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Factoring nonrotative $T^2 \times I$ layers

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Abstract In this note we seek to remedy errors which appeared in [4] and were propagated in subsequent papers.

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1 Introduction

The goal of this note is to highlight two errors which appeared in [4] and to provide substitutes for them. The two incorrect statements are Proposition 5.8 and Part 1 of Theorem 2.2, which is a corollary of Proposition 5.8. The incorrect proofs of both statements appear on pages 365–366 of [4]. (The rest of Theorem 2.2 is unaffected by this mistake and is still valid.) After making a few preliminary definitions, we will explain what the incorrect statements are, why they are wrong, and what can be salvaged.

In this note we assume the ambient manifold M is an oriented, compact 3-manifold and the contact structure ξ on M is oriented and positive, unless otherwise stated. We denote the dividing set of a convex surface Σ by Γ_Σ, and the number of connected components of Γ_Σ by $\#\Gamma_\Sigma$.

1.1

First we recall the classification of *nonrotative* tight contact structures on $T^2 \times [0,1]$. Fix an oriented identification $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$, so we may talk about *slopes* of essential curves on T^2. We will denote $T_t = T^2 \times \{t\}$ and the slope of Γ_{T_t} by s_t. Let ξ be a tight contact structure on $T^2 \times [0,1]$ with convex boundary. Then ξ is said to be *nonrotative* if all convex surfaces parallel to T_0 (or T_1) have dividing curves of the same slope; otherwise ξ is said to be *rotative.* An annulus A in a nonrotative $(T^2 \times [0,1], \xi)$ is *horizontal* if it is convex with Legendrian boundary, and each component of $\Gamma_{T_0} \sqcup \Gamma_{T_1}$ intersects ∂A exactly once. Note we may need to modify $\xi|_{T_0 \sqcup T_1}$ using Giroux's Flexibility Theorem (see [2]) — such modifications will usually be made in this note without explicit mention of the Flexibility Theorem.

Recall the following, which is Lemma 5.7 of [4].

Proposition 1.1 *The set of isotopy classes, rel boundary, of nonrotative tight contact structures on $T^2 \times I$ with a fixed convex boundary, where $s_0 = s_1 = \infty$, $\#\Gamma_{T_0} = 2n_0$, $\#\Gamma_{T_1} = 2n_1$, and the characteristic foliation consists of horizontal Legendrian rulings, is in 1-1 correspondence with isotopy classes of dividing curves Γ_A on the horizontal annulus A, rel ∂A, which consist of $n_0 + n_1$ arcs which connect among the $2(n_0 + n_1)$ fixed endpoints on ∂A ($2n_0$ along T_0 and $2n_1$ along T_1), at least two of which are nonseparating.*

A connected component δ of Γ_A is *nonseparating* if $A \setminus \delta$ is connected.

1.2

Let (M, ξ) be a tight contact manifold. We define a *nonrotative outer layer* of (M, ξ) to be a toric annulus $T^2 \times [0,1] \subset M$ for which:

- T_1 is a boundary component of M,
- $(T^2 \times [0,1], \xi|_{T^2 \times [0,1]})$ is nonrotative, and
- $\#\Gamma_{T_0} = 2$, $\#\Gamma_{T_1} = 2n \geq 2$.

Assume (M, ξ) admits a factorization $M = (T^2 \times [0,1]) \cup M_0$, where $T^2 \times [0,1]$ is a nonrotative outer layer. It was claimed (Proposition 5.8 of [4]) that such a factorization is unique up to isotopy, but this is hardly the case. There is a small amount of flexibility in the factorization process, arising out of one case which was forgotten in the "proof" of Proposition 5.8 of [4]. Also, in Part 1 of Theorem 2.2 of [4], it was claimed that if $(T^2 \times [0,1], \xi)$ is a tight contact manifold with convex boundary and $s_0 \neq s_1$, then there exists a *unique* factorization $T^2 \times [0,1] = (T^2 \times [0, \frac{1}{3}]) \cup (T^2 \times [\frac{1}{3}, \frac{2}{3}]) \cup (T^2 \times [\frac{2}{3}, 1])$, where $T_{\frac{i}{3}}$, $i = 0,1,2,3$ are convex, $T^2 \times [0, \frac{1}{3}]$ and $T^2 \times [\frac{2}{3}, 1]$ are nonrotative, and

$\#\Gamma_{T_{1/3}} = \#\Gamma_{T_{2/3}} = 2$. The existence of such a factorization is still valid, but the uniqueness (purportedly a corollary of Proposition 5.8) does not hold. Potential sources of this nonuniqueness will be explained in Sections 3.1 and 3.2.

In general, it appears that the mechanism of factoring the nonrotative outer layer is a rather subtle one, and the following problem does not have a complete solution at this moment:

Problem 1.2 *Classify tight contact structures on $T^2 \times [0,1]$ with convex boundary, in the case $\#\Gamma_{T_0}$ and $\#\Gamma_{T_1}$ are greater than 2.*

In this paper, we will provide partial results towards the mechanism of factorization. In Section 2, we introduce the notion of *disk-equivalence* and prove the following theorem:

Theorem 1.3 *Any two nonrotative outer layers of a tight contact manifold (M,ξ) corresponding to the same torus boundary component of M are disk-equivalent.*

Theorem 1.3 has the advantage that it is a general theorem which is sufficient for many purposes. For example, the proofs of gluing theorems in [5], which mistakenly used Proposition 5.8 of [4], can be easily patched by using Theorem 1.3 — we did not need the full strength of the (incorrect) Proposition 5.8. This will be explained in Section 2.2.

The drawback of Theorem 1.3 is that the full set of nonrotative outer layers $T^2 \times I$ for a tight contact manifold (M,ξ) may not be all the toric annuli disk-equivalent to the initial outer layer. In Section 3 we exhibit two extreme cases: the *shufflable case*, where all the disk-equivalent toric annuli are represented, and the *universally tight case*, where the full set of nonrotative outer layers is substantially smaller.

There are two general strategies for analyzing the factorization process. The easier strategy is to probe the tight contact structure on (M,ξ) *externally*. This involves attaching nonrotative $T^2 \times I$ layers from outside (called *templates*), and weighing their effect on the resulting glued-up contact manifold. The key is to keep track of the layers which glue to give tight contact manifolds, as well as those which glue to give overtwisted contact manifolds. The other strategy is an internal probe, called *state traversal*, explained in [6]. This internal probe, although usually more difficult to implement in practice, yields more complete information than that of *template attaching*. In this note, we shall restrict ourselves to the (much easier) template method. State traversal should yield a complete solution to Problem 1.2, but the combinatorics seem highly nontrivial.

2 General case

In this section, we prove the general result on nonrotative outer layers, namely Theorem 1.3. Theorem 1.3 has the advantage that it has a nice formulation in terms of *disk-equivalence* which is useful in practice. It also admits a relatively elementary proof using template attaching.

2.1

Consider two nonrotative outer layers $N = T^2 \times [0,1]$ and $N' = (T^2 \times [0,1])'$ of (M, ξ), where $T_1 = T_1'$ is a boundary component of M. Let A and A' be the corresponding horizontal annuli with $\partial A = \delta_0 \sqcup \delta_1$ and $\partial A' = \delta_0' \sqcup \delta_1'$. After sliding δ_1' along $T_1 = T_1'$ if necessary, we may assume that $\delta_1' = \delta_1$ and $\Gamma_A \cap \delta_1 = \Gamma_{A'} \cap \delta_1$. Now, we say A and A' (or N and N') are *disk-equivalent* if there exist embeddings $\phi : A \hookrightarrow D^2$ and $\phi' : A' \hookrightarrow D^2$ where $\phi(\delta_1) = \phi'(\delta_1) = \partial D^2$ and $\phi|_{\delta_1} = \phi'|_{\delta_1'}$, such that Γ_{D^2} on D^2, obtained from $\phi(\Gamma_A)$ by connecting the two endpoints of $\phi(\Gamma_A \cap \delta_0)$ via an arc in $D^2 \setminus \phi(A)$, and Γ'_{D^2}, obtained similarly from $\phi'(\Gamma_{A'})$, are isotopic rel ∂D^2.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 Consider the factorization $M = N \cup M_0$, where $N = T^2 \times [0,1]$ is a nonrotative outer layer and $A_{[0,1]}$ is its horizontal annulus. We prove that $A'_{[0,1]}$ corresponding to another nonrotative outer layer N' is disk-equivalent to $A_{[0,1]}$. Write $\partial A_{[a,b]} = \delta_a \sqcup \delta_b$.

Let $\mathcal{T}_{A_{[0,1]}}$ (resp. $\mathcal{T}$) be the set of isotopy classes of nonrotative tight contact structures $(T^2 \times [1,2], \zeta)$ with a fixed boundary characteristic foliation and $\#\Gamma_{T_2} = 2$, which glue to $(N = T^2 \times [0,1], \xi|_N)$ to yield a tight contact structure on $T^2 \times [0,2]$ which is I-invariant (resp. a tight contact structure on $M \cup (T^2 \times [1,2])$). Here, the *$I$-invariant* tight contact manifold is isomorphic to an invariant neighborhood of a convex surface T_2 (or T_0). By Proposition 1.1, a nonrotative $(T^2 \times [1,2], \zeta)$ is characterized by the dividing set of its horizontal annulus $A_{[1,2]}$. Any $\Gamma_{A_{[1,2]}}$ will have exactly two endpoints along δ_2 and exactly two nonseparating arcs. Associate to $\mathcal{T}_{A_{[0,1]}}$ (resp. $\mathcal{T}$) the corresponding set of isotopy classes $\mathcal{A}_{A_{[0,1]}}$ (resp. $\mathcal{A}$) of $\Gamma_{A_{[1,2]}}$. Let $A_{[0,2]} = A_{[0,1]} \cup A_{[1,2]}$ be the horizontal annulus for $T^2 \times [0,2]$, where we assume that $A_{[0,1]}$ and $A_{[1,2]}$ have common boundary δ_1. Now, $\Gamma_{A_{[1,2]}} \in \mathcal{A}_{A_{[0,1]}}$ if and only if $\Gamma_{A_{[0,2]}}$ consists of exactly two parallel nonseparating arcs. Clearly, $\mathcal{A}_{A_{[0,1]}} \subset \mathcal{A}$, since the I-invariance of $T^2 \times [0,2]$ implies there is a contact diffeomorphism $(M,\xi) \cup ((T^2 \times$

$[1,2]), \zeta) \simeq (M_0, \xi|_{M_0})$. Of course, $\mathcal{A}$, unlike $\mathcal{A}_{A_{[0,1]}}$, depends on the ambient (M, ξ), and $\mathcal{A} - \mathcal{A}_{A_{[0,1]}}$ may or may not contain certain $\Gamma_{A_{[1,2]}}$ for which $\Gamma_{A_{[0,2]}}$ contains (necessarily homotopically essential) closed curves. See Figure 1 for various possibilities of $\Gamma_{A_{[1,2]}}$.

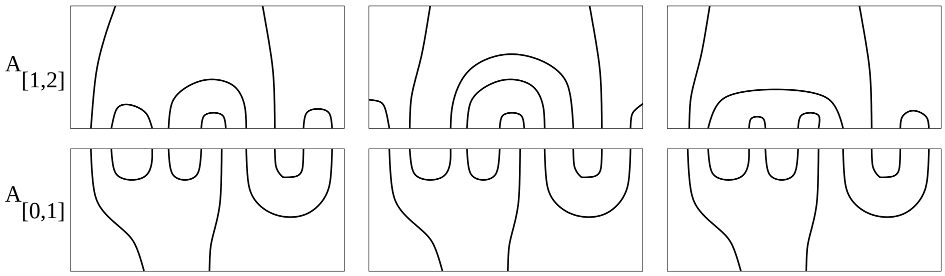

Figure 1: In all the figures, the sides are identified. The right-hand $\Gamma_{A_{[1,2]}}$ is in $\mathcal{T}_{A_{[0,1]}}$, the left-hand diagram is not in $\mathcal{T}$, and it cannot be determined simply by looking at $A_{[0,2]}$ whether the middle is in $\mathcal{T} - \mathcal{T}_{A_{[0,1]}}$.

The induction is done by fixing $(M_0, \xi|_{M_0})$ and inducting on $\#\Gamma_{T_1} = 2n$ over the space of all nonrotative outer layers $N = T^2 \times [0,1]$ with $\#\Gamma_{T_0} = 2$. Note that all nonrotative $N = T^2 \times [0,1]$ with $\#\Gamma_{T_0} = 2$ can be embedded inside an I-invariant neighborhood of T_0 by *folding* (see Section 5.3 of [4]), so all contact structures on $M_0 \cup N$ constructed this way are tight. When $n = 1$, the nonrotative outer layer is clearly unique. Therefore, assume the theorem is true for $\#\Gamma_{T_1} = 2n$, and we prove it for $\#\Gamma_{T_1} = 2(n+1)$. There are two cases: either $\Gamma_{A_{[0,1]}}$ has at least two ∂-parallel curves or there is only one ∂-parallel curve.

Suppose first that there are at least two ∂-parallel curves on $A_{[0,1]}$. Let γ be an arc on $A_{[1,2]}$ whose endpoints are consecutive points of $\Gamma_{A_{[0,1]}} \cap \delta_1$, ie, γ is *∂-parallel.* If the endpoints of γ coincide with the endpoints of a ∂-parallel curve of $A_{[0,1]}$, then, for any completion of γ to a dividing set $\Gamma_{A_{[1,2]}} \supset \gamma$, the gluing $A_{[0,1]} \cup A_{[1,2]}$ corresponds to an overtwisted contact structure. On the other hand, if the endpoints of γ are not (i) the two endpoints of the nonseparating curves of $\Gamma_{A_{[0,1]}}$ and not (ii) the two endpoints of a ∂-parallel curve of $\Gamma_{A_{[0,1]}}$, then γ can be completed into some $\Gamma_{A_{[1,2]}} \in \mathcal{A}$. We now summarize the completability of γ to an element in $\mathcal{A}$: unknown if endpoints are (i), no if endpoints are (ii), and yes otherwise. (Here "unknown" means that it depends on whether adding an extra π-twisting $T^2 \times I$ layer to M_0 yields a tight contact structure or an overtwisted contact structure.) Now, since there are at least two ∂-parallel curves of $\Gamma_{A_{[0,1]}}$, there are at least two ∂-parallel

γ which cannot be completed to an element of $\mathcal{A}$, and at least one of them must have the same endpoints as a ∂-parallel curve of $\Gamma_{A'_{[0,1]}}$. (This follows from repeating the same argument for $A'_{[0,1]}$ instead of $A_{[0,1]}$.) Thus, there is a common ∂-parallel position for both $A_{[0,1]}$ and $A'_{[0,1]}$. Now, attach a horizontal annulus with $2n$ nonseparating curves and one ∂-parallel dividing curve γ right next to the common ∂-parallel position of $A_{[0,1]}$ and $A'_{[0,1]}$ as in Figure 2, and use the inductive step.

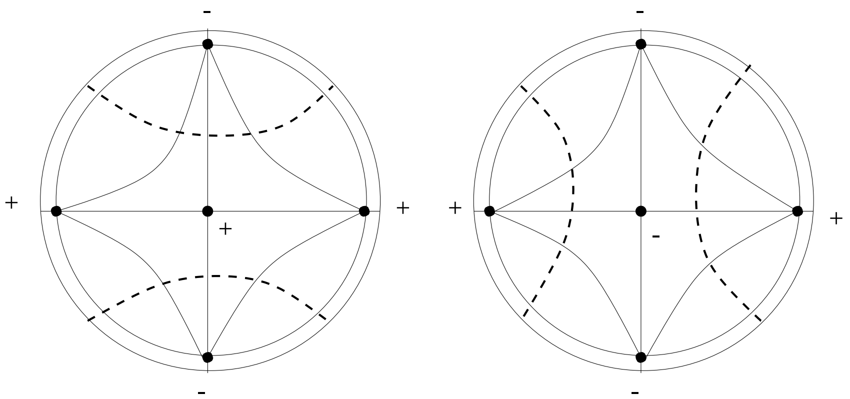

Figure 2: Inductive step

Suppose now that there exists only one ∂-parallel arc of $\Gamma_{A_{[0,1]}}$. Then the two nonseparating curves must be consecutive (ie, one of the regions of $A_{[0,1]}$ divided by these two curves does not contain any other dividing curves), and all the separating curves must be nested concentrically around the one ∂-parallel dividing curve. See Figure 3. The ∂-parallel arc γ on $A_{[1,2]}$ satisfying (ii) is at the center (solid line), and γ satisfying (i) is given by dotted lines. Then $\Gamma_{A'_{[0,1]}}$ satisfies one of the following:

- $\Gamma_{A'_{[0,1]}} = \Gamma_{A_{[0,1]}}$.
- The positions of (i) and (ii) are reversed.
- Positions (i), (ii) for $\Gamma_{A_{[0,1]}}$ are both (ii) for $\Gamma_{A'_{[0,1]}}$.

In each case, $\Gamma_{A_{[0,1]}}$ and $\Gamma_{A'_{[0,1]}}$ are disk-equivalent. □

Note that Theorem 1.3 does not completely address exactly which nonrotative outer layers exist for a given (M,ξ).

Corollary 2.1 *Given two factorizations $M = N \cup M_0$ and $M = N' \cup M'_0$ of a tight (M,ξ), where N, N' are nonrotative outer layers corresponding*

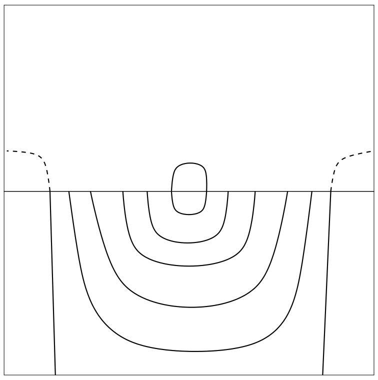

Figure 3: Only one ∂-parallel dividing curve. The bottom annulus is $A_{[0,1]}$ and the top one is $A_{[1,2]}$.

to the same torus boundary component of M, there exists an isomorphism $(M_0, \xi|_{M_0}) \simeq (M_0', \xi|_{M_0'})$.

Proof The actual isomorphism is not an arbitrary isomorphism, but an isotopy in the following sense. Let $(T^2 \times [1,2], \zeta)$ be an element of $\mathcal{T}_{A_{[0,1]}}$ as in the proof of Theorem 1.3. Then there exists a contact isotopy of $(M_0, \xi|_{M_0})$ to $(M,\xi) \cup (T^2 \times [1,2], \zeta)$ inside $M \cup (T^2 \times [1,2])$. This is clear from the I-invariance of $N \cup (T^2 \times [1,2])$. Now we claim that the disk-equivalence of $A_{[0,1]}$ and $A'_{[0,1]}$ implies that $N' \cup (T^2 \times [1,2])$ is I-invariant, thus proving the contact isotopy of $(M_0', \xi|_{M_0'})$ to $(M,\xi) \cup (T^2 \times [1,2], \zeta)$ inside $M \cup (T^2 \times [1,2])$. Write $A'_{[0,2]} = A'_{[0,1]} \cup A_{[1,2]}$, $\partial A_{[0,2]} = \delta_0 \sqcup \delta_2$, and $\partial A'_{[0,2]} = \delta_0' \sqcup \delta_2'$. We then complete $A_{[0,2]}$ (resp. $A'_{[0,2]}$) by attaching a disk D and (resp. D') along δ_0 (resp. δ_0'). By the disk-equivalence, the dividing sets on $A_{[0,2]} \cup D$ and $A'_{[0,2]} \cup D'$ are identical and consist of exactly one ∂-parallel arc along $\delta_2 = \delta_2'$. This in turn implies that, after removing D' from $A'_{[0,2]} \cup D'$, $\Gamma_{A'_{[0,2]}}$ consists of exactly two nonseparating arcs. This completes the proof. □

2.2

In this section we seek to remedy some tightness proofs in [5] which were affected by the misuse of unique factorizations for nonrotative outer layers. The situation we are interested in is the following. Let (M,ξ) be a contact manifold and $T \subset M$ an incompressible torus. Using state traversal in [5] and [6], we want to determine whether (M,ξ) is tight. When we use this method, we start with T convex and for which it is easy to determine that $(M\backslash T, \xi|_{M\backslash T})$ is tight. Successively we find T' isotopic to and disjoint from T, and ask whether $(M\backslash T', \xi|_{M\backslash T'})$ is tight. If yes, then we let T' be the new T, and continue. If tightness is preserved for all possible T', then (M,ξ) is tight. Usually, the initial state consists of $\#\Gamma_T = 2$, but, during the course of the state transitions, $\#\Gamma_{T'}$ may become large. The following theorem allows us to avoid these more complicated states.

Theorem 2.2 *It is sufficient to verify the following in order to prove the tightness of the contact manifold (M,ξ) using state tranversal:*

(1) *$\xi|_{M\backslash T'}$ is tight for every convex T' with $\#\Gamma_{T'} = 2$, obtained from T via a sequence of bypass moves, each of which leaves $\#\Gamma = 2$.*

(2) *Let T'' be a convex torus isotopic to T with tight $\xi|_{M\backslash T''}$. Let $T^2 \times [-0.5, 0.5] \hookrightarrow M$ be a toric annulus with $T_0 = T''$ and nonrotative $T^2 \times [-0.5, 0]$ and $T^2 \times [0, 0.5]$. Then there exists an extension to $T^2 \times [-1, 1] \hookrightarrow M$ where $T^2 \times [-1, 0]$ and $T^2 \times [0, 1]$ are nonrotative outer layers in $M\backslash T''$. In particular, $\#\Gamma_{T_{-1}} = \#\Gamma_{T_1} = 2$.*

Proof The smallest state transition unit $T \rightsquigarrow T''$ consists of attaching a bypass along T to obtain T''. Hence, every pair T, T'' of isotopic tori is related by a sequence of bypass attachments. Suppose that $(M\backslash T', \xi|_{M\backslash T'})$ is tight for every convex T' with $\#\Gamma_{T'} = 2$, obtained from T via a sequence of bypass moves which do not change $\#\Gamma$. Observe that if $T = \Sigma_0 \rightsquigarrow \Sigma_1 \rightsquigarrow \cdots \rightsquigarrow \Sigma_k$ is the sequence of bypass moves which extricates the original T from a candidate overtwisted disk, then there will exist intervals $\Sigma_i \rightsquigarrow \cdots \rightsquigarrow \Sigma_j$ where $\#\Gamma_{\Sigma_i} = \#\Gamma_{\Sigma_j} = 2$ and $\#\Gamma_{\Sigma_l} > 2$ inbetween, or half-intervals $\Sigma_i \rightsquigarrow \cdots \rightsquigarrow \Sigma_k$, where $\#\Gamma_{\Sigma_i} = 2$ and $\#\Gamma_{\Sigma_l} > 2$ thereafter. We will prove that the state transitions when $\#\Gamma > 2$ are rather superficial, and that $(M \setminus \Sigma_i, \xi|_{M\setminus\Sigma_i}) \simeq (M \setminus \Sigma_j, \xi|_{M\setminus\Sigma_j})$.

We inductively assume the following:

(A) T'' is one of the Σ_l between Σ_i and Σ_j (or Σ_k).

(B) $(M \setminus T'', \xi|_{M\setminus T''})$ is tight.

(C) There exist nonrotative layers $T^2 \times [-1, 0]$, $T^2 \times [0, 1]$ with $T_0 = T''$ and $\#\Gamma_{T_{-1}} = \#\Gamma_{T_1} = 2$, and such that $T^2 \times [-1, 1]$ is I-invariant.

(D) There is an isomorphism

$$(M \setminus \Sigma_i, \xi|_{M\setminus\Sigma_i}) \simeq (M \setminus (T^2 \times [-1, 1]), \xi|_{M\setminus(T^2\times[-1,1])}).$$

Let $A_{[-1,0]}$ and $A_{[0,1]}$ be the horizontal annuli corresponding to $T^2 \times [-1,0]$ and $T^2 \times [0,1]$.

Let $(T^2 \times [-0.5,0])'$ be the layer between $\Sigma_l = T''$ and Σ_{l+1}. It is nonrotative because $\#\Gamma_{\Sigma_l} > 2$ and we are considering a single bypass move from Σ_l to Σ_{l+1}. The hypotheses of the theorem guarantee an extension to $(T^2 \times [-1,0])'$, a nonrotative outer layer of $M \backslash T''$. There also exists a nonrotative outer $(T^2 \times [0,1])'$ on the other side of T''. Call the corresponding new horizontal annuli $A'_{[-1,0]}$ and $A'_{[0,1]}$. (Also let $A'_{[-1,1]} = A'_{[-1,0]} \cup A'_{[0,1]}$.)

The key is to prove that the new layer $(T^2 \times [-1,1])'$ containing Σ_{l+1} is I-invariant. This is done by completing $A_{[-1,0]}$ to a disk D_1, $A_{[0,1]}$ to a disk D_2, and likewise forming D'_1 and D'_2 from $A'_{[-1,0]}$ and $A'_{[0,1]}$. If we put D_1 and D_2 together to form S^2 so the dividing curves match up, then there is exactly one dividing curve, since $\Gamma_{A_{[-1,1]}}$ consists of two parallel nonseparating curves. (The corresponding toric annulus is I-invariant.) Now use Theorem 1.3 to see that $D'_1 \cup D'_2$ must also consist of exactly one dividing curve, due to disk-equivalence. Now, $\Gamma_{A'_{[-1,1]}}$ is obtained by removing two small disks from $D'_1 \cup D'_2$, each containing a short arc of the dividing set. Therefore, $\Gamma_{A'_{[-1,1]}}$ must consist of parallel nonseparating curves. This proves that Condition C of the inductive step also holds for Σ_{l+1}. Next, Condition D is satisfied, since

$$(M \setminus \Sigma_i, \xi|_{M\setminus\Sigma_i}) \simeq (M \setminus (T^2 \times [-1,1]), \xi|_{M\setminus(T^2\times[-1,1])}),$$

and

$$(M \setminus (T^2 \times [-1,1]), \xi|_{M\setminus(T^2\times[-1,1])}) \simeq (M \setminus (T^2 \times [-1,1])', \xi|_{M\setminus(T^2\times[-1,1])'}),$$

due to Corollary 2.1. Condition B is now obvious, since $(M \setminus \Sigma_{l+1}, \xi|_{M\setminus\Sigma_{l+1}})$ is obtained from $(M \setminus \Sigma_i, \xi|_{M\setminus\Sigma_i})$ by folding. □

The following suffices for the purposes of gluing in [5].

Corollary 2.3 *Let $M = (T^2 \times [0,1])/\sim$ be a T^2-bundle over S^1, obtained by identifying $T_0 \sim T_1$, and let ξ be a contact structure on M. If $\xi|_{T^2\times[0,1]}$ is a rotative tight contact structure, then $\xi|_M$ is tight if Condition 1 of Theorem 2.2 is satisfied.*

Proof Let $T = T_0 = T_1$. Then $\xi|_{M\setminus T}$ is rotative and any pair of nonrotative layers $(T^2 \times [0,0.1]) \sqcup (T^2 \times [0.9,1])$ can be extended to a pair of nonrotative outer layers $(T^2 \times [0,0.2]) \sqcup (T^2 \times [0.8,1])$ using bypasses and the Imbalance Principle [4]. Moreover, for each state transition $T \rightsquigarrow T''$, if $\xi|_{M\setminus T}$ is rotative, then so is $\xi|_{M\setminus T''}$. □

3 Special cases

In this section we assume the following:

Extendability Condition Let (M,ξ) be a tight contact manifold with convex boundary ∂M, one component of which is a torus T. Assume there exists a factorization $M=(T^2\times[-1,1])\cup M_0$, where $T_1=T$, $s_{-1}=0$, $s_1=-\infty$, $\Gamma_{T_{-1}}=2$, $\Gamma_{T_1}>2$, and every convex torus in $T^2\times[-1,1]$ parallel to T_{-1} (or T_1) has slope s satisfying $-\infty\leq s\leq 0$.

Let us call such a $T^2\times[-1,1]$ a *rotative outermost layer.* Note that the Extendability Condition is very similar to the "quasi-pre-Lagrangian" condition in Colin [1].

3.1

Here we present the first sources of nonuniqueness of nonrotative outer layers. Suppose (M,ξ) is universally tight and satisfies the Extendability Condition. Consider a rotative outermost layer $T^2\times[-1,1]\subset M$, where $s_1=\infty$ and $s_{-1}=0$. Consider a factorization of $T^2\times[-1,1]$ into $T^2\times[-1,0]$ and $T^2\times[0,1]$, where the first is a *basic slice* (ie, contactomorphic to $(T^2\times[-1,0],\xi)$ with convex boundary, $\#\Gamma_{T_{-1}}=\#\Gamma_{T_0}=2$, $s_{-1}=0$, $s_0=-\infty$, and every convex surface parallel to T_0 has dividing curves of slope s satisfying $-\infty\leq s\leq 0$) and the second is a nonrotative outer layer. Let $A_{[0,1]}$ be the horizontal annulus for $T^2\times[0,1]$ and $A_{[-1,0]}$ be the "horizontal annulus" for $T^2\times[-1,0]$ in the sense that A is convex with efficient Legendrian $\partial A_{[-1,0]}=\delta_{-1}\sqcup\delta_0$ of slope 0 on T_{-1} and T_0. Here, a closed curve γ on a convex surface Σ is *efficient* if $\gamma\pitchfork\Gamma_\Sigma$ and the geometric intersection number $|\gamma\cap\Gamma_\Sigma|$ equals the actual number of intersection points. Let $\eta_1,\cdots,\eta_k$ be the 'innermost' dividing curves on $A_{[0,1]}\cup A_{[-1,0]}$, ie, there exists an arc from η_i to δ_{-1} which intersects no other dividing curve except perhaps for closed essential dividing curves on $A_{[-1,0]}$ (if they exist). Then the various nonrotative outer layers are obtained by truncating some η_i.

3.2

Next we consider the following situation, which we call the *shufflable case.*

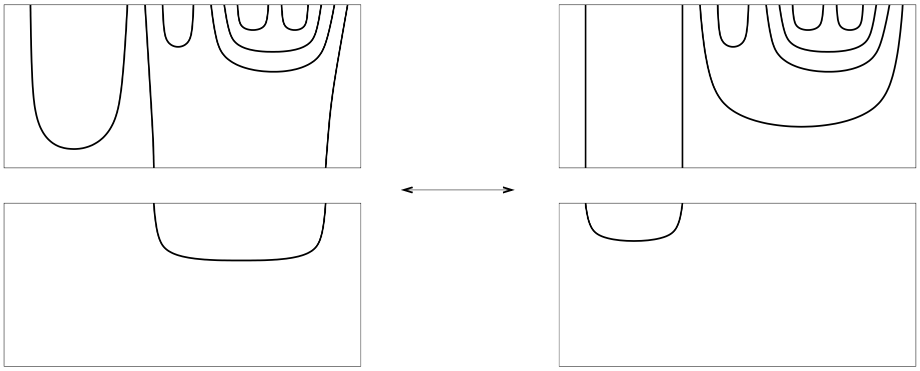

Figure 4: Equivalence in the universally tight case. The top annulus is $A_{[0,1]}$ and the bottom annulus is $A_{[-1,0]}$.

Assumption Let (M, ξ) be a tight contact manifold with convex boundary and T a torus component of ∂M. Suppose there exists a layer $T^2 \times [-2, 1] \subset M$ with $T_1 = T$, for which $s_{-2} = \frac{1}{2}$, $s_{-1} = 0$, $s_0 = s_1 = \infty$, $\#\Gamma_{T_{-2}} = \#\Gamma_{T_{-1}} = \#\Gamma_{T_0} = 2$, and $\#\Gamma_{T_1} = 2n$. Let $T^2 \times [-2, -1]$ and $T^2 \times [-1, 0]$ be basic slices, and let $T^2 \times [0, 1]$ be a nonrotative outer layer. Moreover, assume that the relative Euler classes of $T^2 \times [-2, -1]$ and $T^2 \times [-1, 0]$ are $\pm(1, 1)$, $\mp(1, 1)$, respectively. These two basic layers can be switched via a contact isotopy, which is called *shuffling* in [4]. Therefore, if we have such a $T^2 \times [-2, 1]$-layer, we say we are in the *shufflable case.*

In the shufflable case, the rotative outermost layer is certainly not unique, as can be seen from Figure 5. In other words, there is a clear equivalence relation, where the dividing curve configuration for $A_{[-1,0]}$ is substituted by the other possibility (ie, coming from $A_{[-2,-1]}$ after shuffling).

If we combine moves described in Section 3.1 with the moves described in Figure 5, it is clear that *all* the configurations of $A_{[0,1]}$ disk-equivalent to the initial one are realized. Combining this with Theorem 1.3, we obtain the following:

Proposition 3.1 *Let (M, ξ) be a tight contact manifold with convex boundary ∂M and let T be a torus component of ∂M. Suppose M is shufflable along T. If we fix a nonrotative outer layer $N = T^2 \times [0, 1]$ with $T_1 = T$ and let $A_{[0,1]}$ be its horizontal annulus, then the set of isotopy classes of nonrotative outer layers (rel boundary) for (M, ξ) along T is in 1-1 correspondence with the set of isotopy classes of dividing multicurves (rel boundary) disk-equivalent to $\Gamma_{A_{[0,1]}}$.*

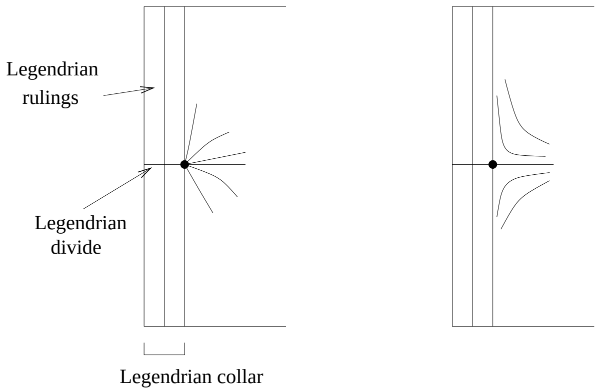

Figure 5: Equivalences in the shufflable case

3.3

The following is the analog of Proposition 1.1 for rotative outermost layers.

Lemma 3.2 *Let $(M = T^2 \times [-1,1], \xi)$ be a rotative outermost layer. Then there exists a unique dividing set $\Gamma_{A_{[-1,1]}}$, modulo closed curves which are parallel to the boundary.*

Proof We take $s_{-1} = 0$, $s_1 = \infty$, and $\#\Gamma_{T_1} > 2$. As in the proof of Theorem 1.3, consider the set $\mathcal{T}$ of nonrotative tight contact structures $(T^2 \times [1,2], \zeta)$ with $\#\Gamma_{T_2} = 2$, which glue to $(M = T^2 \times [-1,1], \xi)$ to yield a tight contact structure on $T^2 \times [-1,2]$. The key difference between this case and Theorem 1.3 is that it is possible to determine $\mathcal{T}$ and its corresponding $\mathcal{A}$ precisely. That is, $\mathcal{A}$ consists of all $\Gamma_{A_{[1,2]}}$ for which $\Gamma_{A_{[-1,2]}}$ does not have any homotopically trivial dividing curves. — in other words, the "unknown" gluings which produced the middle configuration in Figure 1 are now known to be tight gluings. Elements of $\mathcal{A}$ correspond to $(T^2 \times [1,2], \zeta)$, whose attachment makes $T^2 \times [-1,2]$ either into a *basic slice* or adds extra twisting by a multiple of π.

Now, we want to prove that if $A_{[-1,1]}$ and $A'_{[-1,1]}$ are two horizontal annuli for $T^2 \times [-1,1]$, then $A_{[-1,1]} = A'_{[-1,1]}$ modulo parallel closed essential curves. This is proved by induction on $\#\Gamma_{T_1}$. If $\#\Gamma_{T_1} = 2$, then there are two possibilities for $\Gamma_{A'_{[-1,1]}}$ modulo parallel closed essential curves, corresponding to the two possible positions for ∂-parallel dividing curves. In this step only, we attach templates which are basic slices $(T^2 \times [1,2], \zeta')$ (not nonrotative layers) with $s_1 = \infty$ and $s_2 = 0$, and corresponding "horizontal" annuli $A_{[1,2]}$. The two basic slices are also distinguished by the positions of the ∂-parallel dividing curves along δ_1. (As before, we are assuming that $\partial A_{[-1,1]} = \delta_{-1} \sqcup \delta_1$ and $\partial A_{[1,2]} = \delta_1 \sqcup \delta_2$. Note they have a common boundary δ_1.) Since the gluing is tight if and only if a closed homotopically trivial curve does not appear on $A_{[-1,2]}$, the two possible $\Gamma_{A'_{[-1,1]}}$ can be distinguished using templates.

Next, assume inductively that the claim holds for $\#\Gamma_{T_1} = 2n$. Let $\#\Gamma_{T_1} = 2(n+1)$. Now any arc γ on $A_{[1,2]}$ with consecutive endpoints on $\delta_1 \cap \Gamma_{A_{[-1,1]}}$ can be extended to some $\Gamma_{A_{[1,2]}} \in \mathcal{A}$, if and only if the endponts of γ are not the endpoints of a ∂-parallel dividing curve of $A_{[-1,1]}$. This implies that the set of ∂-parallel curves must be the same for $A_{[-1,1]}$ and $A'_{[-1,1]}$. We then reduce to the case $\#\Gamma_{T_1} = 2n$ in the same manner as in the proof of Theorem 1.3. □

3.4

The argument in Section 3.3 generalizes to the case where (M, ξ) is universally tight.

Proposition 3.3 *If (M, ξ) is universally tight and satisfies the Extendability Condition, and ∂M is an incompressible torus, then any two rotative outermost layers are contact diffeomorphic.*

Proof In this case, we can apply the same template matching as in Lemma 3.2. Let $N = T^2 \times [-1, 1]$ be an outermost rotative layer with $T_1 = \partial M$, and $A_{[-1,1]}$ the corresponding horizontal annulus. Let $\mathcal{A}$ be the set of configurations on $A_{[1,2]}$, corresponding to nonrotative $T^2 \times [1, 2]$ for which $M \cup (T^2 \times [1, 2])$ remains tight. We claim that $\mathcal{A}$ once again is the set of $\Gamma_{A_{[1,2]}}$ for which no homotopically trivial dividing curves appear after merging with $A_{[-1,1]}$. Note that there might be some attachments of $T^2 \times [1, 2]$ for which the twisting increases by a multiple of π when we compare $T^2 \times [-1, 1]$ and $T^2 \times [-1, 2]$. This happens when homotopically nontrivial closed curves are created on $A_{[-1,2]}$. The tightness is guaranteed by Colin's gluing theorem for universally tight contact structures along incompressible tori (see [1]). Finally, $\mathcal{A}$ is sufficient to recover $A_{[-1,1]}$. This proves that any two rotative outermost layers are contact diffeomorphic. □

3.5

We make some remarks. Although we were able to corral in the nonrotative outer layers up to disk-equivalence using Theorem 1.3, the *exact set* of allowable nonrotative outer layers for a fixed (M,ξ) with torus boundary is much more difficult to determine.

One of the difficulties (though by no means the only one) is our inability to answer the following question:

Question Let (M,ξ) be a tight contact manifold with a fixed convex torus boundary component T, and let $T^2\times I\subset M$ be a nonrotative outer layer with $T_1=T$. If $T^2\times I$ can be extended to a rotative toric annulus inside M, then can any other nonrotative outer layer $(T^2\times I)'$ in M with $T_1'=T$ be extended to a rotative toric annulus inside M?

If such a statement is true, it can be proved only by probing deeper into the manifold. In other words, nonrotative outer layers do not always exhibit purely superficial data.

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