ISSN 1364-0380 (on line) 1465-3060 (printed)

Geometry & Topology
Volume 9 (2005) 935–970
Published: 25 May 2005

Symplectomorphism groups and isotropic skeletons

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Abstract

The symplectomorphism group of a 2–dimensional surface is homotopy equivalent to the orbit of a filling system of curves. We give a generalization of this statement to dimension 4. The filling system of curves is replaced by a decomposition of the symplectic 4–manifold (M, ω) into a disjoint union of an isotropic 2–complex L and a disc bundle over a symplectic surface Σ which is Poincare dual to a multiple of the form ω. We show that then one can recover the homotopy type of the symplectomorphism group of M from the orbit of the pair (L, Σ). This allows us to compute the homotopy type of certain spaces of Lagrangian submanifolds, for example the space of Lagrangian $\mathbb{RP}^2 \subset \mathbb{CP}^2$ isotopic to the standard one.

AMS Classification numbers Primary: 57R17

Secondary: 53D35

Keywords: Lagrangian, symplectomorphism, homotopy

Proposed: Leonid Polterovich
Seconded: Yasha Eliashberg, Tomasz Mrowka

Received: 25 June 2004
Revised: 24 September 2004

1 Introduction

1.1 Surface diffeomorphisms

A finite set $L = \{\gamma_i\}$ of simple, closed, transversally intersecting parametrized curves on a surface X *fills* if $X \backslash \{\gamma_i\}$ consists solely of discs. Endow X with a symplectic structure ω, and denote by $\mathcal{L}$ the orbit of L under the action of the symplectomorphism group $\mathcal{S}$. We have the following classical result:

Theorem 1.1 *Let (X, ω) be a symplectic surface. Then the orbit map $\mathcal{S} \to \mathcal{L}$ is a homotopy equivalence.*

Sketch proof We must show that the stabilizer of $\{\gamma_i\}$ in $\mathcal{S}$ (the symplectomorphisms of a disjoint union of discs, fixing their boundaries) is contractible. Moser's lemma allows us to replace symplectomorphisms with diffeomorphisms. (The inclusion is a weak deformation retract.) By the Riemann mapping theorem, these act transitively on the space of complex structures on the disc which are standard at the boundary. This set of complex structures is contractible, and thus the homotopy type is reduced to that of the complex automorphisms of the disc which fix the boundary, a contractible set. $\square$

In this paper we will prove a 4 dimensional analog of this statement. The proof follows a similar outline, although it is of course more difficult to carry out each step. Our tool is again the Riemann mapping theorem, but this time augmented by the theory of J–holomorphic spheres in sphere bundles over surfaces, developed by Gromov, Lalonde and McDuff [7, 9, 13].

1.2 Biran decompositions: a higher dimensional analog of filling systems of curves

Paul Biran [3] recently showed that every Kähler manifold M whose symplectic form lies in a rational cohomology class admits a decomposition

$$M = L \coprod E$$

where L is an embedded, isotropic cell complex and E is a symplectic disc bundle over a hypersurface Σ. L is called an isotropic skeleton of M.

We will argue that a Biran decomposition of a symplectic 4–manifold should be regarded as the 4–dimensional analog of a filling system of curves. Indeed, when M is a surface L is a filling system of curves, Σ is a union of points (one in each disc inside $M\backslash L$) and E is the union of discs.

In higher dimensions we have less understanding of the possible singularities of the spine L and as a result we prove a weaker, more technical result. This requires a bit of machinery to state, however when L is given by a smooth submanifold it reduces to the following.

Definition 1.2 Let (M,ω) be a symplectic 4–manifold with the decomposition $(L, E \to \Sigma)$ such that $L \hookrightarrow M$ is a smooth Lagrangian submanifold of M.

(1) Let $\mathcal{S}(M)$ denote the symplectomorphisms of (M,ω).
(2) Let $\mathcal{L}^{\mathrm{sm}}$ denote the Lagrangian embeddings of $\phi\colon L \hookrightarrow M$ which extend to symplectomorphisms of M.
(3) Let $\mathcal{LE}^{\mathrm{sm}}$ denote the space of pairs (ψ, S) where $\psi \in \mathcal{L}^{\mathrm{sm}}$ and S is a symplectic embedded unparametrized surface which is abstractly symplectomorphic to Σ and disjoint from $\psi(L)$.
(4) Let $\mathcal{E}mb_\omega(\Sigma, E)$ denote the space of unparametrized, embedded symplectic surfaces S in $E \subset M$ such that $\omega[S] = \omega[\Sigma]$.

Theorem 1.3 (The main theorem when L is smooth) *Let (M,ω) be a symplectic 4–manifold with Biran decomposition $(L, E \to \Sigma)$ such that $\phi\colon L \hookrightarrow M$ is a smooth, Lagrangian submanifold. Then $\mathcal{S}(M)$ is homotopy equivalent to $\mathcal{LE}^{\mathrm{sm}}$.*

Moreover there is a fibration $\mathcal{LE}^{\mathrm{sm}} \to \mathcal{L}^{\mathrm{sm}}$ whose fiber is homotopy equivalent to $\mathcal{E}mb_\omega(\Sigma, E)$. When Σ has genus 0, $\mathcal{E}mb_\omega(\Sigma, E)$ is contractible and thus $\mathcal{S}(M)$ is homotopy equivalent to $\mathcal{L}^{\mathrm{sm}}$.

We note that in the case that M is a surface the corresponding symplectic embeddings $\mathcal{E}mb_\omega(\Sigma, E)$ are just the embeddings of each point inside the appropriate disc. Thus in the two-dimensional case $\mathcal{E}mb_\omega(\Sigma, E)$ is always contractible, and one recovers (modulo concerns of L's smoothness) Theorem 1.1. We show that in dimension 4, $\mathcal{E}mb_\omega(\Sigma, E)$ is contractible if Σ is a sphere. However the proof relies heavily on the special properties of J–holomorphic spheres in rational surfaces, and it is unclear how one may generalize it to cases when Σ has higher genus.

In general this theorem, and more generally Theorem 2.6, allow us to separate the problem of understanding the topology of the symplectomorphism group into two parts: embeddings of isotropic skeletons up to symplectic equivalence, and $\mathcal{E}mb_\omega(\Sigma, E)$ the embeddings of symplectic surfaces in a fixed homology class disjoint from the spine. This second problem is universal, depending only on the genus, and self-intersection of Σ but not the ambient manifold M.

Understanding the higher homotopy groups of spaces of Lagrangian embeddings $L \to M$ is quite hard. Eliashberg and Polterovich showed that such spaces are locally contractible, ie the space of Lagrangian embeddings of $\mathbb{R}^2$ into $\mathbb{R}^4$ which are standard at ∞ is contractible [6]. However, prior to this paper the author knows of no computation when both domain and range are closed. Thus, while Theorem 2.6 provides a satisfying generalization it is, at present, difficult to use it to compute much about the symplectomorphism group of a 4 manifold. However we can use it leverage our knowledge of symplectomorphism groups into an understanding of spaces of Lagrangian embeddings – a result which should satisfy in proportion to our previous frustration. We obtain the following corollaries in subsection 2.6, showing that spaces of Lagrangian embeddings due indeed have non-trivial global topology. Each is obtained by combining our result with Gromov's computations of the symplectomorphism groups of $\mathbb{CP}^2$ and $S^2 \times S^2$.

Theorem 1.4 *The space of Lagrangian embeddings of $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$ isotopic to the standard one is homotopy equivalent to $\mathbb{PU}(3)$*

Theorem 1.5 *Let ω be a symplectic form on $S^2 \times S^2$ such that $\omega[S^2 \times pt] = \omega[pt \times S^2]$, then the space of Lagrangian embeddings $S^2 \hookrightarrow S^2 \times S^2$ isotopic to the anti-diagonal is homotopy equivalent to $SO(3) \times SO(3)$.*

Richard Hind has recently proven that every Lagrangian sphere in $S^2 \times S^2$ (with the above symplectic structure) is isotopic to the anti-diagonal [8]. One suspects that his methods probably can be used to show that every Lagrangian $\mathbb{RP}^2$ is isotopic to the standard one. Thus both theorems above should actually compute the homotopy type of the full space of Lagrangian embeddings.

Finally we note that the spine L is smooth only in a few special, though important, decompositions, and those which we apply to obtain the above two corollaries may nearly exhaust them. In general one requires the extra generality and complexity of Theorem 2.6. In particular, one should not hope to compute the space of Lagrangian embeddings $L \to M$ for general domain and target via the methods of this paper, without considerable and yet unseen adaptation.

1.3 Acknowledgements

This paper is a portion of the author's PhD thesis, produced under the helpful guidance of his advisors Dusa McDuff and Dennis Sullivan. The author would particularly like to thank Dusa McDuff for her tireless aid in preparing this article for publication. He would also like to thank the referee for his helpful comments. The author is supported in part by NSF grant DMS-0202553.

2 Definitions and precise statements

2.1 Kan complexes

2.1.1 Motivation: germs and Kan complexes

In what follows we need to understand the space of symplectomorphisms fixing an isotropic skeleton, *and the germ of a neighborhood surrounding it.* This forces us to work with "spaces" of germs of mappings. A germ does not have a specified domain, and as a result most natural topologies on spaces of germs have unwanted pathologies.

While the space of germs is difficult to define, compact families of germs are transparent – there is no difficulty with domains. In this paper we are concerned here with weak homotopy type: the study of compact families. Thus there is no real foundational difficulty. We do however require some linguistic finesse, and Kan complexes, simplicial sets that satisfy the extension condition, provide exactly this.

One can find a straightforward introduction to Kan complexes in J.P. May's book [11]. However on a first reading the reader can safely ignore these subtleties and replace each Kan complex with the appropriate "space". Indeed, when the spine is a smooth Lagrangian manifold every symplectomorphism fixing the spine also fixes its normal bundle, and thus the symplectomorphisms fixing a neighborhood of the spine are a deformation retract of those fixing the spine.

2.1.2 A brief introduction to Kan complexes

Since Kan complexes are perhaps unfamiliar to much of this paper's audience we shall give a brief informal introduction here, based on that in May's book.

The basic example: singular simplices of a topological space Let Δ_n denote the n–simplex. If X is a topological space, the singular n–simplices $\Delta^n(X)$ are the continuous maps

$$g\colon \Delta_n \to X.$$

The singular simplices of X are then given by $\Delta(X) = \bigcup_{n\in\mathbb{N}} \Delta^n(X)$.

Kan complexes are designed to axiomatize those properties of the simplices $\Delta(X)$ that are required to do homotopy theory – to construct homotopy groups, fibrations etc. They will be particularly useful in our case because as mentioned above there will be several ill-defined "spaces" in our discussion whose *simplices are well defined*.

Degeneracy and face operators We begin with a graded set $K = \bigcup K_q$. The grading gives the "dimension" of the simplices. The first piece of extra structure we require are the face and degeneracy operators $\partial_i\colon K_q \to K_{q-1}$ and $s_i\colon K_q \to K_{q+1}$. ∂_i axiomatizes the notion of passing from a singular simplex to its ith face. s_i encodes that of passing from an n–simplex to a degenerate $(n+1)$–simplex. One then has a list of axioms encoding the geometry of these operations

$$\begin{array}{lcll}
\partial_i\partial_j &=& \partial_{j-1}\partial_i & \text{if } i<j \\
s_i s_j &=& s_{j+1}s_i & \text{if } i\le j \\
\partial_i s_j &=& s_{j-1}\partial_i & \text{if } i<j \\
\partial_j s_j &=& \text{identity} & = \partial_{j+1}s_j, \\
\partial_i s_j &=& s_j\partial_{i-1} & \text{if } i>j+1
\end{array}$$

A graded set with face and degeneracy operations is called a *simplicial set*.

The extension condition The second necessary structure is that of *extension*. Suppose one has a continuous map f into X which is defined on the union of n faces (all but one) of Δ_n. Then since Δ_n retracts onto this union, one can extend f to be a singular n–simplex, a continuous map of all of Δ_n into X.

One axiomatizes this by saying that if one has a collection of $(n-1)$–simplices in K_{n-1} whose degeneracy and face maps fit together like this union of n faces, then one can find an n–simplex in K_n filling them in. The extension condition is also called the Kan condition. It is required for homotopy of simplices to form an equivalence relation.

Homotopy groups Given two singular simplices $f, g\colon \Delta_n \to X$, sharing a common boundary, we say that they are homotopic (relative to their boundary) if there is an $(n+1)$–simplex h whose non-degenerate faces are f and g.

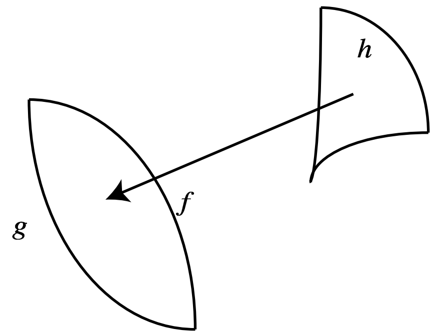

Figure 1: The singular 1–simplices f and g are homotopic because there is a 2–simplex h making each a face.

There is a natural generalization of this definition to the homotopy of simplices in Kan complexes. Again one encodes the phenomena in terms of the face and degeneracy operators. Homotopy of simplices is then an equivalence relation.

Given a based topological space (X, x_0), one can think of $\pi_n(X, x_0)$ as the homotopy classes of singular simplices $f\colon \Delta_n \to X$ such that the faces of Δ_n all map to x_0. In this way one can also define homotopy groups of a Kan complex. One first defines a basepoint by choosing an $x_0 \in K_0$, and considering it together with the sub complex generated by its degeneracy maps. This yields a subcomplex with one completely degenerate simplex in each dimension that we denote also by x_0. Then one can define $\pi_n(K, x_0)$ as the set of homotopy classes of n–simplices in K_n whose faces all lie in x_0. These have a group structure when $n \geq 1$, which is Abelian for $n \geq 2$. One can also define relative Kan homotopy groups, having the expected properties, in an analogous manner.

Kan fibrations A Serre fibration of topological spaces is one that allows "lifting" of homotopies of simplices. A Kan fibration is defined analogously. Just as for spaces, a Kan fibration of based complexes induces a corresponding long exact sequence of homotopy groups,

The geometric realization Given a Kan complex K one can, in a natural way, construct a topological space $|K|$ by constructing a topological simplex for each simplex in K and then ensuring that the faces and and degenerations act appropriately:

$$|K| = \bigcup_{n\in\mathbb{N}} K_n \times \Delta_n / \sim$$

Where $\sim$ is an equivalence relation given by

$$\begin{aligned}(\partial_i k_n, u_{n-1}) &\sim (k_n, \partial_i u_{n-1}),\\ (s_i k_n, u_{n+1}) &\sim (k_n, s_i u_{n+1}),\end{aligned}$$

for $k_n \in K_n, u_{n-1} \in \Delta_{n-1}$, and $u_{n+1} \in \Delta_{n+1}$. $\Delta(|K|)$ is weakly (Kan) homotopy equivalent to K. Similarly, for any topological space X, $|\Delta(X)|$ is weakly homotopy equivalent to X.

2.2 Biran decompositions

Let (M,ω) denote a symplectic 4–manifold. For any symplectic manifold (N,σ), $\mathcal{S}(N,\sigma)$ denotes the symplectomorphisms of (N,σ). If either N or σ is clear from the context they will be omitted. $\mathcal{D}(M)$ denotes the diffeomorphisms of M. Again, if M is clear we will omit it.

Definition 2.1 A smoothly embedded cell complex consists of:

(1) An abstract smooth cell complex C in which the interior of each cell is endowed with a smooth structure.

(2) A continuous map
$$i\colon C \hookrightarrow M$$
which is a smooth embedding when restricted to the interior of each cell in C.

We say that a smoothly embedded cell complex is *isotropic* with respect to a symplectic structure ω, if $i^*(\omega) = 0$ on the interior of each cell.

Definition 2.2 Let (M,ω) be a symplectic manifold. Let J be an almost complex structure compatible with ω. Let Σ_λ a symplectic hypersurface of M, Poincare dual to $\lambda[\omega]$, and such that:

(1) There is a smoothly embedded, isotropic cell complex L_λ disjoint from Σ_λ. In what follows we will call this cell complex an *isotropic skeleton* of M.

(2) $M - L_\lambda$ is an open symplectic disc bundle E over Σ_λ, such that the fibers have area $\frac{1}{\lambda}$ with respect to ω. This bundle is symplectomorphic to the unit disc bundle in the normal bundle to Σ_λ with symplectic form

$$\pi^*\omega|_\Sigma + d(r^2\alpha)/\lambda,$$

where r is the radial coordinate in the fiber, α is the connection 1–form coming from the hermitian metric $\omega(\cdot, J\cdot)$ on the normal bundle, and α is normalized so that its total integral around the boundary of a fiber is $\frac{1}{\lambda}$.

We call such a configuration $(L_\lambda, E \to \Sigma_\lambda)$ a *decomposition* of M.

Theorem 2.3 (Biran [3]) *If M is a smooth, projective variety, then there is a decomposition* $(L_\lambda, E \to \Sigma_\lambda)$.

Biran conjectures that his proofs generalize to any symplectic manifold whose symplectic structure ω has a rational cohomology class. This would rely on Donaldson's result [5] implying that in such a setting one has a symplectic hypersurface Poincare dual to $[\lambda\omega]$.

2.3 Notation and a precise statement of our main theorem

Let $(L, E \to \Sigma)$ be a decomposition of the symplectic 4–manifold (M, ω). Denote the symplectic embeddings of Σ into M by $\mathcal{E}mb_\omega(\Sigma, M)$.

Definition 2.4 Denote by $\mathcal{L}$ the Kan complex of isotropic embeddings of L into M.

An n–simplex in $\mathcal{L}$ is an equivalence class given by:

(1) A neighborhood U of L in M.

(2) A continuous map $\phi\colon \Delta_n \to \mathcal{S}(U, M)$, where $\mathcal{S}(U, M)$ denotes the symplectic embeddings of U into M, which admits an extension to a symplectomorphism of all of M.

Two such pairs (U_1, ϕ_1) and (U_2, ϕ_2) are equivalent if there exists a neighborhood U_3 of L such that $U_3 \subset U_1$, $U_3 \subset U_2$ and

$$\phi_1|_{U_3} = \phi_2|_{U_3}$$

The face maps ∂_i are given by restricting ϕ to the faces of Δ_n. The degeneration maps s_i are given by pre-composing ϕ with the degeneration maps of Δ_n.

Definition 2.5 Denote by $\mathcal{LE}$ the following Kan complex:

an n–simplex consists of a pair $((U,\phi),\psi)$ where:

(1) (U,ϕ) is an n–simplex in $\mathcal{L}$.

(2) A continuous map $\psi\colon \Delta_n \to \mathcal{E}mb_\omega(\Sigma, M)$ such that for each $x \in \Delta_n$, $\psi(x)$ is disjoint from $\phi(L)$.

The face maps ∂_i are given by restricting ϕ and ψ to the faces of Δ_n. The degeneration maps s_i are given by pre-composing ϕ and ψ with the degeneration maps of Δ_n.

Note that $\mathcal{S}(M)$ acts on $\mathcal{LE}$. For any symplectomorphism will carry any simplex of skeletons to another , and will preserves the the homology class of Σ as it is Poincare dual to $\lambda[\omega]$.

This paper is devoted to the proof and application of the following theorem.

Theorem 2.6 *Let (M,ω) be a symplectic $4-$manifold with decomposition $(L, E \to \Sigma)$. Then the orbit map gives a weak homotopy equivalence of $\mathcal{S}(M)$ with $|\mathcal{LE}|$. Moreover there is a fibration $|\mathcal{LE}| \to |\mathcal{L}|$ whose fiber is given by $\mathcal{E}mb_\omega(\Sigma, E)$, the symplectic embeddings of Σ into E. When Σ has genus 0, $\mathcal{E}mb_\omega(\Sigma, E)$ is contractible.*

Here and elsewhere, when we say a map $f\colon X \to Y$ is a homotopy equivalence we mean that:

(1) f gives a bijection between the connected components.

(2) If one chooses a base point x_i in each connected component X_i of X, and chooses $f(x_i)$ to be the basepoint of corresponding connected component Y_i of Y; then the resulting map $f\colon (X_i, x_i) \to (Y_i, f(x_i))$ gives a weak homotopy equivalence.

We do not require that M be Kähler, only that it has a decomposition. However we do restrict ourselves to dimension 4, as we rely heavily on the properties of J–holomorphic spheres in this dimension.

2.4 Conventions

Note that both the symplectomorphisms $\mathcal{S}(M,\omega)$ and the space of pairs $\mathcal{LE}$ are invariant under scaling the symplectic structure by a constant factor. Thus we safely replace ω by $\lambda\omega$ and reduce to the case that the class of Σ and the symplectic form are Poincare dual, and E's fibers have area 1.

Groups and actions Throughout this paper we will be computing and comparing the stabilizers of the action of various groups on various geometric objects. To keep our heads straight it will be helpful to adopt a few notational guidelines.

(1) G_S denotes the elements in the group G which preserve the set S. That is, $G_S = \{g \in G : g(S) \subset S\}$.

(2) $G_{\mathbb{S}}$ denotes the elements in the group G which fix the set S. That is, $G_{\mathbb{S}} = \{g \in G : g(s) = s, \forall s \in S\}$. This notation is required only in the next subsection.

(3) $G_{\overline{S}}$ denotes the elements in the group G which fix both a set S and a framing of that set. These are

$$\{g \in G : \exists \text{ a neighborhood} N_S \supset S : g(s) = s, \forall s \in N_S\}.$$

We endow $G_{\overline{S}}$ with the direct limit topology and say that these are the elements of G which fix a framing of S.

(4) If we fix or preserve more than one set we will denote this by separating the two with a comma. For example: $G_{X,\widetilde{Y}}$ denotes the elements in G which preserve X and fix both Y and a framing.

2.5 Simplifications when L is smooth: Proof of Theorem 1.3

Proof If the skeleton of the decomposition L is a smooth, Lagrangian, submanifold one does not have to introduce Kan complexes. In this case $\mathcal{S}(M)$ is homotopy equivalent to $\mathcal{L}\mathcal{E}^{\mathrm{sm}}$ (see Definition 1.2).

The difference between this result (Theorem 1.3) and the more general statement of Theorem 2.6 is as follows: Theorem 2.6 states that $\mathcal{S}_{\overline{L},\Sigma}$, the symplectomorphisms which fix a framing of L and preserve Σ, are contractible, while Theorem 1.3 requires that $\mathcal{S}_{\mathbb{L},\Sigma}$, the symplectomorphisms which only fix L and preserve Σ, are contractible.

We will now show that the natural map $\mathcal{S}_{\overline{L},\Sigma} \to \mathcal{S}_{\mathbb{L},\Sigma}$ is a homotopy equivalence, and thus that Theorem 1.3 follows from Theorem 2.6. If $\phi \in \mathcal{L}^{\mathrm{sm}}$, every extension of ϕ to a symplectomorphism of M induces the same map on the normal bundle N_L of L. This is an immediate consequence of the corresponding linear statement: If (V,ω) is a symplectic vector space with Lagrangian subspace L, ϕ_1 and ϕ_2 are linear symplectic automorphisms of V such that $\phi_1|_L = \phi_2|_L$ then each induces the same map $V/L \to V/\phi_i(L)$.

Thus Weinstein's neighborhood theorem (in parameters) implies that the map $\mathcal{L} \hookrightarrow \Delta(\mathcal{L}^{\mathrm{sm}})$ is a weak homotopy equivalence. The actions of $\mathcal{S}(M)$ on each space yield a morphism of fibrations:

$$\begin{array}{ccc} \Delta(\mathcal{S}_{\overline{L}}) & \xrightarrow{i_1} & \Delta(\mathcal{S}_{\mathbb{L}}) \\ \downarrow & & \downarrow \\ \Delta(\mathcal{S}) & \xrightarrow{(\mathrm{id})} & \Delta(\mathcal{S}) \\ \downarrow & & \downarrow \\ \mathcal{L} & \xrightarrow{i_2} & \Delta(\mathcal{L}^{\mathrm{sm}}) \end{array}$$

By applying the 5–Lemma (Lemma 7.7) to the associated long exact sequence of Kan homotopy groups we see that $\Delta(\mathcal{S}_{\overline{L}}) \overset{i_1}{\hookrightarrow} \Delta(\mathcal{S}_{\mathbb{L}})$ must also be a homotopy equivalence, so $\mathcal{S}_{\overline{L}} \to S_{\mathbb{L}}$ and $\mathcal{S}_{\overline{L},\Sigma} \to \mathcal{S}_{\mathbb{L},\Sigma}$ are also homotopy equivalences. □

2.6 Applications to spaces of Lagrangian embeddings

We now apply Theorem 1.3 to compute spaces of Lagrangian submanifolds in cases where we know the homotopy type of $\mathcal{S}(M)$. We show that:

(1) The space of Lagrangian embeddings of $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$ isotopic to the standard one is homotopy equivalent to $\mathbb{PU}(3)$.

(2) If ω is a symplectic form on $S^2 \times S^2$ such that $\omega[S^2 \times pt] = \omega[pt \times S^2]$ then the space of Lagrangian embeddings $S^2 \hookrightarrow S^2 \times S^2$ isotopic to the anti-diagonal is homotopy equivalent to $\mathrm{SO}(3) \times \mathrm{SO}(3)$.

These are Theorems 1.4 and 1.5 respectively. In [3] Biran computes decompositions of $\mathbb{CP}^2$ and $(S^2 \times S^2, \omega)$ with a symplectic structure such that $\omega[S^2 \times pt] = \omega[pt \times S^2]$.

Proposition 2.7 (Biran)

(1) *$\mathbb{CP}^2$ admits a decomposition where Σ is a quadric (and thus a sphere) and L the standard $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$.*

(2) *$(S^2 \times S^2, \omega)$ with a symplectic structure such that $\omega[S^2 \times pt] = \omega[pt \times S^2]$ admits a decomposition with Σ the diagonal and L the anti-diagonal.*

In each case the decomposition has L a smooth manifold with $H_1(L)$ torsion, and Σ a sphere. Denote the identity component of $\mathcal{S}(M)$ by $\mathcal{S}(M)^\circ$. Similarly we denote the identity component of $\mathcal{L}^{\rm sm}$ by $\mathcal{L}^{\rm sm}_\circ$. In the two decompositions above $\mathcal{L}^{\rm sm}_\circ$ is the space of all Lagrangian embeddings of L isotopic to the original. For in each case $H_1(L)$ is torsion $H_2(M)\otimes\mathbb{R}\to H_2(M,L)\otimes\mathbb{R}$ is surjective, and thus every isotopy of ϕ can be induced by an ambient symplectic isotopy of M.

The orbit map

$$\mathcal{S}(M)^\circ \to \mathcal{L}^{\rm sm}_\circ$$

is a surjective fibration. By Theorem 1.3 its fiber is homotopy equivalent to $\mathcal{E}mb_\omega(\Sigma,E)$. Moreover since Σ is a sphere, Theorem 1.3 states that $\mathcal{E}mb_\omega(\Sigma,E)$ is contractible. Thus the map is a homotopy equivalence.

Finally, since the identity components of $\mathcal{S}(\mathbb{CP}^2)$ and $\mathcal{S}((S^2\times S^2,\omega)$, are homotopy equivalent to $\mathbb{PU}(3)$ and $\mathrm{SO}(3)\times\mathrm{SO}(3)$ respectively [7] Theorems 1.4 and 1.5 follow. In general we have the following corollary:

Corollary 2.8 *Let (M,ω) be a symplectic 4–manifold with the decomposition $(L,E\to\Sigma)$ such that $\phi\colon L\hookrightarrow M$ is a smooth submanifold of M, $H_2(M)\otimes\mathbb{R}\to H_2(M,L)\otimes\mathbb{R}$ is surjective, and Σ is a sphere. Then the space $\mathcal{L}_\phi$ of Lagrangian embeddings isotopic to ϕ is homotopy equivalent to the identity component of $\mathcal{S}(M)$.*

3 Reduction to a problem on rational surfaces

We seek in this section to perform a fiber-wise compactification of (most of) the disc bundle E into a symplectic sphere bundle $\widehat{E}_\epsilon\to\Sigma$. This sphere bundle will have two distinguished symplectic sections Z_0, which is identified with Σ under the compactification, and Z_∞, which is the image of a neighborhood of L.

Then our theorem is reduced to a problem on rational surfaces:

Definition 3.1 Let (X,ω) be a symplectic 4–manifold which is a symplectic sphere bundle over a surface Σ with a symplectic form ω. Let Z_0 be a symplectic section of F such that $[Z_0]\cdot[Z_0]=k\geq 0$. Denote the homology class of the fiber by $[F]$. We say that (X,Z_0,F,ω) *satisfies the cohomology assumption* if $[\omega]=a\cdot PD([Z_0])+b\cdot PD([F])$ with $a,b>0$ both positive real numbers.

Proposition 3.2 (Problem on rational surfaces) *Let (X,Z_0,F,ω) be a symplectic 4–manifold admitting a symplectic fibration by 2–spheres by F, Z_0 be a* *symplectic section of F such that $[Z_0]\cdot[Z_0] = k \geq 0$. Suppose that (X, Z_0, F, ω) satisfies the cohomology assumption. Let Z_∞ be a symplectic section of F which is disjoint from F, and thus has self-intersection $[Z_\infty]\cdot[Z_\infty] = -k$.*

Then the space of symplectomorphisms $\mathcal{S}(X)_{\overline{\infty}}$ of X which fix a framing of Z_∞ acts transitively on the space $\mathcal{E}_\infty$ of unparametrized,embedded symplectic surfaces S of X disjoint from Z_∞ with contractible stabilizer. Moreover, $\mathcal{E}_\infty$ is contractible when Σ has genus 0.

The compactification $\widehat{E}_\epsilon$ serves two roles: it allows us to play in the more comfortable compact terrain, and it converts the problem of computing the stabilizer of an isotropic object (the spine L) to that of a symplectic object (the section Z_∞). For this dual service we pay a price: we cannot compactify all of $M - L$, and must be satisfied with compactifying the complement of neighborhood of L.

The compactification is described in 3.1, the reduction of our Theorem to Proposition 3.2 is described in 3.2. Its proof is described in section 6 once we have developed the necessary background in sections 4 and 5.

3.1 Compactification of $E_{1-\epsilon}$ via symplectic cutting (a la Lerman)

We apply the techniques of [10] to the present situation. Consider $E \subset M$ as the open unit disc bundle in the normal bundle N_Σ to Σ. Denote by $E_{1-\epsilon} \subset E$ the set $\{x \in E : ||x|| \leq 1-\epsilon\}$.

Lemma 3.3 *There is a surjective C^∞ map $\Psi\colon E_{1-\epsilon} \to \widehat{E}_\epsilon$ where $\widehat{E}_\epsilon$ is a symplectic sphere bundle over Σ. Topologically, Ψ is given by the collapse of the boundary circle in each fiber of the disc bundle $E_{1-\varepsilon}$.*

(1) *Ψ is a symplectomorphism on the interior of $E_{1-\epsilon}$.*

(2) *Ψ maps the boundary of $E_{1-\epsilon}$ to a symplectic section Z_∞ of this bundle whose self-intersection is $-k$.*

(3) *Ψ maps the zero section of $E_{1-\epsilon}$ to a symplectic section Z_0 whose self-intersection is k. Moreover the symplectic form ω on $\widehat{E}_\epsilon$ has cohomology class $(1-\epsilon)PD([Z_0]) + \epsilon k PD([F])$ where $[F]$ denotes the class of the fiber of $\widehat{E}_\varepsilon$. Thus $\widehat{E}_\varepsilon$ satisfies the cohomology assumption (Definition 3.1).*

Proof The bundle $E \to \Sigma$ is given as the unit disc bundle in N_Σ in the hermitian metric induced from that on M. Place the coordinates (r, t) on the fiber of $E = D^2$, where r is a radial coordinate $r = |w|$, and the angular coordinate t lies in $[0, 1]$, (ie $t = \frac{\theta}{2\pi}$) Then the symplectic structure on E is given by

$$\pi^* \omega|_\Sigma + d(|w|^2 \alpha),$$

where α is a connection one form. This structure is invariant under the circle action $S(t)$ given by the Hamiltonian function $\mu = |w|^2$.

We now consider the S^1 action $P(t)$ on the product

$$(E \times C, \omega \oplus \tau),$$

where C denotes the complex numbers and τ denotes their standard complex structure, scaled by the constant factor $\frac{1}{\pi}$. The action is given by

$$P(t)(m, z) = (S(t)m, e^{2\pi i t} z),$$

where $P(t)$ is Hamiltonian with function

$$\zeta = \mu + ||z||^2.$$

Let $\widehat{E}$ be the symplectic reduction of $((E \times C, \omega \oplus \tau), P(t))$ along the level set $\zeta = 1 - \epsilon$. The level set

$$\zeta_{1-\epsilon} := \{(m, z) : \zeta(m, z) = 1 - \epsilon\}$$

is the disjoint union of

$$\left\{(m, z) : \mu(m) < 1 - \epsilon, z = e^{2\pi i t} \sqrt{\mu(m) - (1 - \epsilon)}\right\}$$

and
$$\{(m, 0) : \mu(m) = 1 - \epsilon\}$$

where both members of the disjoint union are invariant under the S^1 action. The map $i\colon E_{1-\epsilon} \to E \times C$ given by

$$i(m) = (m, \sqrt{(1 - \epsilon) - \mu(m)}$$

is a symplectic embedding, whose image is contained in the level set $\zeta_{1-\epsilon}$. I claim that the composition of i with the quotient

$$\pi_Q\colon \zeta_{1-\epsilon} \to \zeta_{1-\epsilon}/S^1 = \widehat{E}_\epsilon$$

of $\zeta_{1-\epsilon}$ by $P(t)$ gives a map

$$\psi = \pi_Q \circ i\colon E_{1-\epsilon} \to \widehat{E}_\epsilon$$

with the properties above. □

Remark 3.4 Symplectic structures on rational surfaces are classified up to diffeomorphism by their cohomology class [9]. Thus condition (4) determines the symplectic structure.

3.2 Translation of the main theorem to compactification

This subsection is devoted to carrying out the reduction to rational surfaces described in the introduction to section 3.

$\mathcal{S}_{\overline{L}}$ acts transitively on $\mathcal{E}mb_\omega(\Sigma, E)$ with contractible stabilizer
I claim that the natural map

$$\phi_{(\Sigma)}\colon\ \mathcal{S}_{\widetilde{L}} \to \mathcal{E}mb_\omega(\Sigma, E)$$

is a surjective homotopy equivalence.

Denote by $\mathcal{E}_\varepsilon$ the space of unparametrized, embedded symplectic surfaces S in $E_{1-\epsilon} \subset M$ such that $\omega[S] = \omega[\Sigma]$, and denote by$\mathcal{S}_{\overline{\varepsilon}}$ the symplectomorphisms of M which fix $M \backslash E_{1-\varepsilon}$ (a neighborhood of the isotropic skeleton L). Now consider the action of each $\mathcal{S}_{\overline{\varepsilon}}$ on $\mathcal{E}_\varepsilon$. The resulting orbit maps ϕ_i yield a morphism of direct systems

$$\begin{array}{ccccccc}
\mathcal{S}_{\overline{\varepsilon_1}} & \hookrightarrow & \mathcal{S}_{\overline{\varepsilon_2}} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{S}_{\overline{L}} \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \cdots & & \downarrow \phi \\
\mathcal{E}_{\varepsilon_1} & \hookrightarrow & \mathcal{E}_{\varepsilon_2} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{E}mb_\omega(\Sigma, E)
\end{array}$$

for $\epsilon_i > \epsilon_{i+1} > 0$. I claim that each ϕ_i is a surjective homotopy equivalence, and thus ϕ is as well. Note that the compactification Ψ induces:

(1) A homeomorphism $\Psi_\Sigma\colon\ \mathcal{E}_\varepsilon \to \mathcal{E}^\epsilon_\infty$, the space of unparametrized, embedded symplectic surfaces S in $\widehat{E}_\epsilon$ disjoint from Z_∞.

(2) A homeomorphism $\Psi_S\colon\ \mathcal{S}_{\overline{\varepsilon}} \to \mathcal{S}(\widehat{E}_\epsilon)_{\overline{\infty}}$, the space of symplectomorphisms $\widehat{E}_\epsilon$ which fix a framing of Z_∞.

Suppose we have proven Proposition 3.2. Then $\widehat{E}_\epsilon$ satisfies the cohomology assumption and thus the map $\mathcal{S}(\widehat{E}_\epsilon)_{\overline{\infty}} \to \mathcal{E}^\varepsilon_\infty$ is a surjective fibration with contractible fiber. Thus we have

$$\mathcal{S}_{\overline{\varepsilon}} \leftrightarrow \mathcal{S}(\widehat{E}_\epsilon)_{\overline{\infty}} \longrightarrow \mathcal{E}^\epsilon_\infty \leftrightarrow \mathcal{E}_\varepsilon,$$

where the second map is given by Proposition 3.2, and $\phi_{(\Sigma)}\colon\ \mathcal{S}_{\overline{\varepsilon}} \to \mathcal{E}_\varepsilon$ is a (weak) surjective homotopy equivalence for $0 < \epsilon < 1$. So ϕ must also be a (weak) surjective homotopy equivalence.

$\mathcal{S}(M) \simeq |\mathcal{LE}|$ We consider the action of $\mathcal{S}(M)$ on $\mathcal{LE}$. I claim that the map $\phi_{(L,\Sigma)}\colon \Delta(\mathcal{S}) \to \mathcal{LE}$ is a homotopy equivalence. This will imply that the associated map on geometric realizations $|\phi_{(L,\Sigma)}|\colon |\Delta(\mathcal{S})| \to |\mathcal{LE}|$ is also a homotopy equivalence. For since the natural inclusion $S \hookrightarrow |\Delta(S)|$ is always a homotopy equivalence, so is the composition

$$S \hookrightarrow |\Delta(\mathcal{S})| \to |\mathcal{LE}|.$$

This will prove our main theorem (Theorem 2.6).

Consider then the Kan fibration

$$\pi\colon\ \Delta(\mathcal{S}) \to \mathcal{LE}$$

where $\mathcal{S}_{\overline{L},\Sigma}$ denotes the symplectomorphisms of M which fix a framing of L and preserve Σ. $\mathcal{L}$ is by definition a homogeneous space for $\mathcal{S}$. That $\mathcal{LE}$ is as well follows from the transitive action of $\mathcal{S}_{\overline{L}}$ on $\mathcal{E}mb_\omega(\Sigma, E)$ proved above. Thus it is enough to prove that $\phi_{(L,\Sigma)}$ is a homotopy equivalence when the fibration is restricted to any connected component. The fiber of π is then homotopy equivalent to $\mathcal{S}_{\overline{L},\Sigma}$, and thus contractible.

If Σ is a sphere, $\mathcal{E}mb_\omega(\Sigma, E)$ is contractible $\mathcal{E}mb_\omega(\Sigma, E)$ is given by the direct limit

$$\mathcal{E}_{\varepsilon_1} \hookrightarrow \mathcal{E}_{\varepsilon_2} \hookrightarrow \cdots \hookrightarrow \mathcal{E}mb_\omega(\Sigma, E)$$

as each $\mathcal{E}_{\varepsilon i} \simeq \mathcal{E}_\infty^{\epsilon_i}$ is thus contractible, so must $\mathcal{E}mb_\omega(\Sigma, E)$ be contractible.

4 A softer symplectomorphism group of a rational surface

Definition 4.1 Let (X, ω) be a symplectic 4–manifold admitting a symplectic fibration by 2–spheres by F, which satisfies the cohomology assumption. Let Z_0 be a symplectic section of F such that $[Z_0]\cdot[Z_0] = k \geq 0$. Let Z_∞ be a symplectic section of F which is disjoint from F, and thus has self-intersection $[Z_\infty]\cdot[Z_\infty] = -k$.

We will now construct a large open neighborhood of the symplectomorphisms $\mathcal{S}(X)$ within the diffeomorphism group $\mathcal{D}(X)$. This neighborhood will have the same homotopy type as $\mathcal{S}(X)$, but it will be far easier to work with. In particular, it will be much easier to understand the "action" of this neighborhood on various objects. Various such "softenings" have been used throughout the study the homotopy type of symplectomorphism groups, beginning with Gromov's initial work [7] and continuing with the work of Abreu and McDuff [1, 2]. The particular neighborhood we construct has its roots in McDuff's inflation argument , used previously to classify the symplectic structures on rational surfaces [12].

We abbreviate $\mathcal{S}(X)$ by $\mathcal{S}$ in this section and the next.

Definition 4.2 Denote by $\mathcal{FZ}$ the space of all triples (F_S, S_0, S_∞) where F_S is a symplectic fibration of X by two spheres in class $[F]$ and S_0 and S_∞ are symplectic sections such that $[S_0] = [Z_0]$ and $[S_\infty] = [Z_\infty]$.

Definition 4.3 Denote by $\mathcal{S}^{\text{soft}}$ the diffeomorphisms g of X which preserve H^2 and such that the triple $(g(F), g(Z_0), g(Z_\infty)) \in \mathcal{FZ}$.

Proposition 4.4 *Let Z_0 be a symplectic section of F such that $[Z_0] \cdot [Z_0] = k \geq 0$, and let Z_∞ be a symplectic section of F which is disjoint from Z_0. Then the inclusion $i\colon \mathcal{S}(X) \hookrightarrow \mathcal{S}^{soft}$ is a weak deformation retract.*

Proof First we show that $\mathcal{S} \subset \mathcal{S}^{\text{soft}}$. If $\gamma \in \mathcal{S}$ it is clear that each member of the triple

$$(\gamma(F), \gamma(Z_0,)\gamma(Z_\infty))$$

is symplectic. What is required then is to show that each such γ preserves H_2. γ preserves ω, and thus also the cohomology class

$$[\omega] = (1-\epsilon)PD([Z_0]) + \epsilon k PD([F]).$$

Thus, as $[\omega]$ is Poincare dual to

$$(1-\epsilon)([Z_0]) + \epsilon k([F]),$$

γ preserves this homology class as well. As $[Z_0]$ and $[F_Z]$ together span $H_2(\widehat{E}_\epsilon)$, it is enough for us to show that γ preserves $[F]$, however there is no other spherical class Q with $[Q] \cdot [Q] = 0$ and such that $0 < \omega(Q) < \omega([F])$. Thus γ must fix $[F]$.

Next we show that i is a deformation retract. Let $\psi\colon D^n \to \mathcal{S}^{\text{soft}}$ such that $\psi(\partial D^n) \subset \mathcal{S}$. We will produce a retraction of ψ to a disc of symplectomorphisms, while fixing its boundary.

We do this by first producing a retraction of forms via inflation, and then applying Moser's Lemma:

Definition 4.5 Denote by $\mathcal{P}$ the space of symplectic forms on X which are cohomologous to $[\omega]$ and which restrict to symplectic forms, agreeing with the orientation induced by ω, on each member of the triple (F, Z_0, Z_∞).

Consider the sphere of symplectic forms $\psi^*(\omega) = \bigcup_{x \in D^n} \psi(x)^*(\omega)$. Then $\psi^*(\omega) \subset \mathcal{P}$. One can show by inflating along Z_0 that $\mathcal{P}$ is weakly contractible. The proof is similar to that in [9] except that we must do it more parameters. We give a sketch below. Thus we can find a contraction of $\psi^*(\sigma)$ to the constant sphere. Moser's Lemma then yields a family of diffeomorphisms

$$M_{\psi,t}\colon\ D^n \times I \to \mathcal{D}(M)$$

such that:

(1) $M_{\psi,1}(d)^*(\omega) = \psi_d^\star(\omega)$

(2) $M_{\psi,0}(d) = \mathrm{id}$

(3) $M_{\psi,t}(\partial D) = \mathrm{id}$

The map $M_{\psi,t}^{-1}(d)\psi(d)\colon\ D^n \times I \to \mathcal{S}^{\mathrm{soft}}$ then yields a retraction of ψ into $\mathcal{S}$ as t travels from 0 to 1.

Now we prove that $\mathcal{P}$ is weakly contractible. Let $\phi\colon\ S^n \to \mathcal{P}$ be a sphere of symplectic forms based at ω. Begin by homotoping ϕ so that it is constant in a neighborhood U_b of the basepoint b.

Since the set of closed 2–forms positive on each member of our triple is convex, ϕ admits a homotopy ϕ_t to the constant map through (possibly non-symplectic) closed 2–forms which are positive on the triple (F_Z, Z_0, Z_∞). Let U be a neighborhood of $S^n \times 0$ such that each form $\sigma \in U$ is symplectic. Let

$$\chi\colon\ S^n \times [0,1] \to [0,1]$$

be a continuous function which vanishes on $U_0 \cup U_b \times [0,1]$ and is positive elsewhere.

Denote by σ_Σ an area form on Σ. Then for some $\kappa \gg 0$ each form in $\phi_t + \kappa\chi(x,t)\pi_f^*(\sigma_\Sigma)$ is symplectic [15], and the new homotopy

$$\phi_t^1(x,t) = \phi_t(x,t) + \kappa\chi(x,t)\pi_f^*(\sigma_\Sigma) \quad x \in S^n, t \in [0,1]$$

travels through symplectic forms, positive on our triple.

However, we pay a price: the cohomology classes of the forms $\phi_t^1(x,t)$ lie on the line $[\omega] + s \cdot \kappa[F]$, where $PD(\cdot)$ denotes Poincare duality, and $s \in [0,1]$. We will now alter the homotopy ϕ_t^1 by adding a sufficient multiples of a Thom class of the section Z_0 for each value of t so that the forms in the new homotopy are Poincare dual to a multiple of $[\omega] = aPD([Z_0]) + bPD([F])$. This process is called inflation. Its earliest appearance came in the papers of McDuff on the classification of symplectic structures on ruled surfaces. The version we will use is more refined:

Lemma 4.6 (McDuff [13]) *Let M be a 4–manifold with a compact family of of symplectic structures γ, and a compact family of almost complex structures $J_\gamma\colon \Gamma \to \mathcal{J}$, which make a symplectic curve C with $C \cdot C \geq 0$ holomorphic, and such that $\gamma(x)$ tames $J_\gamma(x)$ for each $x \in \Gamma$.*

Then for each $\beta > 0$ there is a compact family of closed 2–forms $\tau_\gamma\colon \Gamma \to \Omega^2$, supported in an arbitrarily small neighborhood of C, and such that the form $\gamma(x) + \tau_\gamma(x)$ is symplectic, tames $J_\gamma(x)$ and has cohomology class $[\omega] + \beta PD([C])$.

Applying Proposition 7.4 in the Appendix we can find a family $J_\phi\colon S^n \times [0,1] \to \mathcal{J}$ of almost complex structures on M such that $J_\phi(x,t)$ is tamed by $\phi_t^1(x)$, and which make each member of our triple holomorphic. We then apply McDuff's Lemma above $C = Z_0$ and $\beta = \frac{a\kappa}{b}$, and denote the resulting family of 2–forms by $\tau_\phi(x,t)$. The set of forms taming a given almost complex structure is convex. Thus as $\phi_t^1(x)$ and $\phi_t^1(x) + \tau_\phi(x,t)$ both tame $J_\phi(x,t)$ so does $\phi_t^1(x) + s\cdot\tau_\phi(x,t)$ for $s \in [0,1]$. Thus all of these forms are positive on the triple (F_Z, Z_0, Z_∞).

Consider, then, the homotopy ϕ_t^1:

$$\phi_t^2(x) = \phi_t^1(x) + \chi(x,t)\tau_\phi(x,t)$$

Then

$$\begin{aligned}[\phi_t^2(x)] &= aPD([Z_0]) + bPD([F_Z]) + \chi(x,t)\kappa PD([F_Z]) + \chi(x,t)\frac{a\kappa}{b}PD([Z_0]) \\ &= (1 + \kappa\chi(x,t)/b)[\omega]\end{aligned}$$

Thus we have moved our homotopy to one which takes place only in classes which are multiples of $[\omega]$. One can then rescale each part of the homotopy by the appropriate constant factor to obtain a homotopy of our sphere within the original cohomology class

$$[\phi_t^3(x)] = (\phi_t^2(x) + \chi(x,t)\tau_\phi(x,t)/(1 + \kappa\chi(x,t)/b) \qquad \square$$

Remark 4.7 The proof of Proposition 4.4 does not require that (X,ω) satisfies the cohomology assumption. However our later applications will require this assumption.

4.1 $\mathcal{S}^{\mathrm{soft}}$ "acts transitively" on $\mathcal{FZ}$

We now show that the "group" $\mathcal{S}^{\mathrm{soft}}$ "acts transitively" on $\mathcal{FZ}$. We shall see that, as a result, its homotopy type is amenable to computation via J–holomorphic curves.

Lemma 4.8 *For every triple $(S_0, S_\infty, F_S) \in \mathcal{FZ}$ there is a $g \in \mathcal{S}^{\mathrm{soft}}$ such that $(S_0, S_\infty, F_S) = (g(S_0), g(S_\infty), g(F_S))$*

Proof There is a diffeomorphism taking any symplectic fibrations by 2–spheres with fiber in class $[F]$ to any other (see [13]). Call this diffeomorphism g_F.

Next, note that as Z_0 and S_0 have the same self-intersection, their normal bundles are isomorphic. Thus one can find a fiber-preserving diffeomorphism g_0 of a neighborhood of Z_0 to S_0. Then, as the diffeomorphisms of a disc which fix its boundary form a contractible set, we can extend this to diffeomorphism-preserving F_S.

$g_0 \circ g_F(F, Z_0, Z_\infty) = (F_S, S_0, g_0 \circ g_F(Z_\infty))$. We now aim to find a diffeomorphism g_∞ which preserves F_S and S_0 and carries $g_0 \circ g_F(Z_\infty)$ to S_∞. Denote the sections of F_S which miss S_0 by $\mathcal{Z}_2$. These are sections of the disc bundle $F_S - S_0$. Thus $\mathcal{Z}_2$ is contractible, and we may can find an isotopy of (possibly non-symplectic) sections from $g_0 \circ g_F(Z_\infty)$ to S_∞ lying in $\mathcal{Z}_2$. This isotopy may then be induced by a path of diffeomorphisms which preserve both F_S and S_0. Let g_∞ be the end of this path of diffeomorphisms. $g_\infty \circ g_0 \circ g_F$ is then a diffeomorphism carrying (F_Z, Z_0, Z_∞) into (F_S, S_0, S_∞). □

5 J–holomorphic curves on rational surfaces

In this subsection we supply the necessary background from the theory of J–holomorphic curves on symplectic sphere bundles over surfaces. The main geometric ingredient in our proof is the following Proposition:

Proposition 5.1 (Gromov[7], McDuff [13]) *Let (X, F, Z_0, ω) satisfy the cohomology assumption (Definition 3.1). Then for every almost complex structure J tamed by ω there is a J–holomorphic fibration by spheres in class $[F]$.*

Proof This is well known for generic almost complex structures, and if the base Σ is not a sphere. So suppose $\Sigma = S^2$, and let J_n be a sequence of generic almost complex structures approximating J.

We begin by showing that there is a section Z of the fibration F such that $[Z]\cdot[Z]=0$ or 1, and $\omega[Z]\geq\omega[F]$. Note that there are only two topological S^2–bundles over S^2. One is trivial, and the other is not. For the trivial bundle, every section has even self-intersection, and for the non-trivial bundle every section has odd self-intersection.

For the trivial bundle there is a section Z, of zero self-intersection. $[Z]=[Z_0]-\frac{k}{2}[\mathrm{F}]$, and so $\omega[Z]=k(1-\varepsilon)/2$. Thus, as $\omega(F)=(1-\varepsilon)$ and $k\geq 2$, $\omega[Z]\geq\omega[F]$. For the non-trivial bundle there is a section Z such that $[Z]\cdot[Z]=1$. Then $[Z]=[Z_0]-\frac{k-1}{2}[F]$, and $\omega(Z)=(k+1)(1-\varepsilon)/2\geq\omega(F)$, since $k\geq 1$. Let $x_1\in F(\Sigma)$. Then for each J_n there is a unique smooth irreducible curve F_n through x_1. I claim these cannot degenerate in the limit.

For, suppose these degenerate to a cusp curve B. Denote the irreducible components of B by B_i. Then $[B_i]=a_i[Z]+b_i[F]$, where a_i and b_i always have opposite signs. Since $\sum_{i\in I}a_i=0$, there must be one of each sign. Denote the component with positive a_i by B_+ and that with negative a_i by B_-.

Now choose a point $x_2\in F(\Sigma)\backslash C_1$. There cannot be an irreducible curve in class $[F]$ passing through x_2 as $[F]\cdot[B_-]<0$. Thus there must be a cusp curve C passing through it. It too splits into components C_i with the above properties, with C_+ and C_- defined analogously. But then $[C_+]\cdot[B_+]<0$, contradicting positivity of intersection. □

We will most often use it in the following form:

Lemma 5.2 *Let (X,ω) be a symplectic 4–manifold admitting a symplectic fibration by 2–spheres by F, which satisfies the cohomology assumption. Let $\{S_i\}$ be a collection of symplectic curves such that $[S_i]\cdot[F]=1$, J a tamed almost complex structure which preserves each curve. Then there is a J- holomorphic fibration by 2 spheres F_J in the class of $[F]$, such that with respect to this fibration F_J the curves $\{S_i\}$ are symplectic sections.*

Proof By Proposition 5.1 there is a unique J–holomorphic fibration F by 2 spheres in class $[F]$. As J is tamed this fibration is symplectic. By positivity of intersection each fiber must meet each curve transversely, precisely once. □

5.1 The main geometric lemma

Definition 5.3 Denote by $\mathcal{Z}$ the space of pairs (S_0,S_∞) of disjoint symplectic curves in (X,ω) such that $[S_0]=[Z_0]$ and $[S_\infty]=[Z_\infty]$.

Note that there is an inclusion $i\colon \mathcal{E}_\infty \hookrightarrow \mathcal{Z}$, given by identifying $\mathcal{E}_\infty$ with the pairs of disjoint symplectic curves (S_0, S_∞) where $S_\infty = Z_\infty$.

Proposition 5.4 (Main geometric lemma) *The forgetful map $\pi\colon \mathcal{F}\mathcal{Z} \to \mathcal{Z}$ is a fibration with contractible fiber.*

Proof The proof of this proposition will rely heavily on the results in the Appendix on almost complex structures.

We begin by showing that π is a fibration. π is surjective, for as the curves Z_0 and Z_∞ are disjoint from one another we can find a tamed almost complex structure J which makes each curve holomorphic. Lemma 5.2 then provides a fibration F so that $(F, Z_0, Z_\infty) \in \mathcal{F}\mathcal{Z}$.

I claim that π has path lifting: Let B be a polyhedron. We consider $\Phi\colon B \times I \to \mathcal{Z}$, along with a lifting $\Phi_{\text{lift}}\colon B \times 0 \to S\mathcal{F}$. We aim to extend Φ_{lift} to all of $B \times I$. Applying Proposition 7.4 we can find a $\Phi^J\colon B \times I \to \mathcal{J}$, such that $\Phi(b,t)$ is $\Phi^J(b,t)$–holomorphic, and $\Phi_{\text{lift}}(b,0)$ is $\Phi^J(b,0)$–holomorphic. Applying Lemma 5.2 we gain a family of fibrations, $\Phi_{\text{lift}}(b,t)$ extending our original lifting on $B \times 0$.

Finally we show that π has contractible fiber. Denote by $\mathcal{J}_Z$ the tamed almost complex structures which make both Z_0 and Z_∞ holomorphic. It is enough to show that the map

$$\begin{aligned} \rho\colon \mathcal{J}_Z &\to \pi^{-1}(Z_0, Z_\infty) \\ \rho(J) &= (F, Z_0, Z_\infty), \end{aligned}$$

where F is the unique J–holomorphic fibration determined by Lemma 5.2, is a fibration with contractible fiber. For then ρ will be a weak homotopy equivalence, and as $\mathcal{J}_Z$ is also contractible, so must $\pi^{-1}(Z_0, Z_\infty)$ be contractible. We commence with this task.

We first show that ρ is a fibration on its image, ie that it has path lifting: Let B be a polyhedron. Consider $\Phi\colon B \times I \to \pi^{-1}(Z_0, Z_\infty)$, along with a lifting $\Phi_{\text{lift}}\colon B \times 0 \to \mathcal{J}_{0,\infty}$ such that $\Phi(b,0)$ is $\Phi_{\text{lift}}(b,0)$–holomorphic. Then Proposition 7.4 allows us to extend Φ_{lift} to all of $B \times I$.

Let $(F, Z_0, Z_\infty) \in \pi^{-1}(Z_0, Z)$. Then $\rho^{-1}(F, Z_0, Z_\infty) = \mathcal{J}_{\text{FZ}}$, the space of almost complex structures making each member of the triple holomorphic. However, again by applying Proposition 7.4, we see that $\mathcal{J}_{\text{FZ}}$ is non-empty and contractible. Thus ρ is surjective with contractible fiber. □

6 Understanding the orbit of $\mathcal{S}^{\text{soft}}$: Proofs of Contractibility and Transitivity

We now complete the proof of Theorem 2.6 by proving Proposition 3.2:

Proposition (Problem on rational surfaces) *The space $\mathcal{S}(X)_{\overline{\infty}}$, of symplectomorphisms of X which fix a framing of Z_∞, is homotopy equivalent to the space $\mathcal{E}_\infty$ of unparametrized, embedded symplectic surfaces S of X disjoint from Z_∞. Moreover, when Σ has genus 0, $\mathcal{E}_\infty$ is contractible.*

We will show that $\mathcal{S}(X)_{\overline{\infty}}$ acts transitively on $\mathcal{E}_\infty$, and that the stabilizer $\mathcal{S}(X)_{\overline{\infty},0}$ is contractible. Then the fibration:

$$\mathcal{S}(X)_{\overline{\infty},0} \to \mathcal{S}(X)_{\overline{\infty}} \to \mathcal{E}_\infty$$

gives a homotopy equivalence between $\mathcal{S}(X)_{\overline{\infty}}$ and $\mathcal{E}_\infty$.

6.1 $\mathcal{S}_{\overline{\infty}}$ acts transitively on $\mathcal{E}_\infty$

Proof We begin by showing that $\mathcal{S}$ acts transitively on $\mathcal{Z}$. It is enough to show that there is a symplectomorphism carrying (Z_0, Z_∞) to any other pair (Z_0^1, Z_∞^1) in $\mathcal{Z}$. Let J be an almost complex structure leaving both Z_0 and Z_∞ invariant. Apply Lemma 5.2 and denote the resulting fibration by F. Then by Lemma 4.8 there is a $\alpha_1 \in \mathcal{S}^{\text{soft}}$) which carries (F, Z_0, Z_∞) into (F^1, Z_0^1, Z_∞^1). Since $\mathcal{S} \hookrightarrow \mathcal{S}^{\text{soft}}$ is a deformation retract by Proposition 4, there is an isotopy α_t through $\mathcal{S}^{\text{soft}}$ to a symplectomorphism α_0. Applying this isotopy to (Z_0, Z_∞) yields a path of pairs of curves $\alpha_t(Z_0, Z_\infty)$ which begins at $\alpha_1(Z_0, Z_\infty) = (Z_0^1, Z_\infty^1)$ and ends at $\alpha_0(Z_0, Z_\infty)$ within the orbit of (Z_0, Z_∞) under $\mathcal{S}$. One can then induce this path $\alpha_t(Z_0, Z_\infty)$ by a path of symplectomorphisms Ψ_t, constructed by an easy application of Moser's Lemma. Then $\Psi_1\alpha_0(Z_0, Z_\infty) = (Z_0^1, Z_\infty^1)$.

I claim that $\mathcal{E}_\infty \subset \mathcal{Z}$. $\mathcal{S}_{\overline{\infty}}$ are then precisely the symplectomorphisms which preserve $\mathcal{E}_\infty$ and act transitively on this space.

We must show that if Z is a symplectic curve in $\widehat{E}\backslash Z_\infty$, abstractly symplectomorphic to Z_0, then $[Z] = [Z_0]$. As $[Z_0]$ and $[F]$ span $H_2(\widehat{E})$,

$$[Z] = a[Z_0] + b[F].$$

I claim that $b = 0$. For, as Z misses Z_∞,

$$\begin{aligned} 0 &= [Z] \cdot [Z_\infty] \\ &= (a[Z_0] + b[F]) \cdot [Z_\infty] \\ &= a[Z_0] \cdot [Z_\infty] + b[F] \cdot [Z_\infty] \\ &= 0 + b. \end{aligned}$$

Moreover, as the Z and Z_0 are abstractly symplectomorphic,

$$\begin{aligned} \omega[Z] &= \omega[Z_0] \\ a\omega[Z_0] &= \omega[Z_0] \end{aligned}$$

and thus $$a = 1,$$

hence $[Z] = [Z_0]$, and $\mathcal{E}_\infty \subset \mathcal{Z}$. □

6.2 $\mathcal{S}(X)_{\overline{\infty}}$ is homotopy equivalent to $\mathcal{E}_\infty$

Denote by $\mathcal{S}_{\infty,0}$ the symplectomorphisms that preserve both Z_∞ and Z_0. Denote by $\left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0}$ the diffeomorphisms in $\mathcal{S}^{\mathrm{soft}}$ which do the same.

Proposition 6.1 *$\mathcal{S}_{\infty,0} \hookrightarrow \left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0}$ is a homotopy equivalence.*

Proof Since $\mathcal{S}$ acts transitively on $\mathcal{Z}$ (see Section 6.1), the orbit map $\phi\colon \mathcal{S} \to \mathcal{Z}$ is a fibration. Consider

$$\eta\colon \mathcal{S}^{\mathrm{soft}} \to S\mathcal{F} \to \mathcal{Z};$$

η is the composition of two fibrations. The second, $S\mathcal{F} \to \mathcal{Z}$, is a fibration by Proposition 5.4; that the first is a fibration follows immediately from the definitions. The fiber of η is $\left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0}$.

The inclusion $\mathcal{S} \hookrightarrow \mathcal{S}^{\mathrm{soft}}$ yields a morphism

$$\begin{tikzcd} \mathcal{S}_{\infty,0} \arrow[r, hook, "i_1"] \arrow[d] & \left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0} \arrow[d] \\ \mathcal{S} \arrow[r, hook, "i_2"] \arrow[d, "\phi"] & \mathcal{S}^{\mathrm{soft}} \arrow[d, "\eta"] \\ \mathcal{Z} \arrow[r, "(\mathrm{id})"] & \mathcal{Z} \end{tikzcd}$$

of fibrations: i_2 and (id) are homotopy equivalences, thus so is i_1 by the 5–Lemma (Lemma 7.7). □

6.2.1 We utilize J–holomorphic curves

Proposition 6.2 *$\left(\mathcal{S}^{soft}\right)_{\infty,0}$ is homotopy equivalent to $\mathcal{D}_{\infty,0}$, the diffeomorphisms which preserve the fibration F_Z and both sections Z_0 and Z_∞*

Proof Denote by $\mathcal{F}(Z_0,Z_\infty)$ the subset of $S\mathcal{F}$ given by triples (F,S_0,S_∞) where $S_0=Z_0$ and $S_\infty=Z_\infty$. Restricting the fibration of $\mathcal{S}^{\mathrm{soft}}\to\mathcal{F}\mathcal{Z}$ to $\mathcal{F}(Z_0,Z_\infty)$ yields a fibration

$$\mathcal{D}_{\infty,0}\to\left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0}\to\mathcal{F}(Z_0,Z_\infty).$$

As $\mathcal{F}(Z_0,Z_\infty)$ is the fiber of the forgetful fibration $\pi\colon S\mathcal{F}\to\mathcal{Z}$ it is contractible by the main geometric lemma (Proposition 5.4). $\square$

Denote the symplectomorphisms which preserve Z_0 and fix both Z_∞ and its normal bundle by $\mathcal{S}_{\overline{\infty},0}$. Denote the fiber preserving diffeomorphisms which preserve Z_0, and fix both Z_∞ and its normal bundle by $\mathcal{D}_{0,\overline{\infty}}$.

Proposition 6.3 *$\mathcal{S}_{\overline{\infty},0}$ is homotopy equivalent to $\mathcal{D}_{\overline{\infty},0}$.*

The proof is a fairly straightforward application of the previous ideas. One obtains a morphism of fibrations from the inclusions $\mathcal{D}_{0,\overline{\infty}}\hookrightarrow\left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0}$, $\mathcal{S}_{\overline{\infty},0}\hookrightarrow\left(\mathcal{S}^{\mathrm{soft}}\right)_{\infty,0}$, and the "actions" of each. (Those which are not groups still have the analogous fibrations.) Then we apply the 5–lemma and Proposition 6.2. For details see [4].

6.2.2 We use the Riemann mapping theorem in parameters

The contractibility of $\mathcal{S}_{\overline{\infty},0}$ now follows from combining the above Proposition 6.3 with the following:

Lemma 6.4 *$\mathcal{D}_{0,\overline{\infty}}$ is contractible.*

Proof As the elements in $\mathcal{D}_{0,\overline{\infty}}$ fix the section Z_∞, and preserve the fibration, they must in fact preserve each fiber of F_Z. Thus $\mathcal{D}_{0,\overline{\infty}}$ is the space of sections of a bundle over Σ whose fiber consists of the diffeomorphisms of S^2 which fix a point 0 (where Z_0 intersects the fiber) and the neighborhood of another point ∞ (where Z_∞ intersects the fiber). This fiber is thus contractible by the Riemann mapping theorem. The space of sections $\mathcal{D}_{0,\overline{\infty}}$ is thus also a contractible set. $\square$

6.3 If Σ is a sphere, $\mathcal{E}_\infty$ is contractible

Proposition 6.5 *If Σ is a sphere, $\mathcal{E}_\infty$ is contractible.*

Proof Denote by $\mathcal{J}_\infty$ the set of tamed almost complex structures on X which make Z_∞ holomorphic. Denote by $\mathcal{J}^S_\infty$ the space of pairs (J,S) where $J \in \mathcal{J}_\infty$ and $S \in \mathcal{E}_\infty$ such that S is J–holomorphic. We remind the reader that $k = [Z_0]\cdot[Z_0] = [S]\cdot[S]$.

I claim that the projections $\pi_\Sigma\colon \mathcal{J}^S_\infty \to \mathcal{E}_\infty$ and $\pi_J\colon \mathcal{J}^S_\infty \to \mathcal{J}_\infty$ are fibrations with contractible fiber, and thus homotopy equivalences. As $\mathcal{J}_\infty$ is contractible, this will show that $\mathcal{E}_\infty$ must also be contractible.

We first prove the claim for π_Σ. The fiber of π_Σ is the set of tamed complex structures which make both Z_∞ and S holomorphic. As Z_∞ and S form a disjoint pair of symplectic curves this is a contractible set.

We now prove the claim for π_J. Denote the 2 disc by D^2, and let $J \in \mathcal{J}_\infty$. Fix $k+1$ distinct points x_i on Z_∞. By Proposition 5.1 there is a unique J–holomorphic curve F_i in class $[F]$ which passes through x_i. As both S and the F_i are J–holomorphic they must intersect positively. Thus S meets each F_i in precisely one point σ_i. As S misses Z_∞, $\sigma_i \in F_i - x_i \simeq D^2$. Lemma 6.6 below shows that for any $(k{+}1)$–tuple in $\prod_{i=1\dots k+1} F_i - x_i$ there is a unique such curve S. Therefore π_J is a homotopy equivalence and $\mathcal{E}_\infty$ is contractible. □

Lemma 6.6 *Let $J \in \mathcal{J}_\infty$, and let F_i denote $k{+}1$ distinct holomorphic spheres in class $[F]$. If $\Sigma = S^2$, then for any $(k{+}1)$–tuple of points in $\prod_{i=1\dots k+1} F_i - x_i$ there is a unique, smooth J–holomorphic sphere in Z_0 which passes through them.*

Proof For a generic J the moduli space of J–holomorphic spheres through q points has dimension

$$2c_1(T(\widehat{E}))([Z_0]) - 2q - 2.$$

Now $$2c_1(T(\widehat{E_\epsilon}))([Z_0]) = 2(\chi(Z_0) + [Z_0]\cdot[Z_0]) = 4 + 2k$$

so for the dimension to be 0 we need

$$q = k + 1.$$

The Gromov–Witten invariant for this class is 1. Thus there is a J–holomorphic curve Θ through any $k+1$ points. I claim that this curve is unique. Let Θ_1

and Θ_2 be two curves through these $k+1$ points. Then these two curves must coincide by positivity of intersection as $[Z_0]\cdot[Z_0]=k$.

I claim that Θ is always smooth and irreducible. For one can always approximate J by a sequence of complex structures J_n so that the J_n holomorphic curve through these q points Θ_n is smooth. The sequence of curves Θ_n then converges to Θ, and Θ is thus controlled by Gromov compactness. We will now show that the need to

(1) intersect the curves in class $[F]$ positively (curves in class $[F]$ exist for every J tamed by ω by Proposition 5.1), and
(2) intersect Z_∞ positively ($J \in J_\infty$ and thus Z_∞ is a J_∞ holomorphic curve)

eliminate all such nodal curves, save those of the form

$$Z_\infty \cup \bigcup_{i=1}^{k} F_i$$

where the F_i are (possibly repeated) spheres in class F. However curves of this last form are eliminated as well. They have only k fiber curves F_i, they cannot pass through all $k+1$ points. For each point lies off Z_∞ and in a distinct J–holomorphic fiber. Note that this argument fails when the genus of $\Sigma > 0$. For then, q, the number of points required by the dimension formula, is less than k and there is no contradiction.

All nodal curves are of the form $Z_\infty \cup \bigcup_{i=1}^{k} F_i$ The second homology of $\widehat{E}_\epsilon$ is spanned by $[Z_0]$ and the fiber class $[F]$. The class of each irreducible component Θ_i of a curve may thus be written $a_i[Z_0]+b_i[F]$. Each $a_i>0$ as $a_i=[\Theta_i]\cdot[F]$ and each of the classes is represented by a holomorphic curve.

The union of these components lies in class $[Z_0]$, and thus

$$\sum_i (a_i[Z_0]+b_i[F]) = [Z_0].$$

As all the a_i are positive integers, the only possibility which remains is that one $a_i=1$ and the rest vanish. Moreover, for all i such that $a_i=0$, b_i must be positive, as ω evaluated on each component must be positive. Thus we have reduced ourselves to:

(1) A curve in class $[Z_0]-l[F]$ for $l\in\mathbb{Z}, l>0$.
(2) A collection of curves in class $b_i[F]$ $b_i>0$ such that $\sum_i b_i = l$.

Since there is a unique curve through each point in class $[F]$ these curves of "type 2" must be unions of fibers in F. Positivity of intersection with Z_∞ then implies that the only J–holomorphic curve in class $[Z_0] - l[F]$ $(l > 0)$ is Z_∞ itself with $l = k$. □

7 Appendix

7.1 Tamed almost complex structures preserving sub-bundles

In this subsection we collect the results we require about tamed almost complex structures preserving sub-bundles. They are listed below in order of their difficulty. The first two are classical, the last less so, and we provide a proof of it here.

Definition 7.1 If $\pi\colon V \to B$ is a symplectic vector bundle, let $\pi_J\colon \mathcal{J}(V) \to B$ be the bundle such that $\pi_J^{-1}(b)$ are the tamed almost complex structures on $\pi^{-1}(b)$. If $\eta_i \subset V$ are symplectic sub bundles, let $\pi_J\colon \mathcal{J}(V, \eta_1, \eta_2, \ldots) \to B$ be the (possibly locally non-trivial) bundle where $\pi_J^{-1}(b)$ are the tamed almost complex structures on $\pi^{-1}(b)$, which preserve each η_i.

The first result goes back at least to Gromov's seminal paper [7].

Lemma 7.2 *Let $(V, \omega) \to B$ be a symplectic vector bundle over a polyhedron B. Then $\rho\colon \mathcal{J}(V) \to B$ is a bundle with contractible fibers.*

Next we consider almost complex structures preserving a given plane field. This is also a classical result:

Lemma 7.3 *Let $(V, \omega) \to B$ be a 4–dimensional symplectic vector bundle over a polyhedron B. Let $\vartheta \subset V$ be a 2–dimensional symplectic sub-bundle of V. Then $\rho\colon \mathcal{J}(V, \vartheta) \to B$ is a bundle with contractible fibers.*

Let $Q \subset B$ be a sub-polyhedron of B. Then given a section ϕ_Q of $\rho\colon \mathcal{J}(V, \vartheta) \to Q$, we may construct a section ϕ of $\rho\colon \mathcal{J}(V, \vartheta) \to B$ extending ϕ_Q.

Finally we will need to consider the tamed almost complex structures preserving two transverse plane fields. Preserving two planes requires a good deal more work than preserving one, as pairs of symplectic planes have moduli. This is less well known and we include a proof of it here.

Proposition 7.4 *Let $(V,\omega)\to B$ be a 4–dimensional symplectic vector bundle over a polyhedron B. Let $\vartheta_1,\vartheta_2\subset V$ be 2–dimensional symplectic sub-bundles of V, such that ϑ_1,ϑ_2 intersect transversely in each fiber, and the the symplectic orthogonal projection $\pi_{12}^{\perp}\colon\ \vartheta_1\to\vartheta_2$ is orientation-preserving. Let $Q\subset B$ be a sub-polyhedron, and let ϕ_Q be a section of $\rho\colon\ \mathcal{J}(V,\vartheta_1,\vartheta_2)\to B$, defined over Q. Then there is a section ϕ of ρ which extends ϕ_Q.*

Proof Constructing ϕ is equivalent to constructing a section of $\phi^1\oplus\phi^2$ of $J(\vartheta_1)\oplus J(\vartheta_2)$ such that the resulting almost complex structure is tamed by ω. Denote by ϕ_Q^i the sections such that

$$\phi_Q=\phi_Q^1\oplus\phi_Q^2.$$

Constructing ϕ_1 alone is fairly simple, for by Lemma 7.3,

$$J(\vartheta_1)\to B$$

is a bundle with contractible fibers. Thus this bundle admits a section ϕ^1 extending ϕ_Q^1. We now proceed with the problem of constructing ϕ^2 extending ϕ_Q^2.

Our main tool in will be the following lemma in linear algebra, which provides the almost complex structures satisfying our conditions with a convex structure. This will allow to use partitions of unity to construct ϕ^2.

Lemma 7.5 *Let V,P be two symplectic planes in R^4 with symplectic structure ω. Let $\pi_\perp\colon\ P\to V^\perp$ denote the symplectic orthogonal projection. Suppose that $\pi_\perp$ is orientation-preserving. Fix an $\omega-$tame complex structure J_P on P, and $v\in V$, with $v\neq 0$. Denote the space of almost complex structures J_V on V such that $J=(J_P\oplus J_V)$ is ω–tame by $\mathcal{J}_V$.*

Then $\Phi_v\colon\ \mathcal{J}_V\to V$, given by $J_V\to J_V(v)$, gives a homeomorphism of $\mathcal{J}_V$ onto a convex set.

The proof of this lemma is involved, and we defer it to the end of this subsection. For now we concentrate on its application to our argument. It has the following immediate consequence:

Lemma 7.6 *Let s be a section of ϑ_2, non-vanishing over a set $X_s\subset B$. Then there is a section $\phi^2\in\mathcal{J}(\vartheta_2)$ such that ϕ^2 extends ϕ_Q^2 and $\phi^1\oplus\phi^2$ is tamed by ω over X_s.*

Proof As s is non-vanishing, α is determined by its action on s. Lemma 7.5 tells us that the set of allowable choices for $\alpha(s)$ form an open, convex set. As X_s is paracompact, so we can use a partition of unity to construct a section s_α of ϑ_2 over X_s, so that for

$$\phi^2 \colon s \longrightarrow s_\alpha$$
$$\phi^2 \colon s_\alpha \longrightarrow -s$$

the almost complex structure $\phi^1 \oplus \phi^2$ is tamed by ω. □

Proposition 7.4 then follows by applying a partition of unity to a covering X_{s_i} coming from a finite set of sections $\{s_i\}$ of $\Phi_Z^{1*}(\eta)$ such that $\bigcup_i X_{s_i} = \Sigma \times P$. □

7.1.1 Proof of the linear algebra lemma

In this subsection we provide the proof of the promised linear algebra lemma 7.5

Lemma (Linear algebra lemma) *Let V, P be two symplectic planes in R^4 with symplectic structure ω. Let $\pi_\perp \colon P \to V^\perp$ denote the symplectic orthogonal projection. Suppose that $\pi_\perp$ is orientation-preserving. Fix an $\omega-$tame complex structure J_P on P. Denote the space of almost complex structures J_V on V such that $J = (J_P \oplus J_V)$ is ω–tame by $\mathcal{J}_V$.*

Then for each non-zero $v \in V$, $\Phi_v \colon \mathcal{J}_V \to V$ given by $J_V \to J_V(v)$ gives a homeomorphism of $\mathcal{J}_V$ onto a convex set.

Proof We begin by establishing some useful coordinates. Let $\pi \colon P \to V^\perp$ denote the symplectic orthogonal projection to V. We choose a $v_1 \in \mathrm{im}(\pi)$, and denote $\pi^{-1}(v_1)$ by p. Write

$$p = w_1 + v_1$$

where $w_1 \in V^\perp$. Write

$$J_P(p) = w_2 + v_2$$

where $w_2 \in V^\perp, v_2 \in V$. Then

$$J(w_1) = w_2 + v_2 - J_V(v_1).$$

Applying J to both sides of this equation we compute Jw_2:

$$J(w_2) = -w_1 - J_V(s_2) - s_1$$

Throughout this lemma we will suppress ω and just denote the pairing of two vectors p and q by (p,q).

π is orientation-preserving. Thus, as $(w_1, J(w_2)) > 0$, so is $(w_1, w_2) > 0$. To lessen our burden of constants, scale v_1, thus scaling $p = \pi^{-1}(v_1)$, so that

$$(w_1, w_2) = 1$$

This scaling in turn dilates the image of Φ_{v_1}, and thus does not affect its convexity.

As P is symplectic $(w_1, w_2) + (v_1, v_2) > 0$ and thus

$$(v_1, v_2) > -1. \tag{1}$$

Reduction to Cauchy–Schwartz We now commence in earnest. Let $w + v \in V^{\perp} \oplus V = R^4$. What must we require of J_V so that $(w+v, J(w+v)) > 0$ for all such pairs w and v?

$$(w+v, J(w+v)) \;=\; (w, Jw) + (w, Jv) + (v, Jw) + (v, Jv)$$

$(w, Jv) = 0$ as J must preserve V. And we have

$$(w, Jw) + (v, Jv) + (v, Jw).$$

The first two terms are positive. Also $(v, Jw) = (v, q)$ where q is the projection of Jw to V; this term may well be negative. We seek to bound its absolute value in terms of the other two (positive) terms.

We replace v by $-Jv$ throughout the equation. As we seek a bound for all pairs w, v this has no effect on our task. Moreover as $(v, Jv) = (-JJv, -Jv)$ this has no effect on the third term. The second term (v, q) becomes $(-J_V v, q) = (v, J_V q)$.

We seek to show that

$$|(v, J_V q)| \leq (w, Jw) + (v, Jv).$$

As J_V has determinate 1, it preserves $\omega|_V$, thus $(\cdot, J_V \cdot)$ is a (symmetric) inner product on V. Cauchy–Schwartz then implies that

$$|(v, J_V q)| \leq (v, J_V q)^{\frac{1}{2}} (q, J_V q)^{\frac{1}{2}}.$$

If $v = kq$, this bound is achieved. Thus the tamed J_V are precisely those such that

$$(v, J_V v)^{\frac{1}{2}} (q, J_V q)^{\frac{1}{2}} < (w, Jw) + (v, Jv). \tag{2}$$

Understanding the constraint imposed by Cauchy–Schwartz
We now unpack this inequality. Write w as $aw_1 + bw_2$. Then

$$\begin{aligned}(w, Jw) &= (aw_1 + bw_2, J(aw_1 + bw_2)) \\ &= (aw_1 + bw_2, a(w_2 + v_2 - J_V v_1) + b(-w_1 - J_V v_2 - v_1)) \\ &= a^2(w_1, w_2) - b^2(w_2, w_1) \\ &= a^2 + b^2\end{aligned}$$

and

$$(q, J_V q) = (av_2 - aJ_V v_1 - bJ_V v_2 - bv_1, aJ_V s_2 + av_1 + bv_2 - bJ_V v_1).$$

Expanding the right hand side creates sixteen pairings, however some of them are zero, and the four "ab" terms all cancel. Upon summing we are left with

$$(q, J_V q) \quad = \quad \lambda(v_1, J_V s_1) + \lambda(v_2, J_V v_2) - 2\lambda(v_1, v_2)$$

where we denote $(a^2 + b^2)$ by λ.

Our inequality (2) then reads

$$(v, J_V v)^{\frac{1}{2}}(\lambda(v_1, J_V v_1) + \lambda(v_2, J_V v_2) - 2\lambda(v_1, v_2))^{\frac{1}{2}} < \lambda + (v, J_V v).$$

If $v = 0$ the inequality places no restriction on J_V. Thus we may assume that v is not zero. Since the condition $(w+v, J(w+v)) > 0$ is invariant under scaling by a positive constant, we may scale the vector $v+w$ so that $(v, J_V v) = 1$, if we assume that J_V tames ω on V. We do so, and are left with one free parameter $\lambda > 0$.

$$(\lambda(v_1, J_V v_1) + \lambda(v_2, J_V v_2) - 2\lambda(v_1, v_2))^{\frac{1}{2}} < \lambda + 1.$$

Squaring both sides yields

$$\lambda|(v_1, J_V v_1) + (v_2, J_V v_2) - 2(v_1, v_2)| < \lambda^2 + 2\lambda + 1.$$

This may be achieved for all λ if and only if

$$|(v_1, J_V v_1) + (v_2, J_V v_2) - 2(v_1, v_2)| < 4 \tag{3}$$

At this point our proof bifurcates into three cases.

Case 1: $\mathrm{rk}(\pi) = 2$ We may assume that $v = v_1$, and write $J_V v_1 = cv_1 + dv_3$. We now describe the constraints that (3) places on c and d.

$$J_V v_2 = -\frac{c^2+1}{d}v_1 - cv_2$$

Substituting into (3) we get

$$|(d + (c^2+1)/d - 2(v_1, v_2)| < 4.$$

J_V's tameness restricted to V [1] translates to d having the same sign as (s_1, s_2). Thus $(d + (c^2+1)/d)$ and $-2(s_1, s_2)$ have opposite sign, and our inequality is equivalent to

$$|(d + (c^2+1)/d)| < 4 + 2(v_1, v_2).$$

If we denote $4 + 2(s_1, s_2)$ by γ, the set of solutions of this inequality form a disc centered at

$$(c, d) = (0, \gamma/2)$$

with radius $(\gamma^2/4 - 1)^{\frac{1}{2}}$. Since $(v_1, v_2) > -1$ by (1), $\gamma > 2$ and this disc is non-empty.

Case 2: $\mathrm{rk}(\pi) = 1$ In this case v_1 and v_2 are linearly dependent. So choose a $v_3 \in V$ which is independent of V_1, such that $(v_1, v_3) = 1$. Moreover if $v \neq v_1$ we choose v_3 so that $v = kv_3$. Then

$$J_V v_1 = cv_1 + dv_3$$
$$J_V v_3 = -\frac{c^2+1}{d} v_1 - cv_3.$$

As $v_2 = kv_1$, the constraints that (3) places on c and d are much weaker. Substituting into (3) we get

$$|c| < \frac{4}{1+k^2}$$

and no condition on d. Furthermore, v is either v_1 or v_3, and in both cases the resulting sets of vectors $J_V v_1$ (when $v = v_1$) and $J_V v_3$ (when $v = v_3$) form the band $cv_1 + dv_3$ where

$$|c| < \frac{4}{1+k^2}.$$

Thus they form a convex set.

Case 3: $\mathrm{rk}(\pi) = 0$ This case actually requires no proof at all, for here the two planes are orthogonal. Thus for any J_V tamed by $\omega|_V$, $J_V \oplus J_P$ is tamed by ω, and $\mathrm{im}(\Phi_v)$ is a half plane. □

7.2 A non-traditional 5–Lemma

The 5–Lemma is usually presented in the context of chain complexes, and as such it is usually stated as a Lemma about Abelian groups. However its usual proof actually applies in much more generality. As we will require it, we present the more general statement here. The proof is the standard one which the reader can find in [14].

[1] We assumed this when we scaled v so that $(v, Jv) = 1$.

Lemma 7.7 *Let*

$$\begin{array}{ccccccccc}
G_5 & \xrightarrow{\alpha_5} & G_4 & \xrightarrow{\alpha_4} & G_3 & \xrightarrow{\alpha_3} & G_2 & \xrightarrow{\alpha_2} & G_1 \\
{\scriptstyle\gamma_5}\downarrow & & {\scriptstyle\gamma_4}\downarrow & & {\scriptstyle\gamma_3}\downarrow & & {\scriptstyle\gamma_2}\downarrow & & {\scriptstyle\gamma_1}\downarrow \\
H_5 & \xrightarrow{\beta_5} & H_4 & \xrightarrow{\beta_4} & H_3 & \xrightarrow{\beta_3} & H_2 & \xrightarrow{\beta_2} & H_1
\end{array}$$

be a diagram of pointed sets, with each row exact. Suppose that G_3 and G_2 are groups, γ_i makes H_i a G_i–set, and the morphisms α_3 and β_3 respect this structure. Suppose further that γ_1, γ_2, γ_4 and γ_5 are bijections. Then γ_3 is a bijection.

We will most often use the above generalized 5–lemma in the following form, gained by applying it to the long exact sequence of homotopy groups of a fibration. One needs the extra generality to deal with morphisms on π_0 and π_1

Lemma 7.8 *Let*

$$\begin{array}{ccc}
G_0 & \xrightarrow{\phi_0} & F_i \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{\phi_1} & F_1 \\
\downarrow & & \downarrow \\
G_2 & \xrightarrow{\phi_2} & F_2
\end{array}$$

be a morphism of fibrations such that:

(1) *G_i is a group.*

(2) *H_i is a G_i–set.*

(3) *The maps ϕ_1 and ϕ_2 are weak homotopy equivalences.*

Then ϕ_0 is also a homotopy equivalence.

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