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Cobordisms and Reidemeister torsions of homotopy lens spaces

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Abstract

We show that any 3–dimensional homotopy lens space M^3 that is simple-homotopy equivalent to a lens space $L(p,q)$ is topologically s-cobordant to the lens space. It follows that M has the same multi-signature as $L(p,q)$ and the action of $\pi_1(M)$ on the universal cover of M embeds in a free orthogonal action on S^7.

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We study here 3–dimensional manifolds that have a finite cyclic fundamental group. All such manifolds are homotopy-equivalent to lens spaces (as follows, for instance, from [13] or [4]). Whether every such manifold is in fact homeomorphic to some lens space remains an important unresolved problem.

When the fundamental group is trivial, this is just the Poincaré conjecture. A question in some sense complementary to the Poincaré conjecture is whether such 3–manifolds look like lens spaces from the point of view of high-dimensional topology.

It is well known that there are fake lens spaces in high-dimensions, though the Poincaré conjecture is true. Moreover, the methods of surgery theory alone do not go far in ruling out such fake lens spaces in dimension 3. Thus, there are some essentially 3–dimensional features to this question. We shall use methods of geometric topology to give some results regarding this.

A motivation for this work is its possible relevance to the topological spherical space-form problem [17]. Namely, finite group actions on spheres in high-dimensions are fairly well understood. While surgery cannot be used to construct actions on S^3, one may still obtain restrictions on these actions. A key ingredient in understanding these restrictions in dimension 3 is understanding possible Reidemeister torsions and multi-signatures.

In dimensions 5 and above, the methods of surgery give a complete classification of manifolds with odd-order finite cyclic fundamental group and with universal cover a sphere [19]. They are classified by two invariants, the Reidemeister torsion and the multi-signature. The Reidemeister torsion ρ determines the simple-homotopy type, while the multi-signature Δ is an h-cobordism invariant and determines whether two such manifolds are h-cobordant. If both these invariants coincide for two such manifolds, we have an s-cobordism between them, which then enables us to conclude that they are homeomorphic by using the s-cobordism theorem.

In the case of a 3–manifold M with finite cyclic fundamental group, we show that if M is simple-homotopy equivalent to a lens space, then it is in fact s-cobordant to that lens space. In particular, the Reidemeister torsion determines the multi-signature, at least for simple-homotopy lens spaces. We emphasise that this is only a topological s-cobordism, and may not have a smooth structure.

Theorem 0.1 *Suppose M^3 is a 3–manifold with a simple-homotopy equivalence $f\colon M^3 \to L(p,q)$. Then M^3 is s-cobordant to $L(p,q)$.*

As the multi-signature is an h-cobordism invariant, we obtain:

Corollary 0.2 *The multi-signature of M^3 is the same as that of $L(p,q)$.*

A 4–dimensional surgery theoretic approach would require an *a priori* knowledge of both the Reidemeister torsion and the multi-signature. The content of this paper is that geometric considerations tell us that in certain situations only the Reidemeister torsion is a non-trivial invariant in dimension 3. This interplay between surgery theory and geometric topology is perhaps similar in spirit to the work of Cappell and Shaneson [2]. Related work regarding homology cobordisms of homology lens spaces includes that of Edmonds [3], Fintushel and Stern [8], Kwasik and Lawson [10] and Ruberman [15]. The significant novelty here is the use of geometric 3–manifold techniques in addition to the surgery theoretic results.

The s-cobordism theorem in dimension 4 is false in general [2], and even the h-cobordism theorem, whose truth is unknown, would imply the Poincaré conjecture. Hence, we cannot conclude that M is homeomorphic to a lens space. However, by taking the join of the action of $\pi_1(M)$ on its universal cover with an orthogonal action on S^3, we can obtain a 7–dimensional homotopy lens space which is homeomorphic to a lens space (for details see [17]). Thus, we have:

Corollary 0.3 *The action of $\pi_1(M) = \mathbb{Z}/p\mathbb{Z}$ on the universal cover of M embeds in an orthogonal action on S^7.*

To prove our main result, we start with a manifold M and a simple-homotopy equivalence $f\colon M \to L(p,q)$. We first express M as the result of p/q *Dehn surgery* (all our terminology is explained in the next section) on a knot in a homology sphere, with some restrictions on the Alexander polynomial of the knot. To do this, we take the inverse image of the *core* of the lens space under f, which we we show can be taken to be connected. Surgery on this curve gives a homology sphere, and M in turn is obtained by surgery from this homology sphere.

The restriction on the Alexander polynomial is that its image in the quotient $\mathbb{Z}[T,T^{-1}]/\mathbb{Z}[T^p,T^{-p}]$ is 1. This follows from the hypothesis using results of Fox [5], Brody [1] and Turaev [18]. Next, we modify the curve chosen in M (by performing certain surgeries on unknots) to get a description of M as the result of p/q–surgery on a knot K in a homology sphere Σ with Alexander polynomial 1.

By Freedman's theorems [6], Σ bounds a contractible manifold. Further, as K has Alexander polynomial 1, it is $\mathbb{Z}$*–slice* by a result of Freedman and Quinn [7]. Using this, it is easy to construct the required s-cobordism.

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1 Terminology and notation

1.1 Lens spaces

The three-dimensional lens space $L(p,q)$ is the quotient of S^3, which we regard as the unit sphere in $\mathbb{C}^2$, by the cyclic group of order p generated by $\gamma\colon \mathbb{C}^2 \to \mathbb{C}^2$, $\gamma\colon (z_1, z_2) \to (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$.

A more useful description for our purpose for $L(p,q)$ is that it is the 3–manifold obtained from the solid torus $D^2 \times S^1$ by attaching a 2–handle along the curve representing $p\lambda + q\mu$ and then attaching a 3–handle. Here, μ is a curve on the torus that bounds a disc in the solid torus and λ is a curve transversal to it that intersects it once. We call $\{0\} \times S^1$ the *core* of $L(p,q)$.

A third description, in terms of *Dehn surgery*, is given below.

1.2 Dehn Surgery

Suppose K is a knot in a closed 3–manifold M. Then $M \setminus int(N(K))$, where $N(K)$ is a regular neighbourhood of K, has boundary a torus, with a distinguished homology class μ on it that bounds a disc in $N(K)$. If K is homologically trivial, it has a second distinguished class λ, which is dual to μ, that is homologically trivial in $M \setminus N(K)$. If this is not the case we take λ to be any class dual to μ.

The manifold obtained from M by p/q Dehn surgery on K is the manifold $(M \setminus int(N(K))) \coprod_f (D^2 \times S^1)$, where the attaching map $f\colon \partial D^2 \times S^1 \to \partial N(K)$ is chosen so that a curve representing $p\mu + q\lambda$ bounds a disc in $D^2 \times S^1$.

In particular, Dehn surgeries on the unknot in S^3 give lens spaces.

Clearly, given any Dehn surgery, there is a *dual surgery* on a *dual knot* in the resulting manifold that gives the initial manifold. Namely, in performing the Dehn surgery, a solid torus has been deleted and reglued. One can delete the new solid torus and reglue it as before the surgery.

1.3 Alexander polynomial

Let K be a knot in a homology sphere M, or more generally in some manifold M such that $H_1(M \setminus K) = \mathbb{Z}$. Then the group $\mathbb{Z}$ acts on the infinite cyclic cover of $H_1(M \setminus K)$ making this a module over $\mathbb{Z}[T, T^{-1}] = \mathbb{Z}[\mathbb{Z}]$ which we call the *Alexander module*. The order ideal of this module is principal and hence of the form $\langle \Delta(T) \rangle$. We call $\Delta(T)$ the Alexander polynomial of the knot K.

1.4 Reidemeister torsion

The Reidemeister torsion is an invariant of the simple homotopy type of a homology lens space. Suppose M is a homology lens space, with $H_1(M) = \mathbb{Z}/p\mathbb{Z}$. Then the universal abelian cover of M (corresponding to the commutator subgroup of the fundamental group) has a cell decomposition with a $\mathbb{Z}/p\mathbb{Z}$ action, making the cellular chain complex a $\mathbb{Z}/p\mathbb{Z}$–module. We tensor this chain complex with a field that has a $\mathbb{Z}/p\mathbb{Z}$ action on it (for instance the field of fractions of the group ring, or $\mathbb{C}$ with the action coming from a representation of the cyclic group). The simplices of M give a preferred basis for each C_n of the chain complex C_*. If the resulting complex is acyclic, we can take the determinant of the resulting complex to get the Reidemeister torsion. Details can be found in Turaev [18].

1.5 Multi-signature

Given a homotopy lens space M and an identification $H_1(M) = \mathbb{Z}/p\mathbb{Z}$, we have a classifying map $\phi\colon\ M \to K(\mathbb{Z}/p\mathbb{Z}, 1)$. As the equivariant bordism groups are finite, for some n we can find a 4–manifold W with $\partial W = nM$ and a map $\psi\colon\ W \to K(\mathbb{Z}/p\mathbb{Z}, 1)$ such that its restriction to each boundary component is ϕ. Then $\pi_1(M)$ acts on a cover $\widetilde{W}$ of W. Hence for each simple real representation ρ_i of $\pi_1(M)$ the bilinear form on $H^2(\widetilde{W})$ (obtained by taking cup products and evaluating on the fundamental class) gives a bilinear form on a real vector space. This form has a signature S_i. The formal sum $\frac{1}{n}\Sigma S_i\rho_i$, is well defined up to adding copies of the signatures of the right regular representation. This is the *multi-signature*. For details see [19],[17].

1.6 Slice and $\mathbb{Z}$–slice knots

Let Σ be a homology sphere bounding a contractible 4–manifold N. A knot K in Σ is said to be slice in N if it bounds a properly embedded topologically locally flat disc D^2. The knot is $\mathbb{Z}$–slice if in addition the disc can be chosen so that $\pi_1(N \setminus D^2) = \mathbb{Z}$.

2 The first surgery description

We assume henceforth that we have a simple-homotopy lens space M with a map $f\colon M \to L(p,q)$ which is a simple-homotopy equivalence. In this section, we show that M is obtained by p/q–surgery on a knot K' in a homology sphere Σ' whose Alexander polynomial has image 1 in $\mathbb{Z}[T]/\mathbb{Z}[T^p]$. We shall find an appropriate curve in l in M, so that the required Σ' can be obtained by surgery on l. The knot K' will then be the dual curve.

Let c be the core of $L(p,q)$.

Lemma 2.1 *After a homotopy of f, $f^{-1}(c)$ is a connected curve in M.*

Proof It is easy to make f transversal to c. Then $f^{-1}(c)$ is a union of circles. It remains to homotope f so that we get only one component. We do this using a standard technique in 3–manifold topology related to Stallings 'binding ties' [16], as in Jaco [9].

Suppose $f^{-1}(c)$ has more than one component. Let α be an arc joining two components, such that its two end-points have the same image under f and so that the images of neighbourhoods of the two endpoints coincide. We shall modify α so that $f(\alpha)$ represents the trivial element in the fundamental group of $L(p,q) \setminus \gamma$.

To do this, note that as f is a homotopy equivalence, it has degree one, and hence so does its restriction to $M \setminus int(N(f^{-1}(c))$. Thus the restriction induces a surjection on the fundamental group. In particular, there is a closed loop β in $M \setminus int(N(f^{-1}(c))$ whose image $f(\beta)$ in $L(p,q) \setminus \gamma$ is the inverse of $f(\alpha)$ (pushed off along the common image of neighbourhoods of the two endpoints). We replace α by its concatenation with β and push this off $f^{-1}(c)$ to get the required curve.

Now, we first homotope the map on a neighbourhood of the arc, which we identify with $\alpha \times [-2,2] \times [-2,2]$ so that the image of any point is equal to that of its projection onto the arc, ie, $f(x,s,t) = f(x,0,0)$. To do this, first let $g(x,s,t) = f(x, max(0, 2s-2), max(0, 2t-2))$ if $(s,t) \in [-2,2] \times [-2,2]$ and equal to f otherwise. Clearly g is homotopic to f, so we may replace f by g.

Next, by the choice of α, we have a homotopy $H\colon \alpha \times [0,1] \to L(p,q)$ (fixing endpoints) of $f(\alpha)$ to a point with $H((\alpha \setminus \partial\alpha) \times [0,1]$ disjoint from c. Use this to define the map on $\alpha \times [1/2,1] \times 0$ by $f(x,t,0) = H(x, 2-2t)$ and symmetrically on $\alpha \times [-1,-1/2] \times 0$. We identify α with $[0,1]$. Note that $f^{-1}(c)$ contains the 6 segments $\{0,1\} \times [1/2,1] \times \{0\}$, $\{0,1\} \times [-1,-1/2] \times \{0\}$ and $[0,1] \times \{-1/2, 1/2\} \times \{0\}$. Now extend f so that there are no further points in the inverse image of γ. We have reduced the number of components of $f^{-1}(\gamma)$. By repeating this process we are left with only one component. $\square$

Let Σ' be the homology sphere obtained by the surgery on $f^{-1}(c)$ in M that corresponds (under the identification of a neighbourhood $f^{-1}(c)$ with c using f) to a surgery on γ that gives a sphere. Let K' the corresponding knot in Σ'. Let $l' = f^{-1}(c)$.

Lemma 2.2 *The image of the Alexander polynomial of l' in $\mathbb{Z}[H_1(M)]$ is 1.*

Proof The image $p(T)$ of the Alexander polynomial of l' in $\mathbb{Z}[H_1(M)]$ is the so called Fox–Brody invariant, which by results of Brody [1] and Fox [5] is known to depend only on the homology class of l'. Further, results of Turaev [18] show that this depends only on the Reidemeister torsion (given an identification of homology groups). But $f\colon M \to L(p,q)$ is a simple-homotopy equivalence, and $f_*([l']) = [c]$ in homology. It follows that $p(T) = p'(T)$, where $p'(T)$ is the image of the Alexander polynomial of c in $\mathbb{Z}[H_1(L(p,q))]$, and the group rings are identified using f_*. As c is a core, $p(T) = p'(T) = 1$. $\square$

Corollary 2.3 *The image of the Alexander polynomial of the knot K' in the quotient $\mathbb{Z}[T, T^{-1}]/\mathbb{Z}[T^p, T^{-p}]$ is 1.*

3 The second surgery description

We shall now modify the curve l' to get l so that on repeating the constructions of the previous section, with l in the place of l', we get the final surgery description. Thus, we show:

Proposition 3.1 *M can be obtained by p/q surgery on a knot K with Alexander polynomial 1 in a homology sphere Σ.*

We shall modify the curve l' by performing $1/n$ surgeries on unknots disjoint from, but linked with, l'. The manifold obtained after such a surgery is still M. However the curve, which we now call l, is now embedded in M in a different manner in general.

The effect of surgery on a knot in a manifold on the homology of that manifold depends only on the homology class of the knot and the slope of the surgery. More generally, if we perform a surgery on a link, then the resulting homology also depends on the linking (ie, the homology class of each component in the complement of the other components – this may or may not be a non-trivial choice) as well as the slopes of each of the surgeries. In our situation, we need to modify the Alexander module, which is the homology of the universal cyclic cover. We shall perform a surgery in M. This results in infinitely many surgeries in the cyclic cover.

We shall pick a manifold and knot with the same Alexander module, and a sequence of surgeries that kills this. Thus, we construct $(M', \partial M')$ with $\partial M'$ a torus, $H_1(M') = \mathbb{Z}$ and $(M', \partial M')$ having the same Alexander module as $M \setminus l'$ (the Alexander module is simply the homology of the infinite cyclic cover as a $\mathbb{Z}[H_1(M')]$ module). We call this the model. We find a sequence of surgeries on curves $\gamma'_1, \dots \gamma'_k$ in M' such that each intermediate manifold M'_i has homology $\mathbb{Z}$ and the final manifold has trivial Alexander module.

It suffices to show that we can modify l' so that $M \setminus l'$ has the same Alexander module as M'_1. We find an unknot in M, whose lifts have the same homology class and linking as in the model $(M', \partial M')$. Further, as we choose only the slope of the surgery in M, and need all the surgeries in the cover to have the same slopes as in the model, we need to ensure that the surgery locus has the right *framing*. We define linking and framing following lemma 3.3.

The main construction of this section is in lemma 3.3, where we find an unknot in the right homotopy class in $M \setminus l'$. The rest of the section is then devoted to finding a homotopy of this knot to get the right linking and framing in the infinite cyclic cover.

A special case, where the construction is a little simpler, is when the Alexander module is cyclic. As an aid to intuition, we often also give proofs in this special case, which are simpler.

A surgery on an unknot can be used to construct a knot K_0 in S^3 with Alexander polynomial any given Laurent polynomial $A(t)$ satisfying $A(t) = A(t^{-1})$ and such that $A(1) = \pm 1$ and cyclic Alexander module starting with an unknot [11]. Observe that there is a cancelling surgery, thus one that changes the

Alexander polynomial to 1 without changing the manifold. For, in performing the surgery, a solid torus has been removed and replaced by one glued in in a different way. We can cancel this by removing the new solid torus and re-gluing the old one.

In the special case we shall take the cancelling surgery in this case as our model. Denote the knot in $S^3 \setminus K_0$ as γ_0. We perform surgery on an unknot γ in M, so that the image of the unknot in the infinite cyclic cover of $M \setminus l'$ is the same as that of γ_0 in that of $S^3 \setminus K_0$ (in the sense of the next paragraph), so that we get the same result after surgery. More generally, we will construct a model, and perform the same surgeries, in the sense of the next paragraph.

The homology of the complement of the inverse image of γ_0 in the infinite cyclic cover $\widetilde{S^3 \setminus K_0}$ of $S^3 \setminus K_0$ is determined by its homology class, together with the *linking* of its components. Here, the only linking comes from homologically dependent curves, and is determined by the algebraic intersection of a curve distinct from these with a surface which realises this dependency (which we shall call a Seifert surface). More precisely, suppose we delete a link with several components from a 3–manifold. By Lefschetz duality, the complement of the first component has homology only depending on its homology class, and more generally the homology after deleting each successive component is determined by its homology class in the manifold obtained so far. Further, when some curves have been deleted, the homology of their complement surjects onto that of the original manifold. The kernel is generated by relations among the homology classes of the curves added, and is dual to the corresponding Seifert surfaces. Thus the homology of the complement after each successive curve is deleted is determined by its homology class and its intersections with the Seifert surfaces.

Thus, we need to find an unknot representing the same homology class and with the same linking structure as γ_1' in the model $(M', \partial M')$. Finding a curve in the right class is easy, even without our hypothesis. The hypothesis is required to ensure that the curve can be taken to be an unknot.

We first need some lemmas.

Lemma 3.2 *If $H_1(\widetilde{M \setminus l'})$ is cyclic, then it is generated as a $\mathbb{Z}[\mathbb{Z}]$–module by an element of the form $(T^p - 1)[\gamma]$ and hence every element is of this form. In the general case, every element in homology can be represented as $(T^p - 1)[\gamma]$.*

Proof Let $\Delta(T)$ denote the Alexander polynomial of $M \setminus l'$. If the Alexander module is cyclic, $H_1(\widetilde{M \setminus l'}) \cong \mathbb{Z}[\mathbb{Z}]/\langle \Delta(T) \rangle$. Since $\Delta(T) \equiv \pm 1 (mod(T^p - 1))$,

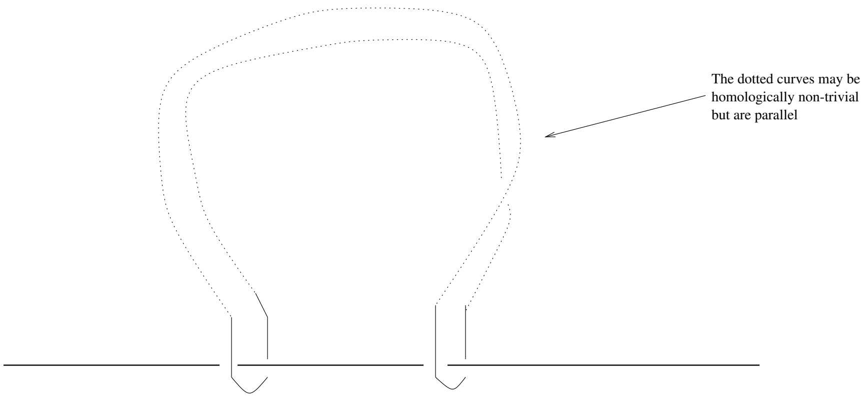

Figure 1: Knots in desired homology class

there is a Laurent polynomial $p(T)$ so that $\Delta(T) = \pm 1 + p(T)(T^p - 1)$. Thus, $p(T)(T^p - 1) \equiv \pm 1 (mod\ \Delta(T))$ generates $H_1(\widetilde{M \setminus l'})$.

In the general case, the Alexander polynomial is the determinant of the presentation matrix (with respect to some system of generators), say A, for the Alexander module (which we call P). As this is congruent to 1 modulo $(T^p - 1)$, the matrix A is invertible on reducing to the module $\mathbb{Z}[T, T^{-1}]/(T^p - 1)$. The claim follows immediately, for, if x is a word in the generators of the Alexander module P, then we have:

$x \equiv Az \mod (T^p - 1)$ for some word z in the generators, or

$x = Az + (T^p - 1)y,\ y \in P$.

But this means that $x = (T^p - 1)y$ in the module P. □

Lemma 3.3 *Given any element $\beta \in H_1(\widetilde{M \setminus l'})$ there is a curve δ in $M \setminus l'$, unknotted in M, which lifts to the universal abelian cover of $M \setminus l'$ so that its lift represents β.*

Proof It follows readily from the previous lemma that β can be expressed as $\beta = (T^p - 1)[\gamma]$, for a lift of some curve γ in $M \setminus l'$. Now take the band connected sum of γ with itself (pushed off using, for instance, the 0–framing) along a curve that goes once around l' (see figure 1 – here the two dotted arcs are parallel, but may represent any homotopy class and knot type). This represents $\beta = (T^p - 1)[\gamma]$ in $H_1(\widetilde{M \setminus l'})$. □

By using the above lemma, we have an unknot in the same homology class as the curve γ_1' in our model space $(M', \partial M')$.

Remark Note that in the above construction, we can choose any γ in a given homotopy class. Further, we can push γ off itself using any other framing.

This, together with some other moves, can be used to change the linking structure of $\beta = (T^p - 1)\gamma$ to get the same linking as in the model space as defined below. Observe that after picking an inverse image of γ, we have a canonical identification of our link with $\mathbb{Z}[T, T^{-1}]$. We have a similar identification in our model space.

Definition 3.4 Suppose M and M' are 3–manifolds with a given identification $H_1(M) = H_1(M')$ and $L \subset M$ and $L' \subset M'$ are links with a given one-to-one correspondence between their components such that corresponding components represent the same elements in homology. Then we say that L and L' have the same linking if each component of L represents the same element in the homology of the complement of the other components as the corresponding element of L' (in the corresponding complement).

The relevant *framing* is special to our situation. The components of the links in the infinite cyclic covers of $M \setminus K$ and the model are identified with $\mathbb{Z}[T, T^{-1}]$. Further, as γ is homologically trivial, it has a preferred framing. This lifts to give framings of each component in the infinite cyclic covers.

Definition 3.5 Given two links $L \subset M$ and $L' \subset M'$ with the same linking with respect to some identification, and with induced framings for each component as above. We say they have the same framing if $ker(H_1(\partial N(L)) \to H_1(P \setminus \mathrm{int}(N(L)))) = ker(H_1(\partial N(L')) \to H_1(P \setminus \mathrm{int}(N(L'))))$ under the identifiaction given by the framings.

We now turn to the linking. Pick a family $T^i\gamma, i \in Q \subset \mathbb{Z}$ of knots generating the Alexander module over $\mathbb{Z}[T, T^{-1}]$, and a family of Seifert surfaces which, together with these generators, give a square presentation matrix for the Alexander module. The boundary of each Seifert surface is a linear combination $\Sigma a(k)T^k\gamma$ of translates of γ. Thus, we have boundary and coboundary maps between the $\mathbb{Z}[T, T^{-1}]$–modules generated by the Seifert surface and the knots. Also pick a link and Seifert surfaces in M' with the same boundary maps. This is possible as the Alexander modules are isomorphic and γ and γ' represent the same homology class.

The linking for each component is determined by the algebraic intersection number with the Seifert surfaces, and thus can be expressed as a linear combination, with $\mathbb{Z}[T,T^{-1}]$ coefficients, of these Seifert surfaces. Namely, a curve β intersects only finitely many translates of a given Seifert surface S. We take these linking numbers as the coefficients of the polynomial. Thus, if $S_j, 1 \leq j \leq k$ form a $\mathbb{Z}[T,T^{-1}]$ basis for the Seifert surface, and $\beta \cdot S$ represents the algebraic intersection number with a surface S, then

$$lk(\beta) = \Sigma_j \Sigma_i (\beta \cdot T^i S_j) T^i S_j.$$

In the case of a cyclic Alexander module, we have a single curve and a single Seifert surface. Note that the total coefficient of the boundary of the Seifert surface F (over all the link components that it intersects) is ± 1, since a curve in the cover is homologically trivial in M, or equivalently, because the Alexander polynomial evaluated at 1 is ± 1. In the general case, we have a co-boundary map, which is the transpose of the presentation matrix. The hypothesis says that this has determinant ± 1 modulo $T^p - 1$

We shall use two moves to change the linking. The first of these can be used to change the linking by any polynomial divisible by $T^p - 1$.

Lemma 3.6 *Suppose the linking in M and the model $(M', \partial M')$ differ by an element in $\mathbb{Z}[T^p, T^{-p}]$. Then we can find an unknot with the same linking as in the model.*

Proof We use a standard construction [11] to construct a knot whose Alexander polynomial is given by surgery on an unknot. Namely, we drag a piece of a curve γ around a knot K and then across itself (see figure 2). If M was a sphere, or a homology sphere, this leads to a change in the homology class of a lift of γ in the Alexander module by $T - 1$. By surgery on an unknot γ constructed using such transformations, we obtain any Alexander polynomial $A(T)$ satisfying $A(1) = \pm 1$ and $A(T) = A(T^{-1})$.

In our situation, this leads to a change in the coboundary by $T^p - 1$ rather than $T - 1$. By first winding around by an element representing T^k, or more generally an arc representing the appropriate element in the Alexander module before making the crossing, one can also change the linking by $T^k(T^p - 1)$, we can change by an multiple of $T^p - 1$ in the module. Finally, by changing the framing along which the curve is pushed off itself in lemma 3.3, we can ensure that the total coefficients are the same. □

This will constitute our first move. Thus, it suffices for us to get the right linking modulo $T^p - 1$.

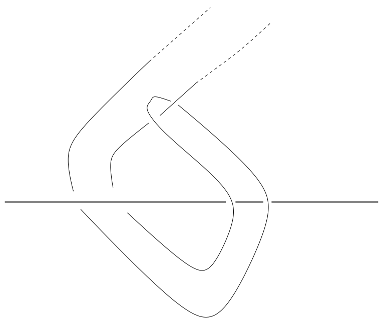

Figure 2: Changing the linking of the knot

Lemma 3.7 *By appropriate choice of γ, we can obtain the same linking modulo $T^p - 1$ as in the model $(M', \partial M')$.*

Proof Note that one can readily change a crossing between γ and $T^k\gamma$ by dragging an arc of γ along a closed curve representing T^k and then crossing γ. This changes the intersection number with F by the coefficient $a(k)$ of $T^k\gamma$ in the boundary ∂F of the Seifert surface. This also changes the linking of β at $T^{p+k}\gamma$ but as the negative of the previous change, so the intersection number here changes by the negative of the coefficient of $T\gamma$ in the boundary of the Seifert surface.

Thus, the linking number with a given Seifert surface has been changed by $a(k) - a(k+p)$, where $a(k)$ is the coefficient of the Seifert surface. We shall see that these moves suffice to get the right linking modulo $T^p - 1$.

For, the linking numbers are determined by the homology classes up to a simultaneous change by the coefficient $a(i)$ of each Seifert surface. But now, as the presentation matrix is invertible modulo $T^p - 1$, such a change in β may be achieved by changing γ modulo $T^p - 1$. Thus, we can ensure that β has the right linking. □

Next, we need to get the right framings, to ensure that a given surgery means the same thing in our case as in the model $(M', \partial M')$. Namely, we can choose what surgery to perform in M. This then results in a surgery on each component in the infinite cyclic cover. In both M and M', we have a natural meridian in the manifold, which corresponds to meridians in the cyclic cover. Further, we have a longitude since the curve chosen is homologically trivial. Thus, we have a longitude for each component in the cover. To ensure that we can choose a surgery that is homologically the same as that in the model, we need to see that the boundary of the Seifert surface in the cover is the same in both cases in terms of the longitude and meridian chosen.

To achieve the right framing, we shall use the above construction along arcs which are homologically trivial in $M \setminus K$ (after closing up by a subarc of γ), though not necessarily so in the universal cyclic cover. This leads to a component in the universal cyclic cover crossing itself. Below we show how the arcs can be chosen.

Observe that if we consider the image of a Seifert surface F under the covering map, the linking is determined by the intersection of the F with the knot, while the framings are the same as the framings in the projection.

Now, consider the intersection of the image of F with a Seifert surface S for the knot. The intersection consists of arcs properly embedded in both surfaces, as well as arcs corresponding to intersections of F with γ. The framing is determined by the linking as well as the the homology class represented by the properly embedded arcs in the universal cyclic cover of the knot complement. Perform a crossings along such an arc (respectively its negative) increases (respectively reduces) the difference between the framing and the longitude (without changing linking numbers in this process as our arc was homologically trivial). Another arc results in the same change provided it represents the same element in the Alexander module.

Now, to ensure that we continue to have an unknot, we may only make a move of the above form along some arc c to β, which results in a change corresponding to $(T^p - 1)c$ to $\gamma = (T^p - 1)\beta$. But we know that any homology class is of this form, and so we can make the desired moves to change framing. Making such changes, we can ensure that the framing in our case is the same as the model $(M', \partial M')$.

Now the surgery in our case corresponding to the cancelling surgery in the model also kills the Alexander polynomial. Thus, in the case where the Alexander module is cyclic, we are done.

In the general case, that the desired sequence of surgeries exists follows form the following (presumably well known) proposition.

Proposition 3.8 *Let P be a homology sphere and K a knot in P. Then, there is a sequence of $1/n$ surgeries along homologically trivial curves in $P \setminus K$ so that K has trivial Alexander polynomial in the final manifold.*

Proof Pick a Seifert surface for K. We shall perform $1/n$ surgeries along curves disjoint from the Seifert surface, which must thus be homologically trivial. Namely, it is well known that there are such curves that form a dual basis to a basis of curves on the Seifert surface with respect to the linking pairing. Hence, one can find curves with any desired combination of linking numbers. Surgery along a curve changes the linking number between a pair of other curves by an amount determined by the linking with the surgery locus. As the entries of the Seifert matrix are linking numbers between curves of the Seifert surface pushed off in two directions, it is easy to see that surgeries on such curves can be used to transform the Seifert matrix, and hence the Alexander polynomial, to that of an unknot. For, surgery on a curve linked once with each of a pair of basis curves on the surface (and unlinked from others) changes their linking, while surgery on a curve linked with just one basis curve changes framing. $\square$

4 Constructing cobordisms

Using the surgery description of the previous section, we shall construct an s-cobordism between M and $L(p,q)$. This is a straightforward application of the following deep theorems [6],[7].

Theorem (Freedman) *Any $\mathbb{Z}$–homology 3–sphere Σ bounds a unique contractible 4–manifold N^4.*

Theorem (Freedman–Quinn) *Suppose that Σ is a homology 3–sphere which bounds a contractible 4–manifold N, and K is a knot in M that has Alexander polynomial 1. Then K bounds a properly embedded, topologically locally flat disc D in N such that $\pi_1(N \setminus D) = \mathbb{Z}$*

Let Σ and K be as in the previous section and take N and D as in the above theorems. Let x_0 be an interior point of D and delete a regular neighbourhood of x_0 in N to get $\hat{N}$. The intersection $A = D \cap \hat{N}$ is a properly embedded, topologically flat annulus in $\hat{N}$ with $\pi_1(\hat{N} \setminus A) = \mathbb{Z}$, and $\hat{N}$ is a cobordism from Σ to S^3. Further, the boundary components of A are $K \subset \Sigma$ and an unknot in the 3–sphere. Now delete a regular neighbourhood $N(A)$ of A in $\hat{N}$ and attach a thickened solid torus $D^2 \times S^1 \times [0,1]$ to $\partial N(A) = S^1 \times S^1 \times [0,1]$, so that the curve representing $p\mu + q\lambda$ bounds discs in the boundary components. Note that this makes sense by construction. This gives an h-cobordism, with the boundary components being obtained by p/q–surgery on K in Σ and an unknot in S^3 respectively. Thus, this is an h-cobordism between M and $L(p,q)$

Finally, as M and $L(p,q)$ are *special complexes* in the sense of Milnor [12], and the Reidemeister torsions are equal under the corresponding identification of fundamental groups, we have an s-cobordism.

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