suppose that $(V, R)$ is a reduced geometric root system, and that $L$ is a lattice in $V$ with $L_{min} \subset L \subset L_{max}$. Then $(L, R, \{H_r\})$ is a semisimple root system.*

Recall that a connected compact Lie group $G$ is said to be *semisimple* if $\pi_1 G$ is finite, or equivalently if the center of $G$ is finite. It is well–known [4] that geometric root systems classify connected semisimple compact Lie groups up to isogeny, ie up to finite covers. A glance at Serre's description of the fundamental group of a connected compact Lie group [31, VIII-10] shows that extra lattice data in a semisimple root system (Proposition 2.25) allow such systems to classify connected semisimple compact Lie groups up to isomorphism.

More generally, root systems classify connected compact Lie groups up to isomorphism: the lattice $\bigcap_r n_r$ in the root system derived from a connected compact Lie group $G$ determines the torus which is the identity component of the center of $G$. In our way of assigning a root system to $G$ (see Section 5), which is dual to the ordinary one (see Remark 5.6), $\bigcap_r n_r$ is the fundamental group of the central torus. This classification theorem is stated in [5, §8, Sect. 9]. It can also be obtained by combining the correspondence between compact Lie groups and reductive complex algebraic groups [29, p247] with the classification of connected reductive complex algebraic groups by *root data* [32]. Springer's root data are essential the same as our root systems.

## 3 The normalizer and reflection extensions

In this section we take a marked reflection torus $(T, W, \{h_\sigma\})$ and construct the associated normalizer extension of $W$ by $T$. Let $\Sigma \subset W$ be the set of reflections in $W$; the group $W$ acts on $\Sigma$ by conjugation, and hence on the free abelian group $\mathbb{Z}[\Sigma]$ generated by $\Sigma$. We first describe an extension of $W$ by $\mathbb{Z}[\Sigma]$, called the *reflection extension* of $W$, which depends only on the structure of $W$ as a reflection group, and then use this in conjunction with the markings $\{h_\sigma\}$ to obtain the normalizer extension.

**3.1 The reflection extension** Write $\Sigma = \coprod_i \Sigma_i$ as a union of conjugacy classes of reflections. For each index $i$ choose a reflection $\tau_i$ in $\Sigma_i$, and let $C_i$ be the centralizer of $\tau_i$ in $W$. Let $L = \pi_1 T$, and let $a_i \in L$ be a nonzero element with $\tau_i(a_i) = -a_i$. It is clear that each element of $C_i$ takes $a_i$ to $\pm a_i$, and that $C_i$ is isomorphic to $\langle \tau_i \rangle \times C_i^\perp$, where $C_i^\perp$ is the subgroup of $C_i$ consisting of elements which fix $a_i$.

As usual, extensions of $W$ by $\mathbb{Z}[\Sigma]$ are classified by elements of $\mathrm{H}^2(W;\mathbb{Z}[\Sigma])$. There are isomorphisms

$$\mathrm{H}^2(W;\mathbb{Z}[\Sigma]) \cong \oplus_i \mathrm{H}^2(W;\mathbb{Z}[\Sigma_i]) \cong \oplus_i \mathrm{H}^2(C_i;\mathbb{Z})$$

where the last isomorphism comes from Shapiro's lemma. Thus in order to specify an extension of $W$ by $\mathbb{Z}[\Sigma]$ it is enough to specify for each $i$ an extension of $C_i$ by $\mathbb{Z}$. We do this as follows. For each $i$ write

$$C_i \cong \langle \tau_i \rangle \times C_i^\perp \cong \mathbb{Z}/2 \times C_i^\perp$$

and consider the obvious extension

$$1 \to \mathbb{Z} \to \mathbb{Z} \times C_i^\perp \to \mathbb{Z}/2 \times C_i^\perp \to 1. \tag{3.2}$$

This extension corresponds to the element of $\mathrm{H}^2(C_i;\mathbb{Z})$ which is a pullback under the projection $C_i \to \mathbb{Z}/2$ of the unique nonzero element in $\mathrm{H}^2(\mathbb{Z}/2;\mathbb{Z})$. By Shapiro's lemma there is a corresponding extension $\rho_i(W)$ of $W$ by $\mathbb{Z}[\Sigma_i]$, and it is easy to see that up to canonical isomorphism this extension of $W$ does not depend on the choice of representative $\tau_i$ for the conjugacy class $\Sigma_i$.

**3.3 Definition** The *reflection extension* $\rho(W)$ of $W$ is the extension of $W$ by $\mathbb{Z}[\Sigma]$ determined by the above collection of extensions of the groups $C_i$; in other words, $\rho(W)$ is the fibrewise product (over $W$) of the extensions

$$1 \to \mathbb{Z}[\Sigma_i] \to \rho_i(W) \to W \to 1.$$

**3.4 The normalizer extension** Given a marked reflection torus

$$\mathcal{T} = (T, W, \{h_\sigma\}),$$

there is a $W$–map

$$f^{\mathcal{T}}\colon\ \mathbb{Z}[\Sigma] \to T$$

which sends a reflection $\sigma$ to $h_\sigma$.

**3.5 Definition** Suppose that $\mathcal{T} = (T, W, \{h_\sigma\})$ is a marked reflection torus. The *normalizer extension* $\nu(T, W, \{h_\sigma\})$ of $W$ by $T$ is defined to be the image of the reflection extension $\rho(W)$ under the map $f^{\mathcal{T}} \colon \mathbb{Z}[\Sigma] \to T$.

**3.6 Remark** The reflection extension of $W$ is determined by an extension class $k \in \mathrm{H}^2(W; \mathbb{Z}[\Sigma])$, and the normalizer extension by $f^{\mathcal{T}}_*(k) \in \mathrm{H}^2(W; T)$.

# 4 The Tits extension

Suppose that $(L, W)$ is a reflection lattice. In this section we derive generators and relations (Section 4.6) for the reflection extension $\rho(W)$ of $W$. To accomplish this, we identify $\rho(W)$ with the *extended Weyl group* constructed by Tits [34]; we call this extended Weyl group the *Tits extension* of $W$ and denote it $\tau(W)$. The presentation for $\rho(W)$ is used both in the proof of Theorem 5.7 and later on in the proof of Proposition 9.12 (Coxeter case).

**4.1 The Tits extension** Let $S = \{s_1, \ldots, s_\ell\}$ be a chosen set of simple reflections in $W$ (see Remark 2.14), and let $\Sigma^*$ be a copy of the set $\Sigma$ of all reflections in $W$. As in Section 3, $W$ acts by conjugation on $\Sigma$ and thus on $\mathbb{Z}[\Sigma]$ or $\mathbb{Z}[\Sigma^*]$.

**4.2 Definition** The *Tits extension* $\tau(W)$ of $W$ is the subgroup of the semidirect product $\mathbb{Z}[\Sigma^*] \rtimes W$ generated by the elements $(s^*, s)$, $s \in S$.

Since the set $S$ generates $W$ [19, 1.5], the natural surjection $\mathbb{Z}[\Sigma^*] \rtimes W \to W$ restricts to a surjection $\tau(W) \to W$. The kernel of this map is contained in $\mathbb{Z}[\Sigma^*]$, and Tits [34, Cor. 2.7] shows that it is exactly $2\mathbb{Z}[\Sigma^*]$, which of course is isomorphic to $\mathbb{Z}[\Sigma]$ as a $W$–module. Under this identification, we can treat $\tau(W)$ as an extension

$$1 \to \mathbb{Z}[\Sigma] \to \tau(W) \to W \to 1 \tag{4.3}$$

of $W$ by $\mathbb{Z}[\Sigma]$. In this section we will prove the following proposition.

**4.4 Proposition** *Suppose that $(L, W)$ is a reflection lattice and that $\Sigma$ is the set of reflections in $W$. Then, as an extension of $W$ by $\mathbb{Z}[\Sigma]$, the reflection extension $\rho(W)$ is isomorphic to the Tits extension $\tau(W)$.*

**4.5 Remark** It follows from Proposition 4.4 that the isomorphism class of the extension $\tau(W)$ is independent of the choice of the set $S$ of simple reflections. This also follows from the fact that any two sets of simple reflections are conjugate in $W$ [19, 1.4].

**4.6 Generators and relations** Let $(m_{i,j})$ be the Coxeter matrix (see Remark 2.14) of $(L, W)$, so that $W$ is generated by simple reflections $\{s_1, \ldots, s_\ell\}$ subject to the relations

$$(s_i s_j)^{m_{i,j}} = 1. \tag{4.7}$$

Note that $m_{i,i} = 1$. Following Tits [34, 0.1], we put these relations in a slightly different form. Write

$$\mathrm{prod}(n; y, x) = \cdots yxyx$$

where the product on the right contains exactly $n$ factors, alternating between $x$ and $y$, and beginning on the right with an $x$. Then $W$ is generated by the elements $s_i$, $1 \le i \le \ell$, subject only to the relations

$$\begin{aligned} s_i^2 &= 1 \\ \mathrm{prod}(m_{i,j}; s_i, s_j) &= \mathrm{prod}(m_{i,j}; s_j, s_i) \text{ for } i \neq j. \end{aligned}$$

In fact the first relation gives $s_i = s_i^{-1}$, while the second one is obtained from (4.7) by the reversible process of moving the leading terms of the expanded expression $(s_i s_j)^{m_{i,j}}$ to the right hand side of the equation and replacing each occurrence of $x^{-1}$ ($x = s_i$ or $s_j$) by $x$ itself.

In [34, 2.6], Tits gives a presentation for $\tau(W)$ parallel to the above presentation for $W$. For notational reasons, think for the moment of $\tau(W)$ as an extension of $W$ by $\mathbb{Z}[\Sigma^*]$. Then $\tau(W)$ is generated by $\mathbb{Z}[\Sigma^*]$ and symbols $q_i$, one for each simple reflection, subject to the following relations:

$$\begin{aligned} q_i^2 &= s_i^* \\ q_i \sigma^* q_i^{-1} &= (s_i \sigma s_i^{-1})^*, \quad \sigma \in \Sigma \\ \mathrm{prod}(m_{i,j}; q_i, q_j) &= \mathrm{prod}(m_{i,j}; q_j, q_i) \text{ for } i \neq j. \end{aligned}$$

With respect to this presentation of $\tau(W)$, the quotient map $\tau(W) \to W$ sends $q_i$ to $s_i$. By Proposition 4.4, these same formulas give a presentation of $\rho(W)$.

The rest of this section is devoted to the proof of Proposition 4.4. We first need an explicit formulation of Shapiro's lemma. As above, if $G$ is a finite group and $H \subset G$ is a subgroup, extensions of $G$ by the permutation module $\mathbb{Z}[G/H]$ correspond bijectively to extensions of $H$ by the trivial $H$–module $\mathbb{Z}$. We will call these two extensions *Shapiro companions* of one another. The following lemma is elementary; see [6, III.8, Ex. 2].

**4.8 Lemma** *Suppose that $G$ is a finite group, $H \subset G$ a subgroup, and*

$$1 \to \mathbb{Z}[G/H] \to E \to G \to 1 \tag{4.9}$$

*an extension of $G$ by the permutation module $\mathbb{Z}[G/H]$. Then the Shapiro companion of (4.9) is obtained by pulling the extension back over $H$ to obtain*

$$1 \to \mathbb{Z}[G/H] \to E' \to H \to 1,$$

*and then taking the quotient of $E'$ by the subgroup of $\mathbb{Z}[G/H]$ generated by the elements $\{x \in G/H, x \neq eH\}$.*

**4.10 Remark** Note that $H$ fixes the coset $eH$ in $G/H$, and so $H$ carries the set $\{x \in G/H, x \neq eH\}$ to itself. This implies that the subgroup of $\mathbb{Z}[G/H]$ generated by this set is a normal subgroup of the group $E'$ above. The quotient

$$\mathbb{Z}[G/H] \;/\; \mathbb{Z}[x \in G/H, x \neq eH]$$

is canonically isomorphic to the trivial $H$–module $\mathbb{Z}$, with generator given by the residue class of $eH$.

We next recall some apparatus from [34]. Let $I$ be a set of indices corresponding to the chosen simple reflections in $W$, and $\mathbf{I}$ the free group on $I$. An element $\mathbf{i} \in \mathbf{I}$ is said to be *positive* if it is a product of elements of $I$; the *length* $l(\mathbf{i})$ is then the number of elements in this product expression. To conform with [34], for each $i \in I$ we will denote the corresponding simple reflection in $W$ by $r_i$ (instead of $s_i$); there is a unique (surjective) homomorphism $r\colon \mathbf{I} \to W$ with $r(i) = r_i$, $i \in I$. A positive word $\mathbf{i} \in \mathbf{I}$ is said to be *minimal* if there does not exist a positive word $\mathbf{j} \in \mathbf{I}$ with $l(\mathbf{j}) < l(\mathbf{i})$ and $r(\mathbf{j}) = r(\mathbf{i})$. If $w \in W$, the *length* of $w$ is defined to be the length of a minimal positive word in $\mathbf{I}$ which maps to $w$ under $r$.

Suppose that $\mathbf{i} = i_1 i_2 \cdots i_m$ is a positive word in $\mathbf{I}$ of length $m$. For each integer $k$ with $1 \leq k \leq m$ set

$$\sigma_k = r(i_1 i_2 \cdots i_{k-1}) \cdot r_{i_k} \cdot r(i_1 i_2 \cdots i_{k-1})^{-1}.$$

Each one of these elements is a reflection in $W$, and the sequence $\sigma_1, \sigma_2, \cdots, \sigma_m$ is said to be the *sequence of reflections associated to the word* $\mathbf{i}$. Recall the following proposition.

**4.11 Proposition** [34, 1.6] *Let $\mathbf{i} \in \mathbf{I}$ be a minimal word, $\sigma \in W$ a reflection, and $n$ the number of times which $\sigma$ appears in the sequence of reflections associated to $\mathbf{i}$. Then $n = 0$ or $1$.*

We need to evaluate the integer $n$ above in a special case.

**4.12 Proposition** *Let $\mathbf{i} \in \mathbf{I}$ be a minimal word, $\sigma \in W$ a reflection, and $n$ the number of times which $\sigma$ appears in the sequence of reflections associated to $\mathbf{i}$. Suppose that $\sigma$ is a simple reflection, that $r(\mathbf{i})$ is a reflection which commutes with $\sigma$, and that $r(\mathbf{i}) \neq \sigma$. Then $n = 0$.*

The proof of this was explained to us by M. Dyer. It depends on the following lemma.

**4.13 Lemma** [17, 1.4] *Suppose that $t = r(i_1 i_2 \cdots i_{2n+1})$ is a reflection in $W$ with $l(t) = 2n+1$. Then $t = r(i_1 \cdots i_n i_{n+1} i_n \cdots i_1)$.*

For the convenience of the reader, we give the argument from [17].

**Proof of Lemma 4.13** Let $x = r(i_n \cdots i_1)$ and $y = r(i_{n+2} \cdots i_{2n+1})$. Then $l(x) = l(y) = n$ and $l(r_{i_{n+1}} x) = l(r_{i_{n+1}} y) = n+1$. Note that $t = t^{-1}$, so that $r_{i_{n+1}} y t = r_{i_{n+1}} y t^{-1} = x$ and $l(r_{i_{n+1}} y t) < l(r_{i_{n+1}} y)$. Applying the Strong Exchange Condition [19, 5.8] to this last observation shows that there is a number $k$ between $n+1$ and $2n+1$ such that

$$x = r_{i_{n+1}} y t = r(i_{n+1} \cdots \widehat{i_k} \cdots i_{2n+1}).$$

Since $l(x) = n$, this is a minimal expression for $x$. Since $l(r_{i_{n+1}} x) > l(x)$ it must be the case that $k = n+1$ and hence that $x = y$. □

**Proof of Proposition 4.12** Write $\mathbf{i} = i_1 i_2 \cdots i_m$, and suppose that $\sigma$ appears among the sequence $\sigma_1, \sigma_2, \cdots, \sigma_m$ of reflections associated to $\mathbf{i}$, that is, suppose that there exists $k$ with $1 \leq k \leq m$ such that

$$\sigma = \sigma_k = r(i_1 \cdots i_{k-1}) r_{i_k} r(i_1 \cdots i_{k-1})^{-1}.$$

Let $t = r(i_1 \cdots i_m)$; it is clear that

$$\begin{aligned} t &= r(i_1 \cdots i_{k-1}) \cdot r_{i_k} \cdot r(i_1 \cdots i_{k-1})^{-1} r(i_1 \cdots i_{k-1}) r(i_{k+1} \cdots i_m) \\ &= \sigma \cdot r(i_1 \cdots i_{k-1}) r(i_{k+1} \cdots i_m). \end{aligned}$$

Choose $i_0 \in I$ so that $\sigma = r_{i_0}$ (recall that $\sigma$ is a simple reflection) and let $\mathbf{j} = i_0 i_1 \cdots i_{k-1} i_{k+1} \cdots i_m$; then $r(\mathbf{j}) = t$ and, since $l(\mathbf{j}) = l(\mathbf{i})$, $\mathbf{j}$ is a minimal word with $r(\mathbf{j}) = t$. Write $\mathbf{j} = j_1 \cdots j_m$, with $j_1 = i_0$, so that $r_{j_1} = \sigma$. For determinant reasons $m = 2q+1$ is odd, and $m > 1$ because $t = r(\mathbf{j}) \neq \sigma$. By Lemma 4.13,

$$\begin{aligned} t &= r_{j_1} \cdot r(j_2 \cdots j_q j_{q+1} j_q \cdots j_2) \cdot r_{j_1} \\ &= \sigma \cdot r(j_2 \cdots j_q j_{q+1} j_q \cdots j_2) \cdot \sigma. \end{aligned}$$

Now consider $\sigma \cdot t \cdot \sigma$. Since $t$ commutes with $\sigma$ and $\sigma^2 = 1$, this element is equal to $t$. In combination with the above equation, we get

$$\begin{aligned} t &= \sigma \cdot t \cdot \sigma \\ &= \sigma \cdot \sigma \cdot r(j_2 \cdots j_q j_{q+1} j_q \cdots j_2) \cdot \sigma \cdot \sigma \\ &= r(j_2 \cdots j_q j_{q+1} j_q \cdots j_2), \end{aligned}$$

which, in contradiction to the assumption that $\mathbf{i}$ is a minimal word, gives $l(t) \le m-2$. This contradiction shows that $\sigma$ does not appear in the list of reflections associated to $\mathbf{i}$. □

Finally, one more calculation from Tits. Let $a\colon \mathbf{I} \to \mathbb{Z}[S^*] \rtimes W$ be the homomorphism determined by $a(i) = (r_i^*, r_i)$, $i \in I$. By definition, the image of $a$ is $\tau(W)$.

**4.14 Lemma** [34, Lemme 2.4] *Let $\mathbf{i} \in \mathbf{I}$ be a positive word of length $m$, and $\sigma_1, \sigma_2, \cdots, \sigma_m$ the sequence of reflections associated to $\mathbf{i}$. Then*

$$a(\mathbf{i}) = \left( \sum_{k=1}^{m} \sigma_k^*, r(\mathbf{i}) \right).$$

**Proof of Proposition 4.4** We identify the kernel of the map $\tau(W) \to W$ with $\mathbb{Z}[\Sigma]$. Then $\tau(W)$ sits inside the semidirect product $\mathbb{Z}[\Sigma^*] \rtimes W$, and there is map of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}[\Sigma] & \longrightarrow & \tau(W) & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}[\Sigma^*] & \longrightarrow & \mathbb{Z}[\Sigma^*] \rtimes W & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

in which the left hand vertical map gives an isomorphism between $\mathbb{Z}[\Sigma]$ and $2\mathbb{Z}[\Sigma^*]$. As in Section 3.1, giving the extension $\tau(W)$ amounts to giving a collection of extensions $\{\tau_i(W)\}$, where $\tau_i(W)$ is an extension of $W$ by $\mathbb{Z}[\Sigma_i]$; the extension $\tau_i(W)$ is obtained from $\tau(W)$ by taking a quotient of $\tau(W)$ by the subgroup of $\mathbb{Z}[\Sigma]$ generated by $\Sigma \setminus \Sigma_i$. Clearly $\tau_i(W)$ lies inside the semidirect product $\mathbb{Z}[\Sigma_i^*] \rtimes W$, and in fact there is a map of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}[\Sigma_i] & \longrightarrow & \tau_i(W) & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}[\Sigma_i^*] & \longrightarrow & \mathbb{Z}[\Sigma_i^*] \rtimes W & \longrightarrow & W & \longrightarrow & 1 \end{array} \tag{4.15}$$

in which as before the left hand vertical map identifies $\mathbb{Z}[\Sigma_i]$ with $2\mathbb{Z}[\Sigma_i^*]$. Pulling the diagram (4.15) back over the inclusion $C_i \to W$ gives a commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}[\Sigma_i] \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{Z}[\Sigma_i^*] \arrow[r] & \mathbb{Z}[\Sigma_i^*] \rtimes C_i \arrow[r] & C_i \arrow[r] & 1 \end{tikzcd} \tag{4.16}$$

Now taking quotients as in Lemma 4.8, in the upper group by $\mathbb{Z}[\Sigma_i \setminus \tau_i]$ and in the lower by $\mathbb{Z}[\Sigma_i^* \setminus \tau_i^*]$, yields:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}\tau_i \arrow[r] \arrow[d] & E_i' \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{Z}\tau_i^* \arrow[r] & \mathbb{Z}\tau_i^* \times C_i \arrow[r] & C_i \arrow[r] & 1 \end{tikzcd} \tag{4.17}$$

It is clear that the left hand vertical map sends $\tau_i$ to $2\tau_i^*$. We need to argue that the upper extension is the one described in (3.2).

Note that every reflection in $W$ is conjugate to a simple reflection [19, 1.14], so we can assume that $\tau_i$ belongs to the set $S$ of simple reflections. Recall that $C_i \cong \langle \tau_i \rangle \times C_i^{\perp}$, where $C_i^{\perp}$ can be identified as the subgroup of $W$ which pointwise fixes $L^-(\tau_i)$. Following Steinberg [33, 1.5] [19, 1.12], $C_i^{\perp}$ is generated by the reflections it contains, and so in particular it is generated by reflections $t$ which commute with $\tau_i$ and are distinct from $\tau_i$. Pick such a $t$, and write $t = r(\mathbf{i})$, where $\mathbf{i}$ is a minimal positive word. By Proposition 4.12, $\tau_i$ does not appear among the list of reflections associated to $\mathbf{i}$. Clearly the image $\alpha$ of $a(\mathbf{i})$ in $\tau_i(W)$ belongs to $E_i$ (see (4.16)); let $\overline{a}(\mathbf{i})$ be the image of $\alpha$ in $E_i'$ (see (4.17)). It now follows from Lemma 4.14 that the image of $\overline{a}(\mathbf{i})$ in $\mathbb{Z}\tau_i^* \times C_i$ is $(0, t)$. The subgroup of $\mathbb{Z}\tau_i^* \times C_i$ generated by these elements as $t$ varies through the reflections in $C_i^{\perp}$ is $\{0\} \times C_i^{\perp}$. Consider the subgroup $E_i''$ of $\mathbb{Z}\tau_i^* \times C_i$ generated by $\{0\} \times C_i^{\perp}$ and by $(\tau_i^*, \tau_i)$. Clearly, $E_i''$ is contained inside the image of $E_i'$ in $\mathbb{Z}\tau_i^* \times C_i$; on the other hand, it is also easy to check that there is a short exact sequence

$$1 \to 2\mathbb{Z}\tau_i^* \to E_i'' \to C_i \to 1.$$

By the five lemma, $E_i'' \cong E_i'$, and it is now simple to verify that $E_i''$ is the extension of $C_i$ described in (3.2). □

# 5 Compact Lie Groups

In this section, we follow the lead of Tits [34] and describe the normalizer of the torus in a connected compact Lie group in terms of a marked reflection torus

(Definition 2.12) associated to the group.

Suppose that $G$ is a connected compact Lie group of rank $r$ with maximal torus $T$, maximal torus normalizer $N = NT$ and Weyl group $W = N/T$. Then $T \cong \mathrm{T}(r)$ and the conjugation action of $W$ on $T$ identifies $W$ with a subgroup of $\mathrm{Aut}(T)$ generated by reflections [18, VII.2.13] [14, 5.16]. In particular, $(T, W)$ is a reflection torus.

For each reflection $\sigma \in W$ we will construct a subgroup $N_\sigma$ of $N$, called the *root subgroup* of $N$ associated to $\sigma$. We will also single out a specific element $h_\sigma$ in $T$, the *marking* associated to $\sigma$.

Given $\sigma$, let $T_0^+(\sigma)$ denote the identity component of the fixed point set of the action of $\sigma$ on $T$. It is easy to see that $T_0^+(\sigma)$ is a torus of rank $r-1$; if $L$ is the lattice corresponding to $T$, then under the isomorphism $\mathrm{T}(1) \otimes L \cong T$, $T_0^+(\sigma)$ is given by $\mathrm{T}(1) \otimes L^+(\sigma)$. Let $G(\sigma)$ denote the centralizer of $T_0^+(\sigma)$ in $G$. Then $G(\sigma)$ is a connected [18, VII.2.8] [14, 7.3] compact Lie subgroup of $G$ with maximal torus $T$ and Weyl group given by the subgroup $\{1, \sigma\}$ of $W$. The normalizer of $T$ in $G(\sigma)$ can be identified with the centralizer of $T_0^+(\sigma)$ in $N$; we denote this group by $N(\sigma)$. The only simply-connected compact Lie group of rank 1 is $\mathrm{SU}(2)$; the center of $\mathrm{SU}(2)$ is $\mathbb{Z}/2$ and the adjoint form is $\mathrm{SO}(3)$. Given this, it follows from the product splitting theorem for compact Lie groups [14, Section 10] that $G(\sigma)$ is isomorphic to one of the following three groups:

(1) A product $\mathrm{SU}(2) \times \mathrm{T}(r-1)$

(2) A product $\mathrm{SO}(3) \times \mathrm{T}(r-1)$

(3) A product $\mathrm{U}(2) \times \mathrm{T}(r-2)$

Recall that $T_0^-(\sigma)$ denotes the identity component of $T^-(\sigma)$; this group is a torus of rank 1.

**5.1 Definition** The *root subgroup* $N_\sigma$ is given by

$$N_\sigma = \{x \in N(\sigma) \mid x \text{ is conjugate in } G(\sigma) \text{ to some } y \in T_0^-(\sigma)\}.$$

**5.2 Lemma** *The set $N_\sigma$ is a closed subgroup of $N(\sigma)$ with two components. The identity component of $N_\sigma$ is the circle $T_0^-(\sigma) = N_\sigma \cap T$, and the natural map $N_\sigma/T_0^-(\sigma) \to N(\sigma)/T \cong \mathbb{Z}/2$ is an isomorphism. If $x, y$ belong to the nonidentity component of $N_\sigma$, then $x^2 = y^2 \in T_0^-(\sigma)$.*

**Proof** The simplest way to prove this is to examine the above three cases. In the first two cases $N_\sigma$ is the normalizer of the maximal torus $T_0^-(\sigma)$ in SU(2) (respectively, SO(3)). In the third case, $N_\sigma$ is the normalizer of the maximal torus $T_0^-(\sigma)$ of the subgroup SU(2) of U(2). □

**5.3 Definition** The *marking* $h_\sigma \in T_0^-(\sigma)$ is the image of the nonidentity component of $N_\sigma$ under the squaring map $x \mapsto x^2$.

**5.4 Lemma** *The element $h_\sigma \in T_0^-(\sigma) \subset T$ is a marking for $\sigma$ in the sense of Definition 2.12.*

**Proof** Again, this is most easily proved by inspection. The element $h_\sigma$ is strongly $\sigma$–negative by construction, and it is easy to see that $2h_\sigma = 0$ (where as usual we write the group operation in the torus additively). Finally, if $\sigma$ is nontrivial mod 2, then $G(\sigma) \cong \mathrm{U}(2) \times \mathrm{T}(r-2)$ and $h_\sigma$ is the nontrivial central element of order 2 in U(2). □

The following statement is a consequence of the naturality of the above constructions.

**5.5 Proposition** *Suppose that $G$ is a connected compact Lie group with maximal torus $T$, maximal torus normalizer $N$, Weyl group $W$, root subgroups $\{N_\sigma\}$, and markings $\{h_\sigma\}$ as above. Suppose that $x \in N$ has image $w \in W$. Then for any reflection $\sigma \in W$,*

$$\begin{aligned} xN_\sigma x^{-1} &= N_{w\sigma w^{-1}} \\ xh_\sigma x^{-1} &= h_{w\sigma w^{-1}} \end{aligned}$$

*In particular, $(T, W, \{h_\sigma\})$ is a marked reflection torus.*

**5.6 Remark** Let $L$ denote $\pi_1 T$ and let $R$ be the root system in $L$ associated to the above marked reflection torus (see Propositions 2.16 and 2.17). It is not hard to identify the dual root system $R^\#$ in $L^\#$ (see Section 2.21) with the root system usually associated to $G$ [31, VIII-8, VI]. The correspondence $G \mapsto R^\#$ produces a bijection between isomorphism classes of connected compact Lie groups and isomorphism classes of root systems (see Section 2.23).

The marking $h_\sigma$ is defined above (Definition 5.3) in terms of $N_\sigma$. However, it is not hard to see that $h_\sigma$ is the only marking for $\sigma$ in the intersection of $T_0^-(\sigma)$ with the square of the nonidentity component of $N(\sigma)$. Since $N(\sigma)$ is the centralizer in $N$ of $T_0^+(\sigma)$, the element $h_\sigma$, and hence the marked reflection torus

$(T, W, \{h_\sigma\})$ and the root system $R^\#$, are determined by the group structure of $N$. In this way we recover the result from [8] and [30] that $G$ is determined up to isomorphism by $N$.

The main result of this section is the following one.

**5.7 Theorem** *Let $G$ be a connected compact Lie group, and $(T, W, \{h_\sigma\})$ the marked reflection torus derived from $G$ (Proposition 5.5). Then the normalizer of $T$ in $G$ is isomorphic to the normalizer extension $\nu(T, W, \{h_\sigma\})$ (see Section 3).*

This is a consequence of the following theorem, due essentially to Tits. See Section 4.6 for the notation "$\mathrm{prod}(n; x, y)$".

**5.8 Theorem** *Suppose that $G$ is a connected compact Lie group with maximal torus $T$, maximal torus normalizer $N$, Weyl group $W$, root subgroups $\{N_\sigma\}$, and markings $\{h_\sigma\}$. Choose a set $\{s_i\}$ of simple reflections in $W$, and let $\{m_{i,j}\}$ be the corresponding Coxeter matrix. For each $i$, choose an element $x_i$ in the nonidentity component of $N_{s_i}$. Then the elements $x_i$ satisfy the following relations:*

$$\begin{aligned} x_i^2 &= h_{s_i} \\ x_i t x_i^{-1} &= s_i(t), \quad t \in T \\ \mathrm{prod}(m_{i,j}; x_i, x_j) &= \mathrm{prod}(m_{i,j}; x_j, x_i) \ \textit{for } i \neq j. \end{aligned} \tag{5.9}$$

**Proof of Theorem 5.7 (given Theorem 5.8)** Let $\nu(W)$ denote the normalizer extension in question. According to Definition 3.5, Proposition 4.4, and Section 4.6, $\nu(W)$ is generated by the torus $T$ together with symbols $q_i$, one for each simple reflection $s_i$, subject to the following relations:

$$\begin{aligned} q_i^2 &= h_{s_i} \\ q_i t q_i^{-1} &= s_i(t) \quad t \in T \\ \mathrm{prod}(m_{i,j}; q_i, q_j) &= \mathrm{prod}(m_{i,j}; q_j, q_i) \ \text{for } i \neq j. \end{aligned} \tag{5.10}$$

By Theorem 5.8, there is a homomorphism $\phi\colon \nu(W) \to N$ which is the identity on $T$ and sends $q_i$ to $x_i$. This is an isomorphism because it is part of a map

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] \arrow[d, "="'] & \nu(W) \arrow[r] \arrow[d, "\phi"'] & W \arrow[r] \arrow[d, "="'] & 1 \\ 1 \arrow[r] & T \arrow[r] & N \arrow[r] & W \arrow[r] & 1 \end{tikzcd}$$

of exact sequences. □

The proof of Theorem 5.8 depends on two auxiliary results. Suppose that $\widetilde{G} \to G$ is a finite covering of connected compact Lie groups, and that $\widetilde{T}$ is a maximal torus in $\widetilde{G}$. Then the image $T$ of $\widetilde{T}$ in $G$ is a maximal torus for $G$, the normalizer $\widetilde{N}$ of $\widetilde{T}$ in $\widetilde{G}$ maps onto the normalizer $N$ of $T$ in $G$, and the induced map $\widetilde{N}/\widetilde{T} \to N/T$ is an isomorphism. In particular, $G$ and $\widetilde{G}$ have isomorphic Weyl groups, which we will denote by the same letter $W$. The lattice corresponding to $\widetilde{T}$ is a subgroup of finite index in the lattice corresponding to $T$, and it follows that an element of $W$ acts as a reflection on $\widetilde{T}$ if and only if it acts as a reflection on $T$. The following statement is clear.

**5.11 Proposition** *Suppose that $p\colon \widetilde{G} \to G$ is a finite covering of connected compact Lie groups with Weyl group $W$. Let $\widetilde{T}$ be a maximal torus in $\widetilde{G}$ and $T \subset G$ the image maximal torus in $G$. For each reflection $\sigma \in W$, let $\widetilde{N}_\sigma$, $\widetilde{h}_\sigma$ (respectively, $N_\sigma$, $h_\sigma$) denote the root subgroup and marking in $\widetilde{G}$ (respectively, $G$) corresponding to $\sigma$. Then $p(\widetilde{N}_\sigma) = N_\sigma$ and $p(\widetilde{h}_\sigma) = h_\sigma$.*

**5.12 Proposition** *Suppose that $G$ is a connected, simply connected compact Lie group with maximal torus $T$, torus normalizer $N$, Weyl group $W$, root subgroups $\{N_\sigma\}$, and markings $\{h_\sigma\}$. Let $\{s_1, \cdots, s_\ell\}$ be a set of simple reflections for $W$. Then if $i$ and $j$ are distinct integers between $1$ and $\ell$, $N_{s_i} \cap N_{s_j} = \{1\}$.*

**Proof** The image of $N_\sigma$ in $W$ is $\{1, \sigma\}$, and so if $i \neq j$ the intersection $N_{s_i} \cap N_{s_j}$ is contained in $T$. Since $N_\sigma \cap T$ is the identity component $T_0^-(\sigma)$ of $T^-(\sigma)$, we must show that for $i \neq j$, $T_0^-(s_i) \cap T_0^-(s_j) = \{0\}$.

Consider the root system $R$ in $L = \pi_1 T$ associated to the marked reflection torus $(T, W, \{h_\sigma\})$ (Remark 5.6). It follows from the fact that $G$ is simply connected that the roots in $R$ (equivalently, the coroots in $R^\#$) span $L$; see [31, VIII.1], where $\Gamma(G)$ denotes what we call $L$ and $\{H_\alpha\}$ denotes the set of coroots in $R^\#$. Let $a_i$ be the root corresponding to $s_i$. Since all of the roots in $R$ are integral linear combinations of the $a_i$ [31, V.8], it is clear that the $a_i$ span $L$. In view of the fact that $G$ is simply connected, the number of simple roots is equal to the rank of $G$, and so $\{a_1, \ldots, a_\ell\}$ form a basis for $L$. It follows immediately that $L^-(s_i) = \langle a_i \rangle$, and so $L \cong \oplus_i L^-(s_i)$. Tensoring with $\mathrm{T}(1)$ gives that $T \cong \oplus_i T_0^-(\sigma_i)$, which gives the desired result. □

**Proof of Theorem 5.8** If the theorem is true for $G$, it is true for the product $G \times \mathrm{T}(n)$. By Proposition 5.11, if the theorem is true for $G$ and $A$ is a finite subgroup in the center of $G$, it is also true for $G/A$. Now any connected compact Lie group $G$ can be written as $(K \times \mathrm{T}(n))/A$, where $K$ is simply connected and $A$ is a finite subgroup of the center of $K \times \mathrm{T}(n)$. It is enough, then, to work in the case in which $G$ is simply connected.

In $\mathrm{SU}(2)$, the square of any element in the nonidentity component of the torus normalizer is equal to the unique nontrivial central element of the group, so it follows from the choice of $h_{s_i}$ that $x_i^2 = h_{s_i}$. Since $x_i \in N$ projects to $s_i \in W$, it is also clear that for $t \in T$, $x_i t x_i^{-1} = s_i(t)$. What remains to check is whether

$$\mathrm{prod}(m_{i,j}; x_i, x_j)\,\mathrm{prod}(m_{i,j}; x_j, x_i)^{-1} = 1 \text{ for } i \neq j. \tag{5.13}$$

(The right hand side of this equation will actually lie in the torus $T$, but we continue to designate the identity element by 1 because it is the identity of the nonabelian group $N$.) We follow [34, 3.3]. Let $x = \mathrm{prod}(m_{i,j} - 1; x_j, x_i)$, and let $k$ denote $j$ or $i$, according to whether $m_{i,j}$ is even or odd. The desired relation can be rewritten

$$x x_j x^{-1} x_k^{-1} = 1. \tag{5.14}$$

Let $w$ be the image of $x$ in $W$. Then $w s_j w^{-1} s_k = 1$ (see Section 4.6), so that $x N_{s_j} x^{-1} \subset N_{s_k}$. Since $x_j \in N_{s_j}$ and $x_k \in N_{s_k}$ it follows that the left hand side of (5.14) lies in $N_{s_k}$.

On the other hand, one can also rewrite relation (5.13) as

$$\mathrm{prod}(m_{i,j}; x_j, x_i)\,\mathrm{prod}(m_{i,j}; x_i, x_j)^{-1} = 1 \text{ for } i \neq j. \tag{5.15}$$

Let $l$ denote $i$ or $j$, according to whether $m_{i,j}$ is even or odd. The same reasoning as before shows that the left hand side of (5.15) lies in $N_{s_l}$. The conclusion is that the right hand side of (5.14) lies in $N_{s_k} \cap N_{s_l}$. Since $G$ is simply connected, we are done by Proposition 5.12. □

## 6 Lattices, tori, and extensions at the prime 2

In this section we copy the results of Sections 2 and 3, replacing $\mathbb{Z}$ by the ring $\mathbb{Z}_2$ of 2–adic integers; however, we do not formulate the notion of a root system over $\mathbb{Z}_2$. The proofs are omitted.

**6.1 Marked complete reflection lattices** A *complete lattice* $\widehat{L}$ is a finitely generated free module over the ring $\mathbb{Z}_2$ of 2–adic integers. A *reflection* on $\widehat{L}$ is an automorphism $\sigma$ of $\widehat{L}$ which is conjugate in $\mathrm{Aut}(\mathbb{Q} \otimes \widehat{L}) \cong \mathrm{GL}(r, \mathbb{Q}_2)$ to a diagonal matrix $\mathrm{diag}(-1, 1, \ldots, 1)$.

**6.2 Definition** Suppose that $\sigma$ is a reflection on a complete lattice $\widehat{L}$. A *strict marking* for $\sigma$ is a pair $(b,\beta)$, where $b \in \widehat{L}$ and $\beta\colon \widehat{L} \to \mathbb{Z}_2$ is a homomorphism such that for any $x \in \widehat{L}$, $\sigma(x) = x + \beta(x)b$. Two strict markings $(b,\beta)$ and $(b',\beta')$ are *equivalent* if $(b,\beta) = (ub, u^{-1}\beta)$ for a unit $u \in \mathbb{Z}_2$. A *marking* for $\sigma$ is an equivalence class $\{(b,\beta)\}$ of strict markings.

**6.3 Remark** As in Remark 2.18, a reflection has two markings if it is trivial mod $2$, and one marking otherwise.

Suppose that $\widehat{L}$ is a complete lattice, $\sigma$ is a reflection on $\widehat{L}$ with marking $\{(b,\beta)\}$, and $w$ is an automorphism of $\widehat{L}$. Let $w \cdot \{(b,\beta)\}$ denote the marking for $w\sigma w^{-1}$ given by $\{(w(b), \beta \circ w^{-1})\}$.

**6.4 Definition** *A complete reflection lattice* is a complete lattice $\widehat{L}$ together with a finite subgroup $W$ of $\mathrm{Aut}(\widehat{L})$ which is generated by the reflections it contains. *A marked complete reflection lattice* is a complete reflection lattice $(\widehat{L}, W)$ together with markings $\{(b_\sigma, \beta_\sigma)\}$, one for each reflection $\sigma$ in $W$, such that for each element $w \in W$ and reflection $\sigma$, $w \cdot \{(b_\sigma, \beta_\sigma)\} = \{(b_{w\sigma w^{-1}}, \beta_{w\sigma w^{-1}})\}$.

**6.5 Marked discrete reflection tori** Let $\breve{\mathrm{T}}(n)$ denote $(\mathbb{Z}/2^\infty)^n$. A *discrete torus* is a discrete group isomorphic to $\breve{\mathrm{T}}(n)$ for some $n$. Any discrete torus $\breve{T}$ gives a complete lattice $\mathrm{Hom}(\breve{\mathrm{T}}(1), \breve{T})$; conversely, a complete lattice $\widehat{L}$ gives a discrete torus $\breve{\mathrm{T}}(1) \otimes \widehat{L}$. These two constructions are inverse to one another up to natural isomorphism, and induce an equivalence between the category of discrete tori and the category of complete lattices.

**6.6 Definition** An automorphism $\sigma$ of a discrete torus $\breve{T}$ is a *reflection* (respectively, *trivial* mod $2$) if the induced automorphism of $\mathrm{Hom}(\breve{\mathrm{T}}(1), \breve{T})$ is a reflection (respectively, trivial mod $2$). An element $x$ of $\breve{T}$ is said to be *strongly $\sigma$–negative* if $x$ lies in the maximal divisible subgroup $\breve{T}_0^-(\sigma)$ of $\breve{T}^-(\sigma)$.

**6.7 Remark** An automorphism $\sigma$ of $\breve{T}$ as above is trivial mod $2$ if and only if it acts as the identity on ${}_2T = \{x \in T \mid 2x = 0\}$. If $T = \breve{\mathrm{T}}(1) \otimes \widehat{L}$, then $x \in \breve{T}$ is strongly $\sigma$–negative if and only if $x \in \breve{\mathrm{T}}(1) \otimes \widehat{L}^-(\sigma)$.

**6.8 Definition** Suppose that $\sigma$ is a reflection on a discrete torus $\breve{T}$. A *marking* for $\sigma$ is an element $h \in \breve{T}$ such that

(1) $h$ is strongly $\sigma$–negative (Remark 6.7),

(2) $2h = 0$, and

(3) $h \neq 0$ if $\sigma$ is nontrivial mod 2.

**6.9 Definition** *A discrete reflection torus* is a discrete torus $\check{T}$ together with a finite subgroup $W$ of $\mathrm{Aut}(\check{T})$ which is generated by the reflections it contains. *A marked discrete reflection torus* is a discrete reflection torus $(\check{T}, W)$ together with markings $h_\sigma \in \check{T}$, one for each reflection $\sigma$ in $W$, such that for $w \in W$, $h_{w\sigma w^{-1}} = w(h_\sigma)$.

**6.10 Proposition** *If $\widehat{L}$ is a complete lattice, there is a natural bijection between marked reflection structures on $\widehat{L}$ and marked reflection structures on the discrete torus $\check{\mathrm{T}}(1) \otimes \widehat{L}$.*

**6.11 The reflection extension** (Compare with Section 3.1.) Suppose that $(\check{T}, W)$ is a discrete reflection torus, with associated complete reflection lattice $\widehat{L}$. Let $\Sigma$ be the set of reflections in $W$, and write $\Sigma = \coprod_i \Sigma_i$ as a union of conjugacy classes of reflections. For each index $i$ choose a reflection $\tau_i$ in $\Sigma_i$, let $C_i$ be the centralizer of $\tau_i$ in $W$, and let $a_i \in \widehat{L}$ be a nonzero element with $\tau_i(a_i) = -a_i$. It is clear that each element of $C_i$ takes $a_i$ to $\pm a_i$, and that $C_i$ is isomorphic to $\mathbb{Z}/2 \times C_i^\perp$, where $C_i^\perp$ is the subgroup of $C_i$ consisting of elements which fix $a_i$. Consider the extension

$$1 \to \mathbb{Z} \to \mathbb{Z} \times C_i^\perp \to \mathbb{Z}/2 \times C_i^\perp \to 1. \tag{6.12}$$

By Shapiro's lemma, this gives rise to an extension $\rho_i(W)$ of $W$ by $\mathbb{Z}[\Sigma_i]$, and it is easy to see that up to canonical isomorphism this extension of $W$ does not depend on the choice of representative $\tau_i$ for the conjugacy class $\Sigma_i$.

**6.13 Definition** The *reflection extension* $\rho(W)$ of $W$ is the extension by $\mathbb{Z}[\Sigma]$ given by the sum of the extensions $\rho_i(W)$, in other words, by the fibre product of the extensions over $W$.

**6.14 The normalizer extension** Let $(\check{T}, W)$ be a discrete reflection torus, and $\Sigma$ the set of reflections in $W$. Given a marking $\{h_\sigma\}$ for $(\check{T}, W)$, ie a marked reflection torus $\mathcal{T} = (\check{T}, W, \{h_\sigma\})$, there is a $W$–map $f^{\mathcal{T}} \colon \mathbb{Z}[\Sigma] \to \check{T}$ which sends a reflection $\sigma$ to $h_\sigma$.

**6.15 Definition** Suppose that $\mathcal{T} = (\check{T}, W, \{h_\sigma\})$ is a marked discrete reflection torus. The *normalizer extension* $\check{\nu}(T, W, \{h_\sigma\})$ of $W$ by $\check{T}$ is the image under $f^{\mathcal{T}} \colon \mathbb{Z}[\Sigma] \to \check{T}$ of the reflection extension $\rho(W)$.

# 7 Classification of marked complete reflection lattices

In this section we prove Proposition 7.4, which roughly states that, one example aside, every marked complete reflection lattice arises from a marked reflection lattice, ie from a classical root system.

It will be convenient to consider pairs $(A, W)$ in which $A$ is an abelian group and $W$ is a group acting on $A$; one example is a reflection lattice $(L, W)$. It is clear what it means for two such objects to be isomorphic. The product $(A, W) \times (A', W')$ of two is the pair $(A \times A', W \times W')$, with $W \times W'$ acting on $A \times A'$ in the obvious way. Note that the product of two complete and/or marked reflection lattices is naturally a reflection lattice of the same type. If $(L, W)$ is a marked reflection lattice, then $(\mathbb{Z}_2 \otimes L, W)$ is naturally a marked complete reflection lattice.

**7.1 Definition** A marked complete reflection lattice is *of Coxeter type* if it is isomorphic to $(\mathbb{Z}_2 \otimes L, W)$ for some marked reflection lattice $(L, W)$.

Let $(L_\Delta, W_\Delta)$ be the complete reflection lattice of rank 3 associated to the exceptional 2–compact group DI(4) [10, 24].

**7.2 Remark** The group $W_\Delta$ is isomorphic to $\mathbb{Z}/2 \times \mathrm{GL}(3, \mathbb{F}_2)$; the central $\mathbb{Z}/2$ in $W_\Delta$ acts by $-1$ on $L_\Delta$, and $W_\Delta$ acts on $\mathbb{Z}/2 \otimes L_\Delta \cong (\mathbb{Z}/2)^3$ via the natural representation of $\mathrm{GL}(3, \mathbb{F}_2)$. These last two facts essentially determine the action of $W_\Delta$ on $L_\Delta$ [10, Section 4]. Every reflection in $W_\Delta$ is nontrivial mod 2, and so has a unique marking (Remark 6.3); in particular, there is a unique way to give $(L_\Delta, W_\Delta)$ the structure of marked complete reflection lattice.

**7.3 Definition** A marked complete reflection lattice is *of type* DI(4) if it is isomorphic to a product of copies of $(L_\Delta, W_\Delta)$.

**7.4 Proposition** *Every marked complete reflection lattice $(\widehat{L}, W)$ is isomorphic to a product $(\widehat{L}_1, W_1) \times (\widehat{L}_2, W_2)$, in which the first factor is of Coxeter type and the second is of type* DI(4).

This depends on a few lemmas.

**7.5 Definition** A marked complete reflection lattice $(\widehat{L}, W)$ is *rationally of Coxeter type* if there is a reflection lattice $(L, W)$ such that $(\mathbb{Q} \otimes \widehat{L}, W)$ is isomorphic as a reflection lattice to $(\mathbb{Q}_2 \otimes L, W)$.

**7.6 Lemma** *Suppose that $(\widehat{L}, W)$ is a marked complete reflection lattice which is rationally of Coxeter type. Then $(\widehat{L}, W)$ is of Coxeter type.*

**Proof** Since $(\widehat{L}, W)$ is rationally of Coxeter type, there is a lattice $L'$ and an action of $W$ on $L'$ such that $(\mathbb{Q} \otimes \widehat{L}, W)$ is isomorphic to $(\mathbb{Q}_2 \otimes L', W)$; we use this isomorphism to identify $\mathbb{Q} \otimes \widehat{L}$ with $\mathbb{Q}_2 \otimes L'$. Under this isomorphism, $\widehat{L}' = \mathbb{Z}_2 \otimes L'$ is a complete lattice in $\mathbb{Q} \otimes \widehat{L}$ preserved by $W$. Adjust $L'$ by a power of 2 so that $\widehat{L} \subset \widehat{L}'$. Now define $L$ by the pullback diagram:

$$\begin{array}{ccc} L & \longrightarrow & \widehat{L} \\ \downarrow & & \downarrow \\ L' & \longrightarrow & \widehat{L}' \end{array}$$

The group $\widehat{L}'/\widehat{L}$ is a finite 2–group, and the map $L' \to \widehat{L}' \to \widehat{L}'/\widehat{L}$ is surjective, this last by Nakayama's lemma and the fact that $\mathbb{Z}/2 \otimes L' \cong \mathbb{Z}/2 \otimes \widehat{L}'$. It follows that $L'/L$ is isomorphic to $\widehat{L}'/\widehat{L}$, and that $L$ is a lattice in $\mathbb{Q} \otimes L'$ with $\mathbb{Z}_2 \otimes L \cong \widehat{L}$. In particular, $(\widehat{L}, W)$ is isomorphic to $(\mathbb{Z}_2 \otimes L, W)$. The fact that under this isomorphism the marking on $\widehat{L}$ is derived from a marking on $L$ follows from the observation that possible markings of $(\widehat{L}, W)$ correspond bijectively to markings of $(L, W)$. □

**7.7 Lemma** *Suppose that $(\widehat{L}, W)$ is a complete reflection lattice, such that $(\mathbb{Q} \otimes \widehat{L}, W)$ decomposes as a product $(V_1, W_1) \times (\mathbb{Q} \otimes L_\Delta, W_\Delta)$. Then $(\widehat{L}, W)$ decomposes as the product $(\widehat{L} \cap V_1, W_1) \times (L_\Delta, W_\Delta)$ of complete reflection lattices. If $(\widehat{L}, W)$ is marked, then there is a unique way to mark the two factors of the product in such a way that the product decomposition respects the markings.*

**Proof** Write $\mathbb{Q} \otimes \widehat{L} \cong V_1 \times V_2$ and $W \cong W_1 \times W_2$, where $V_2 \cong \mathbb{Q} \otimes L_\Delta$ and $W_2 \cong W_\Delta$. Let $\widehat{L}_i = \widehat{L} \cap V_i$, so that $\widehat{L}_i$ is a complete lattice which is a module over $W_i$ and $\mathbb{Q} \otimes \widehat{L}_i \cong V_i$. By elementary rank considerations, if $\sigma \in W$ is a reflection, then either $\sigma \in W_1$ or $\sigma \in W_2$. Consequently, $W_1$ and $W_2$ are generated by the reflections they contain, and both $(\widehat{L}_1, W_1)$ and $(\widehat{L}_2, W_2)$ are complete reflection lattices. Now adjust the copy of $L_\Delta$ in $V_2$ by multiplication by a power of 2 so that $L_\Delta \subset \widehat{L}_2$ but $L_\Delta$ is not contained in $2\widehat{L}_2$. The induced map $f\colon \mathbb{Z}/2 \otimes L_\Delta \to \mathbb{Z}/2 \otimes \widehat{L}_2$ is then a nontrivial map between $W_2$–modules which are $\mathbb{F}_2$–vector spaces of the same dimension; since $\mathbb{Z}/2 \otimes L_\Delta$ is simple as a $W_2$–module (Remark 7.2), the map $f$ is an isomorphism and it follows from Nakayama's lemma that $\widehat{L}_2$ is isomorphic to $L_\Delta$ as a $W_2$–module. Let $\overline{L} = \mathbb{Z}/2 \otimes \widehat{L}$ and $\overline{L}_i = \mathbb{Z}/2 \otimes \widehat{L}_i$. Consider the two $W$–module maps

$$g\colon\ \widehat{L}_1 \times \widehat{L}_2 \to \widehat{L} \qquad g'\colon\ \overline{L}_1 \times \overline{L}_2 \to \overline{L}.$$

Again by Nakayama's lemma, in order to prove that $g$ is an isomorphism it is enough to prove that $g'$ is an isomorphism, or even, by a rank calculation, that $g'$ is injective. Let $g_i'$ denote the natural map $\overline{L}_i \to \overline{L}$ and let $K$ be the kernel of $g'$. It is clear from the definitions that $g_i'$ is injective (ie, if $x \in \widehat{L}$ and $2x \in \widehat{L}_i$ then $x \in \widehat{L}_i$), so that the two maps $K \to \overline{L}_1$ and $K \to \overline{L}_2$ are injective. Since $W_2$ acts trivially on $\overline{L}_1$, $W_2$ must act trivially on $K$, and so the fact that $K = \{0\}$ follows from the fact that $\overline{L}_2 \cong \mathbb{Z}/2 \otimes L_\Delta$ has no submodules with a trivial $W_2$–action. The quickest way to obtain the statement about markings is to interpret the markings as elements of the associated discrete tori (6.10). □

**7.8 Proposition** *The only irreducible finite reflection groups over $\mathbb{Q}_2$ are the classical Weyl groups, with their standard reflection representations, and the group $W_\Delta$, with the reflection representation derived from* DI(4).

**Proof** This follows from [7]. Clark and Ewing observe that the Schur index of a reflection representation is 1, so that the representation is defined over its character field, that is, the field extension of $\mathbb{Q}$ generated by the characters of the group elements. They then determine the character fields of all of the irreducible complex reflection groups. The only character fields contained in $\mathbb{Q}_2$ are the fields derived from classical Weyl group representations (which are all $\mathbb{Q}$) and the field derived from the reflection representation of $W_\Delta$ (which is $\mathbb{Q}(\sqrt{-7})$). The complex reflection representation of $W_\Delta$ is unique up to conjugacy, which easily implies that the associated $\mathbb{Q}_2$–reflection representation is also unique. □

**Proof of Proposition 7.4** Let $(\widehat{L}, W)$ be a complete reflection lattice. According to the classification of 2–adic rational reflection groups (see Proposition 7.8), we can write $(\mathbb{Q} \otimes \widehat{L}, W)$ as a product $(V_1, W_1) \times (\mathbb{Q} \otimes L_\Delta, W_\Delta)^k$ for some $k \geq 0$, where $W_1$ is a classical Weyl group. Repeated applications of Lemma 7.7 give that $(\widehat{L}, W)$ is equivalent as a marked complete reflection lattice to $(\widehat{L}_1, W_1) \times (L_\Delta, W_\Delta)^k$, where $(\widehat{L}_1, W_1)$ is rationally of Coxeter type. The result follows from Lemma 7.6. □

# 8 Completions of compact Lie groups

In this section we give some results about completions of Lie groups that are used in Section 9. Suppose that $G$ is a connected compact Lie group of rank $r$ and that $X$ is the 2–compact group obtained by taking the 2–completion of $G$ (so that $BX$ is the 2–completion of $BG$). Let $(T, W, \{h_\sigma\})$ be the marked reflection torus obtained from $G$, and $\breve{T}$ the 2–primary torsion subgroup of $T$ (1.11), so that $\breve{T}$ is a discrete torus of rank $r$. It is easy to see that $(\breve{T}, W, \{h_\sigma\})$ is a marked discrete reflection torus. Let $\nu(W)$ be the normalizer extension of $W$ by $T$ determined by $(T, W, \{h_\sigma\})$ (Definition 3.5) and $\breve{\nu}(W)$ the parallel normalizer extension of $W$ by $\breve{T}$ (Definition 6.15). Essentially by construction there is a homomorphism $\breve{\nu}(W) \to \nu(W)$ which lies in a map of exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & \breve{T} \arrow[r] \arrow[d, "\subset"'] & \breve{\nu}(W) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "\parallel"'] & 1 \\ 1 \arrow[r] & T \arrow[r] & \nu(W) \arrow[r] & W \arrow[r] & 1 \end{tikzcd}$$

By Theorem 5.7, the inclusion $T \to G$ extends to a homomorphism $\nu(W) \to G$. Combining these homomorphisms with the completion map $BG \to BX$ gives composite maps $B\breve{T} \to BT \to BG \to BX$ and $B\breve{\nu}(W) \to B\nu(W) \to BG \to BX$, which, in the language of [11], correspond to homomorphisms $\breve{T} \to X$ and $\breve{\nu}(W) \to X$.

**8.1 Lemma** *In the above situation, $\breve{T} \to X$ is a maximal discrete torus for $X$, $\breve{\nu}(W)$ is the normalizer of $\breve{T}$ in $X$, and $\breve{\nu}(W) \to X$ is the natural homomorphism.*

See Aside 9.1 for a description of the maximal discrete torus in a 2–compact group.

**Proof of Lemma 8.1** Consider first the chain of maps

$$B\breve{T} \to BT \to BG \to BX.$$

The left hand map induces an isomorphism on mod 2 homology (see [11, Section 6]), as does the completion map $BG \to BX$. Let $X/\breve{T}$ denote the homotopy fibre of $B\breve{T} \to BX$, $G/T$ the homotopy fibre of $BG \to BX$, and $G/\breve{T}$ the homotopy fibre of $B\breve{T} \to BG$. Let $\widehat{T}$ denote the closure [12, Section 3] of $\breve{T}$ (ie $B\widehat{T}$ is the 2–completion of $B\breve{T}$), so that $B\breve{T} \to BX$ extends canonically to $B\widehat{T} \to BX$, and denote the homotopy fibre of this last map by $X/\widehat{T}$. It follows that all of the maps in the diagram

$$X/\widehat{T} \leftarrow X/\check{T} \leftarrow G/\check{T} \to G/T$$

induce isomorphisms on mod 2 homology. In particular, $X/\widehat{T}$ has finite mod 2 homology, and so $\widehat{T} \to X$ is a monomorphism [11, 3.2, 9.11]. It is straightforward to argue that the Euler characteristic $\chi_{\mathbb{Q}_2}(X/\widehat{T})$ [11, 1.5] is the same as the rational Euler characteristic $\chi_{\mathbb{Q}}(G/T)$; since this last is nonzero [14, 5.3], the map $\widehat{T} \to X$ is a maximal torus for $X$ [12, 2.15]. Since $\check{T} \to \widehat{T}$ is a discrete approximation, $\check{T} \to X$ is a 2–discrete maximal torus for $X$. The statements about $\check{\nu}(W)$ amount to an identification of $W$ with the Weyl group $W_X$ of $X$. There is certainly a map $W \to W_X$, since $W$ maps to the self-equivalences of $BT$ over $BG$ and hence by completion to the self-equivalences of $B\widehat{T}$ over $BX$ (see [11, 9.5]). The map is a monomorphism because the completion map $\mathrm{GL}(r,\mathbb{Z}) \cong \mathrm{Aut}(\pi_1 T) \to \mathrm{Aut}(\pi_1\widehat{T}) \cong \mathrm{GL}(r,\mathbb{Z}_2)$ is injective, and consequently an isomorphism by counting: $\#(W) = \chi_{\mathbb{Q}}(G/T)$ [14, 5.3], while $\#(W_X) = \chi_{\mathbb{Q}_2}(X/\widehat{T})$ [11, 9.5]. □

We will also need to refer to the following results.

**8.2 Proposition** *Suppose that $G$ is a connected compact Lie group, that $X$ is the 2–completion of $G$, and that $\pi$ is a finite 2–group. Then the map which sends a homomorphism $f\colon \pi \to G$ to the composite $B\pi \xrightarrow{Bf} BG \to BX$ induces a bijection between conjugacy classes of homomorphisms $\pi \to G$ and homotopy classes of maps $B\pi \to BX$.*

**Proof** This amounts to combining [9, 1.1] with the fact that the completion map $BG \to BX$ induces a bijection from homotopy classes of maps $B\pi \to BG$ to homotopy classes of maps $B\pi \to BX$. This last follows from [9, 2.4]. □

**8.3 Proposition** *Suppose that $G$ is a connected compact Lie group with maximal torus $T$, and that $\check{S} \subset T$ is a discrete torus with topological closure $\overline{S}$. Let $X$ be the 2–completion of $G$, $G_{\check{S}}$ the centralizer of $\check{S}$ in $G$, $G_{\overline{S}}$ the centralizer of $\overline{S}$ in $G$, and $X_{\check{S}}$ the centralizer of $\check{S}$ in $X$. Then the natural map $G_{\overline{S}} \to G_{\check{S}}$ is an isomorphism, while the natural map $BG_{\check{S}} \to BX_{\check{S}}$ is a 2–completion map.*

**Proof** It is elementary that $G_{\overline{S}} \to G_{\check{S}}$ is an isomorphism. The fact that $BG_{\check{S}} \to BX_{\check{S}}$ is a 2–completion map follows from the main theorem of [25] and the argument in the proof of [9, 2.4]. Note that $X_{\check{S}}$ is 2–complete by [11, 5.7, 6.21]. □

# 9 2–compact groups

In this section we associate a marked discrete reflection torus to a connected 2–compact group $X$ (Proposition 9.10), and show that the associated normalizer extension of the Weyl group is isomorphic to the normalizer of a maximal discrete torus in $X$ (Proposition 9.12).

**9.1 Aside** A *complete torus* $\widehat{T}$ is a 2–compact group with the property that its classifying space $B\widehat{T}$ is equivalent to the 2–completion of the classifying space of an ordinary torus. Any 2–compact group $X$ has a *maximal (complete) torus*, which is technically a complete torus $\widehat{T}$ together with a monomorphism $\widehat{T} \to X$ with the property that the Euler characteristic of $X/\widehat{T}$ is nonzero [11, Section 8]. For such a maximal torus, the space $\mathcal{W}$ of self-maps of $B\widehat{T}$ over $BX$ is homotopically discrete, and its monoid of components is a finite group $W$ called the *Weyl group* of $X$. The monoid $\mathcal{W}$ acts on $B\widehat{T}$, and the Borel construction of this action is denoted $BN(\widehat{T})$; this lies in a fibration sequence

$$B\widehat{T} \to BN(\widehat{T}) \to BW.$$

The loop space of $BN(T)$ is called the *normalizer of* $\widehat{T}$ *in* $X$.

Any complete torus has a *discrete approximation*, which is a discrete torus $\breve{T}$ together with a homomorphism $\breve{T} \to \widehat{T}$ with the property that the induced map $B\breve{T} \to B\widehat{T}$ induces an isomorphism on mod 2 homology. If $\widehat{T} \to X$ is a maximal torus for $X$, then the composite $\breve{T} \to \widehat{T} \to X$ is called a *maximal discrete torus* for $X$. See [12, 3.13] for an explanation of how in this case the approximation $\breve{T} \to \widehat{T}$ can be extended to give a discrete approximation $\breve{N}$ for $N(\widehat{T})$: $\breve{N}$ is a discrete group which lies in a short exact sequence

$$\{1\} \to \breve{T} \to \breve{N} \to W \to \{1\},$$

and it is called the *normalizer of* $\breve{T}$ in $X$.

Suppose that $X$ is a connected 2–compact group of rank $r$ with maximal discrete torus $\breve{T}$, Weyl group $W$, and discrete torus normalizer $\breve{N}$. Then $\breve{T} \cong \breve{\mathrm{T}}(r)$ and the conjugation action of $W$ on $\breve{T}$ identifies $W$ with a subgroup of $\mathrm{Aut}(\breve{T})$ generated by reflections [11, 9.7]. In particular, $(\breve{T}, W)$ is a 2–discrete reflection torus.

For each reflection $\sigma \in W$ we will construct a subgroup $\breve{N}_\sigma$ of $\breve{N}$, called the *root subgroup* associated to $\sigma$. We will also single out a specific element $h_\sigma \in \breve{T}$, the *marking* associated to $\sigma$. The technique is to mimic in homotopy theory the constructions of Section 5.

Given $\sigma$, let $\breve{T}_0^+(\sigma)$ denote the maximal divisible subgroup of $\breve{T}$ on which $\sigma$ acts trivially. It is easy to see that $\breve{T}_0^+(\sigma)$ is a discrete torus of rank $r-1$; if $\widehat{L}$ is the complete lattice corresponding to $\breve{T}$, then under the isomorphism $\breve{T} \cong \breve{\mathrm{T}}(1)\otimes\widehat{L}$, $\breve{T}_0^+(\sigma)$ is given by $\breve{\mathrm{T}}(1)\otimes\widehat{L}^+(\sigma)$. Let $X(\sigma)$ denote the centralizer of $\breve{T}_0^+(\sigma)$ in $X$.

**9.2 Aside** The inclusion homomorphism $i\colon \breve{T}_0^+(\sigma) \to \breve{T} \to X$ is by definition represented by a map $Bi\colon B\breve{T}_0^+(\sigma) \to BX$ of classifying spaces. Note that the domain of $Bi$ is a space of type $K(\pi,1)$. The centralizer $X(\sigma)$ is by definition the 2–compact group whose classifying space $BX(\sigma)$ is the mapping space component $\mathrm{Map}(B\breve{T}_0^+(\sigma), BX)_{Bi}$.

Then $X(\sigma)$ is a connected 2–compact subgroup of $X$ [12, 7.8] with maximal discrete torus $\breve{T}$, and Weyl group given by the subgroup $\{1,\sigma\}$ of $W$ [12, 7.6]. The normalizer of $\breve{T}$ in $X(\sigma)$ can be identified with the algebraic centralizer of $\breve{T}_0^+(\sigma)$ in $\breve{N}$; we denote this group by $\breve{N}(\sigma)$. Note that $\breve{N}(\sigma)$ is an extension of $\mathbb{Z}/2$ by a discrete torus, and so is a 2–discrete toral group [11, 6.5].

**9.3 Aside** Let $A = B\breve{T}_0^+(\sigma)$. The classifying space maps

$$A = B\breve{T}_0^+(\sigma) \xrightarrow{u} B\breve{T} \xrightarrow{v} B\breve{N} \xrightarrow{w} BX$$

induce maps

$$\mathrm{Map}(A, B\breve{T})_u \to \mathrm{Map}(A, B\breve{N})_{vu} \to \mathrm{Map}(A, BX)_{wvu}, \tag{9.4}$$

The mapping space component on the far right is $BX(\sigma)$. By covering space theory, $\mathrm{Map}(A, B\breve{T})_u$ is canonically equivalent to $B\breve{T}$, the equivalence being given by evaluation at the basepoint of $A$. Similarly, $\mathrm{Map}(A, B\breve{N})_{vu}$ is the classifying space of the algebraic centralizer of $\breve{T}_0^+(\sigma)$ in $\breve{N}$. It follows from [12, 7.6] that the composite map in (9.4) represents the inclusion in $X(\sigma)$ of a maximal discrete torus, and that the right-hand map represents the inclusion of the normalizer of this torus.

The only simply-connected 2–compact group of rank 1 is the two-completion $\widehat{\mathrm{SU}}(2)$ of $\mathrm{SU}(2)$ (see [16]); the center of $\widehat{\mathrm{SU}}(2)$ is $\mathbb{Z}/2$ and the adjoint form is the 2–completion $\widehat{\mathrm{SO}}(3)$ of $\mathrm{SO}(3)$. Given this, it follows from the product splitting theorem for $p$–compact groups [13] that $X(\sigma)$ is given by one of the following three possibilities (see [12, 7.7], [21]), where $\widehat{\mathrm{T}}(k)$ denotes a 2–complete torus of rank $k$ and $\widehat{\mathrm{U}}(2)$ is the 2–completion of $\mathrm{U}(2)$:

(1) A product $\widehat{\mathrm{SU}}(2) \times \widehat{\mathrm{T}}(r-1)$

(2) A product $\widehat{\mathrm{SO}}(3) \times \widehat{\mathrm{T}}(r-1)$

(3) A product $\widehat{\mathrm{U}}(2) \times \widehat{\mathrm{T}}(r-2)$

Recall that $\breve{T}_0^-(\sigma)$ denotes the maximal divisible subgroup of $\breve{T}^-(\sigma)$; this group is a discrete torus of rank 1.

**9.5 Definition** The *root subgroup* $\breve{N}_\sigma$ is given by

$$\breve{N}_\sigma = \{x \in \breve{N}(\sigma) \mid x \text{ is conjugate in } X(\sigma) \text{ to some } y \in \breve{T}_0^-(\sigma)\}.$$

**9.6 Aside** This definition is couched in the language of [11], but what it means concretely is the following. Suppose that $x \in \breve{N}(\sigma)$ has order $2^n$; note that by [11, 6.19], $\breve{N}(\sigma)$ is a union of finite 2–groups. Represent $x$ by $\chi(x)\colon \mathbb{Z}/2^n \to \breve{N}(\sigma)$. Then $x \in \breve{N}_\sigma$ if and only if the composite

$$B\mathbb{Z}/2^n \xrightarrow{B\chi(x)} B\breve{N}(\sigma) \to BX(\sigma)$$

lifts up to homotopy to a map $B\mathbb{Z}/2^n \to B\breve{T}_0^-(\sigma)$.

Given that $X(\sigma)$ is the 2–completion of a compact Lie group, the following statement comes from combining Lemma 5.2 with the discussion in Section 8. Lemma 9.9 is proved similarly.

**9.7 Lemma** *The set $\breve{N}_\sigma$ is a subgroup of $\breve{N}(\sigma)$ isomorphic to an extension of $\mathbb{Z}/2$ by a rank 1 discrete torus. More precisely, $\breve{N}_\sigma \cap \breve{T} = \breve{T}_0^-(\sigma)$, and the natural map $\breve{N}_\sigma/\breve{T}_0^-(\sigma) \to \breve{N}(\sigma)/\breve{T} \cong \mathbb{Z}/2$ is an isomorphism. If $x, y \in \breve{N}_\sigma \setminus \breve{T}_0^-(\sigma)$, then $x^2 = y^2 \in \breve{T}_0^-(\sigma)$.*

**9.8 Definition** The *marking* $h_\sigma \in \breve{T}_0^-(\sigma)$ is defined to be the image of $\breve{N}_\sigma \setminus \breve{T}_0^-(\sigma)$ under the squaring map $x \mapsto x^2$.

**9.9 Lemma** *The element $h_\sigma \in \breve{T}$ is a marking for $\sigma$ in the sense of Definition 6.8.*

**9.10 Proposition** *Suppose that $X$ is a connected 2–compact group with maximal discrete torus $\breve{T}$, discrete torus normalizer $\breve{N}$, Weyl group $W$, root subgroups $\{\breve{N}_\sigma\}$, and markings $\{h_\sigma\}$ as above. Suppose that $x \in \breve{N}$ has image $w \in W$. Then for any reflection $\sigma \in W$,*

$$x\breve{N}_\sigma x^{-1} = \breve{N}_{w\sigma w^{-1}}$$
$$xh_\sigma x^{-1} = h_{w\sigma w^{-1}}$$

*In particular, $(\breve{T}, W, \{h_\sigma\})$ is a marked discrete reflection torus.*

**Proof** As with Proposition 5.5, the idea here is that conjugation with $x$ gives a symmetry of the whole situation, but there is something to check. Let $c$ denote conjugation with $x$, and let $\tau = w\sigma w^{-1}$. All of the arrows in the following commutative diagram are clear, except for the right hand vertical one:

$$\begin{tikzcd}
BT_0^+(\sigma) \arrow[r] \arrow[d, "Bc"'] & B\check{N} \arrow[r] \arrow[d, "Bc"'] & X \arrow[d, "Bc'"'] \\
BT_0^+(\tau) \arrow[r] & B\check{N} \arrow[r] & X
\end{tikzcd}$$

If the right hand arrow exists, then taking mapping spaces (Aside 9.2) gives a commutative diagram

$$\begin{tikzcd}
B\check{N}(\sigma) \arrow[r] \arrow[d, "Bc"'] & BX(\sigma) \arrow[d] \\
B\check{N}(\tau) \arrow[r] & BX(\tau)
\end{tikzcd}$$

which, in view of Definitions 9.5 and 9.8, yields the desired result. The issue then is to extend the self equivalence $Bc$ of $B\check{N}$ to a self equivalence $Bc'$ of $BX$. One way to achieve this is to represent the map $\check{N} \to X$ by a map $U \to V$ of simplicial groups; the map $c$ can then be realized by conjugation with a vertex of $U$, and it extends to an automorphism of $V$ given by conjugation with the image vertex. Another approach is to treat $c$ as part of the conjugation action of $\check{N}$ on itself. Taking classifying spaces translates this into a basepoint-preserving action of $\check{N}$ on $B\check{N}$, an action which is homotopically captured by the associated sectioned fibration over $B\check{N}$ with fibre $B\check{N}$. It is easy to see that this fibration is just the product projection $B\check{N} \times B\check{N} \to B\check{N}$, with section given by the diagonal map. This fibration evidently extends to the product fibration $BX \times BX \to BX$. □

**9.11 Remark** It is natural to conjecture that the above marked discrete reflection torus, or equivalently the associated marked complete reflection lattice (Proposition 6.10), determines $X$ up to equivalence.

The main result of this section is the following one.

**9.12 Proposition** *Suppose that $X$ is a connected 2–compact group, and that $(\check{T}, W, \{h_\sigma\})$ is the marked discrete reflection torus associated to $X$ (see Proposition 9.10). Then the normalizer $\check{N}$ of $\check{T}$ in $X$ is isomorphic to the normalizer extension $\check{\nu}(\check{T}, W, \{h_\sigma\})$ (Definition 6.15).*

The proof breaks up into two cases. A connected 2–compact group $X$ is said to be *of Coxeter type* (respectively, *of type* DI(4)) if the associated complete marked reflection lattice (Propositions 9.10 and 6.10) is of Coxeter type (respectively, of type DI(4)); see Definitions 7.1 and 7.3. Note that this is a property of the underlying complete reflection lattices, or equivalently of the action of $W$ on $\breve{T}$; the markings come along automatically (see Remark 7.2 and the proof of Lemma 7.6).

**9.13 Proposition** *Any connected 2–compact group $X$ can be written as a product $X_1 \times X_2$ of 2–compact groups, where $X_1$ is of Coxeter type and $X_2$ is of type* DI(4).

**Proof** This is a consequence of Proposition 7.4 and [13, 6.1]. □

A connected 2–compact group $X$ is said to be *semisimple* if its center is finite (equivalently, if $\pi_1 X$ is finite). If $X$ is any connected 2–compact group, the universal cover $\widetilde{X}$ is semisimple, and it follows from [13] and [21] that $X$ is obtained from the product of $\widetilde{X}$ and a 2–complete torus by dividing out [11, 8.3] by a finite central subgroup. We express this by saying that $X$ is a *central product* of $\widetilde{X}$ and a torus. The *semisimple rank* of $X$ is the rank of $\widetilde{X}$; or equivalently the rank of $X$ minus the rank of the center of $X$. There are analogous notions for compact Lie groups.

The rank 1 case of the following statement follows from [16]; the rank 2 case is proved by Bauer et al cite[6.1]rKN.

**9.14 Proposition** *If $X$ is a semisimple 2–compact group of rank $\leq 2$, then $X$ is the 2–completion of a connected compact Lie group.*

If $G$ is a semisimple Lie group, the center of $G$ is a finite abelian group $\mathcal{Z}(G)$. It is easy to see that the center of the corresponding 2–compact group $\widehat{G}$ is the quotient of $\mathcal{Z}(G)$ by the subgroup of elements of odd order; for instance, combine the parallel calculations [12, 7.5] and [14, 8.2] with Lemma 8.1. This implies that any central product of $\widehat{G}$ with a 2–complete torus is the 2–completion of some central product of $G$ with a torus. In conjunction with Proposition 9.14, this observation gives the following.

**9.15 Proposition** *If $X$ is a connected 2–compact group of semisimple rank at most 2, then $X$ is the 2–completion of a connected compact Lie group.*

Suppose that $X$ is a connected 2–compact group with marked discrete reflection torus $(\breve{T}, W, \{h_\sigma\})$ and discrete torus normalizer $\breve{N}$ as above. Let $A$ be a subgroup of $\breve{T}$, $X_A$ the centralizer of $A$ in $X$, and $W_A$ the subgroup of $W$ consisting of all elements which leave $A$ pointwise fixed. Then $\breve{T}$ is a maximal discrete torus for $X_A$, and $W_A$ is the Weyl group of $X_A$. The normalizer $\breve{N}_A$ of $\breve{T}$ in $X_A$ is the centralizer of $A$ in $\breve{N}$, or equivalently the pullback over $W_A \subset W$ of the surjection $\breve{N} \to W$. We assume that $X_A$ is connected; this is always true if $A$ is a discrete torus, and the general criterion is in [12, 7.6]. Under this assumption, $W_A$ is generated by the reflections it contains [12, 7.6], and so $(\breve{T}, W_A)$ is a discrete reflection torus. Let $\{h_\sigma\}_{\sigma\in W_A}$ denote the set of markings for the reflections in $W_A$ obtained from the given set $\{h_\sigma\}$ of markings for the reflections in $W$. It is clear that $(\breve{T}, W_A, \{h_\sigma\}_{\sigma\in W_A})$ is a marked reflection torus. For each reflection $\sigma \in W_A$, let $\breve{N}_\sigma$ be the root subgroup for $\sigma$ computed in $X$, and $\breve{N}_{A,\sigma} \subset \breve{N}_A$ the root subgroup for $\sigma$ computed in $X_A$. Given the inclusion $\breve{N}_A \subset \breve{N}$, both of these root subgroups can be considered as subgroups of $\breve{N}$.

**9.16 Lemma** *In the situation described above, $(\breve{T}, W_A, \{h_\sigma\}_{\sigma\in W_A})$ is the discrete marked reflection torus associated to $X_A$ by Proposition 9.10. Moreover, for each reflection $\sigma \in W_A$, $\breve{N}_{A,\sigma} = \breve{N}_\sigma$ as a subgroup of $\breve{N}$.*

**Proof** By Definition 9.8, the first statement follows from the second. Let $B = \breve{T}_0^+(\sigma)$, $X(\sigma)$ the centralizer of $B$ in $X$, and $X_A(\sigma)$ the centralizer of $B$ in $X_A$. Both $X(\sigma)$ and $X_A(\sigma)$ are connected [12, 7.8], and the monomorphism [11, 5.2, 6.1] $X_A \to X$ induces a monomorphism $X_A(\sigma) \to X(\sigma)$. The monomorphism is an equivalence by [12, 4.7], since both $X_A(\sigma)$ and $X(\sigma)$ have Weyl group $\mathbb{Z}/2$ generated by $\sigma$. A similar argument shows that the centralizer $\breve{N}_A(\sigma)$ of $B$ in $\breve{N}_A$ is the same as the centralizer $\breve{N}(\sigma)$ of $B$ in $\breve{N}$: both groups have $\breve{T}$ as a maximal normal divisible abelian subgroup, and the quotient of each group by $\breve{T}$ is $\{1, \sigma\}$. The second statement of the lemma thus follows from Definition 9.5 □

**9.17 Lemma** *Suppose that $G$ is a connected compact Lie group, and $X$ the 2–compact group obtained as the 2–completion of $G$. Let $W$ be the Weyl group of $G$ (or of $X$), $N$ the normalizer of a maximal torus in $G$, $\breve{N}$ the normalizer of a maximal discrete torus $\breve{T}$ in $X$, and $N_\sigma$ (respectively, $\breve{N}_\sigma$) the root subgroup of $N$ (respectively, $\breve{N}$) corresponding to a reflection $\sigma \in W$. Then under the identification given by Lemma 8.1 of $\breve{T}$ as a subgroup of $T$ and $\breve{N}$ as a subgroup of $N$, $\breve{N}_\sigma = \breve{N} \cap N_\sigma$.*

**Proof** Let $T_0^+(\sigma) \subset T$ (respectively, $\breve{T}_0^+(\sigma) \subset \breve{T}$) be as in Section 5 (respectively, as above). It is easy to see that $\breve{T}_0^+(\sigma)$ is the group of 2–primary torsion elements in $T_0^+(\sigma)$, so that $T_0^+(\sigma)$ is the topological closure of $\breve{T}_0^+(\sigma)$ in $T$, and $\breve{T}_0^+(\sigma) = \breve{T} \cap T_0^+(\sigma)$. Let $N(\sigma)$ (respectively, $\breve{N}(\sigma)$) be the centralizer of $T_0^+(\sigma)$ in $N$ (respectively, the centralizer of $\breve{T}_0^+(\sigma)$ in $\breve{N}$). It follows that $\breve{N}(\sigma) = \breve{N} \cap N(\sigma)$. Let $G(\sigma)$ (respectively, $X(\sigma)$) be the centralizer of $T_0^+(\sigma)$ in $G$ (respectively, the centralizer of $\breve{T}_0^+(\sigma)$ in $X$). There is a commutative diagram:

$$\begin{array}{ccccccc} B\breve{T}_0^-(\sigma) & \longrightarrow & BT_0^-(\sigma) & & & & \\ \downarrow & & \downarrow & & & & \\ B\breve{N}(\sigma) & \longrightarrow & BN(\sigma) & \longrightarrow & BG(\sigma) & \longrightarrow & BX(\sigma) \end{array}$$

Again, $\breve{T}_0^-(\sigma)$ is the 2–primary torsion subgroup of $T^-(\sigma)$. Note that $\breve{N}(\sigma)$ is a union of finite 2–groups (Aside 9.6). According to Proposition 8.3 and Proposition 8.2, an element $x \in \breve{N}(\sigma)$ is homotopically conjugate (Aside 9.6) in $X(\sigma)$ to an element of $\breve{T}_0^-(\sigma)$ if and only if $x$ is algebraically conjugate in $G(\sigma)$ to an element of $\breve{T}_0^-(\sigma)$. Since $x$ is of order a power of 2, this last occurs if and only if $x$ is conjugate in $G(\sigma)$ to an element of $T_0^-(\sigma)$. The lemma follows from the definitions of the root subgroups $N_\sigma$ (Definition 5.1) and $\breve{N}_\sigma$ (Definition 9.5). □

**Proof of Proposition 9.12, Coxeter case** Let $(\widehat{L}, W)$ be the marked complete reflection lattice associated to $X$ (Proposition 6.10), and $(L, W)$ a marked reflection lattice such that $(\widehat{L}, W) \cong (\mathbb{Z}_2 \otimes L, W)$. Associated to $(L, W)$ is a marked reflection torus $(T, W, \{h'_\sigma\})$. The group $\breve{T} \cong \widehat{L} \otimes \breve{\mathrm{T}}(1)$ is naturally isomorphic to the 2–primary torsion subgroup of $T \cong L \otimes \mathrm{T}(1)$, and under this identification, $h'_\sigma = h_\sigma$ for each reflection $\sigma \in W$. Consequently, as in Section 8, the normalizer extension $\breve{\nu}(W) = \breve{\nu}(\breve{T}, W, \{h_\sigma\})$ is a subgroup of the extension $\nu(W) = \nu(T, W, \{h_\sigma\})$ (Definition 3.5). Choose a set $\{s_i\}$ of simple reflections in $W$ (see Remark 2.14). By Section 4.6 and Definition 6.15, $\breve{\nu}(W)$ is generated by $\breve{T}$, together with symbols $q_i$, one for each simple reflection $s_i$, subject to the relations (5.10) (with $T$ replaced by $\breve{T}$). As in the proof of Theorem 5.7, in order to show that $\breve{\nu}(W)$ is isomorphic to $\breve{N}$, it is enough to find elements $x_i \in \breve{N}$, one for each simple reflection, such that $x_i$ projects to $s_i \in W$ and such that the $x_i$ satisfy relations (5.9) (with $T$ replaced by $\breve{T}$). For each simple reflection $s_i$, let $\breve{N}_{s_i}$ be the corresponding root subgroup, and $x_i$ some element in $\breve{N}_{s_i} \setminus T_0^-(s_i)$. We claim the elements $x_i$ satisfy the necessary relations. The first relation in (5.9) is clear (Definition 9.8), and the second follows from the fact that $x_i$ by construction projects to $s_i \in W$. The final relation involves pairs $\{x_i, x_j\}$ of the generating elements. Choose such a pair, let $A$ be the rank $r-2$ discrete torus which is the maximal divisible subgroup of $\breve{T}$ on which $s_i$ and $s_j$ act as the identity, and let $X_A$ be the centralizer of $A$ in $X$. Then $X_A$ is connected [12, 9.8] and of semisimple rank 2. In the notation of Lemma 9.16, $x_i$ (respectively, $x_j$) is in the root subgroup $\breve{N}_{A,s_i}$ (respectively, $\breve{N}_{A,s_j}$) of $\breve{N}$ determined by $X_A$, and so in order to verify the final relation between $x_i$ and $x_j$ we can replace $X$ by $X_A$, or in particular, assume that $X$ has semisimple rank 2. In this case $X$ is the 2–completion of a connected compact Lie group (Proposition 9.15; note that this is a crucial step in the proof, which is due to Kitchloo and Notbohm), and the result essentially follows from Lemma 9.17 and the fact that the relation is satisfied in the Lie group case (Theorem 5.8). The only remaining issue is to show that $\{s_i, s_j\}$ is a set of simple reflections for the Weyl group $W_A$ of $X_A$ (note that $W_A$ is the set of elements in $W$ which pointwise fix $A$).

Let $V$ be the codimension 2 subspace of $\mathbb{R} \otimes L$ which is pointwise fixed by $s_i$ and $s_j$; it is easy to see that $W_A$ is the subgroup of $W$ consisting of element which fix $V$ pointwise. The group $W_A$ is generated by the reflections it contains (see [19, Theorem 1.2(c)], or note that $W_A$ is the Weyl group of a connected 2–compact group). The desired result now follows from [19, Theorem 1.10(a)], given that, in the language of [19], a reflection in $W$ fixes $V$ pointwise if and only if the associated root is orthogonal to $V$, that is, it lies in the $\mathbb{R}$–span of the roots corresponding to $s_i$ and $s_j$. □

**Proof of Proposition 9.12,** DI(4) **case** Suppose that $X$ is of type DI(4). Let $x$ be a nontrivial element of order 2 in $\breve{T}$, and $A$ the subgroup of $\breve{T}$ generated by $x$. Note that the Weyl group $W_\Delta$ of $X$ acts transitively on the set of such elements $x$ (Remark 7.2), so that the conjugacy class of $A$ as a subgroup of $X$ does not depend on the choice of $x$. Let $X_A$ be the centralizer of $A$ in $X$, and $W_A$ the group of elements in $W_\Delta$ which fix $x$. The 2–compact group $X_A$ is connected; one quick way to check this is to observe that all of the reflections in $W_\Delta$ are nontrivial mod 2 (Remark 7.2), and so by [12, Section 7] $X_A$ is connected if and only if $W_A$ is generated by reflections. But $W_A$ is generated by reflections; this can be verified directly, or by observing that in the particular case $X = \mathrm{DI}(4)$ [10], the centralizer of such an $A$ is clearly connected, since it is the 2–completion of Spin(7).

By Remark 7.2 and [12, Section 7], the group $W_A$, which is the Weyl group of $X_A$, is isomorphic to $\{\pm 1\} \times P(1,2)$, where $P(1,2)$ is the subgroup of $\mathrm{GL}(3, \mathbb{F}_2)$ given by matrices which agree with the identity in the first column. For cardinality reasons $W_A$ cannot contain $W_\Delta$ as a factor, and so by Proposition 7.4

$X_A$ is of Coxeter type. Since the index of $W_A$ in $W_\Delta$ is 7, which is odd, the fact that Proposition 9.12 is true for $X$ follows from Remark 10.2 and the fact proved above that Proposition 9.12 is true for $X_A$. □

## 10 A centralizer lemma

In this section we verify that the normalizer extension construction given in Definition 6.15 behaves well in a certain sense when it comes to taking centralizers of subgroups of the torus $\breve{T}$. Our proof of Proposition 9.12 in the DI(4) case depends on this behavior.

We put ourselves in the context of Lemma 9.16; in particular, $X$ is a connected 2–compact group with marked reflection torus $(\breve{T}, W, \{h_\sigma\})$ and discrete torus normalizer $\breve{N}$, $A$ is a subgroup of $\breve{T}$, $X_A$ is the centralizer of $A$ in $X$ and $W_A$ is the subgroup of $W$ consisting of elements which leave $A$ pointwise fixed [12, 7.6]. Recall that we are assuming that $X_A$ is connected. According to Lemma 9.16, $(\breve{T}, W_A, \{h_\sigma\}_{\sigma\in W_A})$ is the marked reflection torus associated to $X_A$. Let $\nu(W)$ (respectively, $\nu(W_A)$) denote the normalizer extension of $W$ (respectively, $W_A$) obtained from $(\breve{T}, W, \{h_\sigma\})$ (respectively, $(\breve{T}, W_A, \{h_\sigma\}_{\sigma\in W_A})$.

**10.1 Lemma** *In the above situation, there is a commutative diagram:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \breve{T} & \longrightarrow & \nu(W_A) & \longrightarrow & W_A & \longrightarrow & 1 \\
 & & \| \downarrow & & \downarrow & & \subset\downarrow & & \\
1 & \longrightarrow & \breve{T} & \longrightarrow & \nu(W) & \longrightarrow & W & \longrightarrow & 1
\end{array}$$

*In particular, the extension $\nu(W_A)$ of $W_A$ is the pullback over $W_A \subset W$ of the extension $\nu(W)$ of $W$.*

**10.2 Remark** Let $\breve{N}$ (respectively, $\breve{N}_A$) denote the normalizer of $\breve{T}$ in $X$ (respectively, $X_A$). It is clear from [12, Section 7] that the extension $\breve{N}_A$ of $W_A$ is the pullback over $W_A \subset W$ of the extension $\breve{N}$ of $W$. We can therefore conclude from Lemma 10.1 that if $\breve{N} \cong \nu(W)$, then $\breve{N}_A \cong \nu(W_A)$. More interestingly, suppose that the index of $W_A$ in $W$ is odd, so that (by a transfer argument) the restriction map $\mathrm{H}^2(W;\breve{T}) \to \mathrm{H}^2(W_A;\breve{T})$ is injective and extensions of $W$ by $\breve{T}$ are detected on $W_A$. In this case we can conclude that if $\breve{N}_A \cong \nu(W_A)$, then $\breve{N} \cong \nu(W)$.

**10.3 A double coset formula** The proof of Lemma 10.1 depends upon a double coset formula for Shapiro companions. If $u\colon H \to G$ is an inclusion of finite groups and $k \in \mathrm{H}^2(H;\mathbb{Z})$, write $u_\#(k)$ for the element of $\mathrm{H}^2(G;\mathbb{Z}[G/H])$ that corresponds to $k$ under Shapiro's lemma. If $v\colon K \to G$ is another subgroup inclusion, the problem is to compute $v^*u_\#(k) \in \mathrm{H}^2(K;\mathbb{Z}[G/H])$. Let $x_\alpha$ run through a set of $(K,H)$ double coset representatives in $G$, so that $G$ can be written as a disjoint union

$$G = \coprod_\alpha K x_\alpha H.$$

For each index $\alpha$, let $K_\alpha = K \cap x_\alpha H x_\alpha^{-1}$, and note that there is a natural isomorphism of $K$–modules

$$\mathbb{Z}[G/H] \cong \oplus_\alpha \mathbb{Z}[K/K_\alpha]. \tag{10.4}$$

Write $v_\alpha$ for the map

$$v_\alpha\colon\ K_\alpha \to x_\alpha H x_\alpha^{-1} \xrightarrow{x_\alpha^{-1}(-)x_\alpha} H$$

and $u^\alpha$ for the inclusion $K_\alpha \to K$. Then for each index $\alpha$, $u^\alpha_\#\, v_\alpha^*(k) \in \mathrm{H}^2(K;\mathbb{Z}[K/K_\alpha])$, so that, under isomorphism (10.4), the element $\oplus_\alpha\ u^\alpha_\#\, v^*_\alpha(k)$ lies in $\mathrm{H}^2(K;\mathbb{Z}[G/H])$.

**10.5 Lemma** *In the above situation, for any $k \in \mathrm{H}^2(H;\mathbb{Z})$,*

$$v^*u_\#(k) = \oplus_\alpha\ u^\alpha_\#\, v^*_\alpha(k).$$

**Sketch of proof** Consider the homotopy fibre square

$$\begin{tikzcd}
\coprod_\alpha BK_\alpha \arrow[r, "\coprod Bv_\alpha"] \arrow[d, "\coprod Bu^\alpha"'] & BH \arrow[d, "Bu"'] \\
BK \arrow[r, "Bv"] & BG
\end{tikzcd}$$

in which the vertical maps are covering spaces with fibre $G/H$. An element $k \in \mathrm{H}^2(H;\mathbb{Z})$ corresponds to an extension of $H$ by $\mathbb{Z}$ and thus to a fibration $p\colon BE \to BH$ with fibre $B\mathbb{Z} = S^1$. The Shapiro companion $u_\#(k) \in \mathrm{H}^2(BG;\mathbb{Z}[G/H])$ similarly corresponds to some fibration $(Bu)_\#(BE)$ over $BG$. One checks that $(Bu)_\#(BE)$ is the fibration whose fibre over $x \in BG$ is the product, taken over $y \in (Bu)^{-1}(x)$, of $p^{-1}(y)$. In other words, $u_\#$ is a kind of geometric multiplicative transfer. By inspection, pulling $BE \to BH$ back over $\coprod Bv_\alpha$ and applying the multiplicative transfer $(\coprod Bu^\alpha)_\#$ gives the same result as pulling $(Bu)_\#(BE)$ back over $Bv$. The lemma is the algebraic expression of this. $\square$

**Proof of Lemma 10.1** Let $\Sigma$ be the set of reflections in $W$ and $\Sigma^A$ the set of reflections in $W_A$, so that $\mathbb{Z}[\Sigma]$ is isomorphic as a $W_A$–module to the sum $\mathbb{Z}[\Sigma^A]+\mathbb{Z}[\Sigma']$, where $\Sigma'$ is the complement $\Sigma\setminus\Sigma^A$. Because of the way in which the normalizer extension $\nu(W_A)$ depends on the reflection extension $\rho(W_A)$, it is enough to check that the restriction of the reflection extension $\rho(W)$ to $W_A$ is the sum of $\rho(W_A)$ and a semidirect product extension of $W_A$ by $\mathbb{Z}[\Sigma']$. As in Section 6.11, write $\Sigma = \coprod \Sigma_i$ as a union of $W$–conjugacy classes. Each conjugacy class $\Sigma_i$ can then be written as a union

$$\Sigma_i = \textstyle\coprod_j \Sigma_{i,j}$$

of $W_A$–conjugacy classes. For each $i$ let $\tau_i$ denote a representative of the $W$–conjugacy class $\Sigma_i$, and $\tau_{i,j}$ a representative of the $W_A$–conjugacy class $\Sigma_{i,j}$; the centralizer of $\tau_i$ in $W$ is $C_i$ and the isotropy subgroup of $\tau_{i,j}$ in $W_A$ is $C_{i,j}$. The group $C_i$ has a canonical extension by $\mathbb{Z}$ ((6.12)); if $\tau_{i,j}\in W_A$ then $C_{i,j}$ is the centralizer of $\tau_{i,j}$ in $W_A$ and it too has a canonical extension by $\mathbb{Z}$. Denote the corresponding extension classes by $k_i \in \mathrm{H}^2(C_i;\mathbb{Z})$ and $k_{i,j}\in \mathrm{H}^2(C_{i,j};\mathbb{Z})$. Let $u^i\colon C_i\to W$, $u^{i,j}\colon C_{i,j}\to W_A$, and $v\colon W_A\to W$ be the inclusion maps. In the notation of Section 10.3, the task comes down to checking for each index $i$ the validity of the formula

$$v^*\, u^i_\#(k_i) = \oplus_{\tau_{i,j}\in W_A} u^{i,j}_\#(k_{i,j}) \oplus_{\tau_{i,j}\notin W_A} 0_{i,j}, \tag{10.6}$$

where $0_{i,j}$ denotes the zero element of $\mathrm{H}^2(W_A;\mathbb{Z}[\Sigma_{i,j}])$.

The left hand side of (10.6) can be evaluated with Lemma 10.5. The double cosets $W_A\backslash W/C_i$ in question correspond to the orbits of the conjugation action of $W_A$ on $\Sigma_i$. For each such orbit $\Sigma_{i,j}$ with orbit representative $\tau=\tau_{i,j}$, choose an element $t\in W$ such that $t\tau_i t^{-1}=\tau$; the element $t$ is then the double coset representative. Note that $C_{i,j} = W_A\cap tC_it^{-1}$. We have to consider the maps

$$\begin{aligned} v_{i,j}\colon\ & C_{i,j}\to tC_it^{-1}\xrightarrow{t^{-1}(-)t} C_i\\ u^{i,j}\colon\ & C_{i,j}\to W_A \end{aligned}$$

and compute $u^{i,j}_\#\, v^*_{i,j}(k_i)$. The key observation is that if $\tau$ is not contained in $W_A$, then $C_{i,j}$, which is generated by the reflections it contains [33, 1.5], must lie in $tC_i^{\perp}t^{-1}$; hence $v^*_{i,j}(k_i)=0$. If $\tau$ is contained in $W_A$, then $C_{i,j}$ is the centralizer of $\tau$ in $W_A$, $t^{-1}\tau t=\tau_i$, and $t^{-1}C_{i,j}^{\perp}t\subset C_i^{\perp}$, so that $v^*_{i,j}(k_i) = k_{i,j}$. □

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