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Quantum $SU(2)$ faithfully detects mapping class groups modulo center

Michael H Freedman
Kevin Walker
Zhenghan Wang

MHF and KW: *Microsoft Research, Redmond, WA 98052, USA*
ZW: *Indiana University, Department of Math, Bloomington, IN 47405, USA*
michaelf@microsoft.com, kevin@messagetothefish.net, zhewang@indiana.edu

Abstract

The Jones–Witten theory gives rise to representations of the (extended) mapping class group of any closed surface Y indexed by a semi-simple Lie group G and a level k. In the case $G = SU(2)$ these representations (denoted $V_A(Y)$) have a particularly simple description in terms of the Kauffman skein modules with parameter A a primitive $4r^{\text{th}}$ root of unity ($r = k + 2$). In each of these representations (as well as the general G case), Dehn twists act as transformations of finite order, so none represents the mapping class group $\mathcal{M}(Y)$ faithfully. However, taken together, the quantum $SU(2)$ representations are faithful on non-central elements of $\mathcal{M}(Y)$. (Note that $\mathcal{M}(Y)$ has non-trivial center only if Y is a sphere with $0, 1$, or 2 punctures, a torus with $0, 1$, or 2 punctures, or the closed surface of genus $= 2$.) Specifically, for a non-central $h \in \mathcal{M}(Y)$ there is an $r_0(h)$ such that if $r \geq r_0(h)$ and A is a primitive $4r^{\text{th}}$ root of unity then h acts projectively nontrivially on $V_A(Y)$. Jones' [9] original representation ρ_n of the braid groups B_n, sometimes called the generic q–analog–$SU(2)$–representation, is not known to be faithful. However, we show that any braid $h \neq \text{id} \in B_n$ admits a cabling $c = c_1, \ldots, c_n$ so that $\rho_N(c(h)) \neq \text{id}$, $N = c_1 + \ldots + c_n$.

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1 Introduction

Let Y denote a compact, connected, oriented surface. The mapping class group $\mathcal{M}(Y) \cong \mathrm{Diff}^+(Y)/\mathrm{Diff}_0^+(Y)$ is defined as the orientation preserving diffeomorphisms modulo isotopy. (We do not put base points on boundary components.) Lickorish [13] showed that $\mathcal{M}$ is finitely generated, Hatcher and Thurston [7] showed that $\mathcal{M}$ is finitely presented and explicit presentation have been written down [15]. It is known that $\mathcal{M}$ is always residually finite [6]. Bigelow [3] [4] has shown that $\mathcal{M}$ is a matrix group when $\mathrm{genus}(Y) = 0$ and when Y is closed and $\mathrm{genus}(Y) = 2$. Of course, $\mathcal{M}(T^2) \cong \mathrm{SL}(2, Z)$ is also a matrix group.

In this note we study the quantum $SU(2)$ representations of $\mathcal{M}$. Except when $\mathcal{M}(Y)$ is the trivial group ($Y =$ sphere or disk), all these representations, and in fact all quantum representations of which the authors are aware[1], have kernel because Dehn twists are carried to operators of finite order. We prove, however, that the direct sum of all the quantum $SU(2)$ representations is faithful except on central elements of $\mathcal{M}(Y)$ which are never detected. It is well-known [8] that $Z(\mathcal{M}(Y)) = \{e\}$ unless $Y = S^1 \times I, T^2, T^2 - \mathrm{pt}, T^2 - 2\ \mathrm{pts},\ T^2 \# T^2$ in which case the center is the group generated by the elliptic or hyper-elliptic involution.

These quantum $SU(2)$ representations are an outgrowth of Jones–Witten theory. We use the [5] construction of these representations based on the skein theory of the Kauffman bracket. This construction produces a projective representation $V_A(Y)$ of $\mathcal{M}(Y)$ whenever Kauffman's variable A is a primitive $4r^{\text{th}}$ root of unity. (When A is a primitive $2r^{\text{th}}$ root of unity a quantum–$SO(3)$ representation is the result. All our faithfulness results are true for this family as well. Experts will have no difficulty guessing the proof of this extension: simply restrict the present proof to "even labels".)

First we consider surfaces Y without boundary.

Theorem 1.1 *Let Y be a closed connected oriented surface and $\mathcal{M}(Y)$ its mapping class group. For every non-central $h \in \mathcal{M}$, there is an integer $r_0(h)$ such that for any $r \geq r_0(h)$ and any A a primitive $4r^{\text{th}}$ root of unity, the*

[1] Bigelow's representation is equivalent to the BMW representation but at a generic value. At a generic value Dehn twist has infinite order but unfortunately, generic values lead to infinite dimensional – not quantized – representations except in the genus $= 0$ case. (To see the difference consider admissible labelling of trees and graphs. Even if the label set is infinite, if the labels on valence $= 1$ vertices are fixed then there are only finitely many admissible labellings in the tree case.)

operator $\langle h \rangle\colon V_A(Y) \to V_A(Y)$ is not the identity, $\langle h \rangle \neq 1 \in \mathcal{P}\,\mathrm{End}(V_A)$, the projective endomorphisms. In particular, any infinite direct sum of quantum $SU(2)$ representations faithfully represents these mapping class groups modulo center.

Theorem 1.1 and Theorem 3.3, which treats surfaces with boundary, have a formal corollary outside quantum topology (which was previously known [6].)

Corollary 1.2 *For all compact orientable surfaces Y $\mathcal{M}(Y)$ is residually finite.* □

Proof Exploit the fact that finitely generated matrix groups over $\mathbb{C}$ are residually finite. □

Within quantum topology the theorem also has an immediate corollary.

Corollary 1.3 *Let Y be a closed connected compact orientable surface. Let N be the mapping torus of a non-central $h\colon Y \longrightarrow Y$. Let $\langle\,\rangle_A$ denote the closed 3–manifold invariant associated to $(SU(2), A)$, A a primitive $4r^{\text{th}}$ root of unity. For all $r \geq$ some $r_0(h), |\langle N \rangle_A| < |\langle S^1 \times Y \rangle_A|$.*

Proof In the case of Y–bundles over a circle S^1 the gluing relations for a TQFT imply that $\langle\,\rangle_A$ is simply trace (monodromy) $= \mathrm{tr}\langle h \rangle_A$. If $\langle h \rangle_A \neq \mathrm{id}$ then $|\mathrm{tr}\langle h \rangle_A| < |\,\mathrm{tr}\,\mathrm{id}_{V_A}|$. □

The proof of Theorem 1.1 is relatively simple. If h is a non-central element of $\mathcal{M}(Y)$, then there is an embedded curve α in Y such that α and $h(\alpha)$ are not isotopic. Associated to any curve α on Y there is a operator $T_\alpha\colon V_A(Y) \to V_A(Y)$, and $T_{h(\alpha)} = \langle h \rangle T_\alpha \langle h^{-1} \rangle$. We show that for r sufficiently large T_α is not equal (even projectively) to $T_{h(\alpha)}$. It follows that $\langle h \rangle$ acts projectively nontrivially on $V_A(Y)$.

The rest of the paper is organized as follows. Section 2 reviews the facts about the $SU(2)$ quantum invariants we will need. Section 3 contains the proofs of the main theorems, modulo a topological lemma which is proved in Section 4. Section 5 contains further remarks on the original Jones braid group representation.

Achknowledgements We would like to thank Jorgen Andersen for bringing to our attention the question of the eventual faithfulness of the $SU(2)$ representations and for explaining to us his gauge-theoretic approach to the problem, which he has now brought to completion [2]. (For readers of both papers, we should point out that it is not yet proven that the gauge theory and Kauffman bracket constructions yield the same representations.) We also thank the referee for helpful comments.

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2 Review of $SU(2)$ quantum invariants

In this section we briefly review Kauffman skein modules [11] and the [5] construction of the $SU(2)$ quantum invariants. For more details, see [11] and [5].

The Kauffman skein module of a 3–manifold M is defined to the free vector space generated by isotopy classes of unoriented framed links in M, modulo the Kauffman skein relation and replacing trivial loops with a factor of $d = -A^2 - A^{-2}$. (See Figure 1. Throughout this paper figures follow the "blackboard framing" convention.)

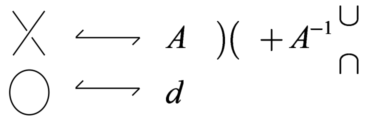

Figure 1: Definition of Kauffman skein module

One can similarly define the Kauffman skein module for a 3–manifold with a finite collection of framed points in its boundary in terms of properly embedded framed 1–submanifolds whose boundary is the given collection of points. Note that for $M = S^3$ any link is equivalent to some multiple of the empty link, so we get a $\mathbb{C}[A, A^{-1}]$ valued invariant of framed links on S^3.

In what follows we specialize to the case

$$A = e^{2\pi i/4r}.$$

(So the Kauffman "polynomial" of a link will actually be a complex number.)

Fact 2.1 *For each $k \leq r-2$ there is a unique skein (finite linear combination of diagrams) P_k in $(B^3, 2k$ points$)$ such that $P_k P_k = P_k$ and P_k is killed by "turn backs". (See Figure 2.)*

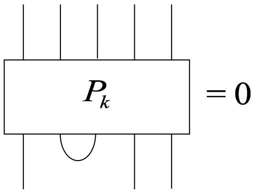

Figure 2: Projector killed by turn-back

It follows that P_k is invariant under a 180 degree rotation (Figure 3), and that P_k is equal to the identity tangle plus terms with turnbacks (Figure 4). P_k is called the *projector* on k strands.

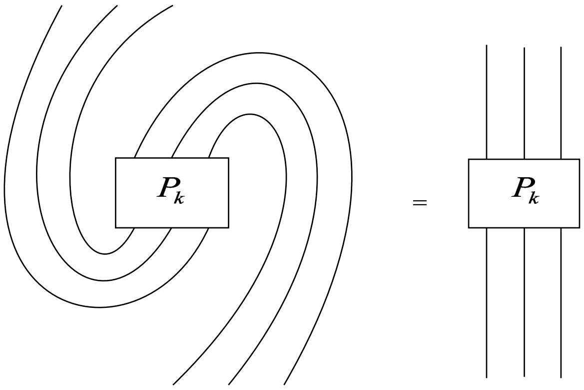

Figure 3: Projector invariant under rotation

Fact 2.2 *For any $n \geq 0$, then identity tangle on n strands can be factored though the sum of projectors $P_0, \ldots, P_{r-2}$. If $n \leq r-2$, then the coefficient of P_n is 1 (Figure 5).*

The fact than only projectors up to $r-2$ are needed is a consequence of A being a $4r^{\text{th}}$ root of 1.

Fact 2.3 *Let b be a braid on k strands and $c(b)$ be the signed number of crossings of b. Then $bP_k = A^{c(b)} P_k$.*

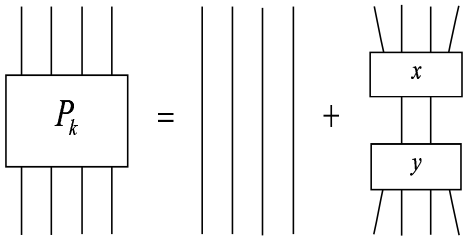

Figure 4: Projector equal to identity plus turn-back terms

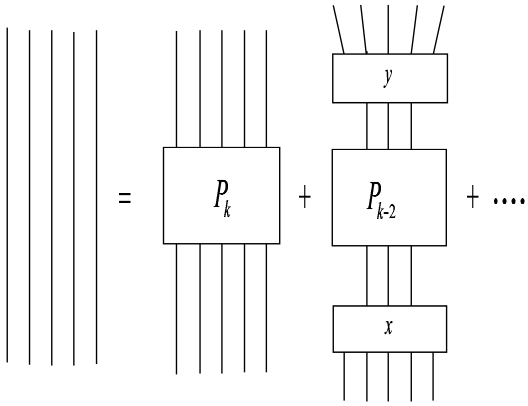

Figure 5: Identity in terms of projectors

Fact 2.3 says that up to scalars, we can absorb a braid into a projector. The proof follows easily from the Kauffman skein relation and Fact 2.1.

Fact 2.4 *Let a, b and c be non-negative integers and let X be a "trivalent vertex" skein as shown in the left hand side of Figure 6. If (a) the three triangle inequalities are satisfied ($a \leq b+c$ etc.), (b) $a+b+c$ is even, and (c) $a+b+c \leq 2r-4$, then X is proportional to the standard diagram on the right hand side of Figure 6. If these conditions are not satisfied then $X = 0$.*

Fact 2.4 follows easily from Fact 2.3 and Figure 4.

Let $G \subset M$ be a trivalent ribbon graph with edges labeled by integers between 0 and $r-2$, such that at each vertex the conditions of Fact 2.4 are satisfied. We will regard G as a shorthand notation for the linear combination of framed links in M obtained by replacing an edge of G labeled by k with P_k, and replacing trivalent vertices with the right hand side of Figure 6.

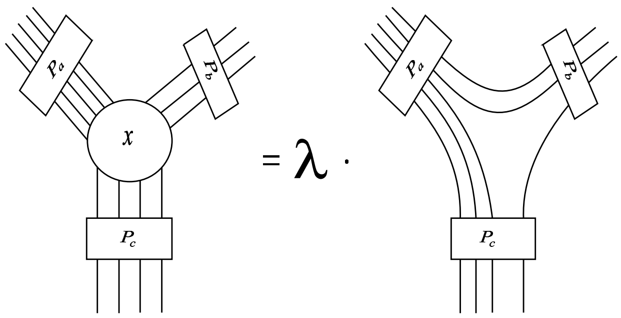

Figure 6: 1–dimensional trivalent vertex space

Let d_k be the value of the skein shown in Figure 7 (unknot labeled by P_k). Let $s_k = cd_k$, where c is a positive real number chosen so that $\sum_{i=0}^{r-2} s_i^2 = 1$.

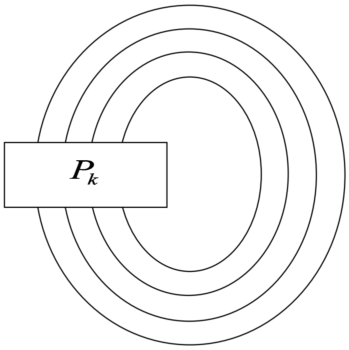

Figure 7: Loop value for projector

In a framed link diagram, a component labeled by ω will mean the linear combination shown in Figure 8.

Fact 2.5 *Framed links with components labeled by ω are invariant under handle slides, balanced stabilization, and the introduction of a circumcision pair. (See Figures 9, 10 and 11.)*

Let L be a framed link in S^3. Let L_ω be the linear combination of labeled framed links obtained by labelling each component of L by ω. It follows from Fact 2.5 that the Kauffman polynomial of L_ω depends only on the 3–manifold described by interpreting L as a surgery diagram, and on the signature of L. For any closed, oriented 3–manifold M and integer n define $Z(M,n)$ to be

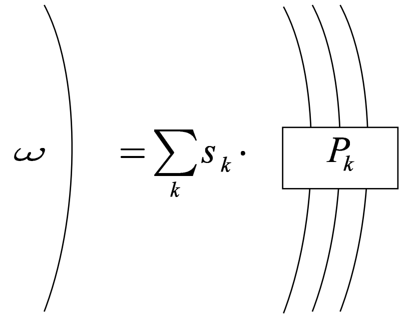

Figure 8: Definition of ω label

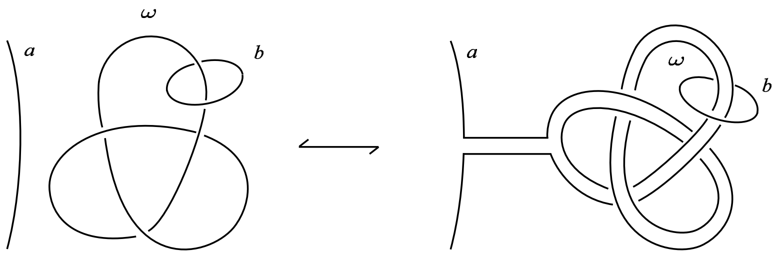

Figure 9: Handle slide invariance

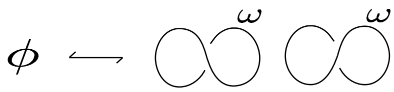

Figure 10: Balanced stabilization invariance

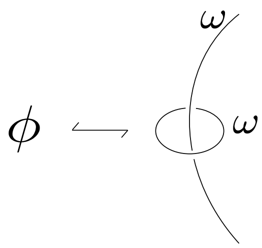

Figure 11: Circumcision pair invariance

this invariant (ie, $Z(M, n)$ is equal to the Kauffman polynomial of L_ω, where

$L \subset S^3$ is any surgery description of M with signature n.) It is easy to see that $Z(M,n) = C^{n-m}Z(M,m)$, where C is the value of the Kauffman polynomial of an unknot with framing 1 (right handed twist).

(Note: n can be interpreted as an equivalence class of framings of the tangent bundle of M, a bordism class of null-bordisms of M, or a p_1–structure on M. See [2], [16] and [5].)

Next we follow the [5] approach to construct a vector space $V(Y)$ for each closed, oriented 2–manifold Y, and an invariant $Z(M) \in V(\partial M)$ for an oriented 3–manifold with boundary. These 2–manifolds and 3–manifolds with boundary should also be equipped with extra structure (framing, null-bordism, or p_1–structure), but we will suppress mention of this since the arguments in the remainder of the paper work even with a projective ambiguity.

Let Y be a closed, oriented 2–manifold. Let $\partial^{-1}Y$ be the set of all isomorphism classes of pairs (M,L), where $\partial M = Y$ and L is a labeled ribbon graph in the interior of M. Let W(Y) be the free vector space generated by $\partial^{-1}Y$. There is a pairing $W(Y) \otimes W(-Y) \to \mathbb{C}$ given by $x \otimes y \mapsto Z(x \cup y)$. Define $V(Y)$ to be the quotient of $W(Y)$ by the annihilator of $W(-Y)$ with respect to this pairing. In other words, $x \sim x'$ if $Z(x \cup y) = Z(x' \cup y)$ for all $y \in W(-Y)$.

If Y is not closed choose a labelling l of the boundary components of Y by integers $0 \le l_c \le r-2$. Let $\widehat{Y}$ be the result of capping off each boundary component of Y by D^2. Define $\partial^{-1}(Y,l)$ to be the set of isomorphism classes of 3–manifold M with ∂M identified with $\widehat{Y}$, and with a properly embedded framed tangle in M which coincides with a standardly embedded copy of P_k in a collar neighborhood of each cap disk, where k is the label assigned to that boundary component of Y by l. We can now define $V(Y;l)$ as above.

The extended mapping class group of Y acts on $\partial^{-1}Y$, and thus on $V(Y)$. The ordinary, non-extended mapping class group of Y has a projective action on $V(Y)$.

The surgery formula for Z shows that $V(Y)$ is spanned by the equivalence classes of links in any single 3–manifold M, $\partial M = Y$. For example, we could take M to be a handlebody H (assuming Y is connected). It then follows from Facts 2.2 and 2.4 that:

Fact 2.6 *Let H be a handlebody with spine S, (ie, S is a 1–complex with vertices at most trivalent, and H is a regular neighborhood of S.) Then $V(\partial H)$ has a basis corresponding to all labellings of the 1–cells of S by integers between 0 and $r-2$, such that the parity and quantum triangle inequalities of Fact 2.4 are satisfied at each vertex of S.*

If Y has non-empty boundary, we get a basis of $V(Y,l)$ by letting $\widehat{Y}$ bound a handlebody H and considering spines of H which meet each cap disk of $\widehat{Y}$ once. Labellings of the spine are constrained to agree with l on 1–cells meeting the boundary.

If Y is closed then $\mathrm{End}(V(Y))$ can be identified with $V(Y \coprod -Y)$, and so is spanned by elements of the form $Z(Y \times I, L)$, where L is a labeled framed link in $Y \times I$. If Y has boundary then $\bigoplus_l \mathrm{End}(V(Y,l))$ can be identified with $V(D(Y))$, where l runs through all labellings of ∂Y and $D(Y) = Y \cup_{\partial Y} -Y$ is the double of Y along its boundary. $D(Y)$ bounds $Y \times I$, and as before $\bigoplus_l \mathrm{End}(V(Y,l))$ is spanned by elements of the form $Z(Y \times I, L)$, where L is a labeled framed link in $Y \times I$. In both cases the action of $\mathrm{End}(\ldots)$ is given in geometric terms by gluing $(Y \times I, L)$ onto a 3–manifold (bounded by Y) representing an element of $V(Y)$ (or $V(Y,l)$).

3 Proof of main theorems

Let Y be a closed, oriented surface, $h\colon Y \to Y$ an orientation preserving homeomorphism, and $V_h\colon V(Y) \to V(Y)$ the action of h on the TQFT vector space.

Proposition 3.1 *Suppose there exists an unoriented simple closed curve $a \subset Y$ such that $h(a)$ is not isotopic (as a set) to a. Then V_h is a multiple of the identity for at most finitely many r. That is, as r increases h is eventually detected.*

Proof Let $C(a) = Z(Y \times I, a \times \{1/2\}) \in V(Y) \otimes V(-Y) = \mathrm{End}(V(Y))$. Define $C(h(a))$ similarly. It's easy to see that $C(h(a)) = V_h C(a) V_h^{-1}$. It therefore suffices to show that $C(a) \neq C(h(a))$.

By Lemma 4.1 there exists a handlebody H bounded by Y such that a bounds an embedded disk in H and $h(a)$ is a non-trivial "graph geodesic" with respect to a spine S of H. Let $Z(H) \in V(Y)$ be the vector determined by H, and $Z(H, h(a)) \in V(Y)$ be the vector determined by the pair $(H, h(a))$. (We can push $h(a)$ into the interior or H.) Then

$$\begin{aligned} & C(a)(Z(H)) = Z(H,a) = d \cdot Z(H), \\ \text{and} \qquad & C(h(a))(Z(H)) = Z(H,h(a)). \end{aligned}$$

It therefore suffices to show that $Z(H, h(a))$ is not a multiple of $Z(H)$.

For each edge e of the spine S, let w_e be the (unsigned) number of times $h(a)$ passes over e. Let m be the maximum of all $w_e + w_f + w_g$ such that e, f and g meet at a vertex of S. Choose r such that $2r - 4 \geq m$.

Let b_w be the basis vector of $V(Y)$ corresponding the labelling w. We claim that $Z(H, h(a)) = \lambda b_w + v$, where $\lambda \neq 0$ and v consists of "lower order" terms – multiples of b_v, where $v_e \leq w_e$ for all edges e of S and $v \neq w$. This follows from Facts 2.2, 2.4 and 2.3. Apply Fact 2.2 at each edge of S. Apply Fact 2.4 at each vertex to see that the result is a linear combination of b_w and lower order terms. Fact 2.3 and the graph geodesic property of $h(a)$ show that the coefficient of b_w is non-zero. On the other hand, $Z(H)$ is the basis vector corresponding to the zero (empty) labelling of S. □

Proof of Theorem 1.1 By Lemma 4.3, non-central elements of the mapping class group must move a simple closed curve, so Theorem 1.1 follows from Proposition 3.1. □

Next we consider the case where Y has boundary. As before, let $h\colon Y \to Y$ be an orientation preserving homeomorphism and

$$V_h \in \bigoplus_{l,l'} \mathrm{Hom}(V(Y,l), V(Y,l'))$$

be the action of h on the TQFT vector spaces.

Proposition 3.2 *Suppose there exists an unoriented, homologically essential simple closed curve $a \subset Y$ such that $h(a)$ is not isotopic to a. Then V_h is a multiple of the identity for at most finitely many r. That is, as r increases h is eventually detected.*

Proof Define operators $C(a)$ and $C(h(a))$ as in the proof of Proposition 3.1. (Note that while $V_h \in \bigoplus_{l,l'} \mathrm{Hom}(V(Y,l), V(Y,l'))$, $C(a)$ and $C(h(a))$ lie in the block diagonal $\bigoplus_l \mathrm{End}(V(Y,l))$.) As before, it suffices to show that $C(a) \neq C(h(a))$.

By Lemma 4.2, $a \times \{1/2\}$ can be extended to a spine of $Y \times I$. Since $h(a)$ is not isotopic in Y to a, $h(a)$ must be isotopic to a graph geodesic distinct from $a \times \{1/2\}$. It follows from Fact 2.6 that $C(a)$ and $C(h(a))$ are (projectively) distinct elements in $V(\partial(Y \times I)) = \bigoplus_l \mathrm{End}(V(Y,l))$, provided r is sufficiently large. □

We can now prove:

Theorem 3.3 *Let Y be a connected orientable surface with boundary and let h be a non-central diffeomorphism of Y. Let $V_h \in \bigoplus_{l,l'} \mathrm{Hom}(V(Y,l), V(Y,l'))$ be the action of h on the TQFT vector spaces. Then V_h is a multiple of the identity for at most finitely many r.*

Proof In light of Proposition 3.2, it suffices to show that any diffeomorphism of Y which fixes all homologically essential simple closed curves lies in the center of the mapping class group. Let h be such a diffeomorphism. Then unless Y is an annulus h cannot permute the boundary components of Y; also h commutes with Dehn twists along homologically essential curves and all "essential" braid twists b ($1/2$ Dehn twists which permute a pair of boundary components) along an essential scc γ which bounds a pair of pants to at least one side. Letting $\mathcal{M}(Y)$ denote the full mapping class group and N the number of boundary components of Y we have a short exact sequence:

$$1 \to \mathcal{M}_0(Y) \to \mathcal{M}(Y) \to \sigma(N) \to 1$$

where $\sigma(N)$ is the permutation group and $\mathcal{M}_0(Y)$ the kernel. If $N = 1$, $\mathcal{M}(Y) = \mathcal{M}_0(Y)$ is generated by Dehn twists along essential sccs and if $N \geq 3$, $\mathcal{M}(Y)$ is generated by Dehn twists along essential sccs together with essential braid twists b as above. In these cases h commutes with a generating set, and therefore all, of $\mathcal{M}(Y)$. When $N = 2$ we need to include some (any) "inessential" braid twist b' along a scc γ' bounding a pair of pants on one side and null bounding on the other side. Since γ' is null homologous, special pleading is now required to prove that $h(\gamma') \approx \gamma'$. We exploit the fact that we may pick any γ' we like so long as it cobounds a pair of points with ∂Y. Choosing γ' amounts to picking a simple arc α between the two components ∂^+ and ∂^- of ∂Y (and then thickening). Choose α so that the geometric intersection numbers are $(\alpha, \beta_0) = 1, (\alpha, \beta_1) = 0, \cdots, (\alpha, \beta_{2g}) = 0$, where $\{\beta_0, \ldots, \beta_{2g}\}$ is a chain a 2 genus $(Y) + 1$ sccs in $\mathrm{int}(Y)$ so that only β's of adjacent indices meet and these meet transversely in a single point and so that ∂^+ is separated from ∂^- by $\bigcup_{i=0}^{2g} \beta_i$ (see Figure 11). Now $(h(\alpha), \beta_i) = (h(\alpha), h(\beta_i)) = (\alpha, \beta_i) = \delta(i)$. It follows that $h(\alpha)$ is isotopic back to α (The isotopy may twist ∂Y.) and that $h(\gamma') \approx \gamma'$. Now the proof can be finished for $N = 2$, as in the case $N \geq 3$, by taking a generating set for $\mathcal{M}(Y)$ consisting of $b' = b'(\gamma')$ together with Dehn twists about essential sccs. □

4 Some topological lemmas

For applications to closed surfaces, we need:

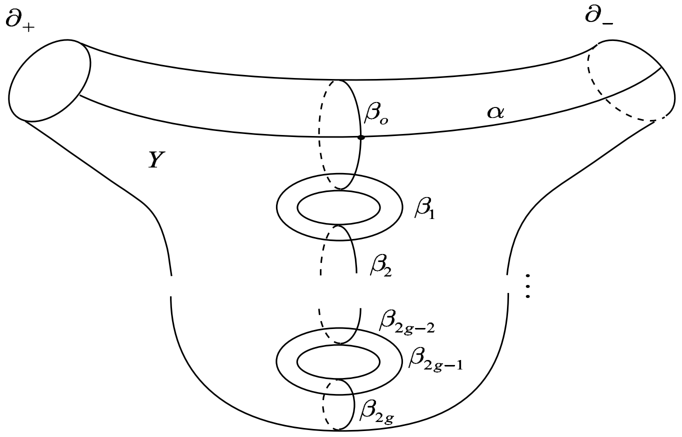

Figure 12

Lemma 4.1 *Let a and b be two non-trivial, non-isotopic simple closed curves on a closed orientable surface Y. Then there exists a pants decomposition of Y such that a is one of the decomposing curves and b is a non-trivial "graph geodesic" with respect to the decomposition. (That is, b does not intersect any curve of the decomposition twice in a row.)*

Proof We will inductively choose a set of decomposing curves on Y, starting with a. At each stage, let Y' denote Y cut along the curves we have chosen thus far, and let b' denote the image of b in Y'. b' is a properly embedded, possibly disconnected, 1–submanifold of Y'.

We say that Y' and b' satisfy Condition X if for each component S of Y' and each component e of $S \cap b'$ either (a) e is non-separating or (b) each component of $S \setminus e$ has genus greater than zero.

Note that initially, when Y' is $Y \setminus a$, Condition X is satisfied (after possibly isotoping b to remove bigons with a). If Y' consists only of pairs of pants (or an annulus if Y was a torus), then Condition X implies the graph geodesic property. Thus it suffices to show that at each stage we can choose an additional decomposing curve such that Condition X is preserved, until we have a pants decomposition.

Choose a component S of Y' which is not a pair of pants or annulus. We will find a simple closed curve (scc) c in S such that $S \setminus c$ still satisfies condition X.

If S has genus greater than zero, let $\bar{S}$ be the closed surface obtained by capping of the boundary of S with disks. Those components of $b' \cap S$ which are: (1) an arc with both endpoints on the same boundary component of S, or (2) a scc, determine a well-defined isotopy class of curves in $\bar{S}$. In case (1) complete the arc to a circle by coning its endpoints in the cap; in case (2) simply include. Choose a curve c in S whose image in $\bar{S}$ does not lie in any of the aforementioned isotopy classes. If the genus of S is ≥ 2, we further require that c is a separating curve. By pushing c across punctured bigons, we may assume that no component of $S \setminus (c \cup b')$ is a punctured bigon (see Figure 13). Thus Condition X is satisfied.

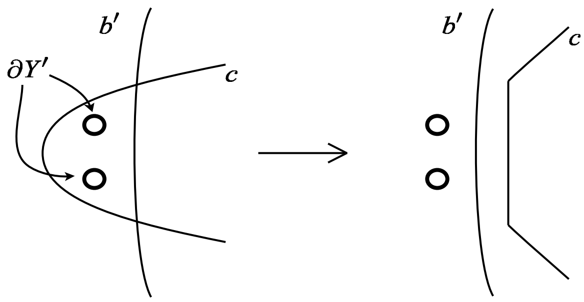

Figure 13: Push across punctured bigon

Note that for a genus 0 surface, Condition X is satisfied if and only if all components of b' are arcs which connect distinct boundary components. Assuming S has four or more punctures, we need to find a scc $c \subset S'$ which is not boundary parallel and meets each arc of b' in at most one point. Cutting along c perpetuates condition X. We use a little geometry here to avoid a greater amount of combinatorics. A well known theorem of Köebe[2][10] represents the edges of any spherical graph by disjoint geodesic arcs of length $< \pi$. Regarding the punctures of S as vertices, represent b' in this way, with the understanding that parallel arcs of b' collapse to a single edge. We call two arcs of b' parallel if they join the same boundary components x and y, and together with an arc in x and an arc in y, bound a rectangle in S. Any great circle γ disjoint from the vertices and containing at least two vertices in each complementary hemisphere is a good choice for c. To find such a γ, start with the great circle γ' determined by any two nonantipotal vertices and perturb it suitably. □

Lemma 4.2 *Let Y be a connected orientable surface with boundary and let a*

[2]Often called Andreev's Theorem.

be a homologically essential simple closed curve in Y. Then a can be extended to a spine of Y.

Proof Cut Y along a and use the classification of surfaces. □

Lemma 4.3 *Suppose Y is a compact oriented surface with or without boundary. Suppose $h\colon Y \longrightarrow Y$ is an orientation preserving homeomorphism, not isotopic to id_Y, which does not change the unparameterised isotopy class of any scc in Y. Then Y is either an annulus, a torus, a torus with ≤ 2 punctures, or the closed surface of genus $= 2$ and h is either the elliptic or hyperelliptic involution.*

Proof If $h\colon Y \longrightarrow Y$ leaves all (unoriented) isotopy classes of scc's invariant then h will commute with all Dehn twists. Since Dehn twists generate $\mathcal{M}(Y)$ [13], $h \in (\mathrm{Center}(\mathcal{M}(Y)) =: Z(\mathcal{M}(Y))$. It is well-known ([8], Theorem 7.5D) that the only surfaces with $Z(\mathcal{M}(Y)) \neq \{e\}$ are $Y = T^2$, $T^2\setminus$ pt., $T^2\setminus$ 2 pts. , $S^1 \times I$, and $T^2 \# T^2$. Furthermore the only nontrivial element of these centers are the elliptic and hyperelliptic involutions respectively. □

5 Further remarks

The Jones representation contains the Burau representation as a particular summand. It is known that the Burau representation is not faithful for B_n with $n \geq 5$. On the other hand, the Jones representation can be obtained by specializing the BMW representation which is faithful for B_n. It seems hard to decide the faithfulness of Jones representation but in this direction, we prove:

Theorem 5.1 *For every braid $h \neq 1 \in B_n$, the n–strand (unframed) braid group $n \geq 2$, there is a cabling $(c_1, \ldots, c_n)$ of h on which the $SU(2)$–Jones representation is nontrivial.*

Proof The Jones representation on B_n when specialized to $A = e^{2\pi i/4r}$, $t = e^{2\pi i/r}$ decomposes as a direct sum of singular and nonsingular pieces. The nonsingular piece is a sum of the $SU(2)$–quantum representations on $V_{1_1 \ldots, 1_n, m}$, the Hilbert space at level $k = r-2$. The subscripts of V are admissible labels at finite punctures and infinity. Cabling produces sums of irreducibles according to a Clebsch–Gordon formula. In particular the Jones representation on the $c_1, \ldots, c_n$ cabling contains as a summand a copy of each admissible $V_{c_1, \ldots, c_n, m}$.

Thus it is sufficient to prove that h acts nontrivially on at least one of these. Theorem 3.3 says that with only finitely many exceptions h is nontrivial in these representations, provided h is not homotopic to the identity in the $n+1-$ punctured sphere, that is $[h] \neq 1 \in (\text{spherical braid group})_{n+1} = SB_{n+1}$.

The proof is not yet finished since the natural morphism $B_n \longrightarrow SB_{n+1}$ has kennel $=$ center(B_n) $=$ $\langle\text{full twist}\rangle$. This "full twist" is Dehn twist about infinity and although this twist is trivial in all $\mathrm{End}((V_{c_1,\ldots,c_n,m})$ its action is computed [11] to be multiplication by the unit scalar $A^{m(m+2)}$. Thus each nontrivial central element is also detected in infinitely many V's. □

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