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Classical and quantum dilogarithmic invariants of flat $PSL(2,\mathbb{C})$–bundles over 3–manifolds

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Abstract

We introduce a family of *matrix dilogarithms*, which are automorphisms of $\mathbb{C}^N \otimes \mathbb{C}^N$, N being any odd positive integer, associated to hyperbolic ideal tetrahedra equipped with an additional decoration. The matrix dilogarithms satisfy fundamental *five-term identities* that correspond to decorated versions of the $2 \to 3$ move on 3–dimensional triangulations. Together with the decoration, they arise from the solution we give of a *symmetrization problem* for a specific family of *basic* matrix dilogarithms, the classical $(N=1)$ one being the Rogers dilogarithm, which only satisfy one special instance of five-term identity. We use the matrix dilogarithms to construct invariant state sums for closed oriented 3–manifolds W endowed with a flat principal $PSL(2,\mathbb{C})$–bundle ρ, and a fixed non empty link L if $N>1$, and for (possibly "marked") cusped hyperbolic 3–manifolds M. When $N=1$ the state sums recover known simplicial formulas for the volume and the Chern–Simons invariant. When $N>1$, the invariants for M are new; those for triples (W,L,ρ) coincide with the quantum hyperbolic invariants defined in [3], though our present approach clarifies substantially their nature. We analyse the structural coincidences versus discrepancies between the cases $N=1$ and $N>1$, and we formulate "Volume Conjectures", having geometric motivations, about the asymptotic behaviour of the invariants when $N \to \infty$.

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1 Introduction

Since its beginning in the eighties, the theory of quantum invariants of links and 3–manifolds has rapidly grown up as a very active domain of research with a large interaction between quite seemingly independent branchs of mathematics and ideas from Quantum Field Theories (QFT) in Physics. They are now organized in a well-structured machinery based on the theory of representations of quantum groups and, more generally, of linear monoidal categories, which is recognized as a very powerful tool for producing 'exact' 3–dimensional QFT (ie, functors from categories of manifold cobordisms towards such linear categories). For reviews, see eg [34, 37]. These exact theories also provide a new predictive power and meaningful framework for the physics ideas they were inspired from.

Nevertheless, in spite of its success and very aesthetic formalism, 3–dimensional 'Quantum Topology' followed until recently a rather divergent path with respect to more classical topological and geometric themes, which were mostly developed during the last decades into Thurston's geometrization program. A conceptual breakthrough was done by Kashaev with his Volume Conjecture [22]. He derived from a family $\{K_N\}$, $N > 1$ being any odd integer number, of conjectural complex valued topological invariants of links in arbitrary closed oriented 3–manifolds, a well-defined family $\{\langle L\rangle_N\}$ of invariants of links L in S^3 [20], later identified by Murakami–Murakami as the values of specific coloured Jones polynomials at the roots of unity $\exp(2i\pi/N)$ [27]. He predicted also that if L is a hyperbolic link, then the asymptotic behaviour of $\langle L\rangle_N$ when $N \to \infty$ recovers the volume of the complement of L. The main motivation for this conjecture is that the asymptotic behaviour of the elementary building blocks of K_N essentially involve classical dilogarithm functions, which are known to be related to the computation of the volume of hyperbolic polyhedra.

In our previous paper [3] we constructed *quantum hyperbolic invariants* (QHI) $H_N(W,L,\rho)$, well-defined possibly up to a sign and an Nth root of unity phase factor. Here $N > 1$ is an odd integer, L is a non empty link in a closed oriented 3–manifold W, and ρ is a flat principal $PSL(2,\mathbb{C})$–bundle over W. These invariants eventually incorporate as a particular case the Kashaev's conjectural ones, by using the trivial flat bundle. A main ingredient of our construction was the use of so called *decorated $\mathcal{I}$–triangulations*, which are particular structured families of oriented hyperbolic ideal tetrahedra with ordered vertices, encoded by their triples of cross-ratio moduli, and equipped with some additional decoration. Each QHI $H_N(W,L,\rho)$ can be expressed as a state sum, ie, the total contraction of a pattern of special automorphisms of $\mathbb{C}^N \otimes \mathbb{C}^N$ associated to the tetrahedra of any such a decorated $\mathcal{I}$–triangulation. However, our understanding of this remarkable family of tensors, in particular of the nature of the decoration of the tetrahedra entering their definition, was not satisfactory (see Remark 5.9 for further comments on this point). As a consequence, also the nature of the QHI remained somewhat obscure.

The first aim of the present paper is to unfold and clarify the structure of these tensors, called here 'quantum' *matrix dilogarithms*. We formalize them and we show their fundamental properties. They are explicitely given automorphisms of $\mathbb{C}^N \otimes \mathbb{C}^N$, $N > 1$ being any odd positive integer, associated to decorated $\mathcal{I}$–tetrahedra. Their main structural property consists in satisfying fundamental *five-term identities*: for every instance, called a *transit*, of an $\mathcal{I}$–decorated version of the $2 \to 3$ bistellar (sometimes called Pachner, or Matveev–Piergallini) local move on 3–dimensional triangulations, the contractions of the two patterns of associated matrix dilogarithms eventually lead to the same tensor up to a determined phase ambiguity. The matrix dilogarithms, as well as the additional decoration on the associated $\mathcal{I}$–tetrahedra, arise from the solution of a *symmetrization problem* for a specific family of *basic* matrix dilogarithms. These are derived from the $6j$–symbols for the cyclic representation theory of a Borel quantum subalgebra $\mathcal{B}_\zeta$ of $U_\zeta(sl(2,\mathbb{C}))$, where $\zeta = \exp(2i\pi/N)$. They satisfy only one particular instance of five-term identity, the *Schaeffer's identity*, with some geometric constraints on the involved cross-ratio moduli. The basic matrix dilogarithms can be considered as natural non-commutative analogues of the classical Rogers dilogarithm. To stress this point, our analysis of the symmetrization problem runs parallel to that for the *classical* case ($N = 1$), where we take the exponential of the classical Rogers dilogarithm as basic dilogarithm. The symmetrization problem makes the technical core of what we call the *semi-local* part of the paper.

Later we face the *global* problems that arise in constructing classical and quantum dilogarithmic invariants based on globally decorated $\mathcal{I}$–triangulations of triples (W, L, ρ) or oriented non compact complete hyperbolic 3–manifolds M of finite volume (for short: *cusped* manifolds).

For triples (W, L, ρ), in the classical case the link is actually immaterial; the dilogarithmic invariant only depends on the pair (W, ρ) and recovers the volume and the Chern–Simons invariant of ρ. On the other hand, in the quantum case it is necessary to incorporate a non empty (arbitrary) "link fixing" in the whole construction, and the invariants are sensitive to the link. They coincide with the QHI, but the present semi-local analysis substantially clarifies their nature.

For cusped manifolds M, in the classical case the dilogarithmic invariant recovers known simplicial formulas for the volume $\mathrm{Vol}(M)$ and the Chern–Simons invariant $\mathrm{CS}(M)$ [33, 28, 32] (see also the recent paper [31]). In the quantum case, the invariants $H_N(M)$ are new. Their construction is clean for the wide class of so called "weakly-gentle" cusped manifolds (see Definition 6.2); for general cusped manifolds it is more tricky. For weakly-gentle cusped manifolds M, we recognize a strong structural coincidence between the classical and the quantum invariants. Both are defined on the same geometric objects, whereas to construct the quantum invariants for general cusped manifolds we have to incorporate systems of arcs that play the role of the link in the closed manifold case. (Note, however, that it is reasonable to ask whether every cusped manifold is weakly-gentle.) This leads us to formulate a version of the Volume Conjecture for weakly-gentle cusped manifolds, relating the asymptotic behaviour of $H_N(M)$ when $N \to \infty$ to $\mathrm{CS}(M)+i\mathrm{Vol}(M)$. Other forms of the Volume Conjecture (for example related to Thurston's hyperbolic Dehn filling theorem) having substantial geometric motivations are also proposed.

By using the matrix dilogarithms as fundamental ingredients, we have developed in [4] a family of exact finite dimensional *quantum hyperbolic field theories* (QHFT). The QHFT are representations in the tensorial category of complex linear spaces of a suitable 2+1 bordism category, based on arbitrary compact oriented 3–manifolds equipped with properly embedded tangles and with flat principal $PSL(2,\mathbb{C})$–bundles having arbitrary holonomy at the meridians of the tangle components. The QHFT incorporate the present dilogarithmic invariants as instances of partition functions.

There is a wide literature about the classical Rogers dilogarithm and the computation of the volume and the Chern–Simons invariant of 3–manifolds equipped with flat $PSL(2,\mathbb{C})$–bundles. In particular, W Neumann's work [28, 29, 30] on this subject has been a fundamental reference and a source of inspiration for us.

Acknowledgement We thank the referee for his remarks and suggestions, that considerably improved the exposition of the paper.

1.1 Description of the paper

In Section 2 we provide the complete statements of our main results; in order to do it, we introduce the necessary apparatus of notions and definitions. This is rather complicated indeed, as it reflects the highly non trivial structure of the matrix dilogarithms and the dilogarithmic invariants. This section is also intended as a sort of self-contained account, without proofs, of the content of the paper. For a deeper understanding, the reader is addressed to the subsequent more technical sections.

In Sections 3, we introduce the *basic* matrix dilogarithms $\mathcal{L}_N$ for every odd integer $N \geq 1$. The necessary quantum algebraic background, in particular the derivation of $\mathcal{L}_N$, $N > 1$, from the representation theory of the Borel quantum algebra $\mathcal{B}_\zeta$, shall be recalled in the Appendix. We formulate the *symmetrization problem*, which roughly asks to modify the basic dilogarithms so as to make them 'transit invariant', satisfying the whole set of five-term relations. In Section 4 and Section 5 we derive the essentially unique solution of this problem, and this leads to the final matrix dilogarithms $\mathcal{R}_N$, with their complicated additional decoration. We show also (Lemma 5.8) that the $\mathcal{R}_N$ coincide with the symmetrized quantum dilogarithms used in [3], which implies that the quantum dilogarithmic invariants of triples (W,L,ρ) considered in Section 6 coincide with the QHI. One aim of Section 4 is to provide, in the simpler case $N=1$, a model for the contructions we need in the quantum case.

In Section 6 we construct and discuss the classical and quantum dilogarithmic invariants for triples (W,L,ρ) and cusped manifolds M.

In Section 7 we construct further invariants called *scissors congruence classes*. The terminology intentionally refers to the background of the 3rd Hilbert problem (see [14, 16] and [29]). The analysis of the relationship with the dilogarithmic invariants is useful to settle out further discrepancies between the classical and quantum cases and to formulate reasonable intermediate questions towards the Volume Conjectures, that we recall at the end of the paper.

2 Statements of the main results

In this section we give the complete statements of the main results, providing the necessary concepts and definitions. First we treat the *semi-local* theory of matrix dilogarithms. Next we consider the construction of invariant dilogarithmic state sums based on *globally* decorated $\mathcal{I}$–triangulations of 3–manifolds.

2.1 Matrix dilogarithms and transit invariance

2.1.1 Flat-charged $\mathcal{I}$–tetrahedra

On the geometric/combinatorial side, the basic building blocks of our constructions are the so called *flat/charged $\mathcal{I}$–tetrahedra* that we are going to define.

An $\mathcal{I}$*–tetrahedron* (Δ, b, w) (see also [3]) consists of

(1) An *oriented* tetrahedron Δ, that we usually represent as positively embedded in $\mathbb{R}^3$ (oriented by its standard basis).

(2) A *branching* b on Δ, that is a choice of edge orientation associated to a total ordering v_0, v_1, v_2, v_3 of the vertices by the rule: each edge is oriented by the arrow emanating from the smallest endpoint. Denote by $E(\Delta)$ the set of b–oriented edges of Δ, and by e' the edge opposite to e. We put $e_0 = [v_0, v_1]$, $e_1 = [v_1, v_2]$ and $e_2 = [v_0, v_2] = -[v_2, v_0]$. These are the edges of the face opposite to the vertex v_3.

(3) A *modular triple*, $w = (w_0, w_1, w_2) = (w(e_0), w(e_1), w(e_2)) \in (\mathbb{C} \setminus \{0,\ 1\})^3$ such that (indices mod$(\mathbb{Z}/3\mathbb{Z})$):

$$w_{j+1} = 1/(1 - w_j)$$

hence

$$w_0 w_1 w_2 = -1\ .$$

This gives a *cross-ratio modulus* $w(e)$ to each edge e of Δ, by imposing that $w(e) = w(e')$ for each edge e.

We say that w is non degenerate if the imaginary parts of the w_j are not equal to zero; in such a case these imaginary parts share the same sign $*_w = \pm 1$.

Complements on $\mathcal{I}$**–tetrahedra** The ordered triple of edges

$$(e_0 = [v_0, v_1], e_2 = [v_0, v_2], e'_1 = [v_0, v_3])$$

departing from v_0 defines a b*–orientation* of Δ. This orientation may or may not agree with the given orientation of Δ. In the first case we say that b is of index $*_b = 1$, and it is of index $*_b = -1$ otherwise.

The 2–faces of Δ can be named and ordered by their opposite vertices. For each $j = 0, \dots, 3$ there are exactly j b–oriented edges incoming at the vertex v_j; hence there are only one source and one sink of the branching. For any 2–face f of Δ the boundary of f is not coherently oriented, only two edges of f have a compatible *prevailing* orientation. In fact, each 2–face f has two orientations; one is the boundary orientation induced by the orientation of Δ, via the convention "last the ingoing normal"; on the other hand, there is the b*–orientation*, that is the orientation of f which induces on ∂f the prevailing orientation among the three b–oriented edges. Remark that the boundary and b–orientations coincide on exactly two 2–faces of Δ.

Consider the half space model of the hyperbolic space $\mathbb{H}^3$. We orient it as an open set of $\mathbb{R}^3$. The natural boundary $\partial\bar{\mathbb{H}}^3 = \mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$ of $\mathbb{H}^3$ is oriented by its complex structure. We realize $PSL(2,\mathbb{C})$ as the group of orientation preserving (ie 'direct') isometries of $\mathbb{H}^3$, with the corresponding conformal action on $\mathbb{CP}^1$. Up to direct isometry, an $\mathcal{I}$–tetrahedron (Δ,b,w) can be realized as an hyperbolic ideal tetrahedron with 4 distinct b–ordered vertices u_0,u_1,u_2,u_3 on $\partial\bar{\mathbb{H}}^3$, in such a way that

$$w_0 = (u_2-u_1)(u_3-u_0)/(u_2-u_0)(u_3-u_1).$$

These 4 points span a 'flat' (2–dimensional) tetrahedron exactly when the modular triple is degenerate (real). When it is non-degenerate, we get a positive embedding of Δ, with its own orientation, onto the corresponding hyperbolic ideal tetrahedron in $\mathbb{H}^3$ iff $*_b *_w = 1$.

Flattenings and integral charges Given any $\mathcal{I}$–tetrahedron (Δ,b,w), we consider an additional decoration made by two $\mathbb{Z}$–valued functions defined on the edges of Δ, called *flattening* and *integral charge* respectively. These functions share the property that *opposite edges take the same value.* Hence it is enough to specify them on the edges e_0,e_1,e_2.

Before we do it, we fix once for ever our favourite standard branch log of the logarithm, which has the imaginary part in $]-\pi,\pi]$. We stress that this log is defined on $\mathbb{C}\setminus\{0\}$, although it is not continuous at the negative real half-line.

Let (Δ,b,w) be an $\mathcal{I}$–tetrahedron, and $f=(f_0,f_1,f_2)$, with $f_i=f(e_i)\in\mathbb{Z}$. Set

$$\mathrm{l}_j = \mathrm{l}_j(b,w,f) = \log(w_j) + i\pi f_j$$

for $j=1,\ 2,\ 3$. We say that (f_0,f_1,f_2) is a *flattening* of (Δ,b,w), and that (Δ,b,w,f) is a *flattened* $\mathcal{I}$–tetrahedron if

$$\mathrm{l}_0+\mathrm{l}_1+\mathrm{l}_2 = 0.$$

We call l_j a *log-branch* of (Δ,b,w) for the edge e_j, and set $\mathrm{l}=(\mathrm{l}_0,\mathrm{l}_1,\mathrm{l}_2)$ for the total log-branch associated to f.

An *integral charge* on a branched tetrahedron (Δ,b) is a function $c=(c_0,c_1,c_2)$, $c_i=c(e_i)\in\mathbb{Z}$, such that $c_0+c_1+c_2=1$. We call the values of c the *charges* of the edges. A flattened $\mathcal{I}$–tetrahedron endowed with an integral charge is said *flat/charged.*

2.1.2 Matrix dilogarithms

Here we define the *matrix dilogarithms* that are associated to the flat/charged $\mathcal{I}$–tetrahedra. First we describe how any function

$$A\colon \mathbb{C}\setminus\{0,1\} \to \mathrm{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

can be interpreted as a function of $\mathcal{I}$–tetrahedra. Later we will give the explicit formulas for the matrix dilogarithms.

We equip $\mathbb{C}^N \otimes \mathbb{C}^N$ with the tensor product of the standard basis of $\mathbb{C}^N$, so that $A = A(x) \in \mathrm{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is given by its matrix elements $A^{\delta,\gamma}_{\beta,\alpha}$, where $\alpha,\ldots,\delta \in \{0,\ldots,N-1\}$. We denote by $\bar{A} = \bar{A}(x)$ the inverse of $A(x)$, with entries $\bar{A}^{\beta,\alpha}_{\delta,\gamma}$.

Take an $\mathcal{I}$–tetrahedron (Δ,b,w). At first, we use the branching to select one cross-ratio modulus, say $x = w_0$. Then we use again the branching in order to establish a one–one correspondence between the 2–faces of Δ and the indices $\alpha,\ldots,\delta$, and write

$$A(\Delta,b,w) := A(w_0)^{*_b}. \tag{1}$$

The idea (see Section 2.1.4) is that when $\mathcal{I}$–tetrahedra are glued along faces, one should be able to form a new tensor by contracting indices corresponding to paired faces. The slots for indices of the resulting tensor are in one–one correspondence with the free faces of the resulting complex. We define the correspondence as follows. Assume that $*_b = 1$. As usual, we name and order the 2–faces by the opposite vertices. So, the ordered faces F_1, F_3 are such that the boundary and b–orientations coincide on them. Set the correspondence $(F_1,F_3) \leftrightarrows (\alpha,\beta)$. Similarly, set $(F_0,F_2) \leftrightarrows (\gamma,\delta)$, where F_0, F_2 are the ordered faces on which the two orientations do not agree. We do the same when $*_b = -1$, but in this case the two orientations agree on F_0 and F_2.

It is very convenient to adopt a pictorial description of this correspondence between automorphisms and $\mathcal{I}$–tetrahedra. First, the tensors $A(x)$ and $\bar{A}(x)$ may be given the graphical encoding shown in Figure 1.

The two figures are normal crossings with an under/over crossing specification and arc orientations. They are decorated with the complex parameter x and integers $\alpha,\ldots,\delta$; we have omitted to draw the arrows on two of the arcs of each crossing, because we stipulate that they are incoming at the central round box. Each figure represents a matrix element; forgetting α,β,γ and δ, we represent the entire automorphisms. We stress that they are *planar* pictures, realized in $\mathbb{R}^2 \cong \mathbb{C}$ with the canonical complex orientation that is used to specify the

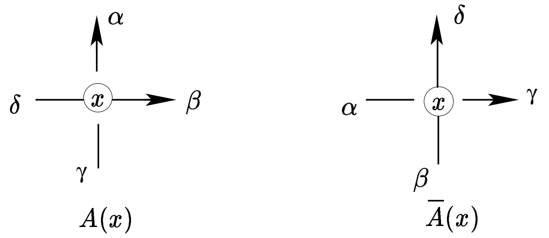

Figure 1: Graphic tensors

index position. Finally we take $\mathcal{I}$–tetrahedra with $w_0 = x$ and $*_b = \pm 1$, and we realize the above graphical encoding of the automorphisms as an *enriched* version of the 1–skeleton of the canonical cell decomposition of $\mathrm{int}(\Delta)$, which is dual to the natural triangulation of Δ (so that each arc of the graph is dual to a determined 2–face of Δ). This is shown in Figure 2. Note that the embedding in Δ of this enriched 1–skeleton is determined by the branching, and, viceversa, the branching contains all the information in order to reconstruct completely (Δ, b, w) itself (this is related to the encoding of branched spines of 3–manifolds via so called normal o–graphs, see [6]).

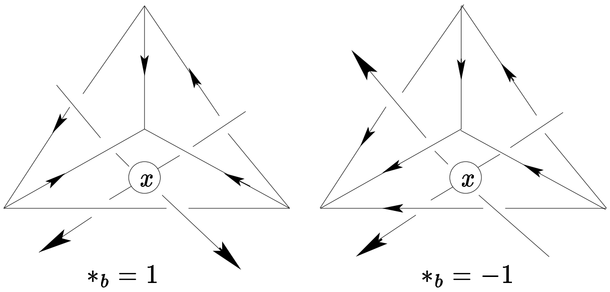

Figure 2: $A(x) = A(\Delta, b, w)$, with $x = w_0$

We can give now the explicit formulas for (the matrix elements of) our matrix dilogarithms $\mathcal{R}_N(\Delta, b, w, f, c)$, associated to flat/charged $\mathcal{I}$–tetrahedra, $N \geq 1$ being any odd positive integer number.

For $N = 1$, we forget the integral charge c, so that $\mathcal{R}_1$ is defined simply on flattened $\mathcal{I}$–tetrahedra. Namely, set

$$\mathcal{R}_1(\Delta, b, w, f) = \exp\biggl(\frac{*_b}{i\pi}\biggl(-\frac{\pi^2}{6} - \frac{1}{2}\int_0^{w_0}\biggl(\frac{\mathrm{l}_0(b,t,f)}{1-t} - \frac{\mathrm{l}_1(b,t,f)}{t}\biggr)\biggr)\biggr)\, dt \quad (2)$$

where any $\mathrm{l}_j(b,t,f)$ is a log-branch as defined in Section 2.1.1. This is just the exponential of a multiple of the lift of the Rogers dilogarithm, discussed in Section 4.1.

For $N = 2m+1 > 1$ and every complex number x set $x^{1/N} = \exp(\log(x)/N)$, where log is the standard branch of the logarithm which has the imaginary part in $]-\pi,\pi]$, as already fixed above (by convention we put $0^{1/N} = 0$). Denote by g the complex valued function, analytic over the complex plane with cuts from the points $x = \zeta^k$ to infinity ($k = 1,\ldots,\ N-1$), defined by

$$g(x) := \prod_{j=1}^{N-1} (1 - x\zeta^{-j})^{j/N}$$

and set $h(x) := g(x)/g(1)$ (we have $g(1) = \sqrt{N}\exp(-i\pi(N-1)(N-2)/12N)$). The function g plays a main role in the cyclic representation theory of a Borel quantum subalgebra of $U_\zeta(sl(2,\mathbb{C}))$ at $\zeta = \exp(2i\pi/N)$ (see the Appendix, and in particular Theorem 8.4).

For any $n \in \mathbb{N}$ and u', $v' \in \mathbb{C}$ satisfying $(u')^N + (v')^N = 1$, put

$$\omega(u',v'|n) = \prod_{j=1}^{n} \frac{v'}{1 - u'\zeta^j}.$$

The functions ω are periodic in their integer argument, with period N. Given a flat/charged $\mathcal{I}$–tetrahedron (Δ, b, w, f, c), set

$$w'_j = \exp((1/N)(\log(w_j) + (f_j - *_b c_j)(N+1)\pi i)).$$

We define

$$\mathcal{R}_N(\Delta, b, w, f, c) = \left((w'_0)^{-c_1}(w'_1)^{c_0}\right)^{\frac{N-1}{2}} (\mathcal{L}_N)^{*_b}(w'_0, (w'_1)^{-1}) \tag{3}$$

where (recall that $N = 2m+1$)

$$\mathcal{L}_N(u',v')^{i,j}_{k,l} = h(u')\ \zeta^{kj+(m+1)k^2}\ \omega(u',v'|i-k)\ \delta(i+j-l)$$

and δ is the N–periodic Kronecker symbol, ie, $\delta(n) = 1$ if $n \equiv 0 \bmod(N)$, and $\delta(n) = 0$ otherwise. Note that for every $N \geq 1$, the exponent $*_b$ in (2) and (3) is coherent with that in (1). The formula for $\mathcal{L}_N^{-1}$ is given in Proposition 8.6.

2.1.3 Transit configurations

We define now the *transit configurations*, that is the suitable $\mathcal{I}$–flat/charged versions of the $2 \to 3$ bistellar (Pachner or Matveev–Piergallini) local move on 3–dimensional triangulations, that will eventually support the fundamental five-term identities between matrix dilogarithms.

It is useful to fix some general notation for triangulations of (compact) 3–dimensional polyhedra. A triangulation, say T, can be considered as a finite family of *abstract* tetrahedra with a fixed identification rule of some pairs of abstract 2–faces, such that, after the identification, each 2–face is common to at most two tetrahedra of T. We also assume that each abstract tetrahedron is oriented, and that the face identifications reverse the orientation, so that the resulting polyhedron is also oriented. Denote by $E(T)$ the set of edges of T, by $E_\Delta(T)$ the whole set of edges of the associated abstract tetrahedra, and by $\epsilon_T\colon E_\Delta(T) \longrightarrow E(T)$ the natural identification map.

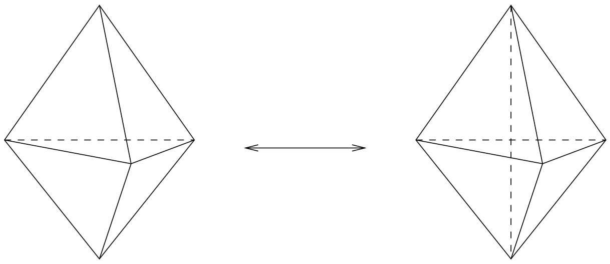

Figure 3: The bare $2 \to 3$ move between singular triangulations

Consider the $2 \leftrightarrow 3$ move shown in Figure 3. We have two triangulations T and T' (by 2 and 3 tetrahedra respectively) of a same oriented polyhedron, and each tetrahedron inherits the induced orientation. Assume that each tetrahedron of T and T' is $\mathcal{I}$–flat/charged. We have to specify the "semi-local" constraints satisfied by each ingredient of the decoration: branchings, modular triples, flattenings and integral charges. By "semi-local" we mean that the constraints hold between the decorations of different triangulations of the same topologically trivial support, where the decorations are given in a purely local way on each tetrahedron. First of all we require that the local branchings on T and T' fit well on common edges, and so define globally branched triangulations (T,b) and (T',b').

We start by defining the *$\mathcal{I}$–transits*, then we will treat the flattening and integral charge transits. A $2 \to 3$ $\mathcal{I}$–transit $(T,b,w) \to (T',b',w')$ consists of a bare $2 \to 3$ move $T \to T'$ that extends to a branching move $(T,b) \to (T',b')$, ie, the two branchings coincide on the 'common' edges of T and T'. Moreover the modular triples have the following behaviour. For each common edge $e \in E(T) \cap E(T')$ we have

$$\prod_{a \in \epsilon_T^{-1}(e)} w(a)^* = \prod_{a' \in \epsilon_{T'}^{-1}(e)} w'(a')^* \tag{4}$$

where $* = \pm 1$ according to the b–orientation of the abstract tetrahedron containing a (respectively a').

Note that (4) implies that the product of the $w'(a')^*$ around the "new" edge of T' is equal to 1. So the inverse $3 \to 2$ $\mathcal{I}$–transits are defined in the very same way, providing that this last condition is verified on T'.

One particular instance of $\mathcal{I}$–transit is shown in Figure 4. Note that in this case all $*_b$ are equal to 1; x, y etc. denotes the cross-ratio modulus w_0 of the corresponding tetrahedron. Assume that all the modular triples are non degenerate, and share the same sign $*_w = 1$. Then we have an oriented *convex* hyperbolic ideal polyhedron with 5 vertices, endowed with two different geometric triangulations by two (respectively three) positively embedded non degenerate ideal tetrahedra. This situation corresponds to a *scissors congruence* relation between polyhedra in $\mathbb{H}^3$. The transit condition (4), including the exponents $*_b$, is the natural algebraic extension to situations including arbitrarily oriented ideal tetrahedra, where the convexity is lost and there are possible overlappings.

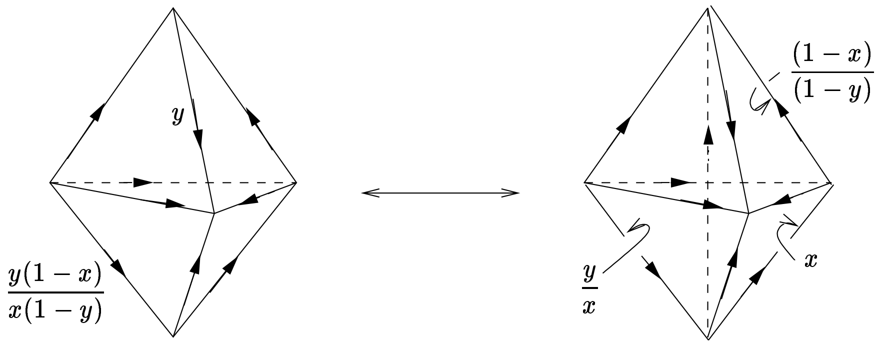

Figure 4: A particular instance of $\mathcal{I}$–transit

Next we define the notion of a $2 \leftrightarrow 3$–transit for flattened $\mathcal{I}$–tetrahedra. Consider a $2 \to 3$ $\mathcal{I}$–transit $(T, b, w) \to (T', b', w')$ as above. The idea is just to take formally the log of the relation (4). Give a flattening to each tetrahedron of the initial configuration, and denote by $\mathrm{l}\colon E_\Delta(T) \to \mathbb{C}$ the corresponding log-branch function on T. A map $f'\colon E_\Delta(T') \to \mathbb{Z}$ defines a $2 \to 3$ *flattening transit* $(T, b, w, f) \to (T', b', w', f')$ if for each common edge $e \in E(T) \cap E(T')$ we have

$$\sum_{a \in \epsilon_T^{-1}(e)} * \, \mathrm{l}(a) = \sum_{a' \in \epsilon_{T'}^{-1}(e)} * \, \mathrm{l}'(a') \tag{5}$$

where $* = \pm 1$ according to the b–orientation of the tetrahedron that contains a (respectively a').

It is easily seen that the flattening transits actually define flattened $\mathcal{I}$–tetrahedra, and that the sum of the values of l' about the new edge of T' is always equal to zero. So the inverse $3 \to 2$ flattening transits are defined in the same way, except that we also require that this last condition holds. Remark that the flattenings of a flattening transit associated to a given $\mathcal{I}$–transit $(T,b,w) \to (T',b',w')$ actually define a flattening transit for every $\mathcal{I}$–transit $(T,b,u) \to (T',b',u')$, if w and w' are non degenerate on the abstract tetrahedra involved in the move and u (respectively u') is a modular triple sufficiently close to w (respectively w').

It remains to define the transits for the integral charges. These (like the single charge itself) do not depend on the modular triples, and even not on the signs $*_b$. A $2 \to 3$ branched move $(T,b,c) \to (T',b',c')$ between charged tetrahedra defines an *integral charge transit* if for each common edge $e \in E(T) \cap E(T')$ we have

$$\sum_{a \in \epsilon_T^{-1}(e)} c(a) = \sum_{a' \in \epsilon_{T'}^{-1}(e)} c'(a'). \tag{6}$$

This implies that the sum of the charges around the new edge of T' is always equal to 2. So we require that this last property is satisfied when we define the inverse $3 \to 2$ charge transits.

The $2 \to 3$ *flat/charged $\mathcal{I}$–transits* are defined by assembling the above definitions.

2.1.4 Five term relations

Here we describe the *contraction* of patterns of automorphisms of $\mathbb{C}^N \otimes \mathbb{C}^N$ associated to patterns of $\mathcal{I}$–tetrahedra.

Let Q be any oriented triangulated 3–dimensional compact polyhedron. For simplicity, we assume that Q is connected. As already said, a triangulation T of Q can be considered as a finite family of abstract tetrahedra Δ^i, with orientation reversing identifications of some pairs of abstract 2–faces. Assume that T is equipped with a *global* branching b. This means that b is a system of orientations of the edges of T that restricts to a branching b^i on each Δ^i (hence, the face identifications are compatible with these local branchings). Assume moreover that each Δ^i is given a structure of $\mathcal{I}$–tetrahedron (Δ^i, b^i, w^i), and that we have a function $A\colon \mathbb{C}\setminus\{0,1\} \to \mathrm{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$. The correspondence $A(\Delta^i, b^i, w^i) = A(w_0^i)^{*_{b^i}}$ in (1) gives us a pattern of automorphisms of $\mathbb{C}^N \otimes \mathbb{C}^N$.

A *state* of (T, b, w) is a function which associate to every 2–simplex t of the 2–skeleton of T an integer $s(t) \in \{0, \ldots, N-1\}$. So, every state determines a matrix entry for each $A(\Delta^i, b^i, w^i)$. As two tetrahedra Δ^k, Δ^l induce opposite orientations on a common face t, the index $s(t)$ is "down" for one of $A(\Delta^k, b^k, w^k)$ or $A(\Delta^l, b^l, w^l)$, while it is "up" for the other. By applying the Einstein's rule of "summing on repeated indices", we get the contraction, or *trace*, of this pattern of tensors. We denote this trace by

$$\prod_{\Delta\subset T} A(\Delta, b, w). \tag{7}$$

The type of the resulting tensor depends on the free 2–faces, and their boundary and b–orientations. This trace construction can be very effectively figured out (in the style of spin networks), if we look at the enriched interior 1–skeleton of the cell decomposition dual to the triangulation. For example, in Figure 5 we show the graphical representation (following Figure 2) of the contractions of tensors corresponding to the two patterns of $\mathcal{I}$–tetrahedra involved in Figure 4.

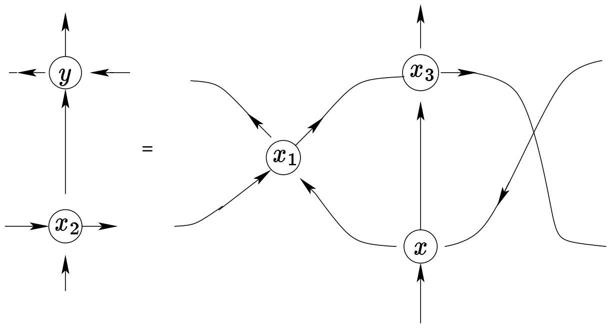

Figure 5: Matrix Schaeffer's identity ($x_1 = y/x$, $x_2 = y(1-x)/x(1-y)$, and $x_3 = (1-x)/(1-y)$)

Now we can state the main results about the semi-local structure of the matrix dilogarithms. First, remark that if we change the branching of a flat/charged $\mathcal{I}$–tetrahedron (Δ, b, w, f, c) by a permutation $p \in S_4$ of its vertices, we get another flat/charged $\mathcal{I}$–tetrahedron $(\Delta, b', w', f', c') = p(\Delta, b, w, f, c)$, where for each edge e of Δ we have $w'(e) = w(e)^{\epsilon(p)}$, $f'(e) = \epsilon(p) f(e)$ and $c'(e) = c(e)$, $\epsilon(p)$ being the signature of p.

There are two main statements, strictly related one to each other. The first describes the good behaviour of the matrix dilogarithms with respect to the above action of S_4 on flat/charged $\mathcal{I}$–tetrahedra (roughly speaking, it says that the matrix dilogarithms are *symmetric*); note that when the branching changes, a different member of the modular triple is selected in the definition of the matrix dilogarithms. The second statement concerns the fundamental five term identities. We give an unified statement for all odd $N \geq 1$, however, it is understood that for $N = 1$ we can forget the integral charges, as they do not enter the definition of the classical dilogarithm $\mathcal{R}_1$. Remark that, as any transit is, in particular, a branching transit, the traces of the two patterns of associated matrix dilogarithms are tensors of the same type.

Theorem 2.1 *Up to a possible sign or Nth root of unity phase factor, the following properties hold true:*

(1) **Symmetry** *For any permutation $p \in S_4$, $\mathcal{R}_N(p(\Delta, b, w, f, c))$ is conjugated to $\mathcal{R}_N(\Delta, b, w, f, c)$, via matrices that depend only on p and the branching b.*

(2) **Transit invariance** *For any $2 \rightarrow 3$ flat/charged $\mathcal{I}$–transit (T, b, w, f, c) $\rightarrow (T', b', w', f', c')$, the traces of the two patterns of associated matrix dilogarithms lead to the same tensor. In formula:*

$$\prod_{\Delta \subset T} \mathcal{R}_N(\Delta, b, w, f, c) \equiv_N \pm \prod_{\Delta' \subset T'} \mathcal{R}_N(\Delta', b', w', f', c')$$

where $\equiv_N$ means equality up to multiplication by Nth roots of unity.

We have given here a qualitative formulation of (1); for a more definite statement, including explicit formulas of the conjugation matrices, see Corollary 5.6.

2.2 Dilogarithmic invariant state sums of globally flat/charged $\mathcal{I}$–triangulations

In this paper, we consider global applications of the matrix dilogarithms either to compact closed oriented 3–manifolds W equipped with a flat principal $PSL(2,\mathbb{C})$–bundle ρ, or to cusped hyperbolic 3–manifolds M, equipped with the holonomy ρ of the hyperbolic structure (see [4] for a wider range of applications).

2.2.1 Globally flat charged $\mathcal{I}$–triangulations of (W,ρ) or M

The first step is to look for *global* $\mathcal{I}$–triangulations of such equipped manifolds. These are possibly singular (see Section 6 for more information about triangulations) globally branched triangulations (T,b) of W or M, such that each tetrahedron Δ^i of T is equipped with a modular triple w^i, making it an $\mathcal{I}$–tetrahedron (Δ^i, b^i, w^i). Moreover, we require that at every edge e of T, we have the following *edge compatibility condition*:

$$\prod_{a\in\epsilon_T^{-1}(e)} w^j(a)^{*_{b^j}} = 1 \tag{8}$$

where $*_{b^j} = \pm 1$ according to the b^j–orientation of the tetrahedron Δ^j that contains a. Note that (8) is the relation satisfied by the cross-ratio moduli at the edge produced by a $2\to 3$ $\mathcal{I}$–transit. So the edge compatibility condition is natural to have a class of triangulations which is *stable* for the $2\to 3$ transits. On the other hand, it is necessary in order to construct hyperbolic 3–manifolds by gluing hyperbolic ideal tetrahedra.

In the case of pairs (W,ρ) these $\mathcal{I}$–triangulations always exist, and can be obtained via the *idealization* of $\mathcal{D}$*–triangulations*, two notions introduced in [3]. We recall them briefly. Let (Δ,b,z) be a branched tetrahedron endowed with a $PSL(2,\mathbb{C})$–valued 1–cocycle z. We write $z_j = z(e_j)$ and $z_j' = z(e_j')$. For instance, the cocycle relation on the 2–face opposite to v_3 reads $z_0z_1z_2^{-1} = 1$. We say that (Δ,b,z) is *idealizable* if

$$u_0 = 0,\ u_1 = z_0(0),\ u_2 = z_0z_1(0),\ u_3 = z_0z_1z_0'(0)$$

are 4 distinct points in $\mathbb{C}\subset\mathbb{CP}^1 = \partial\bar{\mathbb{H}}^3$. These 4 points span a (possibly flat) hyperbolic ideal tetrahedron with ordered vertices.

Let (W,ρ) be as above. We consider the pair (W,ρ) up to orientation preserving homeomorphisms of W and flat bundle isomorphisms of ρ. Equivalently, ρ is identified with a conjugacy class of representations of the fundamental group of W in $PSL(2,\mathbb{C})$.

A $\mathcal{D}$*–triangulation* of (W,ρ) consists of a triple $\mathcal{T} = (T,b,z)$ where: T is a triangulation of W; b is a global branching of T; z is a $PSL(2,\mathbb{C})$–valued 1–cocycle on (T,b) representing ρ and such that (T,b,z) is *idealizable*, ie, all its abstract tetrahedra (Δ^i,b^i,z^i) are idealizable.

If (Δ,b,z) is idealizable, for all $j=0,1,2$ one can associate to e_j the cross-ratio modulus $w_j\in\mathbb{C}\setminus\{0,1\}$ of the hyperbolic ideal tetrahedron spanned by

(u_0, u_1, u_2, u_3). We call the $\mathcal{I}$–tetrahedron (Δ, b, w) with $w = (w_0, w_1, w_2)$ the *idealization* of (Δ, b, z).

For any $\mathcal{D}$–triangulation $\mathcal{T} = (T, b, z)$ of (W, ρ), its *idealization* $\mathcal{T}_{\mathcal{I}} = (T, b, w)$ is given by the family $\{(\Delta^i, b^i, w^i)\}$ of idealizations of the (Δ^i, b^i, z^i). It is a fact (see [3], and also Section 6 for more details) that the idealization of any $\mathcal{D}$–triangulation is an $\mathcal{I}$–triangulation, that is it verifies the edge compatibility condition (8). Moreover, every pair (W, ρ) admits $\mathcal{D}$–triangulations.

The situation is more subtle for cusped manifolds M. It is well-known that every such a manifold M admits *quasi geometric* geodesic triangulations by immersed hyperbolic ideal tetrahedra of non-negative volume (see Remark 2.3 (2)). The volume of M is just given by the sum of the volumes of these tetrahedra. Hence, a quasi geometric geodesic triangulation of M possibly contains flat tetrahedra of null volume, but there are strictly positive ones. With the usual notation, such a triangulation gives rise to a pair (T, w), where each modular triple has non-negative imaginary part, and is only cyclically ordered.

Definition 2.2 A cusped manifold M is said to be *gentle* if it admits a quasi geometric geodesic triangulation (T, w) such that T admits a global branching b. In such a case, set $w' = w^{*_b}$. Then (T, b, w') is said to be a *quasi geometric $\mathcal{I}$–triangulation* of M. For each non degenerate $\mathcal{I}$–tetrahedron of such an $\mathcal{I}$–triangulation, we have $*_b *_{w'} = 1$.

Remarks 2.3 (1) To be "gentle" is a somewhat demanding assumption. Nevertheless, many cusped manifolds are gentle. The simplest example is the complement of the figure-eight knot in the three-sphere. For the sake of simplicity, in the present section we state the results under this assumption. However, in Section 6 we show that the same conclusions hold under the much milder assumption to be "weakly-gentle" (see Definition 6.2). In fact, it is reasonable to ask whether every cusped manifold is weakly-gentle. If not, the construction of the *quantum* invariants for general cusped manifolds is more tricky (see Definition 6.3).

(2) We recall a basic procedure to construct quasi geometric geodesic triangulations of a given cusped manifold M. We start with the *Epstein–Penner canonical cell decomposition* of M [17]. This is obtained by identifying pairs of boundary faces of a finite number of convex ideal hyperbolic polyhedra $\{G_j\}$, each having a finite number of faces. Fix a total ordering of the vertices of each G_j and use it, as usual, to triangulate G_j without adding new vertices. If the orderings match on the paired faces, we eventually get a geodesic triangulation of M by strictly positive ideal tetrahedra, which naturally inherits a global

branching from the total vertex orderings on the G_j. If the orderings do not agree on some pair of identified faces, we have to introduce some degenerate tetrahedra to get a (quasi geometric) triangulation. This triangulation does not inherit a global branching from the construction, but it might nonetheless support some branching.

From now on, in the present section, we consider either pairs (W,ρ), or gentle cusped manifolds M equipped with $\mathcal{I}$–triangulations as just described.

The notion of *global* flattening on an $\mathcal{I}$–triangulation of (W,ρ) or M is obtained by imposing that the associated log-branches formally satisfy, at each edge e of T, the log of the edge compatibility condition (8). More precisely

$$\sum_{a\in\epsilon_T^{-1}(e)} *\ \mathrm{l}(a)=0. \tag{9}$$

Again, this is the natural constraint to get a class of triangulations which is stable with respect to the flattening transits.

Arguing in the same way for the integral charge transits, one would require that the sum of the charges around every edge of T is equal to 2. But a simple 'Gauss–Bonnet' argument on each triangulated sphere making the link of a vertex of a triangulation T of (W,ρ) shows that such tentative global integral charges *do not exist* (for the triangles of such a link triangulation would inherit charges c such that the $c\pi$ should behave like the angles of a *flat* triangulation of the 2–sphere). A way to overcome this problem is to fix an arbitrary non empty link L in W (considered up to ambient isotopy) and to incorporate this *link fixing* in all the constructions. This eventually leads to the following notion of $\mathcal{D}$–triangulation for a triple (W,L,ρ). A *distinguished triangulation* of (W,L) is a pair (T,H) such that T is a triangulation of W and H is a *Hamiltonian* subcomplex of the 1–skeleton of T which realizes the link L (Hamiltonian means that H contains all the vertices of T). A *$\mathcal{D}$–triangulation* $\mathcal{T}=(T,H,b,z)$ for a triple (W,L,ρ) consists of a $\mathcal{D}$–triangulation (T,b,z) for (W,ρ) such that (T,H) is a distinguished triangulation of (W,L). An $\mathcal{I}$–triangulation for (W,L,ρ) is the idealization of a $\mathcal{D}$–triangulation of (W,L,ρ). Finally we can state the notion of global integral charge:

Let X be either a triple (W,L,ρ) or a gentle cusped manifold M, and $\mathcal{T}_{\mathcal{I}}$ be an $\mathcal{I}$–triangulation of X. A *global integral charge* on $\mathcal{T}_{\mathcal{I}}$ is a collection of integral charges on the tetrahedra of $\mathcal{T}_{\mathcal{I}}$ such that the sum of the charges around every edge of T not belonging to H is equal to 2, while the sum of the charges around every edge in H is equal to 0 ($H=\emptyset$ when $X=M$).

2.2.2 Invariant state sums

Let $(\mathcal{T}_{\mathcal{I}}, f, c)$ be a globally flat/charged $\mathcal{I}$–triangulation of X. We can associate to each tetrahedron the corresponding matrix dilogarithm $\mathcal{R}_N(\Delta^i, b^i, w^i, f^i, c^i)$, and take the trace as in (7), that we denote $\mathcal{R}_N(\mathcal{T}_{\mathcal{I}}, f, c)$. As there are no free 2–faces, we get a scalar. We can give a more familiar state sum description of this scalar. Recall that a state of T is function defined on the 2–simplices of the 2–skeleton of T, with values in $\{0,\dots,N-1\}$. Any such a state α determines a matrix element $\mathcal{R}_N(\Delta^i, b^i, w^i, f^i, c^i)_\alpha$ for each matrix dilogarithm $\mathcal{R}_N(\Delta^i, b^i, w^i, f^i, c^i)$. Set

$$\mathcal{R}_N(\mathcal{T}_{\mathcal{I}}, f, c)_\alpha = \prod_i \mathcal{R}_N(\Delta^i, b^i, w^i, f^i, c^i)_\alpha.$$

Then

$$\mathcal{R}_N(\mathcal{T}_{\mathcal{I}}, f, c) = \sum_\alpha \mathcal{R}_N(\mathcal{T}_{\mathcal{I}}, f, c)_\alpha. \tag{10}$$

Finally, we can state the main global results about the classical and quantum dilogarithmic invariants. As for Theorem 2.1, we give unified statements for all odd $N \geq 1$, but for $N = 1$ we can forget the integral charge and work directly with flattened $\mathcal{I}$–triangulations of (W,ρ) or M. On the other hand, in the quantum case the link L is encoded by the global integral charge, which is *entirely responsible for the link contribution to the state sums.* Remark also that both global flattenings and integral charges induce a cohomology class in $H^1(X;\mathbb{Z}/2\mathbb{Z})$, which is transit invariant. The invariants depend on the choice of these classes. Here we prefer to normalize the choice, by requiring that these classes are trivial. The corresponding flat/charged $\mathcal{I}$–triangulations are said to be (cohomologically) *normalized.*

Theorem 2.4 *Let X be either a triple (W,L,ρ) or a gentle cusped manifold M. We have:*

(1) *X admits normalized globally flat/charged $\mathcal{I}$–triangulations $(\mathcal{T}_{\mathcal{I}}, f, c)$.*

(2) *Let v be the number of vertices of T ($v = 0$ for M). For every odd integer $N \geq 1$, the value of the state sum $N^{-v}\mathcal{R}_N(\mathcal{T}_{\mathcal{I}}, f, c)$ does not depend on the choice of the normalized flat/charged $\mathcal{I}$–triangulation of X, possibly up to a sign and multiplication by Nth roots of unity. Hence, up to the same ambiguity, it defines* a dilogarithmic invariant $H_N(X)$.

A direct consequence of the proof of Theorem 5.7 is that for $N \equiv 1 \bmod(4)$, $N > 1$, the invariants $H_N(X)$ have no sign ambiguity. The existence of global flattenings and charges in (1) is based on previous results of Neumann about the combinatorics of 3–dimensional triangulations. For proving (2), we consider the $\mathcal{I}$–decorated versions of few other local moves on 3–dimensional triangulations, besides the $2 \leftrightarrow 3$ one, and the corresponding matrix dilogarithm identities. Then we show that arbitrary flat/charged $\mathcal{I}$–triangulations of X can be connected via a finite sequence of such transits together with $2 \leftrightarrow 3$ transits. The reader is addressed to Section 6 for more information on these invariants.

3 Basic matrix dilogarithms and the symmetrization problem

We use the interpretation of automorphisms $A(x) \in \mathbb{C}^N \otimes \mathbb{C}^N$ as functions of $\mathcal{I}$–tetrahedra, where $x \in \mathbb{C} \setminus \{0,1\}$, and the notions of transits and five term identities introduced in Section 2.1.

Definition 3.1 A *basic matrix dilogarithm of rank* N is a map $\mathcal{L}\colon\ \mathbb{C}\setminus\{0,\ 1\} \rightarrow \mathrm{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$ which satisfies the five-term identity shown in Figure 5, providing that all the modular triples are non degenerate and have imaginary parts of the same sign. We call this particular five-term identity with these constraints on the cross-ratio moduli the *matrix Schaeffer's identity.*

Recall that Figure 5 corresponds to the $\mathcal{I}$–transit of Figure 4, where all the tetrahedra have the same index $*_b = 1$. Note that the Schaeffer's identity holds exactly, with no phase ambiguity.

The family $\{\mathcal{L}_N\}$ We introduce here the explicit family $\{\mathcal{L}_N\}$ of *basic* matrix dilogarithms of rank N used in this paper. Recall that N is an odd positive integer.

The classical dilogarithm $\mathcal{L}_1$ Definition 3.1 is modeled on the fundamental functional identity satisfied by the classical Rogers dilogarithm. As usual, denote by log the standard branch of the logarithm, with imaginary part in $]-\pi,\pi]$. The *Rogers dilogarithm* is the function over $\mathbb{C}$, complex analytic over $\mathfrak{D} = \mathbb{C} \setminus \{(-\infty;0) \cup (1;+\infty)\}$, defined by

$$\mathrm{L}(x) = -\frac{\pi^2}{6} - \frac{1}{2}\int_0^x \left(\frac{\log(t)}{1-t} + \frac{\log(1-t)}{t}\right)\, dt \qquad (11)$$

where we integrate first along the path $[0;1/2]$ on the real axis and then along any path in $\mathfrak{D}$ from $1/2$ to x. Here we add $-\pi^2/6$ so that $\mathrm{L}(1)=0$. When $|x-1/2|<1/2$ we may also write L as

$$\mathrm{L}(x) = -\frac{\pi^2}{6} + \frac{1}{2}\log(x)\log(1-x) + \sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

The sum in the right-hand side is the power series expansion in the open unit disk $|x|<1$ of the *Euler dilogarithm* Li_2, defined by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt$$

and complex analytic over $\mathbb{C}\setminus(1;+\infty)$. For a detailed study of the dilogarithm functions and their relatives, see [24] or the review [38]. The function L is related to the *Bloch–Wigner dilogarithm*

$$\mathrm{D}_2(x) = \mathrm{Im}\big(\mathrm{Li}_2(x)\big) + \arg(1-x)\log|x| \tag{12}$$

which is obtained by adding to $\mathrm{Im}(\mathrm{Li}_2(x))$ a correction term that compensates its jump along the branch cut $(1;+\infty)$. The function $\mathrm{D}_2(x)$ is a real analytic continuation of $\mathrm{Im}(\mathrm{Li}_2(x))$ on $\mathbb{C}\setminus\{0,1\}$, and it is continuous (but not differentiable) at 0 and 1. It gives the volume of $\mathcal{I}$–tetrahedra by the formula

$$\mathrm{Vol}(\Delta, b, w) = *_b\mathrm{D}_2(w_0)$$

and we have the 6–fold symmetry relations

$$\mathrm{D}_2(w_0) = \mathrm{D}_2(w_1) = \mathrm{D}_2(w_2) = -\mathrm{D}_2(w_0^{-1}) = -\mathrm{D}_2(w_1^{-1}) = -\mathrm{D}_2(w_2^{-1}). \tag{13}$$

Moreover, if we apply the formula (12) to the $\mathcal{I}$–transit of Figure 4 we get the five-term functional relation

$$\mathrm{D}_2(y) + \mathrm{D}_2\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big) = \mathrm{D}_2(x) + \mathrm{D}_2(y/x) + \mathrm{D}_2\Big(\frac{1-x}{1-y}\Big) \tag{14}$$

when $x \neq y$. All the other five term relations obtained by changing the branching in Figure 4 also hold true, due to (13).

One would like to think of the Rogers dilogarithm L as the natural complex analytic analogue of $\mathrm{D}_2(x)$. But L verifies similar five-term relations only by putting strong restrictions on the variables. Namely, the analog of (14) is the classical Schaeffer's identity

$$\mathrm{L}(x) - \mathrm{L}(y) + \mathrm{L}(y/x) - \mathrm{L}\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big) + \mathrm{L}\Big(\frac{1-x}{1-y}\Big) = 0 \tag{15}$$

which for real x, y holds only when $0<y<x<1$. This identity characterizes the Rogers dilogarithm: if $f(0;1)\to\mathbb{R}$ is a 3 times differentiable function satisfying (15) for all $0 < y < x < 1$, then $f(x) = k\mathrm{L}(x)$ for a suitable constant k (see eg [15], Appendix). By analytic continuation, the relation (15) holds true for complex parameters x, y, providing that the imaginary part of y is non-zero and x lies inside the triangle formed by 0, 1 and y. This is equivalent to all variables having imaginary parts with the same sign, as in Definition 3.1. We set

$$\mathcal{L}_1(x) = \exp((1/\pi i)\mathrm{L}(x)).$$

Clearly $\mathcal{L}_1$ is a basic matrix dilogarithm of rank 1. We take the exponential in order to unify the treatment of the classical and quantum $(N > 1)$ cases.

The quantum dilogarithms $\mathcal{L}_N$ Let $N = 2m+1 > 1$. Recall the notation introduced in Subsection 2.1.2. We put $u \in \mathbb{C} \setminus \{0,1\}$, $v = 1-u$, and define

$$\mathcal{L}_N(u)_{k,l}^{i,j} = \mathcal{L}_N(u^{\frac{1}{N}}, v^{\frac{1}{N}})_{k,l}^{i,j} = h(u^{\frac{1}{N}})\ \zeta^{kj+(m+1)k^2}\ \omega(u^{\frac{1}{N}}, v^{\frac{1}{N}}|i-k)\ \delta(i+j-l). \tag{16}$$

Up to a different parametrization, the function $\mathcal{L}_N(u', v')$ is the Faddeev–Kashaev's matrix of $6j$–symbols for the cyclic representation theory of a Borel quantum subalgebra $\mathcal{B}_\zeta$ of $U_\zeta(sl(2,\mathbb{C}))$, where $\zeta = \exp(2i\pi/N)$ (see Remarks 5.9 and 8.5). We prove in Section 5 that $\mathcal{L}_N(u)$ is actually a basic matrix dilogarithm of rank N, as in Definition 3.1. We can state now the following problem.

Symmetrization problem for $\mathcal{L}_N$ *For every N, find a suitable* symmetrized *version $\mathcal{R}_N$ of $\mathcal{L}_N$ which satisfies all the instances of five-term identities, for all transit configurations, and without any constraint on the modular triples.*

It turns out that the solution of this problem is strictly related to the study of a suitable *uniformization* of $\mathcal{L}_N$ and to the behaviour of $\mathcal{L}_N$ with respect to the *tetrahedral symmetries*. The flattenings and integral charges arise naturally from this solution.

4 The symmetrization problem for $\mathcal{L}_1$

4.1 Uniformization

We use the "uniformization $\mathrm{mod}(\pi^2\mathbb{Z})$" R of L due to W Neumann [29, 30] (see also the recent [31]).

Let us recall its definition. Let $\widehat{\mathbb{C}} = \widehat{\mathbb{C}}_{00} \cup \widehat{\mathbb{C}}_{01} \cup \widehat{\mathbb{C}}_{10} \cup \widehat{\mathbb{C}}_{11}$, where $\widehat{\mathbb{C}}_{\varepsilon\varepsilon'}$ $(\varepsilon, \varepsilon' = 0,1)$ is the Riemann surface of the function defined on $\mathfrak{D} = \mathbb{C} \setminus \{(-\infty;0) \cup (1;+\infty)\}$ by

$$x \mapsto (\log(x) + \varepsilon i\pi, \log((1-x)^{-1}) + \varepsilon' i\pi).$$

Thus $\widehat{\mathbb{C}}$ is the ramified abelian covering of $\mathbb{C} \setminus \{0,1\}$ obtained from $\mathfrak{D} \times \mathbb{Z}^2$ by the identifications

$$\begin{aligned} \{(-\infty;0) + i0\} \times \{p\} \times \{q\} &\sim \{(-\infty;0) - i0\} \times \{p+2\} \times \{q\} \\ \{(1;+\infty) + i0\} \times \{p\} \times \{q\} &\sim \{(1;+\infty) - i0\} \times \{p\} \times \{q+2\}. \end{aligned}$$

Here $(-\infty;0) \pm i0$ comes from the upper/lower fold of $\mathfrak{D}$ with respect to $(-\infty;0)$, and similarly for $(1;+\infty) \pm i0$. The function

$$\mathrm{l}(x;p,q) = (\log(x) + pi\pi, \log((1-x)^{-1}) + qi\pi) \tag{17}$$

is well-defined and analytic on $\widehat{\mathbb{C}}$. Consider the following lift on $\widehat{\mathbb{C}}$ of the Rogers dilogarithm L, defined in (11):

$$\mathrm{R}(x;p,q) = \mathrm{L}(x) + \frac{i\pi}{2}(p\log(1-x) + q\log(x)). \tag{18}$$

It is known that:

Lemma 4.1 *The formula (18) defines an analytic map* $\mathrm{R}\colon \widehat{\mathbb{C}} \to \mathbb{C}/\pi^2\mathbb{Z}$.

The idea of interpreting x as a modulus of a hyperbolic ideal tetrahedron, and p, q as additional decorations, comes from [29]. We implement this idea in the set up of $\mathcal{I}$–tetrahedra formalized in Subsection 2.1. Given an $\mathcal{I}$–tetrahedron (Δ, b, w), let us consider a $\mathbb{Z}$–valued function f of the edges of Δ such that, for every edge, $f(e) = f(e')$. As for $w = (w_0, w_1, w_2)$, we write $f = (f_0, f_1, f_2)$ with the ordering given by the branching b. Then we set

$$\mathrm{R}(\Delta, b, w, f) = \mathrm{R}(w_0; f_0, f_1).$$

4.2 Tetrahedral symmetries

Let (Δ, b, w, f) be as in Section 4.1. Here we analyze under which condition on f the function $\mathrm{R}(\Delta, b, w, f)$ respects the tetrahedral symmetries. Note that if f is a flattening of (Δ, b, w) and w is non degenerate, then f is a flattening of (Δ, b, u) for every modular triple u sufficiently close to w.

By acting with a permutation $p \in S_4$ on the vertices of Δ, one passes from b to a new branching b'. This gives (Δ, b', w', f'), with $w'(e) = w(e)^{\epsilon(p)}$ and $f'(e) = \epsilon(p)f(e)$ for any edge e of Δ, where $\epsilon(p)$ is the signature of p. Beware that all these data are renamed according to the new ordering of the vertices given by b'. We have:

Lemma 4.2 *For any non-degenerate enriched $\mathcal{I}$–tetrahedron (Δ, b, w, f), the identities*

$$\mathrm{R}(\Delta, b', u', f') = \epsilon(p)\ \mathrm{R}(\Delta, b, u, f) \mod(\pi^2/6)\mathbb{Z}$$

hold true for every permutation p and for every modular triple u sufficiently close to w if and only if f is a flattening of (Δ, b, w). These identities are also satisfied if w is degenerate and we replace u by w.

Proof The basic remark (already made in [28]) is that the Rogers dilogarithm L has symmetries only up to some elementary functions. Indeed, by differentiating both sides of each identity we see that

$$\begin{aligned}
\mathrm{L}\big((1-x)^{-1}\big) &= \mathrm{L}(x) - \varepsilon(i\pi/2)\log(1-x) + (\pi^2/6)\\
\mathrm{L}(1-x^{-1}) &= \mathrm{L}(x) - \varepsilon(i\pi/2)\log(x) - (\pi^2/6)\\
\mathrm{L}(x^{-1}) &= -\mathrm{L}(x) + \varepsilon(i\pi/2)\log(x)\\
\mathrm{L}(1-x) &= -\mathrm{L}(x) - (\pi^2/6)\\
\mathrm{L}\big(x/(x-1)\big) &= -\mathrm{L}(x) + \varepsilon(i\pi/2)\log(1-x) - (\pi^2/3)
\end{aligned} \tag{19}$$

when $\mathrm{Im}(x) \neq 0$, with $\varepsilon = 1$ if $\mathrm{Im}(x) > 0$ and $\varepsilon = -1$ if $\mathrm{Im}(x) < 0$. A straightforward computation shows that these relations imply:

$$\begin{aligned}
\mathrm{R}\big((1-x)^{-1}; p, q\big) &= \mathrm{R}(x; -\varepsilon - p - q, p) + (\pi^2/6) + (p\pi^2/2)\\
\mathrm{R}(1-x^{-1}; p, q) &= \mathrm{R}(x; q, -\varepsilon - p - q) - (\pi^2/6) - (q\pi^2/2)\\
\mathrm{R}(x^{-1}; p, q) &= -\mathrm{R}(x; -p, p + q - \varepsilon) - (p\pi^2/2)\\
\mathrm{R}(1-x; p, q) &= -\mathrm{R}(x; -q, -p) - (\pi^2/6)\\
\mathrm{R}\big(x/x-1; p, q\big) &= -\mathrm{R}(x; p + q - \varepsilon, -q) - (\pi^2/3) + (q\pi^2/2)
\end{aligned} \tag{20}$$

under the same assumption. Lemma 4.1 implies that these relations are still valid up to π^2 when $x \in \mathbb{R}\setminus\{0,1\}$. We get the result by renaming the variables according to the branching. For instance, in the first equality, setting $(x; -\varepsilon - p - q, p) = (u_0; f_0, f_1)$ we have $\big((1-x)^{-1}; p, q\big) = (u_1; f_1, f_2)$, which is obtained from $(u_0; f_0, f_1)$ after the permutation (012). □

4.3 Complete five term relations

Recall the notion of $2 \to 3$ flattening transit from Subsection 2.1.4.

Lemma 4.3 *Let $(T, b, w, f) \to (T', b', w', f')$ be a $2 \to 3$ flattening transit, such that $(T, b, w) \to (T', b', w')$ is the $\mathcal{I}$–transit configuration of Figure 4, without any constraint on the moduli w and w'. Then we have*

$$\sum_{\Delta \subset T} \mathrm{R}(\Delta, b, w, f) = \sum_{\Delta' \subset T'} \mathrm{R}(\Delta', b', w', f') \mod(\pi^2\mathbb{Z})\ . \tag{21}$$

Proof This lemma is equivalent to Proposition 2.5 of [30] (see also [31]). It is based on a clever analytic continuation argument that we reproduce for the sake of completeness, and because it will be reconsidered in the proof of Theorem 5.7 (quantum case). Denote by $(\Delta^i, b^i, w^i, f^i)$ the flattened $\mathcal{I}$–tetrahedron opposite to the i-th vertex (for the ordering induced by b). The moduli give us a point

$(w_0^0, w_0^1, w_0^2, w_0^3, w_0^4) = (x, y, y/x, y(1-x)/x(1-y), (1-x)/(1-y)) \in (\mathbb{C}\setminus\{0,1\})^5$.

Let $\mathfrak{G} \subset (\mathbb{C} \setminus \{0,1\})^5$ be the set of such points. Consider the map

$$F\colon\ \widehat{\mathbb{C}}^5 = \prod_{i=0}^{i=4} \{(w_0^i; f_0^i, f_1^i)\} \longrightarrow (\mathbb{C} \setminus \{0,1\})^5$$

defined by forgetting the f_j^i. Note that the log-branch functions

$$\mathrm{l}_j^i\colon\ \widehat{\mathbb{C}} = \{(w_0^i; f_0^i, f_1^i)\} \to \mathbb{C}$$

are all analytic, by (17) and the fact that $\mathrm{l}_2^i = -\mathrm{l}_0^i - \mathrm{l}_1^i$ on each flattened $\mathcal{I}$–tetrahedron. Moreover, the relations (5) are linear identities between the l_j^i, with $* = 1$ for each summand. Hence they define an *analytic subset* $\widehat{\mathfrak{G}}$ of $F^{-1}(\mathfrak{G})$.

Denote by $\mathfrak{G}^+ \subset \mathfrak{G}$ the space where the w_0^i have positive imaginary parts (what follows could be done with the subset where the w_0^i have negative imaginary parts). From Figure 6 and the above description in terms of x and y, we see that the points of $\mathfrak{G}^+$ are characterized by the property that x lies inside the triangle formed by 0, 1 and y with $\mathrm{Im}(y) > 0$, so that $\mathfrak{G}^+$ is connected and contractible. Moreover, if we let $\mathrm{Im}(x)$ and $\mathrm{Im}(y)$ go towards 0 with $0 < \mathrm{Re}(y) < \mathrm{Re}(x) < 1$, we come to the subset of $\mathfrak{G}$ where $0 < y < x < 1$ with real x and y. We know that the Schaeffer's identity

$$\mathrm{L}(x) - \mathrm{L}(y) + \mathrm{L}(y/x) - \mathrm{L}(\frac{1-x^{-1}}{1-y^{-1}}) + \mathrm{L}(\frac{1-x}{1-y}) = 0$$

holds on this subset. Since it is contained in the frontier of $\mathfrak{G}^+$, and the left-hand side of the Schaeffer's identity is analytic on $\mathfrak{G}^+$, we deduce by analytic continuation that the latter holds true on the whole of $\mathfrak{G}^+$.

Next we describe $\widehat{\mathfrak{G}} \cap F^{-1}(\mathfrak{G}^+)$. In $\mathfrak{G}^+$ the imaginary parts of the w_0^i are positive, so this is also the case for all the other moduli of the $\mathcal{I}$–transit configuration of Figure 4. Hence for any edge $e \in E(T) \cap E(T')$ we get

$$\sum_{a \in \epsilon_T^{-1}(e)} \log(w(a)) = \sum_{a' \in \epsilon_{T'}^{-1}(e)} \log(w'(a')). \tag{22}$$

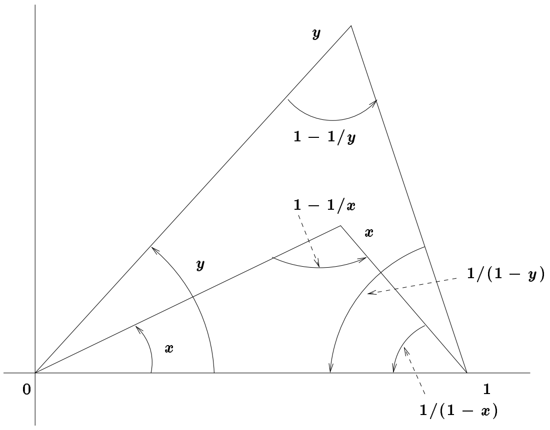

Figure 6: Position of x with respect to y in $\mathfrak{G}^+$, and the associated moduli

This implies that the relations (5) are valid over $F^{-1}(\mathfrak{G}^+)$ if and only if the flattening functions $f\colon E_\Delta(T) \to \mathbb{Z}$ and $f'\colon E_\Delta(T') \to \mathbb{Z}$ verify

$$\sum_{a \in \epsilon_T^{-1}(e)} f(a) = \sum_{a' \in \epsilon_{T'}^{-1}(e)} f'(a'). \tag{23}$$

Let us write (21) over $\widehat{\mathfrak{G}} \cap F^{-1}(\mathfrak{G}^+)$. The dilogarithmic terms of each side are respectively $\mathrm{L}(y) + \mathrm{L}(1 - x^{-1}/1 - y^{-1})$ and $\mathrm{L}(x) + \mathrm{L}(y/x) + \mathrm{L}(1 - x/1 - y)$, which are equal due to the Schaeffer's identity. A straightforward computation using (22) shows that the logarithmic terms at each side are equal if and only if we have :

$$\begin{gathered} f_1^0 - f_0^2 - f_1^2 + f_0^3 + f_1^3 = 0 \\ -f_1^1 + f_1^2 - f_1^3 = 0 \\ f_0^0 - f_1^3 + f_1^4 = 0 \\ -f_0^1 + f_0^3 + f_1^3 - f_0^4 - f_1^4 = 0 \\ f_0^2 - f_0^3 + f_0^4 = 0. \end{gathered} \tag{24}$$

Solving this system and using $f_0^i + f_1^i + f_2^i = -1$ (the w_j^i have positive imaginary parts), we find some of the relations (23). Hence the identity (21) is true over $\widehat{\mathfrak{G}} \cap F^{-1}(\mathfrak{G}^+)$. Since $\widehat{\mathfrak{G}}$ is an analytic subset of $\widehat{\mathbb{C}}^5$, we deduce from Lemma 4.1 that (21) is also true on the whole of $\widehat{\mathfrak{G}}$ up to π^2. □

We can state now the solution of the symmetrization problem for $\mathcal{L}_1$:

Theorem 4.4 *Let $(T, b, w, f) \to (T', b', w', f')$ be any $2 \to 3$ flattening transit.*

Then we have

$$\sum_{\Delta\subset T} *\,\mathrm{R}(\Delta,b,w,f) = \sum_{\Delta'\subset T'} *\,\mathrm{R}(\Delta',b',w',f') \mod(\pi^2\mathbb{Z}) \tag{25}$$

where $*=\pm 1$ according to the b–orientation of Δ (resp. Δ').

Proof By Lemma 4.3, the theorem holds true for the special transit of Figure 4. Any other $2\leftrightarrow 3$ transit is obtained from this one by changing the branching. Correspondingly, let us apply Lemma 4.2 to (21). We find *local* defects, one for each tetrahedron, which are integer multiples of $\pi^2/6$. We claim that these defects *globally* compensate. As before, denote by Δ^i the tetrahedron opposite to the i-th vertex in Figure 4. Any change of branching is obtained as a composition of the transpositions (01), (12), (23) and (34) of the vertices. The following table describes for each Δ^i the defect induced by these transpositions:

	Δ^1	Δ^3	Δ^0	Δ^2	Δ^4
(01)	0	$\frac{f_0^3\pi^2}{2}$	0	$\frac{f_0^2\pi^2}{2}$	$\frac{f_0^4\pi^2}{2}$
(12)	0	$-\frac{\pi^2}{3}-\frac{f_1^3\pi^2}{2}$	$\frac{f_0^0\pi^2}{2}$	0	$-\frac{\pi^2}{3}-\frac{f_1^4\pi^2}{2}$
(23)	$-\frac{\pi^2}{3}-\frac{f_1^1\pi^2}{2}$	0	$-\frac{\pi^2}{3}-\frac{f_1^0\pi^2}{2}$	0	$\frac{f_0^4\pi^2}{2}$
(34)	$\frac{f_0^1\pi^2}{2}$	0	$\frac{f_0^0\pi^2}{2}$	$\frac{f_0^2\pi^2}{2}$	0

Note that the reduction mod(2) of the relations (23) are always satisfied over $\widehat{\mathfrak{G}}$. So this table shows that for any change of the branching in Figure 4 the symmetry defects at both sides of (25) are the same up to π^2. □

Remarks 4.5 Dealing with the classical "commutative" dilogarithm one can prove Theorem 4.4 without using the tetrahedral symmetries (see [31], up to some differences in the set up). On the other hand, the path we have followed displays the interesting "local defects vs global compensations" phenomenon. This path is strictly analogous to what we shall do in the quantum case. As already remarked in [29], the proof of Lemma 4.3 shows that the flattening transits realize the most general relations between enriched $\mathcal{I}$–tetrahedra for which the identities (25) are universaly true, that is independently of the specific values of w and w'.

Finally, as in (2) we set

$$\mathcal{R}_1(\Delta,b,w,f) = \exp((*_b/i\pi)\mathrm{R}(\Delta,b,w,f))$$

The map $\mathcal{R}_1$ gives us the symmetrized matrix dilogarithm of rank 1. Clearly it satisfies the conclusion of Theorem 2.1.

5 The symmetrization problem for $\mathcal{L}_N$, $N>1$

The Appendix collects some quantum algebraic facts used in the present section. In Section 5.1, we describe the lifted matrix Schaeffer's identity for the matrix $\widehat{\mathcal{L}}_N$ obtained in Subsection 8.2 of the Appendix. In Section 5.2 we compute the tetrahedral symmetries of $\widehat{\mathcal{L}}_N$ and we prove Theorem 2.1 for $N>1$.

As before, let $N=2m+1>1$ be any odd positive integer, and denote by log the standard branch of the logarithm, which has the imaginary part in $]-\pi,\pi]$. For any complex number $x\neq 0$ write $x^{1/N}=\exp((1/N)\log(x))$. We denote $\zeta=\exp(2i\pi/N)$. Remark that $\zeta^{m+1}=-\exp(i\pi/N)$, so that $\zeta^{N(m+1)}=1$. For any $u\in\mathbb{C}\setminus\{0,1\}$ and $p\in\mathbb{Z}$ define

$$u'_p=u'_0\ \zeta^{(m+1)p}=\exp((1/N)(\log(u)+p(N+1)\pi i)). \tag{26}$$

We can lift $\mathcal{L}_N$, given in (16), over the Riemann surface $\widehat{\mathbb{C}}$ of Section 4 by setting

$$\widehat{\mathcal{L}}_N(u;p,q)^{k,l}_{i,j}=\mathcal{L}_N(u'_p,v'_{-q})^{k,l}_{i,j}=\frac{g(u'_p)}{g(1)}\ \zeta^{il+(m+1)i^2}\ \omega(u'_p,v'_{-q}|k-i)\ \delta(k+l-j) \tag{27}$$

for any $(u;p,q)\in\{\mathbb{C}\setminus\{(-\infty,0)\cup(1,+\infty)\}\}\times\mathbb{Z}^2$, where $v=1-u$. The matrix $\widehat{\mathcal{L}}_N(u;p,q)$ is invertible, with inverse given in Proposition 8.6. Remark that if $u\in(-\infty,0)$ then $\widehat{\mathcal{L}}_N(u+i0;p,q)=\widehat{\mathcal{L}}_N(u-i0;p+2,q)$ because

$$(1/N)(\log(u+i0)+p(N+1)\pi i)=(1/N)(\log(u-i0)+(p+2)(N+1)\pi i)-2\pi i.$$

On another hand, if $u\in(1,+\infty)$ then u'_p lies on the ray $\{t\zeta^{(m+1)p}\}$, $t>1$. As this is a branch cut of the function g in (27), we have

$$\widehat{\mathcal{L}}_N(u+i0;p,q)=\zeta^{-(m+1)p}\widehat{\mathcal{L}}_N(u-i0;p,q+2).$$

Hence, denoting by U_N the multiplicative group of Nth roots of unity, we see that the matrix valued map $\hat{\mathcal{L}}_N\colon\widehat{\mathbb{C}}\to\mathrm{M}_{N^2}(\mathbb{C}/U_N)$ is complex analytic (compare with Lemma 4.1). Recall that $\equiv_N$ denotes the equality up to multiplication by Nth roots of unity.

5.1 Lifted basic five term relation

We say that (Δ, b, w, a) is an *enriched* $\mathcal{I}$–tetrahedron if (Δ, b, w) is an $\mathcal{I}$–tetrahedron and a is a $\mathbb{Z}$–valued function on the edges of Δ such that $a(e) = a(e')$ for every pair of opposite edges e and e'. We identify a with (a_0, a_1, a_2), where $a_j = a(e_j)$ and the ordering of the edges is induced by the branching b. Similarly to (26), given a we define Nth roots of the moduli by

$$w'_j = w'_{a_j} = \exp((1/N)(\log(w_j) + a_j(N+1)\pi i)). \tag{28}$$

We call $w'\colon E(\Delta) \to \mathbb{C} \setminus \{0,1\}$ the *Nth-branch* of w for a (for short: Nth–branch map), and its values are the *Nth–root moduli*. We write

$$\tau = -w'_0 w'_1 w'_2. \tag{29}$$

Definition 5.1 Consider a $2 \to 3$ $\mathcal{I}$–transit $(T^0, b^0, w^0) \to (T^1, b^1, w^1)$ whose underlying branching transit is as in Figure 4. Suppose that we have a map a^0 that enriches the tetrahedra of T^0 involved in the move. A map a^1 that enriches those in T^1 defines a *Nth–branch transit* if for each common edge $e \in E(T^0) \cap E(T^1)$ we have

$$\prod_{\tilde{e}^0 \in \epsilon_{T^0}^{-1}(e)} (w^0)'(\tilde{e}^0) = \prod_{\tilde{e}^1 \in \epsilon_{T^1}^{-1}(e)} (w^1)'(\tilde{e}^1) \tag{30}$$

where the identification map $\epsilon_{T^i}\colon E_\Delta(T^i) \to E(T^i)$ is as in Subsection 2.1.3.

It is easily seen that (30) implies that the Nth–roots of unity τ in (29) are the same for all tetrahedra. Also, the product of the Nth–root moduli about the new edge of T^1 is equal to τ^2.

For any enriched $\mathcal{I}$–tetrahedron (Δ, b, w, a) we define

$$\widehat{\mathcal{L}}_N(\Delta, b, w, a) = (\widehat{\mathcal{L}}_N)^{*_b}(w_0; a_0, a_1) = (\mathcal{L}_N)^{*_b}(w'_0, (w'_1)^{-1}). \tag{31}$$

The notion of five term identity (in particular the matrix Schaeffer's one) naturally lifts to enriched $\mathcal{I}$–tetrahedra and Nth–branch transits. In the rest of this section we prove:

Theorem 5.2 *The matrix Schaeffer's identity corresponding to any Nth–branch $\mathcal{I}$–transit holds true for the tensors $\widehat{\mathcal{L}}_N(\Delta, b, w, a)$, with furthermore no restriction on the cross-ratio moduli.*

Two remarks are in order. First, when the moduli of the tetrahedra involved in the move satisfy the conditions of Definition 3.1, the Nth–root moduli given by the standard log (ie, with $a \equiv 0$) make a $2 \leftrightarrow 3$ Nth–branch transit. So this theorem implies that the matrix $\mathcal{L}_N$, as defined in (16), *is a basic matrix dilogarithm of rank* N. Second, here we impose a specific branching transit because we have not yet analyzed the symmetries of $\widehat{\mathcal{L}}_N(\Delta, b, w, a)$. We shall relax this assumption in Section 5.2.

Proof of Theorem 5.2 Denote by $(\Delta^i, b^i, w^i, a^i)$ the enriched $\mathcal{I}$–tetrahedron opposite to the i-th vertex in Figure 7. Using Figure 2 we see that the associated (Schaeffer's) five-term identity reads

$$\widehat{\mathcal{L}}_N(\Delta^1, b^1, w^1, a^1)_{23}\ \ \widehat{\mathcal{L}}_N(\Delta^3, b^3, w^3, a^3)_{12} =$$

$$\widehat{\mathcal{L}}_N(\Delta^4, b^4, w^4, a^4)_{12}\ \ \widehat{\mathcal{L}}_N(\Delta^2, b^2, w^2, a^2)_{13}\ \ \widehat{\mathcal{L}}_N(\Delta^0, b^0, w^0, a^0)_{23}.$$

Both sides are operators acting on $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$. The indices show the tensor factor on which the $\widehat{\mathcal{L}}_N$ act, for instance $Y_1^{-1} Z_2^{-1} Y_2 = Y^{-1} \otimes Z^{-1}Y \otimes \mathrm{id}_{\mathbb{C}^N}$, and so on.

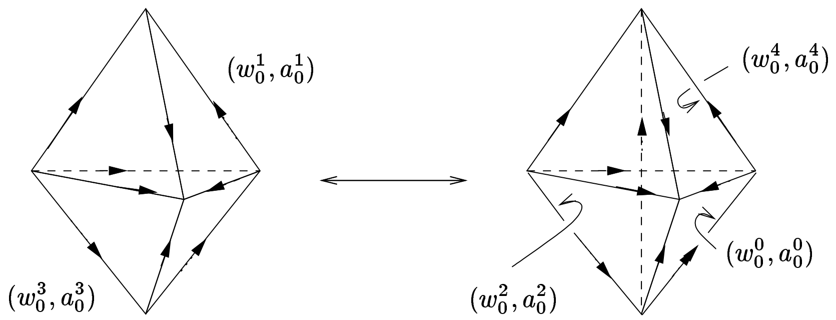

Figure 7: The enriched $\mathcal{I}$–transit supporting the matrix Schaeffer's identity

The splitting formula induced by Theorem 8.4 gives

$$\mathcal{L}_N(w_0', (w_1')^{-1}) = \Upsilon \ \cdot\ \Psi(-Y^{-1} \otimes Z^{-1}Y) \tag{32}$$

where

$$\Upsilon = \frac{1}{N} \sum_{i,j=0}^{N-1} \zeta^{ij}\ Z^{-i} \otimes Y^j$$

and

$$\Psi(-Y^{-1} \otimes Z^{-1}Y) = \frac{g(w_0')}{g(1)} \sum_{t=0}^{N-1} \prod_{s=1}^{t} \frac{(w_0')^{-1}(w_1')^{-1}}{1-(w_0')^{-1}\zeta^{-s}}\ (-Y^{-1} \otimes Z^{-1}Y)^t \tag{33}$$

is obtained by reversing the computation after formula (51). A remarkable fact is that Ψ is a solution (in fact, the unique up to multiplication by scalars) of the functional relation

$$\Psi(\zeta^{-1}A) = \Psi(A)\ \left(\frac{1-(w_0')^{-1}(w_1')^{-1}A}{(w_0')^{-1}}\right) = \Psi(A)\ \left(w_0' - (w_1')^{-1}A\right) \qquad (34)$$

where $A = -Y^{-1} \otimes Z^{-1}Y$. By (32) we have to prove:

$$\Upsilon_{23}\Psi_{23}^1\ \Upsilon_{12}\Psi_{12}^3 = \Upsilon_{12}\Psi_{12}^4\ \Upsilon_{13}\Psi_{13}^2\ \Upsilon_{23}\Psi_{23}^0 \qquad (35)$$

where the Ψ^i are given by (33) for each enriched tetrahedron $(\Delta^i, b^i, w^i, a^i)$, and we omit their matrix arguments for simplicity. The first step is to split this relation into the pentagon relation (43) for Υ, and a five term identity for Ψ that we shall consequently prove. Write

$$U = -Y_1^{-1}Z_2^{-1}Y_2 = -\zeta^{-1}(XZ)_1^{-1}X_2 \quad , \quad V = -Y_2^{-1}Z_3^{-1}Y_3$$

where the matrices X, Y and Z are defined in the proof of Theorem 8.4. By commuting the variables we easily verify that

$$\begin{aligned}
\Psi_{13}^2(-\zeta^{-1}(XZ)_1^{-1}X_3)\ \Upsilon_{23} &= \Upsilon_{23}\ \Psi_{13}^2(-\zeta^{-1}(XZ)_1^{-1}Z_2^{-1}X_3) \\
&= \Upsilon_{23}\ \Psi_{13}^2\left(-(-\zeta^{-1}(XZ)_1^{-1}X_2)(-\zeta^{-1}(XZ)_2^{-1}X_3)\right) \\
&= \Upsilon_{23}\ \Psi_{13}^2(-UV)
\end{aligned}$$

$$\begin{aligned}
\Psi_{12}^4(-\zeta^{-1}(XZ)_1^{-1}X_2)\ \Upsilon_{13} &= \Upsilon_{13}\ \Psi_{12}^4(-\zeta^{-1}(XZ)_1^{-1}X_2(XZ)_3) \\
&= \Upsilon_{13}\ \Psi_{12}^4(U(XZ)_3) \\
\Psi_{12}^4(-\zeta^{-1}(XZ)_1^{-1}X_2)\ \Upsilon_{23} &= \Upsilon_{23}\ \Psi_{12}^4(-\zeta^{-1}(XZ)_1^{-1}X_2(XZ)_3^{-1}) \\
&= \Upsilon_{23}\ \Psi_{12}^4(U(XZ)_3^{-1}).
\end{aligned}$$

Then the right-hand side of (35) is equal to (for simplicity, we only indicate some of the matrix arguments)

$$\begin{aligned}
\Upsilon_{12}\Psi_{12}^4\ \Upsilon_{13}\Psi_{13}^2\ \Upsilon_{23}\Psi_{23}^0(V) &= \Upsilon_{12}\Psi_{12}^4\ \Upsilon_{13}\Upsilon_{23}\ \Psi_{13}^2(-UV)\ \Psi_{23}^0(V) \\
&= \Upsilon_{12}\Upsilon_{13}\ \Psi_{12}^4(U(XZ)_3)\Upsilon_{23}\ \Psi_{13}^2(-UV)\ \Psi_{23}^0(V) \\
&= \Upsilon_{12}\Upsilon_{13}\Upsilon_{23}\ \Psi_{12}^4(U)\ \Psi_{13}^2(-UV)\ \Psi_{23}^0(V).
\end{aligned}$$

and the left-hand side immediately gives

$$\Upsilon_{23}\Psi_{23}^1\ \Upsilon_{12}\Psi_{12}^3 = \Upsilon_{23}\Upsilon_{12}\ \Psi_{23}^1\Psi_{12}^3.$$

As Υ is a linear representation of the canonical element S_ζ of the algebra $Q(\mathcal{B}^0_\zeta)$ (see Section 8.1), it is a solution of the pentagon relation (43). So we are left to show that

$$\Psi^1_{23}(V)\Psi^3_{12}(U) = \Psi^4_{12}(U)\Psi^2_{13}(-UV)\Psi^0_{23}(V) \quad . \tag{36}$$

We first prove that $\Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(V)^{-1}$ commutes with UV. For that it is enough to observe that $UV = \zeta VU$ and to use (34). Namely,

$$\begin{aligned}
&\Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(V)^{-1}\ (UV)\\
&\qquad = \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ (UV)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\qquad = \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ (UV)\ \Psi^3_{12}(\zeta U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\qquad = \Psi^4_{12}(U)^{-1}\ (UV)\ \Psi^1_{23}(\zeta^{-1}V)\ \Psi^3_{12}(\zeta U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\qquad = (UV)\ \Psi^4_{12}(\zeta U)^{-1}\ \Psi^1_{23}(\zeta^{-1}V)\ \Psi^3_{12}(\zeta U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}.
\end{aligned}$$

Moreover, (34) allows us to turn the last four terms into

$$\begin{aligned}
&\Psi^4_{12}(U)^{-1}\ \big((w'_0)^4 - \zeta((w'_1)^4)^{-1}\ U\big)\ \Psi^1_{23}(\zeta^{-1}V)\ \Psi^3_{12}(\zeta U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&= \Psi^4_{12}(U)^{-1}\ \big(\Psi^1_{23}(\zeta^{-1}V)\ (w'_0)^4 - \zeta((w'_1)^4)^{-1}\ \Psi^1_{23}(V)\ U\big)\\
&\qquad\qquad \times\Psi^3_{12}(\zeta U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \big(((w'_0)^1 - ((w'_1)^1)^{-1}\ V)(w'_0)^4 - \zeta((w'_1)^4)^{-1}\ U\big)\\
&\qquad\qquad \times\Psi^3_{12}(\zeta U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \big(\Psi^3_{12}(\zeta U)\ \big((w'_0)^1(w'_0)^4 - \zeta((w'_1)^4)^{-1}\ U\big) -\\
&\qquad\qquad ((w'_1)^1)^{-1}(w'_0)^4\ \Psi^3_{12}(U)\ V\big)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(\zeta U)\ \big((w'_0)^1(w'_0)^4 - \zeta((w'_1)^4)^{-1}\ U -\\
&\qquad\qquad ((w'_1)^1)^{-1}(w'_0)^4\big((w'_0)^3 - \zeta((w'_1)^3)^{-1}\ U\big)\ V\big)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}.
\end{aligned}$$

Since there is an Nth-branch transit, we have (see Figure 7):

$$(w'_1)^0(w'_0)^4 = (w'_1)^1 \quad , \quad (w'_2)^3(w'_0)^1 = (w'_2)^4 \quad , \quad (w'_0)^0(w'_1)^4 = (w'_1)^3. \tag{37}$$

Together with the relations (29), the last two imply $(w'_0)^1(w'_0)^4 = (w'_0)^3(w'_0)^0$. Then, in the last expression, the term between parenthesis reads

$$\big((w'_0)^3 - \zeta((w'_1)^3)^{-1}\ U\big)\ \big((w'_0)^0 - ((w'_1)^0)^{-1}\ V\big).$$

By applying (34) two more times we eventually find

$$\begin{aligned}
&\Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(V)^{-1}\ (UV) =\\
&\qquad\qquad (UV)\ \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(V)^{-1}.
\end{aligned}$$

This shows that $P(-UV) = \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(V)^{-1}$ is a linear functional of $-UV$ (there must be a $-$ sign in front of UV, because $U^N = V^N = -(UV)^N = -\mathrm{Id}_{\mathbb{C}^N}$). To conclude the proof, it is enough to show that $P(-UV)$ satisfies

$$P(-\zeta^{-1}UV) = P(-UV)\ \big((w'_0)^2 - ((w'_1)^2)^{-1}\ (-UV)\big). \tag{38}$$

Indeed, this equation defines $P(-UV)$ as well as $\Psi^2_{13}(-UV)$ up to a scalar, and by Lemma 8.6 we know that for each i the matrix $\widehat{\mathcal{L}}_N(\Delta^i, b^i, w^i, a^i)$ has determinant 1. Consider the change of variable $V \to \zeta^{-1}V$ in $P(-UV)$. We have

$$\begin{aligned}
&\Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(\zeta^{-1}V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\quad= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \big((w'_0)^1 - ((w'_1)^1)^{-1}\ V\big)\ \Psi^3_{12}(U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\quad= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \big((w'_0)^1\ \Psi^3_{12}(U) - ((w'_1)^1)^{-1}\Psi^3_{12}(\zeta^{-1}U)\ V\big)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\quad= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \big((w'_0)^1 - ((w'_1)^1)^{-1}\big((w'_0)^3 - ((w'_1)^3)^{-1}\ U\big)V\big)\\
&\qquad\qquad\times \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\quad= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \big(\Psi^0_{23}(\zeta^{-1}V)^{-1}\big((w'_0)^1 - ((w'_1)^1)^{-1}(w'_0)^3\big)\\
&\qquad\qquad+((w'_1)^1)^{-1}((w'_1)^3)^{-1}\ \Psi^0_{23}(V)^{-1}\ UV\big).
\end{aligned}$$

Again, since we have an Nth-branch transit the following relations hold true:

$$(w'_0)^3 = (w'_0)^2(w'_0)^4 \quad , \quad (w'_0)^1 = (w'_0)^0(w'_0)^2 \quad , \quad (w'_1)^2 = (w'_1)^1(w'_1)^3.$$

Together with the first relation in (37), the first above gives $((w'_1)^1)^{-1}(w'_0)^3 = (w'_0)^2((w'_1)^0)^{-1}$. So the term between parenthesis is equal to

$$\Psi^0_{23}(\zeta^{-1}V)^{-1}\big((w'_0)^0 - ((w'_1)^0)^{-1}\ V\big)(w'_0)^2 + ((w'_1)^2)^{-1}\ \Psi^0_{23}(V)^{-1}\ UV$$

and we find

$$\begin{aligned}
&\Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(\zeta^{-1}V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(\zeta^{-1}V)^{-1}\\
&\qquad= \Psi^4_{12}(U)^{-1}\ \Psi^1_{23}(V)\ \Psi^3_{12}(U)\ \Psi^0_{23}(V)^{-1}\ \big((w'_0)^2 - ((w'_1)^2)^{-1}(-UV)\big).
\end{aligned}$$

This proves (38), whence the theorem. □

5.2 Tetrahedral symmetries

As in Section 5.1, we define an enriched $\mathcal{I}$–tetrahedron (Δ, b, w, a) from a flat/charged one (Δ, b, w, f, c) by putting $a = f - *_b c$. In that case, (28) gives $\tau = \zeta^{-*_b(m+1)}$ in (29).

Recall that the symmetry group on four elements numbered from 0 to 3 is generated by the transpositions (01), (12) and (23). We saw in Section 4 that if we change the branching of a flattened $\mathcal{I}$–tetrahedron (Δ, b, w, f) by a permutation (ij) of its vertices, we get another flattened $\mathcal{I}$–tetrahedron $(ij)(\Delta, b, w, f)$. This behaviour naturally extends to flat/charged $\mathcal{I}$–tetrahedra.

Let S and T be the $N \times N$ invertible square matrices with entries

$$T_{i,j} = \zeta^{i^2(m+1)}\delta(i+j) \quad , \quad S_{i,j} = N^{-1/2}\zeta^{ij}.$$

We have $S^4 = \mathrm{id}_{\mathbb{C}^N}$ and $S^2 = \zeta'(ST)^3$ for some Nth–root of unity ζ'. Hence the matrices S and T define a projective N–dimensional representation Θ of $SL(2,\mathbb{Z})$. The following proposition describes the tetrahedral symmetries of $\widehat{\mathcal{L}}_N(\Delta, b, w, f, c)$ in terms of Θ.

Proposition 5.3 *Let (Δ, b, w, f, c) be a flat/charged $\mathcal{I}$–tetrahedron with $*_b = +1$. We have*

$$\begin{aligned}
\widehat{\mathcal{L}}_N\big((01)(\Delta, b, w, f, c)\big) &\equiv_N (w_0')^{\frac{1-N}{2}}\ T_1^{-1}\ \widehat{\mathcal{L}}_N(\Delta, b, w, f, c)\ T_1\\
\widehat{\mathcal{L}}_N\big((12)(\Delta, b, w, f, c)\big) &\equiv_N (w_1')^{\frac{N-1}{2}}\ S_1^{-1}\ \widehat{\mathcal{L}}_N(\Delta, b, w, f, c)\ T_2\\
\widehat{\mathcal{L}}_N\big((23)(\Delta, b, w, f, c)\big) &\equiv_N (w_0')^{\frac{1-N}{2}}\ S_2^{-1}\ \widehat{\mathcal{L}}_N(\Delta, b, w, f, c)\ S_2
\end{aligned}$$

where $w_i' = (w_i)'_{f_i - *_b c_i}$, $T_1 = T \otimes 1$ etc., and we write $\equiv_N$ for the equality up to multiplication by Nth roots of unity. Moreover, for any enriched non degenerate $\mathcal{I}$–tetrahedron (Δ, b, w, a), these identities hold true for (Δ, b, u, a) with u sufficiently close to w if and only if $a = f - *_b c$ is a flat/charge for (Δ, b, w).

To prove this result, we use some formulas described in the Appendix. The ambiguity up to Nth roots of unity is due to Lemma 8.3 (i) and (iii) and a similar identity. In Figure 8 we show these symmetry relations (up to scalars) by using the graphic tensors of Subsection 2.1.2; there we put $a = (f, c)$, $\bar{T} = T^{-1}$ and $\bar{S} = S^{-1}$. Note that the matrices T and S and their inverses act as duality morphisms. We need the following inversion formula:

Lemma 5.4 *Suppose that $w_0'w_1'w_2' = -\tau$. Then we have*

$$\prod_{j=1}^{n} \frac{(w_1')^{-1}}{1 - w_0'\zeta^j} \cdot \prod_{j=1}^{N-n} \frac{w_2'}{1 - (w_0')^{-1}\zeta^{j-1}} = \zeta^{-(m+1)(N-n)(N-n-1)}\ \tau^{-n}.$$

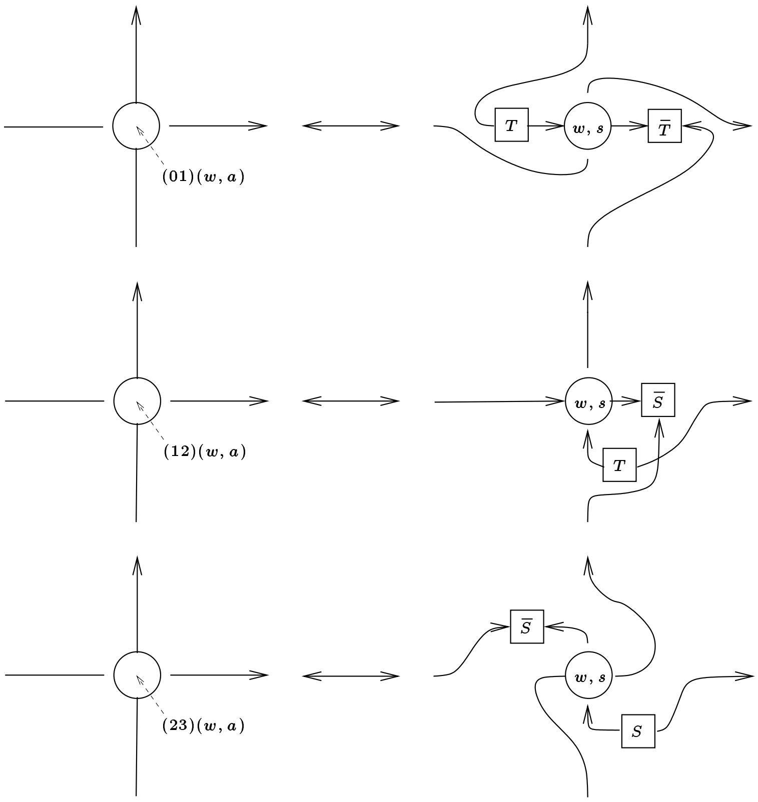

Figure 8: The symmetry relations of $\widehat{\mathcal{L}}_N$ for flat/charged $\mathcal{I}$–tetrahedra

Proof This consists of the following straightforward computation:

$$\prod_{j=1}^{n}\frac{(w_1')^{-1}}{1-w_0'\zeta^j}\cdot\prod_{j=1}^{N-n}\frac{w_2'}{1-(w_0')^{-1}\zeta^{j-1}}=\frac{w_2(w_1')^{-n}(w_2')^{-n}}{\prod_{j=1}^{n}(1-w_0'\zeta^j)\prod_{j=1}^{N-n}(1-(w_0')^{-1}\zeta^{j-1})}$$

$$=\frac{w_2(w_1')^{-n}(w_2')^{-n}(-1)^{N-n}}{\zeta^{(m+1)(N-n)(N-n-1)}(w_0')^{n-N}\prod_{j=1}^{n}(1-w_0'\zeta^j)\prod_{j=1}^{N-n}(1-w_0'\zeta^{1-j})}$$

$$=-\zeta^{-(m+1)(N-n)(N-n-1)}\ \tau^{-n}\ w_2w_0\ (1-w_0)^{-1}$$
$$=\zeta^{-(m+1)(N-n)(N-n-1)}\ \tau^{-n}.$$ □

Proof of Proposition 5.3 By (31) and Proposition 8.6 we have

$$\begin{aligned}
\widehat{\mathcal{L}}_N\big((01)(\Delta,b,w,f,c)\big)^{i,l}_{k,j} &= (\widehat{\mathcal{L}}_N)^{-1}((w_0')^{-1},w_2')^{i,l}_{k,j}\\
&= [(w_0')^{-1}]\ \frac{g(1)}{g((w_0')^{-1})}\ \zeta^{-ij-(m+1)i^2}\ \delta(k+j-l)\ \prod_{s=1}^{k-i}\frac{1-(w_0')^{-1}\zeta^{s-1}}{w_2'}\\
&\equiv_N (w_0')^{\frac{1-N}{2}}\ \frac{g(w_0')}{g(1)}\ \zeta^{-ij-(m+1)i^2}\ \delta(k+j-l)\ \zeta^{(m+1)(i-k)^2}\ \prod_{s=1}^{i-k}\frac{(w_1')^{-1}}{1-w_0'\zeta^s}\\
&\equiv_N (w_0')^{\frac{1-N}{2}}\ \frac{g(w_0')}{g(1)}\ \zeta^{-il+(m+1)k^2}\ \delta(k+j-l)\ \prod_{s=1}^{i-k}\frac{(w_1')^{-1}}{1-w_0'\zeta^s}
\end{aligned}$$

where we use Lemma 8.3 (i) and Lemma 5.4 in the second equality ($\tau = \zeta^{-(m+1)}$). Now, this may be written as

$$\begin{aligned}
&(w_0')^{\frac{1-N}{2}}\ \sum_{i',k'=0}^{N-1}\zeta^{-(m+1)(i')^2}\delta(i+i')\ \zeta^{(m+1)(k')^2}\delta(k+k')\\
&\qquad\qquad\times\ \frac{g(w_0')}{g(1)}\ \zeta^{i'l+(m+1)(i')^2}\ \delta(j-k'-l)\ \prod_{s=1}^{k'-i'}\frac{(w_1')^{-1}}{1-w_0'\zeta^s}\\
&= (w_0')^{\frac{1-N}{2}}\ \sum_{i',k'=0}^{N-1}(T^{-1})^{i'}_i\ \widehat{\mathcal{L}}_N(w_0',(w_1')^{-1})^{k',l}_{i',j}\ T^k_{k'}
\end{aligned}$$

which proves the first relation. The third comes from a very similar computation:

$$\begin{aligned}
&\widehat{\mathcal{L}}_N\big((23)(\Delta,b,w,f,c)\big)^{k,j}_{i,l} = (\widehat{\mathcal{L}}_N)^{-1}((w_0')^{-1},w_2')^{k,j}_{i,l}\\
&\equiv_N (w_0')^{\frac{1-N}{2}}\ \frac{g(w_0')}{g(1)}\ \zeta^{-kj+(m+1)i^2}\ \delta(i+l-j)\ \prod_{s=1}^{k-i}\frac{(w_1')^{-1}}{1-w_0'\zeta^s}\\
&\equiv_N (w_0')^{\frac{1-N}{2}}\ \frac{g(w_0')}{g(1)}\ \big(N^{-1}\sum_{l'=0}^{N-1}\zeta^{l'(i+l-j)}\big)\ \zeta^{-kj+(m+1)i^2}\ \delta(i+l-j)\prod_{s=1}^{k-i}\frac{(w_1')^{-1}}{1-w_0'\zeta^s}\\
&\qquad\equiv_N (w_0')^{\frac{1-N}{2}}\ \sum_{j',l'=0}^{N-1}(N^{-1/2}\zeta^{ll'})\ (N^{-1/2}\zeta^{-jj'})\\
&\qquad\qquad\qquad\times\ \frac{g(w_0')}{g(1)}\ \zeta^{il'+(m+1)i^2}\ \delta(j'-k-l')\ \prod_{s=1}^{k-i}\frac{(w_1')^{-1}}{1-w_0'\zeta^s}\\
&\qquad\equiv_N (w_0')^{\frac{1-N}{2}}\ \sum_{j',l'=0}^{N-1}(S^{-1})^{j'}_j\ \widehat{\mathcal{L}}_N(w_0',(w_1')^{-1})^{k,l'}_{i,j'}\ S^l_{l'}.
\end{aligned}$$

where we use again Lemma 8.3 (i) and Lemma 5.4 in the first equality. The second symmetry relation is more sophisticated. Consider the right-hand side. It gives

$$\begin{aligned}
&\sum_{i',l'=0}^{N-1} (S^{-1})_i^{i'}\ \widehat{\mathcal{L}}_N(w_0',(w_1')^{-1})_{i',j}^{k,l'}\ T_{l'}^l \\
&= \sum_{i',l'=0}^{N-1} \left(N^{-1/2}\zeta^{-ii'}\right)\left(\zeta^{(m+1)(l')^2}\delta(l+l')\right) \\
&\qquad\qquad \times \frac{g(w_0')}{g(1)}\,\zeta^{i'l'+(m+1)(i')^2}\delta(k+l'-j)\prod_{s=1}^{k-i'}\frac{(w_1')^{-1}}{1-w_0'\zeta^s} \\
&= N^{-1/2}\zeta^{(m+1)l^2}\,\frac{g(w_0')}{g(1)}\,\delta(j+l-k)\sum_{i'=0}^{N-1}\zeta^{-i'l+(m+1)(i')^2-ii'}\prod_{s=1}^{k-i'}\frac{(w_1')^{-1}}{1-w_0'\zeta^s} \\
&= N^{-1/2}\zeta^{(m+1)(l^2-k^2)}\,\frac{g(w_0')}{g(1)}\,\delta(j+l-k)\sum_{i'=0}^{N-1}\zeta^{i'(k-i-l)}\prod_{s=1}^{i'-k}\frac{1-(w_0')^{-1}\zeta^{s-1}}{w_2'} \\
&= \frac{g(w_0')}{g(1)}\,N^{-1/2}\zeta^{(m+1)(l^2-k^2)+k(k-i-l)}\,\delta(j+l-k) \\
&\qquad\qquad \times \sum_{i'=0}^{N-1}\left(\zeta^{k-i-l}(w_2')^{-1}\right)^{i'-k}\prod_{s=1}^{i'-k}\left(1-(w_0')^{-1}\zeta^{s-1}\right) \\
&= \frac{g(w_0')}{g(1)}\,N^{-1/2}\zeta^{(m+1)(l^2-k^2)+k(k-i-l)}\,\delta(j+l-k)\ f(0,(w_0')^{-1}\zeta^{-1}|(w_2')^{-1}\zeta^{j-i})
\end{aligned}$$

where the function $f(x,y|z)$ is defined in the proof of Proposition 8.6. As described there, we have

$$\frac{f(0,(w_0')^{-1}\zeta^{-1}|(w_2')^{-1}\zeta^{j-i})}{f(0,(w_0')^{-1}\zeta^{-1}|(w_2')^{-1})} = \prod_{s=1}^{j-i}\frac{1-(w_2')^{-1}\zeta^{s-1}}{w_1'\zeta^{-(m+1)}\zeta^s}$$

where we note that $-(w_0')^{-1}(w_2')^{-1}\zeta^{-1} = w_1'\zeta^{-(m+1)}$. Simplifying the powers of ζ with the help of the Kronecker symbol $\delta(j+l-k)$, we see immediately that the right-hand side reads

$$N^{-1/2}\,\frac{g(w_0')}{g(1)}\,f(0,(w_0')^{-1}\zeta^{-1}|(w_2')^{-1})\,\zeta^{-il-(m+1)i^2}\,\delta(j+l-k)\prod_{s=1}^{j-i}\frac{1-(w_2')^{-1}\zeta^{s-1}}{w_1'}$$

A very similar computation to Lemma 8.3 (iii) shows that

$$f(0,y|z) \equiv_N \frac{(-yz)^{\frac{N-1}{2}}\,g(1)}{g(y^{-1}/\zeta)g(z/\zeta)}$$

for $z^N = 1/(1-y^N)$. This gives

$$f(0,(w_0')^{-1}\zeta^{-1}|(w_2')^{-1}) \equiv_N \frac{(-(w_0')^{-1}(w_2')^{-1})^{\frac{N-1}{2}}\ g(1)}{g(w_0')\ g((w_2')^{-1}/\zeta)}$$

$$\equiv_N \frac{(w_1')^{\frac{N-1}{2}}\ g(1)}{g(w_0')g((w_2')^{-1})}\ \frac{w_1'}{1-(w_2')^{-1}}$$

where we use Lemma 8.2 in the last equality. Hence, noting that $|g(1)| = N^{1/2}$, we find

$$N^{-1/2}\ \frac{g(w_0')}{g(1)}\ f(0,(w_0')^{-1}\zeta^{-1}|(w_2')^{-1}) \equiv_N N^{-1}(w_1')^{\frac{N-1}{2}}\frac{w_1'}{1-(w_2')^{-1}}\ \frac{g(1)}{g((w_2')^{-1})}$$

$$\equiv_N (w_1')^{\frac{1-N}{2}}\ \frac{g(1)\ [(w_2')^{-1}]}{g((w_2')^{-1})}$$

with $[x] = N^{-1}(1-x^N)/(1-x)$ as in Proposition 8.6. So

$$\sum_{j',k'=0}^{N-1} (S^{-1})_i^{i'}\ \widehat{\mathcal{L}}_N(w_0',(w_1')^{-1})_{i',j}^{k,l'}\ T_{l'}^l$$

$$\equiv_N (w_1')^{\frac{1-N}{2}}\ \frac{g(1)\ [(w_2')^{-1}]}{g((w_2')^{-1})}\ \zeta^{-il-(m+1)i^2}\ \delta(j+l-k)\ \prod_{s=1}^{j-i} \frac{1-(w_2')^{-1}\zeta^{s-1}}{w_1'}$$

$$= (w_1')^{\frac{1-N}{2}}\ (\widehat{\mathcal{L}}_N)^{-1}((w_2')^{-1},w_1')_{j,l}^{i,k} \quad .$$

This proves the second symmetry relation. The last claim follows from the fact that we need $\tau = \zeta^{-(m+1)}$ in all the above computations. □

Let us recall the matrix $\mathcal{R}_N$ from Subsection 2.1.2:

Definition 5.5 For each odd integer $N > 1$, the *symmetrized matrix dilogarithm* of flat/charged $\mathcal{I}$–tetrahedra is defined as

$$\begin{aligned}\mathcal{R}_N(\Delta,b,w,f,c) &= \left((w_0')^{-c_1}(w_1')^{c_0}\right)^{\frac{N-1}{2}}\ (\widehat{\mathcal{L}}_N)^{*_b}(w_0;f_0-*_bc_0,f_1-*_bc_1)\\ &= \left((w_0')^{-c_1}(w_1')^{c_0}\right)^{\frac{N-1}{2}}\ (\mathcal{L}_N)^{*_b}(w_0',(w_1')^{-1})\end{aligned}$$

where $w_i' = (w_i)'_{f_i-*_bc_i}$.

Note that the log of the scalar $(w_0')^{-c_1}(w_1')^{c_0}$ is of the same form as the function we add to the classical Rogers dilogarithm to define the uniformized dilogarithm $\mathrm{R}(x;p,q)$ in (18). We have the following result, which is the precise form of (1) in Theorem 2.1:

Corollary 5.6 *For any flat/charged $\mathcal{I}$–tetrahedron (Δ,b,w,f,c) with $*_b = +1$, the following symmetry relations hold true:*

$$\begin{array}{l}\mathcal{R}_N\big((01)(\Delta,b,w,f,c)\big) \equiv_N \pm\, T_1^{-1}\ \mathcal{R}_N(\Delta,b,w,f,c)\ T_1\\ \mathcal{R}_N\big((12)(\Delta,b,w,f,c)\big) \equiv_N \pm\, S_1^{-1}\ \mathcal{R}_N(\Delta,b,w,f,c)\ T_2\\ \mathcal{R}_N\big((23)(\Delta,b,w,f,c)\big) \equiv_N \pm\, S_2^{-1}\ \mathcal{R}_N(\Delta,b,w,f,c)\ S_2 \quad .\end{array}$$

where $\equiv_N$ means equality up to multiplication by Nth roots of unity.

Proof For the first equality, on the left hand side the scalar factor in Definition 5.5 reads

$$\big((w_0')^{-1})^{-c_2}((w_2')^{-1})^{c_0}\big)^{\frac{N-1}{2}} = \big((w_0')^{-c_1+1}((w_0'w_2')^{-c_0})\big)^{\frac{N-1}{2}}$$

because $c_0+c_1+c_2=1$. As $w_0'w_1'w_2' = -\zeta^{-(m+1)}$, this is equal to $(w_0')^{\frac{N-1}{2}}$ times $((w_0')^{-c_1}((w_1')^{c_0}))^{\frac{N-1}{2}}$ up to sign and multiplication by Nth roots of unity. By Proposition 5.3, this coincides with the scalar factor on the right hand side. The same computation works for the other transpositions. □

5.3 Complete five term relations

Recall from Subsection 2.1.3 the notion of *flat/charged* $\mathcal{I}$–transit. It is immediate that a flat/charged $\mathcal{I}$–transit associated to the branching configuration of Figure 4 defines an Nth–branch transit (see Definition 5.1) for the induced enriched $\mathcal{I}$–tetrahedra. Finally we can prove the statement (2) of Theorem 2.1:

Theorem 5.7 *For every odd integer $N>1$, the tensors $\mathcal{R}_N$ satisfy the five-term identities associated to arbitrary flat/charged $\mathcal{I}$–transits with no constraints on the cross ratio moduli, possibly up to sign and multiplication by Nth roots of unity. Moreover, the flat/charged $\mathcal{I}$–transits realize the minimal relations between enriched $\mathcal{I}$–tetrahedra for this property to hold.*

By Proposition 5.3, the last claim shows that having the *whole set* of five-term identities is equivalent to having the symmetry relations (and similarly when $N=1$ after replacing the flat/charge with flattenings). On the other hand, the analyticity of $\mathcal{R}_N(\Delta,b,w,a)$ for all branchings b implies only $a = f+\lambda c$ for an arbitrary $\lambda\in\mathbb{C}$, where f is a flattening and c an integral charge. This, in turn, is enough for the matrix Schaeffer's identity.

Proof Consider first a transit whose branching is as in Figure 4. By Theorem 5.2 and Definition 5.5, in that case the statement is true if we remove the powers of $(w_0')^{-c_1}(w_1')^{c_0}$ from the $\mathcal{R}_N$. We claim that the products P_0 and P_1 of these scalars at both sides of the transit are also equal, up to a sign if $N \equiv 3 \bmod(4)$.

Denote by Δ^j the tetrahedron opposite to the j-th vertex (for the ordering of the vertices induced by the branching); do the same for the log-branches and integral charges. Recall the notation l_i^j for the log-branch of the i-th edge of Δ^j. Then P_0 reads

$$\exp\left(\frac{N-1}{2}\left(-c_1^1\mathrm{l}_0^1 + c_0^1\mathrm{l}_1^1 - c_1^3\mathrm{l}_0^3 + c_0^3\mathrm{l}_1^3\right)\right)$$

times $(-1)^{\frac{N-1}{2}(f_0^1c_1^1+f_1^1c_0^1+f_0^3c_1^3+f_1^3c_0^3)}$, and P_1 is

$$\exp\left(\frac{N-1}{2}\left(-c_1^0\mathrm{l}_0^0 + c_0^0\mathrm{l}_1^0 - c_1^2\mathrm{l}_0^2 + c_0^2\mathrm{l}_1^2 - c_1^4\mathrm{l}_0^4 + c_0^4\mathrm{l}_1^4\right)\right)$$

times $(-1)^{\frac{N-1}{2}(f_0^0c_1^0+f_1^0c_0^0+f_0^2c_1^2+f_1^2c_0^2+f_0^4c_1^4+f_1^4c_0^4)}$. Consider the sums in the above exponentials. They are formally the same as minus the *sums of logarithmic terms* at both sides of (21) (just replace each log of a modulus by the corresponding log-branch, and similarly each flattening by a charge). The proof of Proposition 4.4 tells us that when the log of the moduli satisfy (22), these sums are equal when the relations (23) are true. Since the relations (5) are formally identical to (22), and the transit of integral charges is defined by the identities (23), we deduce that $P_0 = \pm P_1$. So the statement is true for a flat/charged $\mathcal{I}$–transit with a branching as in Figure 4.

We get the result for all the other instances of $2 \leftrightarrow 3$ flat/charged $\mathcal{I}$–transits by using Lemma 5.6. Indeed, note that the matrices S and T always act on the input spaces of the tensors $\mathcal{R}_N$, whereas their inverses always act on the output spaces. As the composition of the $\mathcal{R}_N$ is defined by the *oriented* graphs described in Subsection 2.1.2, the matrix action compensates on the common faces of two tetrahedra. So the five-term relation corresponding to any $2 \leftrightarrow 3$ flat/charged $\mathcal{I}$–transit is conjugated to the one for Figure 4.

By the proof of Theorem 5.2, we know that the lifted matrix Schaeffer's identity for $\widehat{\mathcal{L}}_N$ holds true if and only if the relations (30) are valid. This implies that: for each enriched $\mathcal{I}$–tetrahedra (Δ, b, w, a) we have $w_0'w_1'w_2' = -\tau$ for a fixed root of unity τ, each a must be of the form $a = f + \lambda c$, $\lambda \in \mathbb{Z}$, for some flattening f and integral charge c, and f and c transit as usual. We claim that the whole set of five-term identities hold simultaneously if and only if $\lambda = -*_b$. For instance, when $\tau = 1$ (that is $\lambda = 0$, the case of flattened $\mathcal{I}$–tetrahedra), we see easily that the first relation in Proposition 5.3 becomes

$$\widehat{\mathcal{L}}_N\big((01)(\Delta, b, w, f)\big) \equiv_N M_1\ (w_0')^{\frac{1-N}{2}}\ T_1^{-1}\ \widehat{\mathcal{L}}_N(\Delta, b, w, f)\ T_1\ M_1^{-1}$$

where M is the $N\times N$–matrix with entries $M^i_j = \zeta^{(m+1)i}\,\delta(i-j)$. The second relation becomes even more complicated. The same phenomenon happens whenever we do not put $a = f - *_b c$. In general there is no global compensation of these further matrix actions when considering symmetries on the lifted matrix Schaeffer's identity for flattened $\mathcal{I}$–tetrahedra. So we are forced to consider flat/charged $\mathcal{I}$–tetrahedra to get the versions of the lifted matrix Schaeffer's identity for all instances of enriched $\mathcal{I}$–transits. □

As promised in the Introduction, the next lemma shows that for $N>1$ the tensors $\mathcal{R}_N$ coincide, up to an Nth root of unity, with the symmetrized quantum dilogarithms used in [3] (see Definition 3.2 in that paper, where the Nth root moduli $\underline{w}'_i$ below are given by Nth roots of certain ratios of cocycle parameters, as explained in Remark 5.9).

Lemma 5.8 *For any flat/charged $\mathcal{I}$–tetrahedron we have*

$$\begin{aligned}\mathcal{R}_N(\Delta,b,w,f,c)^{i,j}_{k,l} &= \left((w'_0)^{-c_1}(w'_1)^{c_0}\right)^{\frac{N-1}{2}}\,\zeta^{(m+1)c_1(i-k)-(m+1)^2 f_1 c_0}\\ &\quad\times \frac{g(\underline{w}'_0)}{g(1)}\,\zeta^{kj+(m+1)k^2}\,\delta(i+j-l)\prod_{s=1}^{i-k-(m+1)c_0}\frac{(\underline{w}'_1)^{-1}}{1-\underline{w}'_0\zeta^s}\end{aligned}$$

where $\underline{w}'_n = (w_n)'_{f_n}$ and $w'_n = (w_n)'_{f_n - *_b c_n}$ for $n = 0,1,2$.

Proof Consider the right hand side. We have

$$\zeta^{(m+1)c_1(i-k)}\prod_{s=1}^{i-k-(m+1)c_0}\frac{(\underline{w}'_1)^{-1}}{1-\underline{w}'_0\zeta^s} = \prod_{s=1}^{i-k}\frac{(w'_1)^{-1}}{1-w'_0\zeta^s}\prod_{s=1}^{-(m+1)c_0}\frac{(\underline{w}'_1)^{-1}}{1-\underline{w}'_0\zeta^s}$$

and Lemma 8.2 implies

$$\frac{g(\underline{w}'_0)}{g(1)}\prod_{s=1}^{-(m+1)c_0}\frac{(\underline{w}'_1)^{-1}}{1-\underline{w}'_0\zeta^s} = \frac{g(w'_0)}{g(1)}\,\zeta^{(m+1)^2 f_1 c_0}.$$

Gathering these formulas we find the result. □

Remark 5.9 In [3] we started with the Faddeev-Kashaev's matrix of $6j$–symbols for the cyclic representation theory of a Borel quantum subalgebra $\mathcal{B}_\zeta$ of $U_\zeta(sl(2,\mathbb{C}))$, where $\zeta = \exp(2i\pi/N)$. We associated this matrix to a tetrahedron equipped with a 1–cocycle taking values in the Borel subgroup B of upper triangular matrices of $SL(2,\mathbb{C})$. This was possible due to a very natural parametrization of the cyclic irreducible representations of $\mathcal{B}_\zeta$ (see Remark 8.5). Only *a posteriori* we noticed that the relevant parameters were certain ratios of the matrix entries of the cocycle values, that corresponded to the cross-ratio moduli of determined hyperbolic ideal tetrahedra. We used the *idealization* procedure to transfer in the set up of $\mathcal{I}$–triangulations the results and the computations previously obtained in terms of cocycle parameters.

However, the fundamental objects were just the quantum basic matrix dilogarithms $\mathcal{L}_N$. Studying them directly is far from a mere rephrasing of the results in [3]. As a by-product, here we show that the symmetrization is intrisically related to the algebra $\mathcal{B}_\zeta$. Also, in [3] a choice of flattening was *hidden* in a preliminary choice at hand of the Nth roots of the cocycle parameters; the integral charge was intended as a way to contruct *another* tensor from $\mathcal{L}_N$, rather than a way to evaluate the *same* matrix valued function on different *variable* branches of the Riemann surface $\widehat{\mathbb{C}}$, globally organized by the combination of flattenings and charges.

6 Classical and quantum dilogarithmic invariants

In this section we define the state sum invariants based on the matrix dilogarithms $\mathcal{R}_N$. As this construction is a generalization and a refinement of the QHI's one, we shall often refer to [3]. For the sake of clarity, we begin with some general facts about 3–manifold triangulations.

A few generalities on 3–manifold triangulations Consider a tetrahedron Δ with its usual triangulation with 4 vertices, and let Γ_Δ be the interior of the 2–skeleton of the dual cell decomposition. A *simple* polyhedron P is a 2–dimensional compact polyhedron such that each point of P has a neighbourhood which can be embedded into an open subset of Γ_Δ. A simple polyhedron P has a natural stratification given by its singularities; P is *standard* if all the strata of this stratification are open cells; depending on the dimension, we call them *vertices, edges* and *regions.*

Every compact 3–manifold Y (which for simplicity we assume connected) with non-empty boundary has *standard spines* [9], that is standard polyhedra P together with an embedding in $\mathrm{Int}(Y)$ such that Y is a regular neighbourhood of P. Moreover, Y can be reconstructed from any of its standard spines. Since we shall always work with combinatorial data encoded by triangulations/spines, which define the corresponding manifold only up to PL-homeomorphisms, we shall systematically forget the underlying embeddings.

A *singular triangulation* of a 3–dimensional polyhedron Q is a triangulation for which self-adjacencies and multiple adjacencies of 3–simplices along 2–faces are allowed. This shall be simply called a *triangulation* of Q. For any Y as above, let us denote by $Q(Y)$ the space obtained by collapsing each connected component of ∂Y to a point. A (topological) *ideal triangulation* of Y is a triangulation T of $Q(Y)$ such that the vertices of T are precisely the points of $Q(Y)$ corresponding to the components of ∂Y. By removing small open neigbourhoods of the vertices of $Q(Y)$, any ideal triangulation leads to a cell decomposition of Y by *truncated tetrahedra*, which induces a (singular) triangulation on ∂Y. If Y is oriented, the tetrahedra of any triangulation inherit the induced orientation. From now on we will consider only oriented manifolds.

For any ideal triangulation T of Y, the 2–skeleton of the *dual* cell-decomposition of $Q(Y)$ is a standard spine $P(T)$ of Y. This procedure can be reversed, so that we can associate to each standard spine P of Y an ideal triangulation $T(P)$ of Y such that $P(T(P)) = P$. Hence standard spines and ideal triangulations are dual equivalent viewpoints which we will freely intermingle.

Consider now a compact closed 3–manifold W. For any $r_0 \geq 1$, let Y be the manifold obtained by removing r_0 disjoint open balls from W. By definition $Q(Y) = W$ and any ideal triangulation of Y is a singular triangulation of W with r_0 vertices. It is easily seen that all triangulations of W come in this way from ideal triangulations.

In Figure 9 and Figure 10 we recall some elementary moves on the triangulations and simple spines of a polyhedron $Q(Y)$. They are called the (respectively dual) $2 \to 3$ move, bubble move, and $0 \to 2$ move. The bubble move consists in replacing a 2–simplex by the cone on a 2–sphere triangulated by two 2–simplices. It is a fact (see [25, 35]) that standard spines of the same compact oriented 3–manifold Y with boundary and with at least two vertices (which, of course, is a painless requirement) may always be connected by means of a finite sequence of the dual $2 \to 3$ move and its inverse. The dual result holds for ideal triangulations. In order to handle triangulations of closed 3–manifolds we need a move which allows us to vary the number of vertices, like the bubble move.

We say that a triangulation T of a compact closed 3–manifold W is *quasi regular* if all the edges have distinct vertices. Every W admits quasi-regular triangulations.

Global branching A global branching on an ideal triangulation T of Y (see Section 2.1.4) can be defined in terms of the dual orientations of the regions of the standard spine $P(T)$, which are dual to the edges of T. In fact, a global branching gives the spine $P(T)$ a structure of oriented *branched surface*

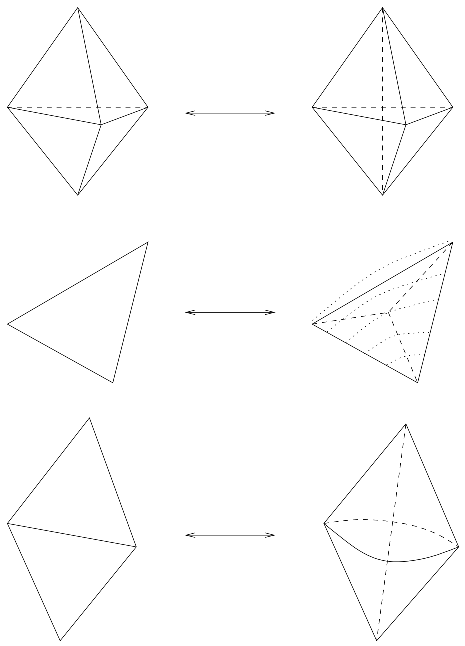

Figure 9: The moves between singular triangulations

embedded in Y (this also justifies the term "branching" - this point of view is widely developed in [6]). Every Y has triangulations supporting a branching [6, Theorem 3.4.9].

We introduced in Subsection 2.2.1 explicitely described $\mathcal{I}$–*triangulations* for compact closed pairs (W, ρ) or gentle cusped manifolds M. This notion extends straightforwardly to (topological) ideal triangulations of any compact oriented Y.

It is easy to see that the S_4–action on $\mathcal{I}$–tetrahedra defined before Theorem 2.1 extends to $\mathcal{I}$–triangulations: a global change of the branching turns any $\mathcal{I}$–triangulation (T, b, w) into another $\mathcal{I}$–triangulation (T, b', w').

Pseudo-developing maps We describe here an important geometric object associated to any $\mathcal{I}$–triangulation, that clarifies the role of the edge compatibility condition (8). Given an $\mathcal{I}$–triangulation (T, b, w) of Y, lift T to a cellulation $\widetilde{T}$ of the universal covering $\widetilde{Y}$, and fix a base point $\tilde{x}_0$ in the 0–skeleton of $\widetilde{T}$; denote by x_0 the projection of $\tilde{x}_0$ onto Y. For each tetrahedron in $\widetilde{T}$ that contains $\tilde{x}_0$, use the moduli of its projection in T to define an hyperbolic ideal tetrahedron, by respecting the gluings in $\widetilde{T}$. Doing similarly with the vertices

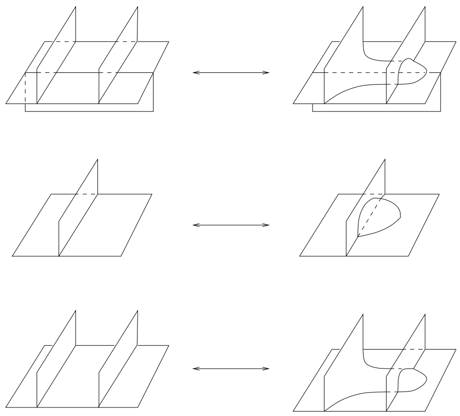

Figure 10: The moves on standard spines

adjacent to $\tilde{x}_0$ and so on, we construct an image in $\bar{\mathbb{H}}^3$ of a complete lift of T in $\widetilde{T}$, having one tetrahedron in each $\pi_1(Y,x_0)$–orbit. The edge compatibily implies that for any two paths of tetrahedra in $\widetilde{T}$ having a same starting point, we get the same end point. This construction extends to a piecewise-linear map $D\colon \widetilde{Y} \to \bar{\mathbb{H}}^3$, equivariant with respect to the action of $\pi_1(Y,x_0)$ and $PSL(2,\mathbb{C})$. So we eventually find: a representation $\tilde{\rho}\colon \pi_1(Y,x_0) \to PSL(2,\mathbb{C})$ satisfying $D(\gamma(x)) = \tilde{\rho}(\gamma)D(x)$ for each $\gamma \in PSL(2,\mathbb{C})$; a piecewise-straight continuous section of the flat bundle $\widetilde{Y} \times_{\tilde{\rho}} \bar{\mathbb{H}}^3 \to Y$, with structural group $PSL(2,\mathbb{C})$ and total space the quotient of $\widetilde{Y} \times \bar{\mathbb{H}}^3$ by the diagonal action of $\pi_1(Y,x_0)$ and $\tilde{\rho}$. The map D behaves formally as a developing map for a $(PSL(2,\mathbb{C}),\mathbb{H}^3)$–structure on Y (see eg [5, Ch. B] for this notion). By the arbitrary choices we made, only the conjugacy class ρ of $\tilde{\rho}$ is well defined, and D is only defined up to the left action of $PSL(2,\mathbb{C})$. This class is preserved by $\mathcal{I}$–transits.

Flat/charged $\mathcal{I}$–transits In Subsection 2.1.3 we defined the notion of a $2 \leftrightarrow 3$ flat/charged $\mathcal{I}$–transit. We need to discuss this notion for the other triangulation local moves. We stipulate that whenever they apply to a global triangulation, everything remains unchanged outside the support of the given move. For clarity we repeat the definitions also for the $2 \to 3$ move.

An *$\mathcal{I}$–transit* $(T,b,w) \to (T',b',w')$ of $\mathcal{I}$–triangulations of the same 3–manifold Y consists of a bare triangulation move $T \to T'$ that extends to a branching move $(T,b) \to (T',b')$, ie, the two branchings coincide on the 'common' edges of T and T'; moreover the modular triples have the following behaviour.

For a $2 \to 3$ move we require that for each common edge $e \in E(T) \cap E(T')$ we have

$$\prod_{a\in\epsilon_T^{-1}(e)} w(a)^* = \prod_{a'\in\epsilon_{T'}^{-1}(e)} w'(a')^* \tag{39}$$

where $*=\pm 1$ according to the b–orientation of the abstract tetrahedron containing a (respectively a').

For the $0\to 2$ and bubble move we require that for each edge $e\in E(T')$ we have

$$\prod_{a'\in\epsilon_{T'}^{-1}(e)} w'(a')^* = 1. \tag{40}$$

The $\mathcal{I}$–transits for negative $3\to 2$ moves are defined in the same way, and for negative $2\to 0$ and bubble moves w' is defined by simply forgetting the moduli of the two disappearing tetrahedra.

Consider a $2\to 3$ $\mathcal{I}$–transit $(T,b,w)\to(T',b',w')$ as in Figure 4. Give a flattening to each tetrahedron of the initial configuration, and denote by $\mathrm{l}\colon E_\Delta(T)\to\mathbb{C}$ the corresponding log-branch function on T. A map $f'\colon E_\Delta(T')\to\mathbb{Z}$ defines a $2\to 3$ *flattening transit* $(T,b,w,f)\to(T',b',w',f')$ if for each common edge $e\in E(T)\cap E(T')$ we have

$$\sum_{a\in\epsilon_T^{-1}(e)} *\,\mathrm{l}(a) = \sum_{a'\in\epsilon_{T'}^{-1}(e)} *\,\mathrm{l}'(a') \tag{41}$$

where $*=\pm 1$ according to the b–orientation of the tetrahedron that contains a (respectively a').

A map $f'\colon E_\Delta(T')\to\mathbb{Z}$ defines a $0\to 2$ (respectively bubble) *flattening transit* if for each edge $e\in E(T')$ we have

$$\sum_{a'\in\epsilon_{T'}^{-1}(e)} *\,\mathrm{l}'(a') = 0. \tag{42}$$

For negative $2\to 0$ and bubble moves the flattening transits are defined by simply forgetting the flattenings of the two disappearing tetrahedra.

Note that the relations (42) mean that the two new tetrahedra have *the same* log-branches, for their b–orientations are always opposite (and similarly for (40)).

We saw in Subsection 2.2.1 that we need to fix an (arbitrary) non empty link L in W in order to remove an obstruction to the existence of global integral charges on triangulations of W. So, we have to refine all the transits in order to make the set of distinguished triangulations (T,H) for (W,L) stable with respect to them. First, a negative $3 \to 2$ or $2 \to 0$ move is admissible if and only if the disappearing edge does not belong to the link. Moreover (see [3, Section 4.1.1] for details):

We say that a bubble move $(T,H) \to (T',H')$ on a 2–face t of T is *distinguished* if t contains an edge e of H, and H' is defined by replacing e in H with the two new edges of T' making with e the boundary of a 2–face. We have a *bubble charge transit* $(T,H,c) \to (T',H',c')$ if the sum of charges: stays equal about the two edges of t distinct from e; goes from 0 to 2 about e; is equal to 0 about the two new edges of H'; is equal to 2 for the remaining edge of the two new tetrahedra of T'.

Such bubble transits preserve the Hamiltonian property of the link realization. The flat/charged $\mathcal{I}$–transits are just obtained by assembling the above definitions.

$\mathcal{D}$–triangulations of pairs (W,ρ) This notion, as well the *idealization* procedure, was introduced in Subsection 2.2.1. Recall that quasi-regular triangulations of W support idealizable $\mathcal{D}$–triangulations for any pair (W,ρ) (in fact *generic* 1-cocycles on a quasi-regular triangulation are idealizable). Moreover, there is the following natural notion of:

$\mathcal{D}$–transits Let $(T_0,b_0) \to (T_1,b_1)$ be a transit of branched triangulations of W and $z_k \in Z^1(T_k;PSL(2,\mathbb{C}))$, $k=0,1$. We have a *cocycle transit* $(T_0,z_0) \leftrightarrow (T_1,z_1)$ if z_0 and z_1 agree on the common edges of T_0 and T_1. This makes an *idealizable cocycle transit* if both z_0 and z_1 are idealizable 1–cocycles, and in this case we say that $(T_0,b_0,z_0) \leftrightarrow (T_1,b_1,z_1)$ is a $\mathcal{D}$–transit. It is easy to see that z_0 and z_1 as above represent the same flat bundle ρ. We have:

Proposition 6.1 ([3], Proposition 2.16) *Consider a fixed pair (W,ρ), and denote by $\mathcal{I}$ the idealization map $\mathcal{T} \to \mathcal{T}_{\mathcal{I}}$ on the $\mathcal{D}$–triangulations of (W,ρ). For any $\mathcal{D}$–transit $\mathfrak{d}$ there exists an $\mathcal{I}$–transit $\mathfrak{i}$ (respectively for any $\mathfrak{i}$ there exists $\mathfrak{d}$) such that $\mathfrak{i} \circ \mathcal{I} = \mathcal{I} \circ \mathfrak{d}$.*

Similarly, the natural behaviour of $\mathcal{D}$–triangulations with respect to branching changes dominates, via the idealization, the one of $\mathcal{I}$–triangulations.

Globally flat/charged $\mathcal{I}$–triangulations for either triples (W,L,ρ) or gentle cusped manifolds M were defined in 2.2.1. Let us discuss now *arbitrary* cusped manifolds.

General cusped manifolds Let M be an arbitrary cusped manifold. Given a quasi-geometric geodesic triangulation (T,w) of M, there exists a finite sequence $T \to \ldots \to T'$ of positive $2 \to 3$ moves such that T' supports a global branching b' [6, Theorem 3.4.9]. For each move, we can define the transit of cross-ratio moduli by (39), with $* = 1$ everywhere.

Definition 6.2 We say that M is *weakly-gentle* if there exists such a sequence $T \to \ldots \to T'$ of positive $2 \to 3$ moves that lifts to a sequence of transits $(T,w) \to \ldots \to (T',w')$. In that case, we call $(T',b',(w')^{*_b})$ *an $\mathcal{I}$–triangulation* of M. For any flattening f' for $(w')^{*_b}$ and integral charge c', we say that $(T',b',(w')^{*_b},f',c')$ is a flat/charged $\mathcal{I}$–triangulation of M.

In this definition, the exponent $*_b$ means that the cross-ratio moduli of a tetrahedron are turned to the inverse if its branching orientation is negative. The authors do not know any example of non weakly-gentle cusped manifolds. Roughly speaking, M is not weakly-gentle if, in order to give any quasi-geometric geodesic triangulation of M a global branching, we are forced to introduce new interior vertices by performing some bubble moves. Dealing with the *quantum* state sums, this gives rise to a technical difficulty similar to the one that leads to the "link fixing" for closed manifold W. This motivates the following definition.

Definition 6.3 If M is a cusped manifold that is not weakly-gentle, we call a *marking* of M the choice of an edge l in the canonical Epstein–Penner cell decomposition of M. A *flat/charged $\mathcal{I}$–triangulation* $(T',H',b',(w')^{*_b},f',c')$ *of* (M,l) consists of: a triangulation T' of the polyhedron $Q(M)$, obtained as in Definition 6.2 from a quasi-geometric geodesic triangulation of M via positive $2 \to 3$ moves *and bubble moves*; a Hamiltonian subcomplex H' of the 1–skeleton of T' with one edge l_1 isotopic to l, and the union of the other edges isotopic to a second copy l_2 of l (we summarize this property by saying that (T',H') is a *distinguished* triangulation of (M,l)); a flattening f' for $(w')^{*_b}$; a collection c' of integral charges on the abstract tetrahedra of T', such that the sum of the charges is equal to 4 about l_1, 0 about each edge of l_2, and 2 about the other edges of T'.

If M is weakly-gentle we set $l = \emptyset$, so that the present definition incorporates Definition 6.2.

From now on, an $\mathcal{I}$–triangulation of a cusped manifold M always mean one as in Definition 6.3, possibly forgetting the marking and the flat/charge.

Let us say that a $\mathbb{Z}$–valued decoration of the abstract edges of an $\mathcal{I}$–triangulation is a *rough* integral charge, flattening, or flat/charge if it is *locally* of that form, on each tetrahedron. Recall that a flat/charged $\mathcal{I}$–tetrahedron (Δ, b, w, f, c) defines an enriched $\mathcal{I}$–tetrahedron (Δ, b, w, a) as in Section 5.1 by putting $a_j = f_j - *_b c_j$, $j = 1, 2, 3$.

Lemma 6.4 *Any rough flat/charge whose associated Nth–branch map satisfies the relations (30) comes from a pair (f, c), where f is a global flattening and c a global integral charge.*

Proof For any rough flattening f and integral charge c, any other is of the form $f' = f + b$ or $c' = c + d$, where b satisfies $b_0 + b_1 + b_2 = 0$, and similarly for d. So any rough flat/charge a locally appears as $a = f - *_b c = f' - *_b c''$ for a suitable rough integral charge c''. In particular we can assume that f is a global flattening. But imposing now the condition (30) for all $N > 1$ simultaneously, we realize that c is necessarily a global integral charge. □

Cohomological normalization of flattenings and charges The reductions mod(2) of both global flattenings and integral charges induce cohomology classes in $H^1(W; \mathbb{Z}/2\mathbb{Z})$ or $H^1(M; \mathbb{Z}/2\mathbb{Z})$. Moreover, in the case of a cusped manifold M, both the log-branches and integral charges induce classes in $H^1(\partial M; \mathbb{Z})$, where ∂M denotes the toric boundary components of the natural compactification of M. These classes are defined as follows. For any mod(2) (respectively integral) 1–homology class a in W or M (respectively b in ∂M), realize it by a disjoint union of (respectively oriented and essential) closed paths transverse to the triangulation and 'without back-tracking', ie, such that they never depart from a 2–face of a tetrahedron (respectively 1–face of a triangle) by which they entered. Then the mod(2) sum of the flattenings or charges of the edges that we encounter when following such paths in W or M define the value of the corresponding class on a. Similarly, the *signed* sum of the log-branches or charges of the edges whose ends are vertices that we encounter when following such paths on ∂M define the value of the corresponding class on b; for each vertex v the sign is $*_b$ (respectively $-*_b$) if, with respect to v, the path goes in the direction (respectively opposite to the one) given by the orientation of ∂M. In Lemma 4.12 of [3] we proved that the transits of global integral charges for pairs (W, L) preserve these cohomology classes. This result extends immediately to cusped manifolds M, to flattenings for both (W, ρ) or M, and also to log-branches for ∂M. So, from now on, we normalize the global flattenings and integral charges by requiring that *all these classes are trivial.* (Otherwise, the dilogarithmic invariants would just depend on them).

Theorem 6.5

(1) *Every pair* (W,ρ) *or cusped manifold* M *has* $\mathcal{I}$*–triangulations* $\mathcal{T}_{\mathcal{I}}$*, and every such triangulation admits global flattenings* f.

(2) *For every triple* (W,L,ρ) *or pair* (M,l) *there exist distinguished* $\mathcal{I}$*–triangulations, and every such triangulation admits global integral charges* c.

The first claim in (1) is essentially proved in Subsection 2.2.1. For (W,ρ) take any quasi-regular triangulation with an idealizable cocycle representing ρ. For non-weakly-gentle cusped manifolds, it is enough to use generic bubble transits in the sequence of moves considered in Definition 6.3 (see the proof of Theorem 6.8). The first claim in (2) for triples (W,L,ρ) is a result of [3]. The existence of global integral charges for (W,L) is shown in Chapter 2 of [1]. For pairs (M,l), see the proof of Theorem 6.8. The general existence of global flattenings and integral charges are slight adaptations of earlier results of W Neumann (first claim of Theorem 4.2 in [30], and Theorem 2.4 i) in [28] respectively).

Let $(\mathcal{T}_{\mathcal{I}},f)$ be a flattened $\mathcal{I}$–triangulation of (W,ρ) or M. Let $(\mathcal{T}_{\mathcal{I}},f,c)$ be a flat/charged $\mathcal{I}$–triangulation of (W,L,ρ) or (M,l). By using the symmetrized classical dilogarithm $\mathcal{R}_1$, we can define the state sum

$$\mathcal{R}_1(\mathcal{T}_{\mathcal{I}},f)=\textstyle\prod_i \mathcal{R}_1(\Delta^i,b^i,w^i,f^i)$$

which reads as the exponential of $1/i\pi$ times a signed sum of uniformized Rogers dilogarithms. By using the quantum matrix dilogarithms $\mathcal{R}_N$, $N>1$, we can define the state sums

$$\mathcal{R}_N(\mathcal{T}_{\mathcal{I}},f,c)=\textstyle\sum_\alpha\prod_i \mathcal{L}(\Delta^i,b^i,w^i,f^i,c^i)_\alpha$$

where the sum is over the states of $\mathcal{T}_{\mathcal{I}}$. Let us denote by v the number of 'interior' vertices of a given triangulation ($v=0$ in the case of a weakly-gentle cusped manifold). Put $X=(W,\rho)$ or M, and $\widetilde{X}=(W,L,\rho)$ or (M,l) (recall that $l=\emptyset$ if M is weakly-gentle). The main result of this section is:

Theorem 6.6

(1) *The value of* $\mathcal{R}_1(\mathcal{T}_{\mathcal{I}},f)$ *does not depend, up to sign, on the choice of the flattened* $\mathcal{I}$*–triangulation* $(\mathcal{T}_{\mathcal{I}},f)$*. Hence, up to sign,* $H_1(X):=\mathcal{R}_1(\mathcal{T}_{\mathcal{I}},f)$ *is a well defined invariant called the* classical dilogarithmic invariant *of* X.

(2) *For every odd* $N>1$*, the value of* $N^{-v}\mathcal{R}_N(\mathcal{T}_{\mathcal{I}},f,c)$ *does not depend, up to sign and multiplication by* N*th roots of unity, on the choice of the flat/charged* $\mathcal{I}$*–triangulation* $(\mathcal{T}_{\mathcal{I}},f,c)$*. Hence* $H_N(\widetilde{X}):=N^{-v}\mathcal{R}_N(\mathcal{T}_{\mathcal{I}},f,c)$ *is a well defined invariant, up to the same ambiguity, called* a quantum dilogarithmic invariant *of* $\widetilde{X}$.

We note that the normalization N^{-v} of the quantum state sums comes from the behaviour of $\mathcal{R}_N$ for bubble flat/charged $\mathcal{I}$–transits.

The first step is to extend the proof of the transit invariance of the state sums to the other transits, besides the $2 \to 3$ one.

Lemma 6.7 *For any odd $N \geq 1$ and any flattened (respectively flat/charged) $\mathcal{I}$–transit between flattened (respectively flat/charged) $\mathcal{I}$–triangulations of X (respectively $\widetilde{X}$), the traces of the two patterns of associated matrix dilogarithms of rank N are equal, possibly up to a sign and an Nth root of unity phase factor.*

Proof Consider an $\mathcal{I}$–triangulation of X, and a flattening transit that lifts the $\mathcal{I}$–transit of Figure 4. As usual, denote by Δ^j the tetrahedron opposite to the j-th vertex for the ordering of the vertices induced by the branching. Do a further $2 \to 3$ flattening transit with Δ^0 and Δ^2. A mirror image of Δ^4 appears, which together with Δ^4 forms the final configuration of a $0 \to 2$ flattening transit. The other two new flattened $\mathcal{I}$–tetrahedra have the same decorations and gluings as Δ^1 and Δ^3. Hence Theorem 4.4 implies that $0 \leftrightarrow 2$ transits correspond to identities of the form (above for Δ^4):

$$\mathcal{R}_1(\Delta, b, w, f)\ \mathcal{R}_1^{-1}(\bar{\Delta}, b, w, f) = 0.$$

Here $\bar{}$ denotes the opposite orientation; as remarked before Proposition 6.1, the mirror moduli and flattenings are the same. Finally, we note that the final configuration of a bubble move is just obtained by gluing two more faces in the final configuration of a $0 \to 2$ move, and the moduli and flattenings behave well with respect to this gluing (that is the relations (40) and (42) are satisfied). So we get the very same relation for a bubble flattening transit. This proves the statement for $\mathcal{R}_1$.

We prove the statement for $\mathcal{R}_N$, $N > 1$, and the $0 \leftrightarrow 2$ and bubble flat/charged $\mathcal{I}$–transits via the very same arguments. The corresponding two term relations are:

$$\mathcal{R}_N(\Delta, b, w, f, c)\ \mathcal{R}_N(\bar{\Delta}, b, w, f, c) \equiv_N \mathrm{Id}_{\mathbb{C}^N} \otimes \mathrm{Id}_{\mathbb{C}^N}$$

$$\mathrm{Trace}_i\big(\mathcal{R}_N(\Delta, b, w, f, c)\ \mathcal{R}_N(\bar{\Delta}, b, w, f, c)\big) \equiv_N N\ \cdot\ \mathrm{Id}_{\mathbb{C}^N}$$

where $\bar{}$ is as above, and the trace is over one of the tensor factors in the first relation. The trace appears for the bubble transits because, as we said above, the final configuration of the underlying moves are obtained from $0 \leftrightarrow 2$ ones by gluing two more faces. Again, we use the fact that the charges behave well under this gluing. □

As the state sums are fully transit invariant, the main theorem follows from the following triangulation connectedness theorem.

Theorem 6.8

(1) *Any two flattened $\mathcal{I}$–triangulations of X can be connected via a finite number of transits of flattened $\mathcal{I}$–triangulations.*

(2) *Any two flat/charged $\mathcal{I}$–triangulations of $\widetilde{X}$ can be connected via a finite number of transits of flat/charged $\mathcal{I}$–triangulations.*

Remarks 6.9 (1) A version of (1) was proved independently by Neumann in [31].

(2) For triples (W, L, ρ), this theorem is a genuine refinement of the main result of [3]. In that paper, we gave a complete proof of a weaker connectedness result, enough to show that the QHI are well defined, but not to get the invariance of the scissors congruence classes discussed in Section 7. The present strong version was mentioned in Section 4.5 of [3] without proof. Along with the proof of Theorem 6.8 we shall stress the main differences with respect to [3].

In the course of the proof we shall use the following two results. The first ensures the connectedness of the space of *branched* (topological) ideal triangulations of an arbitrary compact oriented 3–manifold, and the second is an uniqueness (rigidity) result for $\mathcal{I}$–triangulations of cusped manifolds with maximal volume.

Theorem 6.10 [10] *For any two branched triangulations T and T' of a same compact oriented 3–manifold Y, there exists a finite sequence $T \to \ldots \to T'$ made of $2 \to 3$, $0 \to 2$, bubble moves or their inverses.*

Theorem 6.11 (See [13] or [19]) *Let (T, b, w) be an arbitrary $\mathcal{I}$–triangulation of a cusped 3–manifold M. Then, among all $\mathcal{I}$–triangulations supported by (T, b), this is the only one such that the algebraic sum of the volumes of its $\mathcal{I}$–tetrahedra equals Vol(M).*

Proof of Theorem 6.8 In Proposition 4.27 of [3] we proved for triples (W, L, ρ) a version of Theorem 6.8 *up to branching changes*; we included in the definition of a $\mathcal{D}$–triangulation that it was *quasi-regular*, and we used *only* $\mathcal{I}$–triangulations covered by such $\mathcal{D}$–triangulations, via the idealization. Moreover, the flattenings were hidden (see Remark 5.9). We note that for pairs (W, ρ) the proof is easier because there is no link.

The proof of Theorem 6.8 up to branching changes is enough for a weaker version of Theorem 6.6 (1), where the invariants are defined up to a sign and multiplication by 12-th roots of unity when $N = 1$. We have only to apply Proposition 4.2. On the other hand, to get the statement with the weaker ambiguity we must show that even the branching changes can be realized by means of a sequence of transits, and then exploit the global compensations occuring when doing them (see the proof of Theorem 4.4). For that, we use Theorem 6.10.

The use of $\mathcal{I}$–triangulations dominated, via the idealization, by quasi-regular $\mathcal{D}$–triangulations makes the proof simpler because these triangulations support *generic* idealizable $PSL(2,\mathbb{C})$–valued 1–cocycles, and we have room for choosing paths of transits between their idealizations. In [3] we proved first that quasi-regular $\mathcal{D}$–triangulations can be connected by (quasi-regular) $\mathcal{D}$–transits, and we applied Lemma 6.1. Here we want to do it for $\mathcal{I}$–triangulations of a triple (W,L,ρ) dominated by non necessarily quasi-regular $\mathcal{D}$–triangulations, as it can actually happen when ρ is non trivial, and develop arguments that eventually apply also to $\mathcal{I}$–triangulations of cusped manifolds, that are not dominated by any $\mathcal{D}$–triangulation. The key point is to show that we can get round of accidental stops to the $\mathcal{I}$–transits. We do it as follows.

Suppose we have a sequence of branched moves between two $\mathcal{I}$–triangulations supported by T and T', such that some of them do not allow an $\mathcal{I}$–transit (for instance, when $x = y$ in Figure 4); call them *bad moves*. Take the first, m. Dually (at the level of standard spines), we are in a situation like in Figure 11, where B is a 3–cell with the two regions R' and R'' lying on its boundary (immersed) 2–sphere. The shaded region R is "bad", because one among the cross-ratio moduli attached to its corners belongs to the forbidden values $\{0,1\}$ (recall that the vertices are dual to tetrahedra). Just before m, do a (positive) bubble $\mathcal{I}$–transit by gluing a disk in the *interior* of B, which we call a *capping disk*. For distinguished triangulations of triples (W,L,ρ), this bubble move takes place near the link H, and is as explained before Proposition 6.1. In particular, one, R_H, of the two regions on which we glue the capping disk is dual to an edge of H, and after the move, the capping disk as well as the new region locally "opposite" to it and adjacent to R_H are dual to edges of H. (Recall that H is Hamiltonian, so that R_H exists).

A key point is that there is one degree of freedom in choosing the bubble $\mathcal{I}$–transit. So, for a generic choice of it we can slide via successive $2 \to 3$ $\mathcal{I}$–transits a portion of the capping disk in the position shown at the left of Figure 12, and also achieve the move m' as an $\mathcal{I}$–transit (we have to keep track of the region R_H, for otherwise the link H would split). Of course, now the cross-ratio moduli attached to the tetrahedra dual to the vertices of that configuration, ie, at all the corners of the regions $R_1, \dots, R_5$, are altered compared to those already present in Figure 11. By continuing the initial sequence between the spines $P(T)$ and $P(T')$, the regions involved in the (dual) moves following m can intersect the capping disk. However, the arguments of Proposition 4.23 in [3] show that the capping disk is not an obstruction for doing these subsequent moves, so that we eventually reach the very same position as in $P(T')$ by sliding "under" the capping disk (this is possible essentially because the moves are purely local). Hence, by applying the same procedure each time we meet a bad move and using the genericity of bubble $\mathcal{I}$–transits, we get a sequence of $\mathcal{I}$–transits from T to a triangulation T''. We can see that the standard spine $P(T'')$ dual to T'' is obtained from $P(T)$ just by gluing successively some (non adjacent) 2–disks, each one corresponding to a capping disk. Again by using the arguments of Proposition 4.23 in [3], we can remove them one after the other via further $\mathcal{I}$–transits, and eventually get a sequence of $\mathcal{I}$–transits from T to T'.

If the so obtained $\mathcal{I}$–triangulation supported by T' is not the one we started with, note that the very same arguments imply that we can find sequences of $\mathcal{I}$–transits from T' to a quasi-regular triangulation T_0, thus supporting two distinct $\mathcal{I}$–decorations. As there exist $\mathcal{D}$–triangulations supported by T_0 that dominate each of these $\mathcal{I}$–decorations, we can apply the proof of Proposition 4.27 in [3] to see that they can be connected via $\mathcal{I}$–transits.

Finally, the argument for the invariance with respect to the choice of integral charge given in Proposition 4.27 of [3] works word-by-word also for the flattenings, because flattenings and integral charges on the same triangulation are both affine spaces over the *same* integral lattice. This follows directly from their construction, see the references after Theorem 6.5. This concludes the proof for the case of closed manifolds.

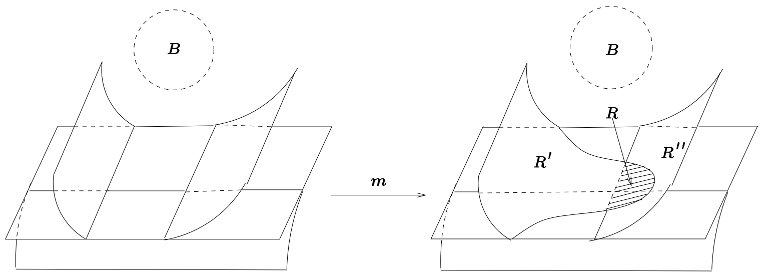

Figure 11: A bad move

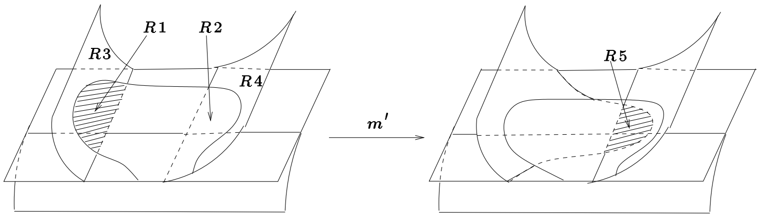

Figure 12: The capping disk of a bubble move turns the bad move into a good one

For ideal triangulations of cusped manifolds M, the same arguments allow us to prove the following preliminary result:

Let $\mathcal{T}_\mathcal{I} = (T, b, w)$ and $\mathcal{T}'_\mathcal{I} = (T', b', w')$ be two $\mathcal{I}$–triangulations of M (quasi geometric if M is gentle, as in Definition 6.3 otherwise). Then there exists a finite sequence of $\mathcal{I}$–transits connecting $\mathcal{T}_\mathcal{I}$ to $\mathcal{T}''_\mathcal{I} = (T', b', w'')$, that is another $\mathcal{I}$–triangulation supported by (T', b').

We stress that we possibly include also bubble transits, adding interior vertices. The volume of $\mathcal{T}_\mathcal{I}$, which is the algebraic sum of the volumes of its hyperbolic ideal tetrahedra, is not altered by $\mathcal{I}$–transits and the transits used for defining $\mathcal{I}$–triangulations of general (non gentle) cusped manifolds. So the volume of $\mathcal{T}_\mathcal{I}$, $\mathcal{T}'_\mathcal{I}$ and $\mathcal{T}''_\mathcal{I}$ is that of the hyperbolic manifold M, and Theorem 6.11 yields $w' = w''$.

We can include flattenings to the transits. However, the sum of the charges about one edge of the initial triangle involved in a bubble move goes from 2 to 4 (there was no link inside M), whereas in the final configuration this sum is equal to 0 for two of the new edges, and is 2 for all the other edges. Topologically, this means choosing two of the new edges to form an unknotted and properly embedded arc in the natural compactification of M (with one boundary torus at each end) that passes through the new "interior" vertex. Hence, starting from $\mathcal{T}_\mathcal{I}$, when we use a charge transit we obtain a rough integral charge which is not a global one as defined in Section 2.2.1.

If M is weakly-gentle, as $\mathcal{T}'_\mathcal{I}$ does not contain interior (non singular) vertices, any sequence of flat/charged $\mathcal{I}$–transits connecting $\mathcal{T}_\mathcal{I}$ to $\mathcal{T}'_\mathcal{I}$ eventually gives a true global integral charge.

If M is not weakly-gentle, we consider only distinguished $\mathcal{I}$–triangulations $(\mathcal{T}_\mathcal{I}, H)$ and $(\mathcal{T}'_\mathcal{I}, H')$ of (M, l), for a marking l of M. We do bubble charge transits as usual, on 2–simplices that contain an edge of l_2. Any sequence of flattening $\mathcal{I}$–transits connecting $(\mathcal{T}_{\mathcal{I}}, f)$ to $(\mathcal{T}'_{\mathcal{I}}, f')$ can be lifted to a sequence $(\mathcal{T}_{\mathcal{I}}, H, f, c) \rightarrow \ldots \rightarrow (\mathcal{T}'_{\mathcal{I}}, H', f'', c'')$ between distinguished flat/charged $\mathcal{I}$–triangulations of (M, l).

It remains to show the invariance with respect to the choice of flattening and integral charge on a fixed $\mathcal{I}$–triangulation of M. We do this as for closed manifolds. Namely, as already said above, the sets of flattenings and integral charges on a given ideal triangulation are both affine spaces over the same integral lattice, and the generators of this lattice depend only the local combinatorics of the triangulation (each generator is associated to the abstract star of an edge). □

Remark 6.12 (No discrepancy for weakly-gentle cusped manifolds) The flattenings of $\mathcal{I}$–triangulations of weakly-gentle cusped manifolds are just $-*_b$ times the integral charges. Thus, we can take $f - *_b c = -2 *_b c$ for the flat/charges of the matrix dilogarithms in $\mathcal{R}_N(\mathcal{T}_{\mathcal{I}}, f, c)$, $N > 1$. Hence, the classical and quantum dilogarithmic invariants of weakly-gentle cusped manifolds are defined on the very same geometric supports, the flattened $\mathcal{I}$–triangulations.

Remark 6.13 (Common features of classical and quantum invariants) We defined $\mathcal{R}_1(\mathcal{T}_{\mathcal{I}}, f)$ by taking the *exponential of* $(1/i\pi)$ *times* a sum of uniformized Rogers dilogarithms. In this way, for every $N \geq 1$, we have:

(1) $H_1(X)$ and $H_N(\widetilde{X})^N$, $N > 1$, are well defined up to sign ambiguity;

(2) $H_1(X)^* = H_1(X^*)$ and $H_N(\widetilde{X})^* = H_N(\widetilde{X}^*)$, $N > 1$, where * denotes the complex conjugation, and X^* (respectively $\widetilde{X}^*$) denotes $(-W, \rho^*)$ or $-M$ (respectively $(-W, L, \rho^*)$ or $(-M, l)$). For $N > 1$ this is a consequence of Proposition 4.29 in [3]. For $N = 1$ this follows easily from the behaviour of the Rogers dilogarithms under complex conjugation of its argument.

Remark 6.14 (On the phase ambiguity of $H_N(\tilde{X})$, $N > 1$) There is only a sign ambiguity for the flat/charged $\mathcal{I}$–transit corresponding to Figure 4. So we could expect that the ambiguity up to Nth roots of unity do vanish for *all* the transits in Corollary 5.7, due to global compensations (see Theorem 4.4 and Remark 4.5). However, although this happens for a certain non trivial subset of transits, in general this is not the case.

Indeed, by using Lemma 5.8, a statement of Proposition 5.3 with the full Nth root of unity dependence follows from Proposition 6.4 in the Appendix of [3], for a specific choice of flattenings of the involved moduli. The symmetry defects appearing there are formally the exponentials of terms of the *very same* form as those coming from Lemma 4.2, so that we get a table as in Section 4.3, where the defects are replaced with powers of ζ depending linearly on the charges. But we cannot deduce the global compensations, because, as $\zeta = \exp(2i\pi/N)$ and $N > 1$, we cannot work with the charges mod(2). At the time of this writing the authors do not see any way to renormalize $\mathcal{R}_N$ to avoid this discrepancy. Understanding the *geometric meaning* of the phase ambiguity is an important open question about the quantum dilogarithmic invariants.

6.1 $H_1(X)$ and Cheeger–Chern–Simons classes

For every pair (W,ρ) set

$$\mathbf{R}(W,\rho) := \mathrm{CS}(\rho) + i\ \mathrm{Vol}(\rho) \in \mathbb{C}/(\pi^2\mathbb{Z})$$

where $\mathrm{CS}(\rho)$ and $\mathrm{Vol}(\rho)$ are respectively the Chern-Simons invariant and the volume of ρ. As we consider ρ up to isomorphisms of flat bundles, it may be identified with the conjugacy class of its holonomy representations. We refer to Chapter 10-11 of [16] and the references therein for details on these notions.

Meyerhoff extended in [26] the definition of CS to cusped manifolds M, so that we can consider again

$$\mathbf{R}(M) := \mathrm{CS}(M) + i\ \mathrm{Vol}(M) \in \mathbb{C}/(\pi^2\mathbb{Z}).$$

The following result holds (we use the notation of Section 4):

Theorem 6.15 *Let X be either a pair (W,ρ) or a cusped manifold M, and $(\mathcal{T},f)$ be a flattened $\mathcal{I}$–triangulation of X. Then*

$$\mathbf{R}(X) = \sum_{\Delta\subset T} *\ \mathrm{R}(\Delta,b,w,f) \mod(\pi^2\mathbb{Z}).$$

Hence $H_1(X)$ is equal to $\exp\big((1/i\pi)\mathbf{R}(X)\big)$, where both invariants are defined up to a sign.

This theorem is proved in [31], using earlier deep results of Dupont–Sah [14], and Dupont [15].

Our proof that

$$H_1(X) = \exp\bigg((1/i\pi)\sum_{\Delta\subset T} *\ \mathrm{R}(\Delta,b,w,f)\bigg)$$

is a well defined invariant of X is independent of its identification with $\exp\big((1/i\pi)\mathbf{R}(X)\big)$. This proof shows that $\sum_{\Delta\subset T} * \mathrm{R}(\Delta,b,w,f) \mod(\pi^2\mathbb{Z})$ itself does not depend on the choice of $(\mathcal{T},f)$. It is based on direct geometric manipulations of decorated triangulations, and is structurally the same for the whole family of classical and quantum dilogarithmic invariants. On the other hand, in the classical case one can adopt the slightly different point of view of looking for *simplicial formulas* for the already known classical invariant $\mathbf{R}(X)$ (thus obtaining, by the way, that the values of these formulas do not depend on the choice of the combinatorial support).

7 Scissors congruence classes

We construct further invariants of 3–manifolds called *scissors congruence classes*, that belong to suitably defined *(pre)-Bloch-like groups.* Later we discuss some problems about the relations between these invariants and the dilogarithmic ones.

Fix one *base* oriented tetrahedron Δ. By varying the respective decorations, we get the sets of $\mathcal{D}$–tetrahedra $\{*_b(\Delta,b,z)\}$, $\mathcal{I}$–tetrahedra $\{*_b(\Delta,b,w)\}$, flattened $\mathcal{I}$–tetrahedra $\{*_b(\Delta,b,w,f)\}$ and flat/charged $\mathcal{I}$–tetrahedra $\{*_b(\Delta,b,w,f,c)\}$. Let us call them generically *$\mathcal{A}$–tetrahedra.* We will specify $\mathcal{A}=\mathcal{D},\mathcal{I},\mathcal{I}_f,\mathcal{I}_{fc}$ when necessary. We also denote by $\mathcal{A}$ the set of all $\mathcal{A}$–tetrahedra and by $\mathbb{Z}[\mathcal{A}]$ the free $\mathbb{Z}$–module generated by $\mathcal{A}$. We stipulate that the sign of $\mathcal{A}$–tetrahedra is compatible with the algebraic sum in $\mathbb{Z}[\mathcal{A}]$, ie, $-(-(\Delta,b,z))=(\Delta,b,z)$, and so on.

Any instance of $\mathcal{A}$–transit naturally induces a linear relation between the involved $\mathcal{A}$–tetrahedra. Consider the relations on $\mathbb{Z}[\mathcal{A}]$ generated by the five term identities corresponding to all instances of $2\leftrightarrow 3$ $\mathcal{A}$–transit. Denote by $\mathcal{P}(\mathcal{A})$ the resulting quotient of $\mathbb{Z}[\mathcal{A}]$. We call $\mathcal{P}(\mathcal{A})$ the *$\mathcal{A}$–(pre)-Bloch-like group.*

In the previous sections we described the behaviour of $\mathcal{A}$–tetrahedra with respect to the tetrahedral symmetries. The identification of $\mathcal{A}$–tetrahedra related by these symmetries gives new relations in $\mathbb{Z}[\mathcal{A}]$, whence a quotient map $\mathcal{P}(\mathcal{A})\to\mathcal{P}'(\mathcal{A})$. Clearly there are forgetful maps $\mathcal{P}(\mathcal{I}_{fc})\to\mathcal{P}(\mathcal{I}_f)\to\mathcal{P}(\mathcal{I})$, and similar ones for the $\mathcal{P}'(\mathcal{A})$, making a commutative diagram with the quotient maps $\mathcal{P}(\mathcal{A})\to\mathcal{P}'(\mathcal{A})$. If $\mathcal{T}$ is any $\mathcal{A}$–triangulation of any oriented 3–manifold Y, then the formal sum of the $\mathcal{A}$–tetrahedra of $\mathcal{T}$ determines an element $[\mathcal{T}]_{\mathcal{A}}\in\mathcal{P}(\mathcal{A})$ (respectively $[\mathcal{T}]'_{\mathcal{A}}\in\mathcal{P}'(\mathcal{A})$).

Some results of the previous section can be immediately rephrased and somewhat illuminated in the present set up. Noting that every $\mathcal{I}$–tetrahedron is the idealization of a $\mathcal{D}$–tetrahedron, the fact that $\mathcal{D}$–transits dominate $\mathcal{I}$–transits yields:

Proposition 7.1 *The idealization induces a surjective homomorphism $\mathcal{I}_{\mathcal{P}}\colon \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{I})$.*

Also, the results of Section 4 give the following.

Proposition 7.2 *For any element $\sum_i a_i *_{b_i} (\Delta, b_i, w_i, f_i)$ in $\mathcal{P}(\mathcal{I}_f)$ (respectively $\mathcal{P}'(\mathcal{I}_f)$), the formula*

$$\sum_i a_i *_{b_i} \mathrm{R}(\Delta, b_i, w_i, f_i)$$

defines a function $\mathrm{R}\colon \mathcal{P}(\mathcal{I}_f) \to \mathbb{C}/\pi^2\mathbb{Z}$ (respectively $\mathrm{R}'\colon \mathcal{P}'(\mathcal{I}_f) \to \mathbb{C}/(\pi^2/6)\mathbb{Z}$).

Obviously we can also define the function $H_1 = \exp(\mathrm{R}/i\pi)$ on $\mathcal{P}(\mathcal{I}_f)$.

Remark 7.3 We know from Section 4 that the function R' is well defined *only* $\mathrm{mod}(\pi^2/6)\mathbb{Z}$, while the weaker ambiguity in the definition of R is due to remarkable global compensations occurring in the $2 \leftrightarrow 3$ flattened $\mathcal{I}$–transits. This means, in particular, that $\mathcal{P}'(\mathcal{I}_f)$ is a *genuine* quotient of $\mathcal{P}(\mathcal{I}_f)$. Recall, on the contrary, that for the classical pre-Bloch group the tetrahedral symmetries are consequences of the five-term identity [14].

With the usual notation, denote $X = (W,\rho)$ or M, and $\widetilde{X} = (W,L,\rho)$ or (M,l). Let $(\mathcal{T},f)$ be a flattened $\mathcal{I}$–triangulation for X, with the associated element $[(\mathcal{T},f)]_{\mathcal{I}_f} \in \mathcal{P}(\mathcal{I}_f)$. Similarly, let $(\mathcal{T},f,c)$ be a flat/charged $\mathcal{I}$–triangulation for $\widetilde{X}$, with the associated element $[(\mathcal{T},f,c)]_{\mathcal{I}_{fc}} \in \mathcal{P}(\mathcal{I}_{fc})$. Here is the main result of this section:

Theorem 7.4

(1) *The class $[(\mathcal{T},f)]_{\mathcal{I}_f} \in \mathcal{P}(\mathcal{I}_f)$ does not depend on the choice of $(\mathcal{T},f)$. Hence $\mathfrak{c}_{\mathcal{I}_f}(X) = [(\mathcal{T},f)]_{\mathcal{I}_f}$ is a well defined invariant of X, called its $\mathcal{I}_f$–*scissors congruence class.

(2) *The class $[(\mathcal{T},f,c)]_{\mathcal{I}_{fc}} \in \mathcal{P}(\mathcal{I}_{fc})$ does not depend on the choice of $(\mathcal{T},f,c)$. Hence $\mathfrak{c}_{\mathcal{I}_{fc}}(X) = [(\mathcal{T},f,c)]_{\mathcal{I}_{fc}}$ is a well defined invariant of X, called its $\mathcal{I}_{fc}$–*scissors congruence class.

Proof By Theorem 6.8, any two $\mathcal{A}$–triangulations of a same 3–dimensional closed polyhedron can be connected by means of *arbitrary* $\mathcal{A}$–transits, including bubble and $0 \to 2$ transits. In particular, we can realize via transits arbitrary global branching changes. (Without invoking this fact we would get, in an easier way, the weaker result that the class is well defined in $\mathcal{P}'(\mathcal{A})$). Only the five term relations associated to $2 \leftrightarrow 3$ $\mathcal{A}$–transits occur in the definition of $\mathcal{P}(\mathcal{A})$. Hence we have to prove that the relations induced by all instances of the other transits follow from the five term ones. This is done in the following two lemmas (compare with Lemma 6.7). For simplicity we restrict the first to $\mathcal{P}[\mathcal{D}]$.

Given a $\mathcal{D}$–tetrahedron (Δ, b, z), consider the tetrahedron $\Delta!$ obtained by deforming an edge of Δ until it passes through the opposite edge. We still denote b and z the branching and cocycle induced on $\Delta!$. Note that we can identify Δ and $\Delta!$ as *bare* tetrahedra, since one is obtained from the other by a cellular self-homeomorphism.

Lemma 7.5 *The following relation holds in* $\mathcal{P}(\mathcal{D})$*:* $(\Delta, b, z) = (\Delta!, b, z)$.

Proof Let us prove a particular instance of this relation. All the others come in exactly the same way, for there is no restriction on the specific branching we choose in the arguments below. Our arguments are based on a pictorial encoding with decorated tetrahedra, but this is no loss of generality since the corresponding algebraic relations in $\mathcal{P}(\mathcal{D})$ may be thought as between abstract elements.

Consider the sequence of $2 \to 3$ $\mathcal{D}$–transits in Figure 13, starting with an arbitrary cocycle transit. Denote by $\mathcal{D}_i = (b_i, z_i)$ the decoration of Δ_i. We call $(\Delta_5, \mathcal{D}_5)$ and $(\Delta_8, \mathcal{D}_8)$ the two $\mathcal{D}$–tetrahedra glued along two faces in the final configuration (see Figure 14); $(\Delta_8, \mathcal{D}_8)$ is glued to $(\Delta_6, \mathcal{D}_6)$ and $(\Delta_7, \mathcal{D}_7)$ along f_1 and f_2.

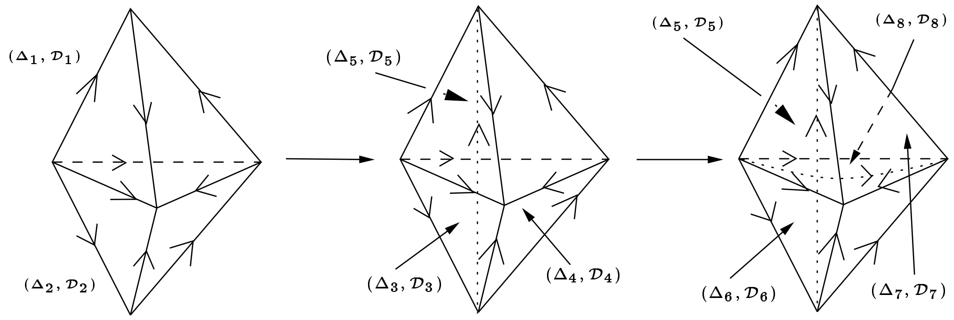

Figure 13: How to produce two term relations

The branchings and the cocycles of $(\Delta_6, \mathcal{D}_6)$ and $(\Delta_7, \mathcal{D}_7)$ are respectively the same as for $(\Delta_1, \mathcal{D}_1)$ and $(\Delta_2, \mathcal{D}_2)$. Hence we may identify $(\Delta_6, \mathcal{D}_6)$ with

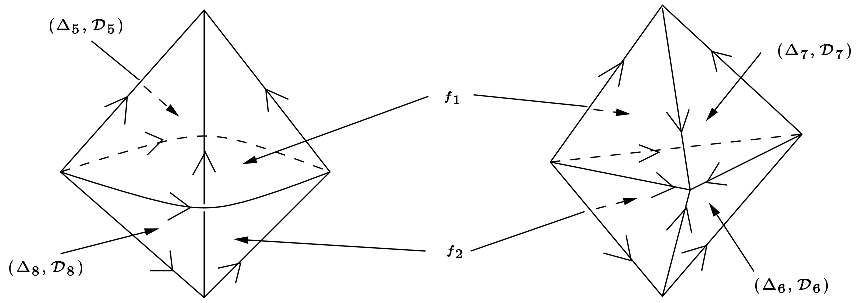

Figure 14: Decomposition of the final configuration in Figure 13

$(\Delta_1, \mathcal{D}_1)$ and $(\Delta_7, \mathcal{D}_7)$ with $(\Delta_2, \mathcal{D}_2)$. As a consequence of the five term relations in $\mathcal{P}(\mathcal{D})$, we deduce that the composition of transits in Figure 13 translates into

$$\begin{aligned}(\Delta_1, \mathcal{D}_1) + (\Delta_2, \mathcal{D}_2) &= (\Delta_5, \mathcal{D}_5) - (\Delta_8, \mathcal{D}_8) + (\Delta_6, \mathcal{D}_6) + (\Delta_7, \mathcal{D}_7) \\ &= (\Delta_5, \mathcal{D}_5) - (\Delta_8, \mathcal{D}_8) + (\Delta_1, \mathcal{D}_1) + (\Delta_2, \mathcal{D}_2)\end{aligned}$$

where we note that $\mathcal{D}_8$ gives a negative orientation to Δ_8. This yields $(\Delta_5, \mathcal{D}_5) - (\Delta_8, \mathcal{D}_8) = 0$. As the mirror image of $(\Delta_5, \mathcal{D}_5)$ is $(\Delta_8, \mathcal{D}_8)$, this proves the lemma. □

Corollary 7.6 *The relations in $\mathbb{Z}[\mathcal{A}]$ corresponding to the $0 \leftrightarrow 2$ and bubble $\mathcal{A}$–transits are consequences of the relations corresponding to the $2 \leftrightarrow 3$ $\mathcal{A}$–transits.*

Proof A $0 \leftrightarrow 2$ $\mathcal{D}$–transit leads to mirror $\mathcal{D}$–tetrahedra $(\Delta_1, \mathcal{D}_1)$ and $(\Delta_2, \mathcal{D}_2)$. So the conclusion for $0 \leftrightarrow 2$ $\mathcal{D}$–transits follows from Lemma 7.5.

If we cut open the final configuration of a bubble $\mathcal{D}$–transit along an interior face, we obtain the final configuration of a $0 \to 2$ $\mathcal{D}$–transit. So, except for what concerns the gluings of the two involved $\mathcal{D}$–tetrahedra, a bubble $\mathcal{D}$–transit is abstractly given by the same data as a $0 \to 2$ $\mathcal{D}$–transit, and the two term relations in $\mathbb{Z}[\mathcal{D}]$ associated to the $0 \leftrightarrow 2$ $\mathcal{D}$–transits and the bubble $\mathcal{D}$–transits are the same.

By using Proposition 7.1 we deduce the result for $\mathcal{P}[\mathcal{I}]$. The very same "mirror argument" of Lemma 7.5 lifts so as to eventually imply the other cases $\mathcal{A} = \mathcal{I}_f$ and $\mathcal{A} = \mathcal{I}_{fc}$: the only difference is that there is one degree of freedom for choosing the flattenings or the charges during a flat/charged $2 \to 3$ transit. This causes no problem because, given the first flattening or charge transit in Figure 13, we can always choose the second so as to produce, again, mirror images in the final configuration. Also, we know from the discussion before Proposition 6.1 that the bubble charge transits depend on the choice of two interior edges in the final configuration, where the sum of the charges is equal to 0. As above, if we cut it open along the interior face enclosed by these two edges, we obtain the final configuration of a $0 \to 2$ charge transit. This concludes the proof. $\square$

It follows from Theorem 7.4 (1) and Proposition 7.2 that

Corollary 7.7 *We have* $H_1(X) = H_1(\mathfrak{c}_{\mathcal{I}_f}(X))$.

This means that the classical dilogarithmic invariants coincide with the values of a function on $\mathcal{P}(\mathcal{I}_f)$ at the points corresponding to the scissors congruence classes. As mentioned in Section 6.1, this function can be identified with the universal second Cheeger–Chern–Simons class for flat $PSL(2,\mathbb{C})$–bundles.

A similar result, (by using $\mathcal{P}(\mathcal{I}_{fc})$ instead of $\mathcal{P}(\mathcal{I}_f)$) is hopeless for the quantum dilogarithmic invariants H_N, $N > 1$, because the formal sums of tetrahedra that represent the points of the *Abelian* group $\mathcal{P}(\mathcal{I}_{fc})$ do not encode any information about 2–face identifications, which, on the contrary, are essential in the definition of the state sums. A way to overcome this problem consists in defining an "augmented" scissors congruence class, belonging to a Bloch-like group of further enriched $\mathcal{I}_{fc}$–tetrahedra, where also the *states* are incorporated in the augmented decorations; such a procedure is described in Section 5 of [2]. However, this appears purely formal and risks hiding more substantial questions. For instance, it makes sense to ask whether the value of a quantum dilogaritmic state sum for a 3–manifold $\mathcal{I}_{fc}$–triangulation does actually only depend on the corresponding $\mathcal{I}_{fc}$–scissors congruence class, though a positive answer would be very surprising. More precisely:

Question 7.8 Let (W_j, L_j, ρ_j), $j = 1, 2$, be triples with the same $\mathcal{I}_{fc}$–scissors congruence classes: $\mathfrak{c}_{\mathcal{I}_{fc}}(W_1, L_1, \rho_1) = \mathfrak{c}_{\mathcal{I}_{fc}}(W_2, L_2, \rho_2)$. Do we have

$$H_N(W_1, L_1, \rho_1) = H_N(W_2, L_2, \rho_2)$$

and similarly after replacing one (W_j, L_j, ρ_j) or both with cusped manifolds ?

On the other hand, we proposed in Section 5 of [3] a 'Volume Conjecture' for cusped manifolds M, saying that the dominant term of the asymptotic expansion, when $N \to \infty$, of $H_N(M)^N$ grows exponentially with N^2, and has a growth rate equal to $H_1(M) = \exp\big((1/i\pi)\mathbf{R}(M)\big)$ (see Subsection 6.1 for details on $\mathbf{R}$). In fact, the existence of the invariants $H_N(M)$, $N \geq 1$, was only conjectural in that paper. Now this conjecture is perfectly consistent for weakly-gentle cusped manifolds. It is mostly motivated by the strong structural coincidence between the classical and quantum invariants of such manifolds (see Remark 6.12).

With similar motivations, we also proposed a Volume Conjecture for sequences (W_n, L_n, ρ_n) of compact hyperbolic 3–manifolds converging geometrically to a cusped manifold M (here, the L_n are the links isotopic to short simple closed geodesics that disappear in the limit opening some cusps, and the ρ_n are the holonomies of the hyperbolic structures on the W_n). By Corollary 7.7, on the path to these conjectures there is the seemingly weaker conjecture:

Conjecture 7.9 *For every weakly-gentle cusped manifold M, the dominant term of the asymptotic expansion of $H_N(M)$ when $N \to +\infty$ only depends on the scissors congruence class $\mathfrak{c}_{\mathcal{I}_{fc}}(M)$. Similarly, for the dominant term of $H_N(W_n, L_n, \rho_n)$ when both N and n tend to $+\infty$.*

Note that the asymptotic behaviour of $H_N(W, L, \rho)$ actually depends, in general, on the link L (see again Section 5 of [3]), whereas $H_1(W, \rho)$ only depends on $\mathfrak{c}_{\mathcal{I}_f}(W, \rho)$, which, for any link L in W, is the image of $\mathfrak{c}_{\mathcal{I}_{fc}}(W, L, \rho)$ via the natural forgetful map.

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8 Appendix

In Section 8.1 we recall some results on Heisenberg doubles. The detailed constructions are given in [1], Chapter 3, but some of them were already announced by Kashaev in [21]. This serves in Section 8.2 to give a representation theoretic formulation of the tensor $\widehat{\mathcal{L}}_N$ defined in Section 5.

8.1 Algebraic preliminaries

Heisenberg doubles. Let $A = (1, \epsilon, m, \Delta, S)$ be a Hopf algebra with unity over a ring k, where 1, ϵ, m, Δ and S are respectively the unit, the counit, the multiplication, the comultiplication and the antipode of A. When A is infinite dimensional we put $k = \mathbb{C}[[h]]$, the ring of formal power series over $\mathbb{C}$ with indeterminate h, and we assume that $A = U_h(\mathfrak{g})$, the quantum universal enveloping algebra (QUE) of a complex finite dimensional Lie algebra $\mathfrak{g}$ (see eg [11, Section 6-8]). Every tensor product below is over k, and when $A = U_h(\mathfrak{g})$ they are implicitely completed in h–adic topology. When A is finite dimensional the dual k–module $A^* = \mathrm{Hom}(A, k)$ naturally inherits a *dual* Hopf algebra structure $A^* = (\epsilon^*, 1^*, \Delta^*, m^*, S^*)$ with

$$\langle x, m(a \otimes b)\rangle = \langle m^*(x), a \otimes b\rangle \quad , \quad \langle x \otimes y, \Delta(a)\rangle = \langle \Delta^*(x \otimes y), a\rangle$$

$$\langle x, 1\rangle = 1^*(x) \ , \quad \langle \epsilon^*, a\rangle = \epsilon(a) \quad , \quad \langle x, S(a)\rangle = \langle S^*(x), a\rangle$$

where $a,\ b \in A,\ x,\ y \in A^*$, and $\langle\ ,\ \rangle\colon A^* \otimes A \to k$ is the canonical pairing. When $A = U_h(\mathfrak{g})$ we have the Drinfeld's notion of QUE-*dual* Hopf algebra $A^* = U_h^*(\mathfrak{g})$ [11, Section 6.3.D-8.3]. This is a QUE-algebra isomorphic to $U_h(\mathfrak{g})$ as a $\mathbb{C}[[h]]$–module, with a dual Hopf algebra structure defined as above.

For any $a \in A$, denote by $\mathrm{ev}_a\colon A^* \to k$ the evaluation map: $\mathrm{ev}_a(x) = \langle x, a\rangle$. Let $\pi_A\colon A \to \mathrm{End}_k(A^*)$ be the homomorphism defined by $\pi_A(a) = (id \otimes \mathrm{ev}_a)\ m^*$, and $\pi_{A^*}\colon A^* \to \mathrm{End}_k(A^*)$ be given by multiplication on the left. The *Heisenberg double* $\mathcal{H}(A)$ of A is the subalgebra of $\mathrm{End}_k(A^*)$ (topologically) generated by the image of π_A and π_{A^*} (this notion seems to have been introduced in [8] and [36]). The image R in $A \otimes A^* \cong \mathrm{End}_k(A)$ of the identity morphism defines an automorphism

$$(\pi_A \otimes \pi_{A^*})(R) = (\epsilon^* \otimes \Delta^*)\ (m^* \otimes \epsilon^*)$$

of $(A^*)^{\otimes 2}$. We say that R is the *canonical element* of $\mathcal{H}(A)$. Given dual (topological) basis $\{e_\alpha\}_\alpha$ and $\{e^\beta\}_\beta$ of A and A^* for the pairing $\langle\ ,\ \rangle$, we write $R = \sum_\gamma e_\gamma \otimes e^\gamma$. Viewed as an element of $\mathrm{End}_k((A^*)^{\otimes 2})$, it has the following remarkable (equivalent) properties:

$$(1 \otimes e_\alpha)\ R = R\ \Delta(e_\alpha) \quad , \quad R\ (e^\alpha \otimes 1) = m^*(e^\alpha)\ R$$
$$R_{12}\ R_{13}\ R_{23} = R_{23}\ R_{12} \tag{43}$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, etc. The last identity is called the *pentagon* relation. The first identity shows that for any linear representation ρ of A in the second tensor factor of the left hand side, R^{-1} induces embeddings of ρ into the tensor product of two linear representations of A described via $\Delta(\rho)$, that is *Clebsch-Gordan operators.* The pentagon relation means that R (here R_{23} in the left hand side) induces matrices of change of basis between the two possible ways of composing such embeddings, that is *matrices of* $6j$*–symbols.* Finally, let us note that we can reconstruct completely $\mathcal{H}(A)$ from the relations (43). In fact, any solution of the pentagon equation uniquely corresponds to a Hopf module over some Hopf algebra [8, Theorem 4.10], [12, Theorem 5.7].

The case of $U_h(b(2,\mathbb{C}))$ Let us now specialize the above considerations to the (positive) quantum Borel subalgebra $A = U_h(b(2,\mathbb{C}))$ of $U_h(sl(2,\mathbb{C}))$. Recall that this is the QUE Hopf algebra over $\mathbb{C}[[h]]$ topologically generated by elements H and D such that $HD - DH = D$, with comultiplication and antipode given by [11, Section 6.4], [23, Section 17]:

$$\begin{array}{l} \Delta(H) = H \otimes 1 + 1 \otimes H \quad , \quad \Delta(D) = 1 \otimes D + D \otimes e^{hH} \\ S(H) = -H \quad , \quad S(D) = -De^{-hH} \quad , \quad \epsilon(H) = \epsilon(D) = 0 \quad , \quad \epsilon(1) = 1. \end{array}$$

(We denote by 1 the identity of $U_h(b(2,\mathbb{C}))$ and $\mathbb{C}[[h]]$). For technical reasons related to some of our choices below, let us introduce $\mathbb{C}((h))$, the field of fractions of $\mathbb{C}[[h]]$, and consider $U_h(sl(2,\mathbb{C}))$ as a $\mathbb{C}((h))$–module. The QUE-dual Hopf algebra $U_h^*(b(2,\mathbb{C}))$ is isomorphic as a topological Hopf algebra over $\mathbb{C}[[h]]$ to the *negative* quantum Borel subalgebra of $U_h(sl(2,\mathbb{C}))$, endowed with the opposite comultiplication [11, Proposition 8.3.2]. Hence $U_h^*(b(2,\mathbb{C}))$ is topologically generated over $\mathbb{C}[[h]]$ by elements $\bar{H}$ and $\bar{D}$ such that $\bar{H}\bar{D} - \bar{D}\bar{H} = -h\bar{D}$, with comultiplication and antipode given by:

$$\begin{array}{l} \Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H} \quad , \quad \Delta(\bar{D}) = 1 \otimes \bar{D} + \bar{D} \otimes e^{-\bar{H}} \\ S(\bar{H}) = -\bar{H} \quad , \quad S(\bar{D}) = -\bar{D}e^{\bar{H}} \quad , \quad \epsilon(\bar{H}) = \epsilon(\bar{D}) = 0 \quad , \quad \epsilon(1) = 1. \end{array}$$

Clearly the map $\bar{H} \to -hH$, $\bar{D} \to D$ is an isomorphism of topological algebras over $\mathbb{C}((h))$. It is shown in [1], Proposition 3.2.5, that the *Heisenberg double* $\mathcal{H}_h(b(2,\mathbb{C}))$ of $U_h(sl(2,\mathbb{C}))$ is isomorphic to the $\mathbb{C}((h))$–algebra topologically generated over $\mathbb{C}[[h]]$ by elements H, D, $\bar{H}$ and $\bar{D}$ such that:

$$\begin{array}{ll} HD - DH = D \quad , & \bar{H}\bar{D} - \bar{D}\bar{H} = -h\bar{D} \\ H\bar{H} - \bar{H}H = 1 \quad , & D\bar{H} = \bar{H}D \\ H\bar{D} - \bar{D}H = -\bar{D} \quad , & D\bar{D} - \bar{D}D = (1-q)\, e^{hH} \end{array}$$

where we put $q = e^{-h}$. Moreover, we can write the canonical element of $\mathcal{H}_h(b(2,\mathbb{C}))$ as

$$R_h = e^{H \otimes \bar{H}}\ (D \otimes \bar{D}; q)_\infty^{-1} \quad . \tag{44}$$

Here we denote by $(x;q)_\infty$ the q*–dilogarithm*, which is the formal power series in $\mathbb{C}(q)[[x]]$ given by

$$(x;q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}(-x)^n}{(q)_n}$$

where $(q)_n = (1-q)\dots(1-q^n)$. The proof of this result is instructive. A remarkable fact is that the term $e^{H\otimes \bar{H}}$ in (44) is the canonical element of the Heisenberg double $\mathcal{H}_h^0$ of the Hopf subalgebra of $\mathcal{H}_h(b(2,\mathbb{C}))$ topologically generated over $\mathbb{C}[[h]]$ by H. Note that in $\mathcal{H}_h^0$ we have $H\bar{H}-\bar{H}H=1$. Then, it is easy to see that the pentagon relation (43) for $e^{H\otimes \bar{H}}$ is a direct consequence of the Baker-Campbell-Hausdorff formula for the complex Lie algebra generated by H and $\bar{H}$. Moreover, the pentagon relation for R_h splits into the product of the one for $e^{H\otimes \bar{H}}$, together with the following q*–dilogarithm equation*:

$$(U;q)_\infty\ \left(\frac{[U,V]}{(1-q)};q\right)_\infty\ (V;q)_\infty = (V;q)_\infty\ (U;q)_\infty \tag{45}$$

where $U = 1\otimes D\otimes \bar{D}$ and $V = D\otimes \bar{D}\otimes 1$.

Integral form of $\mathcal{H}_h(b(2,\mathbb{C}))$ at a root of unity We need to specialize the formal parameter $q=\exp(-h)$ to specific (non zero) complex numbers. For that we must consider the *integral form* of $\mathcal{H}_h(b(2,\mathbb{C}))$. As defined in [1], Definition 3.2.9, this is the $\mathbb{C}[q,q^{-1}]$–algebra $\mathcal{H}_q(b(2,\mathbb{C}))$ generated by elements E,E^{-1}, $\bar{E}$, $\bar{E}^{-1}$, D and $\bar{D}$ such that

$$\begin{array}{lll} EE^{-1} = E^{-1}E = 1 & & \\ DE = qED & , & \bar{D}\bar{E} = q\bar{E}\bar{D} \\ E\bar{E} = q\bar{E}E & , & D\bar{E} = \bar{E}D \\ E\bar{D} = q\bar{D}E & , & D\bar{D} - \bar{D}D = (1-q)\ E. \end{array}$$

This algebra is obtained just by mimicking the commutation relations between the elements D, $\bar{D}$, $E=\exp(hH)$ and $\bar{E}=\exp(-\bar{H})$ of $\mathcal{H}_h(b(2,\mathbb{C}))$, as is usual in quantum group theory [11, Section 9]. The subalgebra of $\mathcal{H}_q(b(2,\mathbb{C}))$ generated by E, E^{-1} and D is isomorphic to the integral form $\mathcal{B}_q$ of $U_h(b(2,\mathbb{C}))$, which has comultiplication, counit and antipode given by:

$$\begin{array}{l} \Delta(E) = E\otimes E \quad , \quad \Delta(D) = E\otimes D + D\otimes 1 \\ \epsilon(E) = 1 \quad , \quad \epsilon(D) = 0 \quad , \quad S(E) = E^{-1} \quad , \quad S(D) = -E^{-1}D. \end{array}$$

So we write $\mathcal{H}(\mathcal{B}_q) = \mathcal{H}_q(b(2,\mathbb{C}))$. For any non zero complex number ϵ, we can now evaluate q in ϵ, thus giving the specialization $\mathcal{H}(\mathcal{B}_\epsilon)$ of $\mathcal{H}_q(b(2,\mathbb{C}))$.

A main problem is that the canonical element $R_h \in \mathcal{H}_h(b(2,\mathbb{C}))$ does not survive this procedure. First because $(D\otimes \bar{D};q)_\infty$ is an infinite sum, which moreover is ill defined when q is a root of unity. Also, $\exp(H\otimes \bar{H})$ cannot be written in terms of the generators of $\mathcal{H}(\mathcal{B}_q)$. However, R_h acts by conjugation as an automorphism of $\mathcal{H}(\mathcal{B}_q)^{\otimes 2}$. We will construct, when $q=\zeta^{-1}$ is a root of unity, a specific element R_ζ of a suitable extension of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})^{\otimes 2}$, that implements the regular part of this action of R_h. We do this in two steps, that we describe below. We fix a primitive Nth root of unity $\epsilon = \zeta^{-1}$ for a positive integer $N>1$.

The action of $\exp(H\otimes \bar{H})$ at a root of unity We have seen that $\exp(H\otimes \bar{H})$ is the canonical element of $\mathcal{H}_h^0$. Now, the specialization in ζ^{-1} of the integral form of $\mathcal{H}_h^0$ is the subalgebra of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$ generated by E and $\bar{E}$ such that $\bar{E}E = \zeta E\bar{E}$. This subalgebra can be endowed with the very same structure of Hopf algebra as for $\mathcal{B}_\zeta$, so we denote it by $\mathcal{B}_\zeta^0$. Moreover, it is a central extension of a Heisenberg double. Indeed, we have an isomorphism (see [1], Section 3.2.3)

$$\mathcal{B}_\zeta^0/(E^N = \bar{E}^N = 1) \cong \mathcal{H}(\mathbb{C}[\mathbb{Z}/N\mathbb{Z}])$$

where $\mathcal{H}(\mathbb{C}[\mathbb{Z}/N\mathbb{Z}])$ is the Heisenberg double of the group algebra $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$ of $\mathbb{Z}/N\mathbb{Z}$, endowed with its usual Hopf algebra structure (see eg [23, Section 3]). Hence the canonical element S_N of $\mathcal{H}(\mathbb{C}[\mathbb{Z}/N\mathbb{Z}])$ is the natural 'periodic' analogue of $\exp(H\otimes\bar{H})$. We can define the image of S_N in the following extension of $(\mathcal{B}_\zeta^0)^{\otimes 2}$. The algebra $\mathcal{B}_\zeta^0$ has no zero divisors, since it is an Ore extension of a polynomial ring [23, Section 4]. Then its center $\mathcal{Z}(\mathcal{B}_\zeta^0)$ is an integral domain and we can consider its quotient ring $Q(\mathcal{Z}(\mathcal{B}_\zeta^0))$. Put

$$Q(\mathcal{B}_\zeta^0) := \mathcal{B}_\zeta^0 \otimes_{\mathcal{Z}(\mathcal{B}_\zeta^0)} Q(\mathcal{Z}(\mathcal{B}_\zeta^0)) \tag{46}$$

and let c_E and $c_{\bar{E}}$ be elements of $\mathcal{Z}(\mathcal{B}_\zeta^0)$ such that $c_E^N = E^N$ and $c_{\bar{E}}^N = \bar{E}^N$ (they exist because $\mathcal{Z}(\mathcal{B}_\zeta^0)$ is integrally closed [11, Proposition 11.1.2]). The image of S_N in $Q(\mathcal{B}_\zeta^0)^{\otimes 2}$ can be written as (see [1], Lemme 3.2.10)

$$S_\zeta = \frac{1}{N}\sum_{i,j=0}^{N-1} \zeta^{ij}\ (E')^i \otimes (\bar{E}')^j \tag{47}$$

where $E' = c_E^{-1}E$, $\bar{E}' = c_{\bar{E}}^{-1}\bar{E} \in Q(\mathcal{B}_\zeta^0)$. Note that the sum $N^{-1}\sum_i \zeta^{ij}(\bar{E}')^j$ is the normalized inverse Fourier transform of $\bar{E}'$. We can verify that the action by conjugation of $\exp(H\otimes\bar{H})$ on $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})^{\otimes 2}$ is the same as the action by conjugation of S_ζ.

The action of $(D\otimes\bar{D};q)_\infty$ at a root of unity This is best seen by considering the behaviour of the q–dilogarithm $(x;q)_\infty$ when $q\to\zeta^{-1}$. Note that it is a solution of the q–difference equation $(1-x)f(qx)-f(x)=0$. When $|q|<1$, $(x;q)_\infty$ converges normally on any compact domain of $\mathbb{C}$ and defines an entire function $E_q(x)$ that may be written in the form of the infinite product $E_q(x) = \prod_{n=0}^\infty(1-xq^n)$. The zeros of $E_q(x)$ are simple and span the set $\{q^{-n},\ n\geq 0\}$. When $|q|>1$, the radius of convergence of $(x;q)_\infty$ is $|q|$, and its sum defines in the open disc of convergence a holomorphic function $e_q(x)$. Moreover, we can continue $e_q(x)$ meromorphically to the whole complex plane, so that $e_q(x) = E_{q^{-1}}(xq^{-1})^{-1}$. The function $e_q(x)$ has no zeros and its poles are simple and span the set $\{q^n\ ,\ n\geq 1\}$.

The remarkable fact is that $(x;q)_\infty$ has essential singularities when q tends to roots of unity, as we explain now. Let $q=\exp(-\varepsilon/N^2)\zeta^{-1}$, where $\mathrm{Re}(\varepsilon)>0$. Recall from Section 2.1.2 and Section 3 the definition of the Euler dilogarithm Li_2 and the function g. When $|x|<1$, the q–dilogarithm $(x;q)_\infty$ has for $\epsilon\to 0$ the following asymptotic behaviour (see eg [7]):

$$(x;q)_\infty = g^{-1}(x)\ \ (1-x^N)^{1/2}\ \exp(-\mathrm{Li}_2(x^N)/\epsilon)\ \ \big(1+\mathcal{O}(\epsilon)\big).$$

Replacing g^{-1} with its power series expansion at $x=0$, we deduce that the 'regular' part of $(D\otimes\bar{D};q)_\infty$ when $q\to\zeta^{-1}$ lies in the vector space $\widehat{\mathcal{H}}(\mathcal{B}_{\zeta^{-1}})$ of formal power series in the generators of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$, with complex coefficients. Similarly to (46), define

$$Q(\widehat{\mathcal{H}}(\mathcal{B}_{\zeta^{-1}})) := \widehat{\mathcal{H}}(\mathcal{B}_{\zeta^{-1}}) \otimes_{\mathcal{Z}(\mathcal{B}^0_\zeta)} Q(\mathcal{Z}(\mathcal{B}^0_\zeta)).$$

Then

$$R_\zeta = S_\zeta \quad g(D\otimes\bar{D}) \tag{48}$$

acts by conjugation on $Q(\widehat{\mathcal{H}}(\mathcal{B}_{\zeta^{-1}}))^{\otimes 2}$ as the regular part of R_h. Denote by r the central element $(1-(D^N\otimes\bar{D}^N))^{1/N}$, viewed as the evaluation at $D^N\otimes\bar{D}^N$ of the power series expansion of $(1-x)^{1/N}$ at $x=0$. We can prove that $r^{(1-N)/2}R_\zeta$ verifies the pentagon relation (43) (the element r serves as a 'determinant-like' normalization, see Proposition 8.6). As for R_h, this relation splits into the product of the pentagon relation for S_ζ, and an identity obtained from (45) by replacing each q–dilogarithm with the evaluation of $r^{(1-N)/2}g$ at certain multiples of $D\otimes\bar{D}$ by central elements. We will not describe this matter here (see Remark 8.1). Rather, we consider below the five term relations induced on the 'cyclic' finite dimensional representations of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$.

Remark 8.1 The function g (respectively the matrix Ψ defined by (32)) is a 'cyclic' analogue of $(x;q)_\infty$, because Lemma 8.2 below (respectively (34)) is a version for $q=\zeta$ of the q–difference equation $(1-x)f(qx)-f(x)=0$, that defines $(x;q)_\infty$ when $f(0)=1$. So the matrix identity (36) is a cyclic version of (45). An equivalent form of it was first announced in [18]. See also [7] for a description starting from the formal asymptotics of $(x;q)_\infty$.

8.2 The $\mathcal{L}_N$ are representations of R_ζ

First we need to introduce some properties of the function g. As before, let $N>1$ be any odd positive integer, and denote by log the standard branch of the logarithm, which has the imaginary part in $]-\pi,\pi]$. For any complex number $x\neq 0$ write $x^{1/N}=\exp((1/N)\log(x))$. As remarked in the Introduction, the function

$$g(x)=\prod_{j=1}^{N-1}(1-x\zeta^{-j})^{j/N}$$

is complex analytic on $\mathfrak{D}_N$, the complex plane with cuts from the points $x=\zeta^k$, $k=1,\dots,N-1$, to infinity.

Lemma 8.2 *For any $x\in\mathbb{C}\setminus\{\zeta^j,\ j=1,\dots,N-1\}$ and $k\in\mathbb{Z}$ we have*

$$g(x\zeta^k)=g(x)\prod_{j=1}^{k}\frac{(1-x^N)^{1/N}}{1-x\zeta^j}.$$

Proof We have

$$g(x\zeta^k) = \prod_{j=1}^{N-1}(1-x\zeta^{k-j})^{k/N} \prod_{j=k+1}^{N-1}(1-x\zeta^{k-j})^{(j-k)/N} \prod_{j=1}^{k}(1-x\zeta^{k-j})^{(j-k)/N}$$

$$= \frac{(1-x^N)^{k/N}}{(1-x\zeta^k)^{k/N}} \prod_{l=1}^{N-k-1}(1-x\zeta^{-l})^{l/N} \left(\prod_{l=N-k+1}^{N}(1-x\zeta^{-l})^{l/N} \prod_{l=0}^{k-1}(1-x\zeta^{l})^{-1} \right)$$

$$= g(x)\ (1-x^N)^{k/N} \prod_{l=1}^{k}(1-x\zeta^l)^{-1} = \ g(x) \prod_{j=1}^{k} \frac{(1-x^N)^{1/N}}{1-x\zeta^j}.$$

Note that in the second equality we used the fact that $\sum_{j=1}^N \log(1-x\zeta^j) = \log(1-x^N)$ if $|x|<1$ and, by analytic continuation, if x belongs to the cut complex plane $\mathfrak{D}_N$. In fact if x lies in the interior of such a half ray, then $x^N \in (1,+\infty)$, so that the imaginary parts are corrected by the same amount on both sides. Hence we find the desired result. □

Denote by $S(x|z)$ the rational function defined on the curve $\{x^N + z^N = 1\}$ by

$$S(x|z) = \sum_{k=1}^{N} \prod_{j=1}^{k} \frac{z}{1-x\zeta^j}.$$

Lemma 8.3 *When both sides are defined the following identities hold true:*

(i) $g(x)\ g(1/x) \equiv_N \dfrac{g(1)^2}{N}\ x^{\frac{1-N}{2}}\ \dfrac{1-x^N}{1-x}$

(ii) $x\ S(x|z\zeta) = (1-z)\ S(x|z)$

(iii) $g(x)\ g(z/\zeta)\ S(x|z) \equiv_N x^{N-1}\ g(1)$

where $\equiv_N$ denotes the equality up to multiplication by Nth roots of unity.

Proof (i) Note that

$$g(x)^N\ g(1/x)^N = x^{\frac{N(1-N)}{2}} \prod_{j=1}^{N-1}(1-x\zeta^{-j})^j\ (x-\zeta^{-j})^j.$$

The product in the right hand side is a polynomial in x of degree $N(N-1)$, with zeros the Nth roots of unity ζ^j, $j \neq 0$, each with multiplicity N. So there is a function $C(x)$, constant up to roots of unity, such that

$$g(x)\ g(1/x) = C(x)\ x^{\frac{(1-N)}{2}}\ \frac{1-x^N}{1-x}.$$

We find $C(x) \equiv_N g(1)^2/N$ by taking the limit $x \to 1$.

(ii) This is a simple computation:

$$x\ S(x|z\zeta) = \sum_{k=1}^{N} \ \prod_{j=1}^{k} \frac{z}{1-x\zeta^j}\ (x\zeta^k - 1 + 1) = S(x|z) - zS(x|z).$$

(iii) Consider the rational function $Q(x) = \prod_{j=0}^{N-1} S(x|z\zeta^j)$. It does only depend on x, because $z^N = 1 - x^N$ and Q is invariant under the substitution $z \mapsto z\zeta$. Remark that

$$S(x\zeta|z) \ = \ \sum_{k=1}^{N} \prod_{j=1}^{k} \frac{z}{1-x\zeta^{j+1}} = \sum_{k=1}^{N} \frac{1-x\zeta}{z} \prod_{j=1}^{k+1} \frac{z}{1-x\zeta^j} = S(x|z)\ \frac{1-x\zeta}{z}.$$

So we have

$$Q(x\zeta) \ = \ Q(x)\ \frac{(1-x\zeta)^N}{z^N} \ = \ Q(x)\ \frac{(1-x\zeta)^N}{1-x^N}.$$

This shows that Q has no zeros, and that its poles are the roots of unity $\{\zeta^j\}$, $j = 1,\ldots,N-1$, where ζ^j has multiplicity j. Hence we find

$$Q(x) = \frac{\prod_{j=1}^{N-1}(1-\zeta^j)^j}{\prod_{j=1}^{N-1}(1-x^{-1}\zeta^j)^j} = g(1)^N\ \frac{x^{\frac{N(N-1)}{2}}}{g(x)^N}$$

where the normalization constant is found by taking $x = 1$, which gives $S(1|0) = 1$ and $Q(1) = 1$. Moreover, by applying (ii) directly to the formula for Q we get

$$Q(x) = S(x|z)^N\ x^{\frac{-N(N-1)}{2}} \ \prod_{j=1}^{N-1} (1 - (z/\zeta)\ \zeta^{-j})^j.$$

Then
$$S(x|z)^N = g(1)^N\ \frac{x^{N(N-1)}}{g(z/\zeta)^N g(x)^N}$$

which is just the Nth power of (iii). □

Consider now a complex linear representation ρ of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$ with support V_ρ, such that: each generator is mapped into $\mathrm{GL}(V_\rho)$, and $\rho(\bar{D})\rho(D) = \zeta\ \rho(D)\rho(\bar{D})$ (ie $\rho(\bar{D}D) = -\rho(E)$). We say that ρ is *cyclic.* One can check that V_ρ is necessarily finite dimensional, with $\dim_{\mathbb{C}}(V_\rho) = N$ if ρ is irreducible. The elements of the center $\mathcal{Z}$ of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$ (in particular the Nth powers of the generators) act as scalars on V_ρ, so we have homomorphisms $\chi_\rho\colon \mathcal{Z} \to \mathbb{C}$ called the *central characters.* We note that ρ induces a pair of cyclic irreducible representations of $\mathcal{B}_{\zeta^{-1}}$, by considering its restriction to the algebras generated by E, D and $\bar{E}$, $\bar{D}$ respectively.

Recall from Section 5 that the matrix valued map $\hat{\mathcal{L}}_N\colon \widehat{\mathbb{C}} \to \mathrm{M}_{N^2}(\mathbb{C}/U_N)$ is complex analytic, where U_N is the multiplicative group of Nth roots of unity. Recall also that $\equiv_N$ denotes the equality up to multiplication by Nth roots of unity, and that the integer m is defined by $N = 2m+1$.

Theorem 8.4 *For any $(u;p,q) \in \widehat{\mathbb{C}}$, there exists a cyclic irreducible linear representation ρ of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$ on $\mathbb{C}^N$ such that $\chi_\rho(D^N)\chi_\rho(\bar{D}^N) = 1-u^{-1}$, and*

$$\widehat{\mathcal{L}}_N(u;p,q) \equiv_N (\rho\otimes\rho)\left(r^{\frac{1-N}{2}} R_\zeta\ (D'\otimes\bar{D}')^{-p(m+1)}\right)$$

where r and R_ζ are defined in (48), $D' = c_D^{-1}D$ and $\bar{D}' = c_{\bar{D}}^{-1}\bar{D}$ for central elements c_D, $c_{\bar{D}} \in \mathcal{H}(\mathcal{B}_{\zeta^{-1}})$ with $c_D^N = D^N$ and $c_{\bar{D}}^N = \bar{D}^N$, and $\epsilon \in \{-1,+1\}$ is defined by $\log(u)+\log(1/(1-u))+\log(1-u^{-1}) = \epsilon\pi i$.

We note that the factor $(D'\otimes\bar{D}')^{-p(m+1)}$ only serves to get the Nth root v'_{-q} of $1-u$ in $\widehat{\mathcal{L}}_N(u;p,q) = \mathcal{L}_N(u'_p, v'_{-q})$. Putting $p=q=0$ we recover the basic matrix dilogarithms $\mathcal{L}_N(u)$ defined in (16). The ambiguity up to Nth roots of unity depends on the arguments of u'_p and v'_{-q}, and is due to the use of Lemma 8.3 (i) and (iii).

Proof Let Z, X and $Y = \zeta^{(m+1)}XZ$ be the $N\times N$–matrices with entries defined as: $Z_i^j = \zeta^i\ \delta(i-j)$, $X_i^j = \delta(i-j-1)$, and $Y_i^j = \zeta^{(m+1)+j}\ \delta(i-j-1)$ in the canonical basis of $\mathbb{C}^N$, where δ is, as usual, the Kronecker symbol $\mathrm{mod}(N)$. Consider a cyclic irreducible representation ρ of $\mathcal{H}(\mathcal{B}_{\zeta^{-1}})$ given by

$$\begin{aligned} \rho(E) &= t_\rho^{-2}\ Z^{-1} \quad , \quad \rho(D) = -\frac{x_\rho}{t_\rho}\ Y^{-1} \\ \rho(\bar{E}) &= s_\rho^{-2}\ Y \quad , \quad \rho(\bar{D}) = \frac{1}{s_\rho y_\rho}\ Z^{-1}Y = \frac{\zeta^{-(m+1)}}{s_\rho y_\rho}\ X \end{aligned}$$

for non zero complex numbers t_ρ, s_ρ, x_ρ and y_ρ. Choose the central elements c_E and $c_{\bar{E}}$ in (47) such that $\rho(c_E) = t_\rho^{-2}\mathrm{Id}_{\mathbb{C}^N}$ and $\rho(c_{\bar{E}}) = s_\rho^{-2}\mathrm{Id}_{\mathbb{C}^N}$. We see immediately that

$$(\rho\otimes\rho)(S_\zeta) = \frac{1}{N}\sum_{i,j=0}^{N-1}\zeta^{ij}\ Z^{-i}\otimes Y^j. \tag{49}$$

Consider $(\rho\otimes\rho)(r^{\frac{1-N}{2}}g(D\otimes\bar{D})\ (D'\otimes\bar{D}')^{-p(m+1)})$. As $g(x)$ is complex analytic on the open unit disk $|x|<1$, by considering its power series expansion $g(x) = \sum_{k=0}^\infty a_k x^k$ at $x=0$ we see that for any unipotent matrix M of order N we have

$$\begin{aligned} \sum_{k=0}^\infty a_k(xM)^k &= \sum_{k=0}^\infty\sum_{s=0}^{N-1}\left(\frac{1}{N}\sum_{t=0}^{N-1}\zeta^{t(k-s)}\right)\ a_k(xM)^k \\ &= \frac{1}{N}\sum_{s,t=0}^{N-1}(\sum_{k=0}^\infty a_k x^k\zeta^{tk})\ \zeta^{-st}M^s \end{aligned}$$

whence

$$g(xM) = \frac{1}{N}\sum_{s,t=0}^{N-1} g(x\zeta^t)\ \zeta^{-st}M^s. \tag{50}$$

This identity may be analytically continued to the cut complex plane $\mathfrak{D}_N$ with respect to the variable x. Now, set $x = \chi_\rho(D^N)\chi_\rho(\bar{D}^N)$. Given $a\in\mathbb{Z}$ choose x_ρ and y_ρ so that $x'_a = -x_\rho/t_\rho s_\rho y_\rho$. Moreover, let the central elements c_D and $c_{\bar{D}}$ verify $\rho(c_D) = -\zeta^{-(m+1)}x_\rho/t_\rho$ and $\rho(c_{\bar{D}}) = 1/s_\rho y_\rho$. We have $(\rho\otimes\rho)(r) = y'_0$, where $y = 1-x$, and

$$\begin{aligned}
&(\rho\otimes\rho)(r^{\frac{1-N}{2}} g(D\otimes\bar{D})\ (D'\otimes\bar{D}')^{-b(m+1)}) \\
&\quad = (y'_0)^{\frac{1-N}{2}}\ g(x'_a(Y^{-1}\otimes Z^{-1}Y))\ (\zeta^{(m+1)}Y^{-1}\otimes Z^{-1}Y)^{-b(m+1)} \\
&\quad = (y'_{-b})^{\frac{1-N}{2}}\ \frac{1}{N}\sum_{k,t=0}^{N-1} g(x'_a\zeta^k)\ \zeta^{-kt}\ (Y^{-1}\otimes Z^{-1}Y)^{t-b(m+1)} \\
&\quad = (y'_{-b})^{\frac{1-N}{2}} g(x'_a)\ \frac{1}{N}\sum_{k,t=0}^{N-1}\prod_{j=1}^{k}\left(\frac{y'_{-b}}{1-x'_a\zeta^j}\right)\ \zeta^{-kt}\ (Y^{-1}\otimes Z^{-1}Y)^t.
\end{aligned}$$

The second equality follows from (50) and the third from Lemma 8.2 together with a sum reordering $t\mapsto t-b(m+1)$. Moreover, by Lemma 8.3 (ii) and (iii) we have

$$\sum_{k=0}^{N-1}\prod_{j=1}^{k}\left(\frac{y'_{-b}}{1-x'_a\zeta^j}\right)\ \zeta^{-kt} \equiv_N \frac{g(1)(x'_a)^{N-1}}{g(y'_{-b}\zeta^{-1})g(x'_a)}\ \prod_{j=1}^{N-t}\frac{1-y'_{-b}\zeta^{j-1}}{x'_a}.$$

On another hand, the lemmas 8.2 and 8.3 (i) give respectively

$$g(y'_{-b}\zeta^{-1}) = g(y'_{-b})\ \prod_{j=1}^{N-1}\frac{x'_0}{1-y'_{-b}\zeta^j} = g(y'_{-b})\ \frac{1-y'_{-b}}{x'_0}$$

$$g(y'_{-b})\ g((y'_{-b})^{-1}) \equiv_N \frac{g(1)^2}{N}\ (y'_{-b})^{\frac{N-1}{2}}\ \frac{x}{1-y'_{-b}}.$$

Since

$$\prod_{j=1}^{N-t}\frac{1-y'_{-b}\zeta^{j-1}}{x'_a} = \prod_{s=t+1}^{N}\frac{1-y'_{-b}\zeta^{-s}}{x'_a} = \prod_{s=1}^{t}\frac{x'_a}{1-y'_{-b}\zeta^{-s}}$$

we find that $(\rho\otimes\rho)(r^{\frac{1-N}{2}}\ g(D\otimes\bar{D})\ (D'\otimes\bar{D}')^{-b(m+1)})$ is equal, up to multiplication by Nth roots of unity, to

$$\frac{g((y'_{-b})^{-1})}{g(1)}\ \sum_{t=0}^{N-1}\prod_{s=1}^{t}\frac{x'_a}{1-y'_{-b}\zeta^{-s}}\ (Y^{-1}\otimes Z^{-1}Y)^t. \tag{51}$$

It is easily seen that $(Y^{-1}\otimes Z^{-1}Y)^t\ {}^{i,j}_{k,l} = \zeta^{-kt-t(t+1)(m+1)}\ \delta(l-j-t)\ \delta(k+t-i)$.

Then the ${}^{i,j}_{k,l}$–component of this sum is

$$\frac{g((y'_{-b})^{-1})}{g(1)} \sum_{t=0}^{N-1} \zeta^{-kt-t(t+1)(m+1)}\ \delta(l-j-t)\ \delta(k+t-i) \prod_{s=1}^{t} \frac{x'_a}{1-y'_{-b}\zeta^{-s}}$$

$$= \frac{g((y'_{-b})^{-1})}{g(1)}\ \zeta^{-k(i-k)}\ \delta(l+k-i-j) \prod_{s=1}^{i-k} \frac{x'_a\zeta^{-s}}{1-y'_{-b}\zeta^{-s}}$$

$$= \frac{g((y'_{-b})^{-1})}{g(1)}\ \zeta^{-k(i-k)}\ \delta(l+k-i-j) \prod_{s=1}^{i-k} \frac{-x'_a(y'_{-b})^{-1}}{1-(y'_{-b})^{-1}\zeta^{s}}$$

$$= \frac{g((y'_{-b})^{-1})}{g(1)}\ \zeta^{-k(i-k)}\ \omega((y'_{-b})^{-1},(-x/y)'_{a+b+\epsilon}|i-k)\ \delta(l+k-i-j)$$

where ϵ is defined as in the statement, replacing u by x. Finally, denote by $\{e_j\}_j$ the canonical basis of $\mathbb{C}^N$. It follows from (49) that $(\rho\otimes\rho)(S_\zeta)(e_i\otimes e_j) = e_i\otimes Y^i e_j$. As $(Y^i)^j_l = \zeta^{(m+1)i^2+ij}\delta(l-i-j)$ we get

$$(\rho\otimes\rho)(r^{\frac{1-N}{2}}\ R_\zeta\ (D'\otimes\bar{D}')^{-b(m+1)})^{i,j}_{k,l}$$

$$\equiv_N \frac{g((y'_{-b})^{-1}))}{g(1)} \sum_{s,t=0}^{N-1} \zeta^{(m+1)s^2+st}\ \delta(k-s)\ \delta(l-s-t) \\ \times\ \zeta^{-s(i-s)}\ \omega((y'_{-b})^{-1},(-x/y)'_{a+b+\epsilon}|i-s)\ \delta(s+t-i-j)$$

$$\equiv_N \frac{g((y'_{-b})^{-1})}{g(1)}\ \zeta^{(m+1)k^2+kj}\ \omega((y'_{-b})^{-1},(-x/y)'_{a+b+\epsilon}|i-k)\ \delta(i+j-l).$$

Renaming the variables as $u = 1/y = 1/(1-x)$, $p = b$ and $q = -a-b-\epsilon$, this is exactly the formula (27) for $\widehat{\mathcal{L}}_N(u;p,q) = \mathcal{L}_N(u'_p, v'_{-q})$. □

Remark 8.5 The isomorphism classes of cyclic irreducible representations of $\mathcal{B}_\zeta$ are in one-one correspondence with the elements of a dense subset of a Borel subgroup B of $PSL(2,\mathbb{C})$; a parametrization is given by $[\nu]\mapsto(x_\nu^N, y_\nu^N)$, where $\chi_\nu(E^N) = x_\nu^{2N}$ and $\chi_\nu(D^N) = x_\nu^N y_\nu^N$. In particular, the representation of $\mathcal{B}_{\zeta^{-1}}$ defined by $\rho(E)$ and $\rho(D)$ in Theorem 8.4 is isomorphic to the *dual* of the representation given by (x_ν^N, y_ν^N). In fact, Theorem 8.4 and the discussion at the begining of Subsection 8.1 show that the $\widehat{\mathcal{L}}_N$ are matrices of $6j$–symbols for the cyclic irreducible representations of $\mathcal{B}_\zeta$: they describe the associativity of the tensor product of such representations (see [3], Appendix). The matrix Schaeffer's identity for $\widehat{\mathcal{L}}_N$ (Theorem 5.2) is a version of the pentagon relation satisfied by the $6j$–symbols.

We conclude this section with a 'unitarity-like' property of $\mathcal{L}_N(u'_p, v'_{-q})$:

Proposition 8.6 *The matrix $\mathcal{L}_N(u'_p, v'_{-q})$ is invertible and has determinant 1. Moreover, we have*

$$\mathcal{L}_N^{-1}(u'_p, v'_{-q}) \equiv_N U \left(\mathcal{L}_N((u'_p)^*, (v'_{-q})^*)^T \right)^* U^{-1}$$

where T is the transposition, $*$ the complex conjugation, and U is the symmetric $N^2 \times N^2$–matrix given by $U^{i,j}_{k,l} = \delta(k+i)\ \delta(l+j)$. In components:

$$\mathcal{L}_N^{-1}(u'_p, v'_{-q})^{i,j}_{k,l} = [u'_p]\ \frac{g(1)}{g(u'_p)}\ \zeta^{-il-(m+1)i^2}\ (w(u'_p\zeta^{-1}, v'_{-q}|k-i))^{-1}\ \delta(k+l-j).$$

Proof By Theorem 8.4, the invertibility of $\mathcal{L}_N(u'_p, v'_{-q})$ follows from that of R_ζ and r, together with the fact that $(\rho\otimes\rho)(D'\otimes\bar{D}')$ is invertible (the representation ρ is cyclic). We compute the determinant as follows. Recall that S_ζ is defined in (47). Since the matrices Z and Y are unipotent of order N (they satisfy $Z^N = Y^N = \mathrm{Id}_{\mathbb{C}^N}$), the eigenvalues of $(\rho\otimes\rho)(S_\zeta)$ span the set of Nth roots of unity ζ^j, $j = 0,\dots,N-1$, each with multiplicity N. So $\det((\rho\otimes\rho)(S_\zeta)) = 1$. Moreover, with the notation of the proof of Theorem 8.4, we see that $\det((\rho\otimes\rho)(r^{\frac{1-N}{2}} g(D\otimes\bar{D})))$ is equal to

$$\begin{aligned}
y^{\frac{N(1-N)}{2}}\ \det\left(g(x'_a(Y^{-1}\otimes Z^{-1}Y))\right) &= y^{\frac{N(1-N)}{2}} \prod_{j=0}^{N-1} g(x'_a\zeta^j)^N \\
&= y^{\frac{N(1-N)}{2}}\ g(x'_a)^N \prod_{j=1}^{N-1}\prod_{i=1}^{j} \frac{y}{(1-x'_a\zeta^i)^N} \\
&= y^{\frac{N(1-N)}{2}} \prod_{j=1}^{N-1}(1-x'_a\zeta^{-j})^{Nj}\ \frac{y^{\frac{N(N-1)}{2}}}{\prod_{j=1}^{N-1}(1-x'_a\zeta^j)^{N(N-j)}} = 1
\end{aligned}$$

where we use Lemma 8.2 in the second equality. As $\det\left((\rho\otimes\rho)(D'\otimes\bar{D}')\right) = \det(Y^{-1}\otimes Z^{-1}Y)) = 1$, by Theorem 8.4 we eventually find $\det(\mathcal{L}_N(u'_p, v'_{-q})) = 1$.

As for the last property, note that we have $g(x^*)^* = (-x)^{\frac{N-1}{2}}\zeta^{\frac{(N-1)(2N-1)}{6}} g(1/x)$. Let us write $[x] = N^{-1}(1-x^N)/(1-x)$; then, Lemma 8.3 *i)* gives

$$\frac{g(x^*)^*}{g(1)^*} = \frac{g(x^*)^*}{g(1)^*}\ \frac{g(1)^*}{g(1)}\ (-1)^{\frac{N-1}{2}}\zeta^{-\frac{(N-1)(2N-1)}{6}} = \frac{g(1/x)\ x^{\frac{N-1}{2}}}{g(1)} \equiv_N [x]\ \frac{g(1)}{g(x)}.$$

On another hand, we have

$$\begin{aligned}
(w((u')^*, (v')^*|n))^* &= \prod_{i=1}^{n} \frac{v'}{1-u'\zeta^{-i}} = \prod_{i=n+1}^{N} \frac{1-u'\zeta^{-i}}{v'} = \prod_{j=0}^{N-n-1} \frac{1-u'\zeta^j}{v'} \\
&= (w(u'\zeta^{-1}, v'|-n))^{-1}
\end{aligned}$$

where, to simplify the notation, we write u' and v' for given Nth roots of u and v. So we get (recall that $N = 2m+1$):

$$\begin{aligned}
\left(U\left(\mathcal{L}_N((u')^*, (v')^*)^T\right)^* U^{-1}\right)^{i,j}_{k,l} &= (\mathcal{L}_N((u')^*, (v')^*)^*)^{-k,-l}_{-i,-j} \\
&\equiv_N [u']\ \frac{g(1)}{g(u')}\ \zeta^{-il-(m+1)i^2}\ (w(u'\zeta^{-1}, v'|k-i))^{-1}\ \delta(k+l-j).
\end{aligned}$$

We show that this is the inverse of $\mathcal{L}_N(u',v')$ as follows. First we have

$$\begin{aligned}&[u'] \sum_{s,t=0}^{N-1} \zeta^{-sj-s^2(m+1)}\ \zeta^{sl+(m+1)s^2}\ \frac{w(u',v'|k-s)}{w(u'\zeta^{-1},v'|i-s)}\ \delta(i+j-t)\ \delta(k+l-t)\\ &\overset{(*)}{=} [u']\delta(i+j-k-l)\zeta^{-i(j-l)}w(u',v'|k-i) \sum_{s=0}^{N-1} \zeta^{(i-s)(j-l)}\ \frac{w(u'\zeta^{k-i},v'|i-s)}{w(u'\zeta^{-1},v'|i-s)}.\end{aligned}$$

Next we observe that the above sum is obtained from

$$f(x,y|z) = \sum_{s=0}^{N-1} \prod_{j=1}^{s} \frac{1-y\zeta^j}{1-x\zeta^j}\ z^s$$

by setting $x = u'\zeta^{k-i}$, $y = u'\zeta^{-1}$ and $z = \zeta^{j-l}$. Straightforward manipulations very similar to that of Lemma 8.3 *ii)* imply that

$$f(x,y|z\zeta) = \frac{1-z}{x-yz\zeta}\ f(x,y|z)$$

$$f(x\zeta,y|z) = \frac{(1-x\zeta)(x-yz)}{z(x-y)}\ f(x,y|z).$$

By iterating these two identities we find

$$\begin{aligned} f(x\zeta^t, x\zeta^{-1}|\zeta^s) &= x^{-[s-1]_N} \prod_{a=1}^{s-1} \frac{1-\zeta^{s-a}}{\zeta^t-\zeta^{s-a}}\ f(x\zeta^t,x\zeta^{-1}|\zeta)\\ &= x^{-[s-1]_N} \prod_{a=1}^{s-1} \frac{1-\zeta^{s-a}}{\zeta^t-\zeta^{s-a}}\ \delta(t)\ f(x,x\zeta^{-1}|\zeta)\\ &= x^{-[s-1]_N}\ \delta(t)\ f(x,x\zeta^{-1}|\zeta)\end{aligned}$$

where $[a]_N$ denotes the rest of the Euclidean division of $a \in \mathbb{N}$ by N. Also,

$$\begin{aligned} f(x,x\zeta^{-1}|\zeta) &= 1 + \frac{(1-x)\zeta}{1-x\zeta} + \frac{(1-x)\zeta^2}{1-x\zeta^2} + \ldots + \frac{(1-x)\zeta^{N-1}}{1-x\zeta^{N-1}}\\ &= (1-x)\left(\frac{1}{1-x} + \frac{\zeta}{1-x\zeta} + \ldots + \frac{\zeta^{N-1}}{1-x\zeta^{N-1}}\right)\\ &= (1-x)\ \frac{d}{dx}\left(-\log(1-x^N)\right) = Nx^{N-1}\frac{1-x}{1-x^N}\ .\end{aligned}$$

Hence $f(u'\zeta^{k-i},u'\zeta^{-1}|\zeta^{j-l}) = x^{-[j-l-1]_N}\ \delta(k-i)\ (u')^{N-1}[u']^{-1}$, which shows that $(*)$ above is equal to $\delta(k-i)\ \delta(j-l)$. This concludes the proof of the last claim. □