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The size of triangulations supporting a given link

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Abstract

Let $\mathcal{T}$ be a triangulation of S^3 containing a link L in its 1–skeleton. We give an explicit lower bound for the number of tetrahedra of $\mathcal{T}$ in terms of the bridge number of L. Our proof is based on the theory of almost normal surfaces.

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1 Introduction

In this paper, we prove the following result.

Theorem 1 *Let $L \subset S^3$ be a tame link with bridge number $b(L)$. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra such that L is contained in the 1–skeleton of $\mathcal{T}$. Then*

$$n > \frac{1}{14}\sqrt{\log_2 b(L)},$$

or equivalently

$$b(L) < 2^{196n^2}.$$

The definition of the bridge number can be found, for instance, in [2]. So far as is known to the author, Theorem 1 gives the first estimate for n in terms of L that does not rely on additional geometric or combinatorial assumptions on $\mathcal{T}$. We show in [13] that the bound for $b(L)$ in Theorem 1 can not be replaced by a sub-exponential bound in n. More precisely, there is a constant $c \in \mathbb{R}$ such that for any $i \in \mathbb{N}$ there is a triangulation $\mathcal{T}_i$ of S^3 with $\leq c \cdot i$ tetrahedra, containing a two-component link L_i in its 1–skeleton with $b(L_i) > 2^{i-2}$.

The relationship of geometric and combinatorial properties of a triangulation of S^3 with the knots in its 1–skeleton has been studied earlier, see [6], [15], [1], [3], [7]. For any knot $K \subset S^3$ there is a triangulation of S^3 such that K is formed by three edges, see [4]. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra and let $K \subset S^3$ be a knot formed by a path of k edges. If $\mathcal{T}$ is shellable (see [3]) or the dual cellular decomposition is shellable (see [1]), then $b(K) \leq \frac{1}{2}k$. If $\mathcal{T}$ is vertex decomposable then $b(K) \leq \frac{1}{3}k$, see [3].

We reduce Theorem 1 to Theorem 2 below, for which we need some definitions. Denote $I = [0,1]$. Let M be a closed 3–manifold with a triangulation $\mathcal{T}$. The i–skeleton of $\mathcal{T}$ is denoted by $\mathcal{T}^i$. Let S be a surface and let $H\colon S \times I \to M$ be an embedding, so that $\mathcal{T}^1 \subset H(S^2 \times I)$. A point $x \in \mathcal{T}^1$ is a *critical point* of H if $H_\xi = H(S \times \xi)$ is not transversal to $\mathcal{T}^1$ in x, for some $\xi \in I$. We call H a $\mathcal{T}^1$*–Morse embedding*, if H is in general position with respect to $\mathcal{T}^1$; we give a more precise definition in Section 5. Denote by $c(H, \mathcal{T}^1)$ the number of critical points of H.

Theorem 2 *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. There is a $\mathcal{T}^1$–Morse embedding $H\colon S^2 \times I \to S^3$ such that $\mathcal{T}^1 \subset H(S^2 \times I)$ and $c(H, \mathcal{T}^1) < 2^{196n^2}$.*

For a link $L \subset \mathcal{T}^1$, it is easy to see that $b(L) \leq \frac{1}{2}\min_H\{c(H,\mathcal{T}^1)\}$, where the minimum is taken over all $\mathcal{T}^1$–Morse embeddings $H\colon S^2 \times I \to S^3$ with $L \subset H(S^2 \times I)$. Thus Theorem 1 is a corollary of Theorem 2.

Our proof of Theorem 2 is based on the theory of almost 2–normal surfaces. Kneser [14] introduced 1–normal surfaces in his study of connected sums of 3–manifolds. The theory of 1–normal surfaces was further developed by Haken (see [8], [9]). It led to a classification algorithm for knots and for sufficiently large 3–manifolds, see for instance [11], [17]. The more general notion of almost 2–normal surfaces is due to Rubinstein [19]. With this concept, Rubinstein and Thompson found a recognition algorithm for S^3, see [19], [22], [16]. Based on the results discussed in a preliminary version of this paper [12], the author [13] and Mijatović [18] independently obtained a recognition algorithm for S^3 using local transformations of triangulations.

We outline here the proof of Theorem 2. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. If $S \subset S^3$ is an embedded surface and $S \cap \mathcal{T}^1$ is finite, then set $\|S\| = \mathrm{card}(S \cap \mathcal{T}^1)$. Let $S_1, \dots, S_k \subset S^3$ be surfaces. A surface that is obtained by joining $S_1, \dots, S_k$ with some small tubes in $M \setminus \mathcal{T}^1$ is called a *tube sum* of $S_1, \dots, S_k$.

Based on the Rubinstein–Thompson algorithm, we construct a system $\tilde\Sigma \subset S^3$ of pairwise disjoint 2–normal 2–spheres such that $\|\tilde\Sigma\|$ is bounded in terms of n and any 1–normal sphere in $S^3 \setminus \tilde\Sigma$ is parallel to a connected component of $\tilde\Sigma$. The bound for $\|\tilde\Sigma\|$ can be seen as part of a complexity analysis for the Rubinstein–Thompson algorithm and relies on results on integer programming.

A $\mathcal{T}^1$–Morse embedding H then is constructed "piecewise" in the connected components of $S^3 \setminus \tilde\Sigma$, which means the following. There are numbers $0 < \xi_1 < \cdots < \xi_m < 1$ such that:

(1) $\|H_0\| = \|H_1\| = 0$.

(2) There is one critical value of $H|[0,\xi_1]$, corresponding to a vertex $x_0 \in \mathcal{T}^0$. The set of critical points of $H|[\xi_m, 1]$ is $\mathcal{T}^0 \setminus \{x_0\}$.

(3) For any $i = 1, \dots, m$, the sphere H_{ξ_i} is a tube sum of components of $\tilde\Sigma$.

(4) The critical points of $H|[\xi_i, \xi_{i+1}]$ are contained in a single connected component N_i of $S^3 \setminus \tilde\Sigma$.

(5) The function $\xi \mapsto \|H_\xi\|$ is monotone in any interval $[\xi_i, \xi_{i+1}]$, for $i = 1, \dots, m-1$.

This is depicted in Figure 1, where the components of $\tilde\Sigma$ are dotted. The components N_i run over all components of $S^3 \setminus \tilde\Sigma$ that are not regular neighbourhoods of vertices of $\mathcal{T}$. Thus an estimate for m is obtained by an estimate

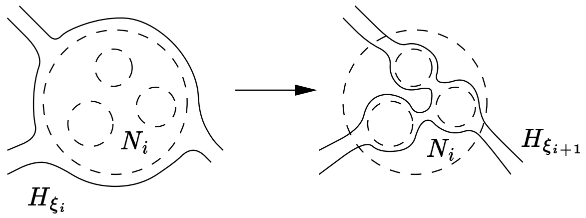

Figure 1: About the construction of H

for the number of components of $\tilde{\Sigma}$. By monotonicity of $\|H_\xi\|$, the number of critical points in N_i is bounded by $\frac{1}{2}\|\partial N_i\| \leq \frac{1}{2}\|\tilde{\Sigma}\|$. This together with the bound for m yields the claimed estimate for $c(H, \mathcal{T}^1)$.

The main difficulty in constructing H is to assure property (5). For this, we introduce the notions of upper and lower reductions. If S' is an upper (resp. lower) reduction of a surfaces $S \subset S^3$, then S is isotopic to S' such that $\|\cdot\|$ is monotonely non-increasing under the isotopy. Let N be a connected component of $S^3 \setminus \tilde{\Sigma}$ with $\partial N = S_0 \cup S_1 \cup \cdots \cup S_k$. We show that there is a tube sum S of $S_1, \ldots, S_k$ such that either S is a lower reduction of S_0, or S_0 is an upper reduction of S. Finally, if H_{ξ_i} is a tube sum of S_0 with some surface $S' \subset S^3 \setminus N$, then $H|[\xi_i, \xi_{i+1}]$ is induced by the lower reductions (resp. the inverse of the upper reductions) relating S_0 with S. Then $H_{\xi_{i+1}}$ is a tube sum of S with S', assuring properties (3)–(5).

The paper is organized as follows. In Section 2, we recall basic properties of k–normal surfaces. It is well known that the set of 1–normal surfaces in a triangulated 3–manifold is additively generated by so-called *fundamental surfaces*. In Section 3, we generalize this to 2–normal surfaces contained in *sub-manifolds* of triangulated 3–manifolds. The system $\tilde{\Sigma}$ of 2–normal spheres is constructed in Section 4, in the more general setting of closed orientable 3–manifolds. In Section 5, we recall the notions of almost k–normal surfaces (see [16]) and of impermeable surfaces (see [22]), and introduce the new notion of split equivalence. We discuss the close relationship of almost 2–normal surfaces and impermeable surfaces. This relationship is well known (see [22], [16]), but the proofs are only partly available. For completeness we give a proof in the last Section 9. In Section 6 we exhibit some useful properties of almost 1–normal surfaces. The notions of upper and lower reductions are introduced in Section 7. The proof of Theorem 2 is finished in Section 8.

In this paper, we denote by $\#(X)$ the number of connected components of a topological space X. If X is a tame subset of a 3–manifold M, then $U(X) \subset M$

denotes a regular neighbourhood of X in M. For a triangulation $\mathcal{T}$ of M, the number of its tetrahedra is denoted by $t(\mathcal{T})$.

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2 A survey of k–normal surfaces

Let M be a closed 3–manifold with a triangulation $\mathcal{T}$. The number of its tetrahedra is denoted by $t(\mathcal{T})$. An *isotopy mod* $\mathcal{T}^n$ is an ambient isotopy of M that fixes any simplex of $\mathcal{T}^n$ set-wise. Some authors call an isotopy mod $\mathcal{T}^2$ a normal isotopy.

Definition 1 Let σ be a 2–simplex and let $\gamma \subset \sigma$ be a closed embedded arc with $\gamma \cap \partial\sigma = \partial\gamma$, disjoint to the vertices of σ. If γ connects two different edges of σ then γ is called a *normal arc*. Otherwise, γ is called a *return*.

We denote the number of connected components of a topological space X by $\#(X)$. Let σ be a 2–simplex with edges e_1, e_2, e_3. If $\Gamma \subset \sigma$ is a system of normal arcs, then Γ is determined by $\Gamma \cap \partial\sigma$, up to isotopy constant on $\partial\sigma$, and e_1 is connected with e_2 by $\frac{1}{2}\left(\#(\Gamma \cap e_1) + \#(\Gamma \cap e_2) - \#(\Gamma \cap e_3)\right)$ arcs in Γ.

Definition 2 Let $S \subset M$ be a closed embedded surface transversal to $\mathcal{T}^2$. We call S *pre-normal*, if $S \setminus \mathcal{T}^2$ is a disjoint union of discs and $S \cap \mathcal{T}^2$ is a union of normal arcs in the 2–simplices of $\mathcal{T}$.

The set $S \cap \mathcal{T}^1$ determines the normal arcs of $S \cap \mathcal{T}^2$. For any tetrahedron t of $\mathcal{T}$, the components of $S \cap t$, being discs, are determined by $S \cap \partial t$, up to isotopy fixed on ∂t. Thus we obtain the following lemma.

Lemma 1 *A pre-normal surface $S \subset M$ is determined by $S \cap \mathcal{T}^1$, up to isotopy mod $\mathcal{T}^2$.* □

Definition 3 Let $S \subset M$ be a pre-normal surface and let k be a natural number. If for any connected component C of $S \setminus \mathcal{T}^2$ and any edge e of $\mathcal{T}$ holds $\#(\partial C \cap e) \leq k$, then S is *k–normal*.

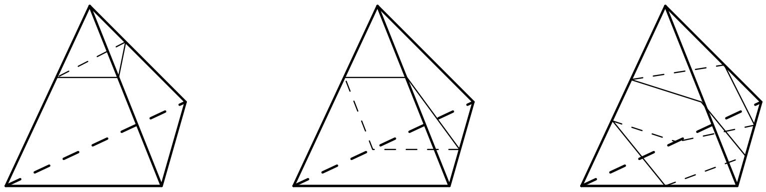

Figure 2: A triangle, a square and an octagon

We are mostly interested in 1– and 2–normal surfaces. Let S be a 2–normal surface and let t be a tetrahedron of $\mathcal{T}$. Then the components of $S \cap t$ are copies of triangles, squares and octagons, as in Figure 2. For any tetrahedron t, there are 10 possible types of components of $S \cap t$: four triangles (one for each vertex of t), three squares (one for each pair of opposite edges of t), and three octagons. Thus there are $10\,t(\mathcal{T})$ possible types of components of $S \setminus \mathcal{T}^2$. Up to isotopy mod $\mathcal{T}^2$, the set $S \setminus \mathcal{T}^2$ is described by the vector $\mathfrak{x}(S)$ of $10\,t(\mathcal{T})$ non-negative integers that indicates the number of copies of the different types of discs occuring in $S \setminus \mathcal{T}^2$. Note that the 1–normal surfaces are formed by triangles and squares only.

We will describe the non-negative integer vectors that represent 2–normal surfaces. Let $S \subset M$ be a 2–normal surface and let $x_{t,1}, \dots, x_{t,6}$ be the components of $\mathfrak{x}(S)$ that correspond to the squares and octagons in some tetrahedron t. It is impossible that in $S \cap t$ occur two different types of squares or octagons, since two different squares or octagons would yield a self-intersection of S. Thus all but at most one of $x_{t,1}, \dots, x_{t,6}$ vanish for any t. This property of $\mathfrak{x}(S)$ is called *compatibility condition.*

Let γ be a normal arc in a 2–simplex σ of $\mathcal{T}$ and t_1, t_2 be the two tetrahedra that meet at σ. In both t_1 and t_2 there are one triangle, one square and two octagons that contain a copy of γ in its boundary. Moreover, each of them contains *exactly one* copy of γ. Let $x_{t_i,1}, \dots, x_{t_i,4}$ be the components of $\mathfrak{x}(S)$ that correspond to these types of discs in t_i, where $i = 1, 2$. Since $\partial S = \emptyset$, the number of components of $S \cap t_1$ containing a copy of γ equals the number of components of $S \cap t_2$ containing a copy of γ. That is to say $x_{t_1,1} + \cdots + x_{t_1,4} = x_{t_2,1} + \cdots + x_{t_2,4}$. Thus $\mathfrak{x}(S)$ satisfies a system of linear Diophantine equations, with one equation for each type of normal arcs. This property of $\mathfrak{x}(S)$ is called *matching condition.* The next claim states that the compatibility and the matching conditions characterize the vectors that represent 2–normal surfaces. A proof can be found in [11], Chapter 9.

Proposition 1 *Let $\mathfrak{x}$ be a vector of $10\,t(\mathcal{T})$ non-negative integers that satisfies both the compatibility and the matching conditions. Then there is a 2–normal surface $S \subset M$ with $\mathfrak{x}(S) = \mathfrak{x}$.* □

Two 2–normal surfaces S_1, S_2 are called *compatible* if the vector $\mathfrak{x}(S_1) + \mathfrak{x}(S_2)$ satisfies the compatibility condition. It always satisfies the matching condition. Thus if S_1 and S_2 are compatible, then there is a 2–normal surface S with $\mathfrak{x}(S) = \mathfrak{x}(S_1) + \mathfrak{x}(S_2)$, and we denote $S = S_1 + S_2$. Conversely, let S be a 2–normal surface, and assume that there are non-negative integer vectors $\mathfrak{x}_1, \mathfrak{x}_2$ that both satisfy the matching condition, with $\mathfrak{x}(S) = \mathfrak{x}_1 + \mathfrak{x}_2$. Then both $\mathfrak{x}_1$ and $\mathfrak{x}_2$ satisfy the compatibility condition. Thus there are 2–normal surfaces S_1, S_2 with $S = S_1 + S_2$. The Euler characteristic is additive, i.e., $\chi(S_1 + S_2) = \chi(S_1) + \chi(S_2)$, see [11].

Remark 1 The addition of 2–normal surfaces extends to an addition on the set of pre-normal surfaces as follows. If $S_1, S_2 \subset M$ are pre-normal surfaces, then $S_1 + S_2$ is the pre-normal surface that is determined by $\mathcal{T}^1 \cap (S_1 \cup S_2)$. The addition yields a semi-group structure on the set of pre-normal surfaces. This semi-group is isomorphic to the semi-group of integer points in a certain rational convex cone that is associated to $\mathcal{T}$. The Euler characteristic is *not* additive with respect to the addition of pre-normal surfaces.

3 Fundamental surfaces

We use the notations of the previous section. The power of the theory of 2–normal surfaces is based on the following two finiteness results.

Proposition 2 *Let $S \subset M$ be a 2–normal surface comprising more than $10\,t(\mathcal{T})$ two-sided connected components. Then two connected components of S are isotopic mod $\mathcal{T}^2$.* □

This is proven in [9], Lemma 4, for 1–normal surfaces. The proof easily extends to 2–normal surfaces.

Theorem 3 *Let $N \subset M \setminus U(\mathcal{T}^0)$ be a sub–3–manifold whose boundary is a 1–normal surface. There is a system $F_1, \dots, F_q \subset N$ of 2–normal surfaces such that*

$$\|F_i\| < \|\partial N\| \cdot 2^{18\,t(\mathcal{T})}$$

for $i = 1, \dots, q$, and any 2–normal surface $F \subset N$ can be written as a sum $F = \sum_{i=1}^{q} k_i F_i$ with non-negative integers $k_1, \dots, k_q$.

The surfaces $F_1, \ldots, F_q$ are called *fundamental*. Theorem 3 is a generalization of a result of [10] that concerns the case $N = M \setminus U(\mathcal{T}^0)$.

The rest of this section is devoted to the proof of Theorem 3. The idea is to define a system of linear Diophantine equations (*matching equations*) whose non-negative solutions correspond to 2–normal surfaces in N. The fundamental surfaces $F_1, \ldots, F_q$ correspond to the Hilbert base vectors of the equation system, and the bound for $\|F_i\|$ is a consequence of estimates for the norm of Hilbert base vectors. Note that in an earlier version of this paper [12], we proved Theorem 3 in essentially the same way, but using handle decompositions of 3–manifolds rather than triangulations.

Definition 4 A *region* of N is a component R of $N \cap t$, for a closed tetrahedron t of $\mathcal{T}$. If $\partial R \cap \partial N$ consists of two copies of one normal triangle or normal square then R is a *parallelity region.*

Definition 5 The *class* of a normal triangle, square or octagon in N is its equivalence class with respect to isotopies mod $\mathcal{T}^2$ with support in $U(N)$.

Let t be a closed tetrahedron of $\mathcal{T}$, and let $R \subset t$ be a region of N. One verifies that if R is not a parallelity region then $\partial R \cap \partial N$ either consists of four normal triangles ("type I") or of two normal triangles and one normal square ("type II"). If R is of type I, then R is isotopic mod $\mathcal{T}^2$ to $t \setminus U(\mathcal{T}^0)$, and any other region of N in t is a parallelity region. As in the previous section, R contains four classes of normal triangles, three classes of normal squares and three classes of normal octagons. If R is of type II, then t contains at most one other region of N that is not a parallelity region, that is then also of type II. A normal square or octagon in t that is not isotopic mod $\mathcal{T}^2$ to a component of $\partial R \cap \partial N$ intersects ∂R. Thus R contains two classes of normal triangles and one class of normal squares.

Let $m(N)$ be the number of classes of normal triangles, squares and octagons in regions of N of types I and II. If N has k regions of type I, then N has $\leq 2(t(\mathcal{T}) - k)$ regions of type II, thus $m(N) \leq 10k + 6(t(\mathcal{T}) - k) \leq 10\, t(\mathcal{T})$. Let $\overline{m}(N)$ be the number of parallelity regions of N. It is easy to see that $\overline{m}(N) \leq \frac{1}{2}\,\#(\partial N \setminus \mathcal{T}^2) \leq \frac{1}{6}\,\|\partial N\| \cdot t(\mathcal{T})$.

Any 2–normal surface $F \subset N$ is determined up to isotopy mod $\mathcal{T}^2$ with support in $U(N)$ by the vector $\overline{\mathfrak{r}}_N(F)$ of $m(N) + \overline{m}(N)$ non-negative integers that count the number of components of $F \setminus \mathcal{T}^2$ in each class of normal triangles, squares and octagons. Let $\gamma_1, \gamma_2 \subset \mathcal{T}^2$ be normal arcs, and let R_1, R_2 be two regions of N with $\gamma_1 \subset \partial R_1$ and $\gamma_2 \subset \partial R_2$. For $i = 1, 2$, let $x_{i,1}, \ldots, x_{i,m_i}$ be the components of $\bar{\mathfrak{x}}_N(F)$ that correspond to classes of normal triangles, squares and octagons in R_i that contain γ_i in its boundary. If $x_{1,1}+\cdots+x_{1,m_1} = x_{2,1}+\cdots+x_{2,m_2}$ then we say that $\bar{\mathfrak{x}}_N(F)$ satisfies the *matching equation* associated to $(\gamma_1, R_1; \gamma_2, R_2)$.

For $i = 1, 2$, R_i contains one class of normal triangles that contain a copy of γ_i in its boundary. If R_i is not a parallelity region, then R_i contains one class of normal squares that contain a copy of γ_i in its boundary. If K_i is of type I, then K_i additionally contains two classes of normal octagons containing a copy of γ_i in its boundary. Thus if R_i is a parallelity region then $m_i = 1$, if it is of type I then $m_i = 4$, and if it is of type II then $m_i = 2$.

For any 2–normal surface $F \subset N$, let $\mathfrak{x}_N(F) \in \mathbb{Z}_{\geq 0}^{m(N)}$ be the vector that collects the components of $\bar{\mathfrak{x}}_N(F)$ corresponding to the classes of normal triangles, squares and octagons in regions of N of types I and II. As in the previous section, the vector $\mathfrak{x}_N(F)$ (resp. $\bar{\mathfrak{x}}_N(F)$) satisfies a *compatibility condition*, i.e., for any region R of N vanish all but at most one components of $\mathfrak{x}_N(F)$ (resp. $\bar{\mathfrak{x}}_N(F)$) corresponding to classes of squares and octagons in R.

Lemma 2 *Suppose that any component of N contains a region that is not a parallelity region. There is a system of matching equations concerning only regions of N of types I and II, such that a vector $\mathfrak{x} \in \mathbb{Z}_{\geq 0}^{m(N)}$ satisfies these equations and the compatibility condition if and only if there is a 2–normal surface $F \subset N$ with $\mathfrak{x}_N(F) = \mathfrak{x}$. The surface F is determined by $\mathfrak{x}_N(F)$, up to isotopy in N mod $\mathcal{T}^2$.*

Proof Let $\gamma \subset N \cap \mathcal{T}^2$ be a normal arc. Let R_1, R_2 be the two regions of N that contain γ. Let $F \subset N$ be a 2–normal surface. Since $\partial F = \emptyset$, the number of components of $F \cap R_1$ containing γ and the number of components of $F \cap R_2$ containing γ coincide. Thus $\bar{\mathfrak{x}}_N(F)$ satisfies the matching equation associated to $(\gamma, R_1; \gamma, R_2)$. We refer to these equations as N–matching equations. We will transform the system of N–matching equations by eliminating the components of $\bar{\mathfrak{x}}_N(F)$ that do not belong to $\mathfrak{x}_N(F)$.

Let $\gamma_1, \gamma_2 \subset \mathcal{T}^2$ be normal arcs, and let R_1, R_2 be two different regions of N with $\gamma_1 \subset \partial R_1$ and $\gamma_2 \subset \partial R_2$. Assume that R_1 is a parallelity region of N. Then $m_1 = 1$, thus the matching equation associated to $(\gamma_1, R_1; \gamma_2, R_2)$ is of the form $x_{1,1} = x_{2,1} + \cdots + x_{2,m_2}$. Hence we can eliminate $x_{1,1}$ in the N–matching equations. For any region R_3 of N and any normal arc $\gamma_3 \subset \partial R_3$, the elimination transforms the matching equation associated to $(\gamma_1, R_1; \gamma_3, R_3)$ into the matching equation associated to $(\gamma_2, R_2; \gamma_3, R_3)$. We iterate the elimination process. Since any component of N contains a region that is not a parallelity region, we eventually transform the system of N–matching equations to a system $\mathfrak{A}$ of matching equations that concern only regions of N of types I and II.

Let $\mathfrak{x} \in \mathbb{Z}_{\geq 0}^{m(N)}$ be a solution of $\mathfrak{A} \cdot \mathfrak{x} = 0$. By the elimination process, there is a unique extension of $\mathfrak{x}$ to a solution $\bar{\mathfrak{x}}$ of the N–matching equations. If $\mathfrak{x}$ satisfies the compatibility condition then so does $\bar{\mathfrak{x}}$, since a parallelity region contains at most one class of normal squares. Now the lemma follows by Proposition 1, that is proven in [11]. □

Proof of Theorem 3 It is easy to verify that if R is a parallelity region then there is only one class of 2–normal pieces in R. If a component N_1 of N is a union of parallelity regions, then N_1 is a regular neighbourhood of a 1–normal surface $F_1 \subset N_1$, that has a connected non-empty intersection with each region of N_1. Any pre-normal surface in N_1 is a multiple of F_1 (thus, is 1-normal), see [8]. We have $\|F_1\| = \frac{1}{2}\|\partial N_1\|$. Thus by now we can suppose that any component of N contains a region that is not a parallelity region.

By Lemma 2, the $\mathfrak{x}$–vectors of 2–normal surfaces in N satisfy a system of linear equations $\mathfrak{A} \cdot \mathfrak{x} = 0$. By the following well known result on Integer Programming (see [21]), the non-negative integer solutions of such a system are additively generated by a finite set of solutions.

Lemma 3 *Let $\mathfrak{A} = (a_{ij})$ be an integer $(n \times m)$–matrix. Set*

$$K = \left(\max_{i=1,\dots,n} \sum_{j=1}^{m} a_{ij}^2 \right)^{1/2} .$$

There is a set $\{\mathfrak{x}_1, \dots \mathfrak{x}_p\}$ of non-negative integer vectors such that $\mathfrak{A} \cdot \mathfrak{x}_i = 0$ for any $i = 1, \dots, p$, the components of $\mathfrak{x}_i$ are bounded from above by mK^m, and any non-negative integer solution $\mathfrak{x}$ of $\mathfrak{A} \cdot \mathfrak{x} = 0$ can be written as a sum $\mathfrak{x} = \sum k_i \mathfrak{x}_i$ with non-negative integers $k_1, \dots, k_p$. □

The set $\{\mathfrak{x}_1, \dots \mathfrak{x}_p\}$ is called *Hilbert base* for $\mathfrak{A}$, if p is minimal.

As in the previous section, if $F \subset N$ is a 2–normal surface and $\mathfrak{x}_N(F)$ is a sum of two non-negative integer solutions of $\mathfrak{A} \cdot \mathfrak{x} = 0$ then there are 2–normal surfaces $F', F'' \subset N$ with $F = F' + F''$. Thus the surfaces $F_1, \dots, F_q \subset N$ that correspond to Hilbert base vectors satisfying the compatibility condition additively generate the set of all 2–normal surfaces in N.

It remains to bound $\|F_i\|$, for $i = 1, \dots, q$. Since F_i is 2–normal and any edge of $\mathcal{T}$ is of degree ≥ 3, we have $\|F_i\| \leq \frac{8}{3}\,\#(F_i \setminus \mathcal{T}^2)$. By the elimination process in the proof of Lemma 2, any component of $\bar{\mathfrak{r}}_N(F_i)$ that corresponds to a parallelity region of N is a sum of at most four components of $\mathfrak{r}_N(F_i)$. By the bound for the components of $\mathfrak{r}_N(F_i)$ in Lemma 3 (with $m = m(N)$ and $K^2 = 8$) and our bounds for $m(N)$ and $\overline{m}(N)$, we obtain

$$\begin{aligned}
\|F_i\| &\leq \frac{8}{3} \cdot (m(N) + 4\,\overline{m}(N)) \cdot \left(m(N) \cdot 2^{\frac{3}{2}m(N)}\right) \\
&\leq \frac{8}{3} \cdot \left(10\,t(\mathcal{T}) + \frac{2}{3}\,\|\partial N\|\,t(\mathcal{T})\right) \cdot 10\,t(\mathcal{T}) \cdot 2^{15\,t(\mathcal{T})} \\
&< (300 + 20\,\|\partial N\|) \cdot t(\mathcal{T})^2 \cdot 2^{15\,t(\mathcal{T})}.
\end{aligned}$$

Using $t(\mathcal{T}) \geq 5$ and $\|\partial N\| > 0$, we obtain $\|F_i\| < \|\partial N\| \cdot 2^{18\,t(\mathcal{T})}$. □

4 Maximal systems of 1–normal spheres

Let $\mathcal{T}$ be a triangulation of a closed orientable 3–manifold M. By Proposition 2, there is a system $\Sigma \subset M$ of $\leq 10\,t(\mathcal{T})$ pairwise disjoint 1–normal spheres, such that any 1–normal sphere in $M \setminus \Sigma$ is isotopic mod $\mathcal{T}^2$ to a component of Σ. We call such a system *maximal*. It is not obvious how to construct Σ, in particular how to estimate $\|\Sigma\|$ in terms of $t(\mathcal{T})$. This section is devoted to a solution of this problem.

Construction 1 Set $\Sigma_1 = \partial U(\mathcal{T}^0)$ and $N_1 = M \setminus U(\mathcal{T}^0)$. Let $i \geq 1$. If there is a 1–normal fundamental projective plane $P_i \subset N_i$ then set $\Sigma_{i+1} = \Sigma_i \cup 2P_i$ and $N_{i+1} = N_i \setminus U(P_i)$. Otherwise, if there is a 1–normal fundamental sphere $S_i \subset N_i$ that is not isotopic mod $\mathcal{T}^2$ to a component of Σ_i, then set $\Sigma_{i+1} = \Sigma_i \cup S_i$ and $N_{i+1} = N_i \setminus U(S_i)$. Otherwise, set $\Sigma = \Sigma_i$.

Since M is orientable, a projective plane P_i is one-sided and $2P_i$ is a sphere. By Proposition 2 and since embedded spheres are two-sided in M, the iteration stops for some $i < 10\,t(\mathcal{T})$.

Lemma 4 $\|\Sigma\| < 2^{185\,t(\mathcal{T})^2}$.

Proof In Construction 1, we have

$$\begin{aligned}
\|\Sigma_{i+1}\| &< \|\Sigma_i\| + 2\|\Sigma_i\| \cdot 2^{18\,t(\mathcal{T})} \\
&< \|\Sigma_i\| \cdot 2^{18\,t(\mathcal{T})+2}
\end{aligned}$$

by Theorem 3 . The iteration stops after $< 10\,t(\mathcal{T})$ steps, thus

$$\|\Sigma\| < \|\Sigma_1\| \cdot 2^{180\,t(\mathcal{T})^2+20\,t(\mathcal{T})} \leq \|\Sigma_1\| \cdot 2^{184\,t(\mathcal{T})^2},$$

using $t(\mathcal{T}) \geq 5$. Since $\|\partial U(\mathcal{T}^0)\|$ equals twice the number of edges of $\mathcal{T}$, we have $\|\Sigma_1\| \leq 4\,t(\mathcal{T})$, and the lemma follows. □

Lemma 5 Σ *is maximal.*

Proof It is to show that any 1–normal sphere $S \subset M \setminus U(\Sigma)$ is isotopic mod $\mathcal{T}^2$ to a component of Σ. Let N be the component of $M \setminus U(\Sigma)$ that contains S. If N contains a 1–normal fundamental projective plane P, then $N = U(P)$ by Construction 1. Thus $S = 2P = \partial N$, which is isotopic mod $\mathcal{T}^2$ to a component of Σ. Hence we can assume that N does not contain a 1–normal fundamental projective plane.

We express S as a sum $\sum_{i=1}^{q} k_i F_i$ of fundamental surfaces in N. Since $\chi(S) = 2$ and the Euler characteristic is additive, one of the fundamental surfaces in the sum, say, F_1 with $k_1 > 0$, has positive Euler characteristic. It is not a projective plane by the preceding paragraph, thus it is a sphere. By construction of Σ, the sphere F_1 is isotopic mod $\mathcal{T}^2$ to a component of Σ, thus it is parallel to a component of ∂N. Hence F_1 is disjoint to any 1–normal surface in N, up to isotopy mod $\mathcal{T}^2$. Thus S is the disjoint union of $k_1 F_1$ and $\sum_{i=2}^{q} k_i F_i$. Since S is connected, it follows $S = F_1$. Thus S is isotopic mod $\mathcal{T}^2$ to a component of Σ. □

We will extend Σ to a system $\tilde{\Sigma}$ of 2–normal spheres. To define $\tilde{\Sigma}$, we need a lemma about 2–normal spheres in the complement of Σ.

Lemma 6 *Let N be a component of $M \setminus U(\Sigma)$. Assume that there is a 2–normal sphere in N with exactly one octagon. Then there is a 2–normal fundamental sphere $F \subset N$ with exactly one octagon and $\|F\| < 2^{189\,t(\mathcal{T})^2}$.*

Proof Let $S \subset N$ be a 2–normal sphere with exactly one octagon. If N contains a 1–normal fundamental projective plane P, then $N = U(P)$ by Construction 1, and any pre-normal surface in N is a multiple of P, i.e., is 1–normal. Thus since $S \subset N$ is not 1–normal, there is no 1–normal fundamental projective plane in N.

We write S as a sum of 2–normal fundamental surfaces in N. Since S has exactly one octagon, exactly one summand is not 1–normal. Since any projective plane in the sum is not 1–normal by the preceding paragraph, at most one summand is a projective plane. Since $\chi(S) = 2$ and the Euler characteristic is additive, it follows that one of the fundamental surfaces in the sum is a sphere F.

Assume that F is 1–normal, i.e., $S \neq F$. The construction of Σ implies that F is isotopic mod $\mathcal{T}^2$ to a component of ∂N. Thus it is disjoint to any 2–normal surface in N. Therefore S is a disjoint union of a multiple of F and of a 2–normal surface with exactly one octagon, which is a contradiction since S is connected. Hence F contains the octagon of S. We have $\|F\| < \|\Sigma\| \cdot 2^{18\, t(\mathcal{T})}$ by Theorem 3. The claim follows with Lemma 4 and $t(\mathcal{T}) \geq 5$. □

The preceding lemma assures that the following construction works.

Construction 2 For any connected component N of $M \setminus U(\Sigma)$ that contains a 2–normal sphere with exactly one octagon, choose a 2–normal sphere $F_N \subset N$ with exactly one octagon and $\|F\| < 2^{189\, t(\mathcal{T})^2}$. Set

$$\tilde{\Sigma} = \Sigma \cup \bigcup_N F_N.$$

Since $\#(\tilde{\Sigma}) \leq 10\, t(\mathcal{T})$ by Proposition 2, it follows $\|\tilde{\Sigma}\| < 10\, t(\mathcal{T}) \cdot 2^{189\, t(\mathcal{T})^2} < 2^{190\, t(\mathcal{T})^2}$.

5 Almost k–normal surfaces and split equivalence

We shall need a generalization of the notion of k–normal surfaces. Let M be a closed connected orientable 3–manifold with a triangulation $\mathcal{T}$.

Definition 6 A closed embedded surface $S \subset M$ transversal to $\mathcal{T}^2$ is *almost k–normal*, if

(1) $S \cap \mathcal{T}^2$ is a union of normal arcs and of circles in $\mathcal{T}^2 \setminus \mathcal{T}^1$, and

(2) for any tetrahedron t of $\mathcal{T}$, any edge e of t and any component ζ of $S \cap \partial t$ holds $\#(\zeta \cap e) \leq k$.

Our definition is similar to Matveev's one in [16]. Note that there is a related but different definition of "almost normal" surfaces due to Rubinstein [19]. Any surface in M disjoint to $\mathcal{T}^1$ is almost 1–normal. Any almost k–normal surface that meets $\mathcal{T}^1$ can be seen as a k–normal surface with several disjoint small tubes attached in $M \setminus \mathcal{T}^1$, see [16]. The tubes can be nested. Of course there are many ways to add tubes to a k–normal surface. We shall develop tools to deal with this ambiguity.

Let $S \subset M$ be an almost k–normal surface. By definition, the connected components of $S \cap \mathcal{T}^2$ that meet $\mathcal{T}^1$ are formed by normal arcs. Thus these components define a pre-normal surface $S^\times$, that is obviously k–normal. It is determined by $S \cap \mathcal{T}^1$, according to Lemma 1. A disc $D \subset M \setminus \mathcal{T}^1$ with $\partial D \subset S$ is called a *splitting disc* for S. One obtains $S^\times$ by splitting S along splitting discs for S that are disjoint to $\mathcal{T}^2$, and isotopy mod $\mathcal{T}^1$.

If two almost k–normal surfaces S_1, S_2 satisfy $S_1^\times = S_2^\times$, then S_1 and S_2 differ only by the choice of tubes. This gives rise to the following equivalence relation.

Definition 7 Two embedded surfaces $S_1, S_2 \subset M$ transversal to $\mathcal{T}^2$ are *split equivalent* if $S_1 \cap \mathcal{T}^1 = S_2 \cap \mathcal{T}^1$ (up to isotopy mod $\mathcal{T}^2$).

If two almost k–normal surfaces $S_1, S_2 \subset M$ are split equivalent, then $S_1^\times = S_2^\times$, up to isotopy mod $\mathcal{T}^2$. In particular, two k–normal surfaces are split equivalent if and only if they are isotopic mod $\mathcal{T}^2$.

Definition 8 If $S \subset M$ is an almost k–normal surface and $S^\times$ is the disjoint union of k–normal surfaces $S_1, \dots, S_n$, then we call S a *tube sum* of $S_1, \dots, S_n$. We denote the set of all tube sums of $S_1, \dots, S_n$ by $S_1 \circ \cdots \circ S_n$.

Definition 9 Let $S = S_1 \cup \cdots \cup S_n \subset M$ be a surface transversal to $\mathcal{T}^2$ with n connected components, and let $\Gamma \subset M \setminus \mathcal{T}^1$ be a system of disjoint simple arcs with $\Gamma \cap S = \partial\Gamma$. For any arc γ in Γ, one component of $\partial U(\gamma) \setminus S$ is an annulus A_γ. The surface

$$S^\Gamma = (S \setminus U(\Gamma)) \cup \bigcup_{\gamma \subset \Gamma} A_\gamma$$

is called the *tube sum of* $S_1, \dots, S_n$ *along* Γ.

If $S_1, \dots, S_n$ are k–normal, then $S^\Gamma \in S_1 \circ \cdots \circ S_n$.

We recall the concept of impermeable surfaces, that is central in the study of almost 2–normal surfaces (see [22],[16]). Fix a vertex $x_0 \in \mathcal{T}^0$. Let $S \subset M$ be a connected embedded surface transversal to $\mathcal{T}$. If S splits M into two pieces, then let $B^+(S)$ denote the closure of the component of $M \setminus S$ that contains x_0, and let $B^-(S)$ denote the closure of the other component. We do not include x_0 in the notation "$B^+(S)$", since in our applications the choice of x_0 plays no essential role.

Definition 10 Let $S \subset M$ be a connected embedded surface transversal to $\mathcal{T}^2$. Let $\alpha \subset \mathcal{T}^1 \setminus \mathcal{T}^0$ and $\beta \subset S$ be embedded arcs with $\partial\alpha = \partial\beta$. A closed embedded disc $D \subset M$ is a *compressing disc* for S with string α and base β, if $\partial D = \alpha \cup \beta$ and $D \cap \mathcal{T}^1 = \alpha$. If, moreover, $D \cap S = \beta$, then we call D a *bond* of S.

Let $S \subset M$ be a connected embedded surface that splits M and let D be a compressing disc for S with string α. If the germ of α in $\partial\alpha$ is contained in $B^+(S)$ (resp. $B^-(S)$), then D is *upper* (resp. *lower*). Let D_1, D_2 be upper and lower compressing discs for S with strings α_1, α_2. If $D_1 \subset D_2$ or $D_2 \subset D_1$, then D_1 and D_2 are *nested.* If $D_1 \cap D_2 \subset \partial\alpha_1 \cap \partial\alpha_2$, then D_1 and D_2 are *independent* from each other.

Upper and lower compressing discs that are independent from each other meet in at most one point.

Definition 11 Let $S \subset M$ be a connected embedded surface that is transversal to $\mathcal{T}^2$ and splits M. If S has both upper and lower bonds, but no pair of nested or independent upper and lower compressing discs, then S is *impermeable.*

Note that the impermeability of S does not change under an isotopy of S mod $\mathcal{T}^1$. The next two claims state a close relationship between impermeable surfaces and (almost) 2–normal surfaces. By an octagon of an almost 2–normal surface $S \subset M$ in a tetrahedron t, we mean a circle in $S \cap \partial t$ formed by eight normal arcs. This corresponds to an octagon of $S^\times$ in the sense of Figure 2.

Proposition 3 *Any impermeable surface in M is isotopic mod $\mathcal{T}^1$ to an almost 2–normal surface with exactly one octagon.*

Proposition 4 *A connected 2–normal surface that splits M and contains exactly one octagon is impermeable.*

We shall need these statements later. As the author found only parts of the proofs in the literature (see [22],[16]), he includes proofs in Section 9.

We end this section with the definition of $\mathcal{T}^1$–Morse embeddings and with the notion of thin position. Let S be a closed 2–manifold and let $H\colon S \times I \to M$ be a tame embedding. For $\xi \in I$, set $H_\xi = H(S \times \xi)$.

Definition 12 An element $\xi \in I$ is a *critical parameter* of H and a point $x \in H_\xi$ is a *critical point* of H with respect to $\mathcal{T}^1$, if x is a vertex of $\mathcal{T}$ or x is a point of tangency of H_ξ to $\mathcal{T}^1$.

Definition 13 We call H a $\mathcal{T}^1$*–Morse embedding*, if it has finitely many critical parameters, to any critical parameter belongs exactly one critical point, and for any critical point $x \in \mathcal{T}^1 \setminus \mathcal{T}^0$ corresponding to a critical parameter ξ, one component of $U(x) \setminus H_\xi$ is disjoint to $\mathcal{T}^1$. The number of critical points with respect to $\mathcal{T}^1$ of a $\mathcal{T}^1$–Morse embedding H is denoted by $c(H, \mathcal{T}^1)$.

The last condition in the definition of $\mathcal{T}^1$–Morse embeddings means that any critical point of H is a vertex of $\mathcal{T}$ or a local maximum resp. minimum of an edge of $\mathcal{T}$.

Definition 14 Let F be a closed surface, let $J\colon F \times I \to M$ be a $\mathcal{T}^1$–Morse embedding, and let $\xi_1, \dots, \xi_r \in I$ be the critical parameters of J with respect to $\mathcal{T}^1$. The *complexity* $\kappa(J)$ of J is defined as

$$\kappa(J) = \# \left(\mathcal{T}^1 \setminus \left(\bigcup_{i=1}^{r} J_{\xi_i} \right) \right).$$

If $\kappa(J)$ is minimal among all $\mathcal{T}^1$–Morse embeddings with the property $\mathcal{T}^1 \subset J(F \times I)$, then J is said to be in *thin position* with respect to $\mathcal{T}^1$. This notion was introduced for foliations of 3-manifolds by Gabai [5], was applied by Thompson [22] for her recognition algorithm of S^3, and was also used in the study of Heegaard surfaces by Scharlemann and Thompson [20].

If $J(F \times \xi)$ splits M and has a pair of nested or independent upper and lower compressing discs D_1, D_2, then an isotopy of J along $D_1 \cup D_2$ decreases $\kappa(J)$, see [16], [22]. We obtain the following claim.

Lemma 7 *Let $J\colon F \times I \to M$ be a $\mathcal{T}^1$–Morse embedding in thin position and let $\xi \in I$ be a non-critical parameter of J. If $J(F \times \xi)$ has both upper and lower bonds, then $J(F \times \xi)$ is impermeable.* □

6 Compressing and splitting discs

Let M be a closed connected 3–manifold with a triangulation $\mathcal{T}$. In the lemmas that we prove in this section, we state technical conditions for the existence of compressing and splitting discs for a surface.

Lemma 8 *Let $S_1, \dots, S_n \subset M$ be embedded surfaces transversal to $\mathcal{T}^2$ and let S be the tube sum of $S_1, \dots, S_n$ along a system $\Gamma \subset M \setminus \mathcal{T}^1$ of arcs. Assume that S splits M, and $\Gamma \subset B^-(S)$. If none of $S_1, \dots, S_n$ has a lower compressing disc, then S has no lower compressing disc.*

Proof Set $\Sigma = S_1 \cup \dots \cup S_n$. Let $D \subset M$ be a lower compressing disc for S. One can assume that a collar of $\partial D \cap S$ in D is contained in $B^-(S)$. Then, since by hypothesis $U(\Gamma) \cap \Sigma \subset B^-(S)$, any point in $\partial D \cap U(\Gamma) \cap \Sigma$ is endpoint of an arc in $D \cap \Sigma$. Therefore there is a sub-disc $D' \subset D$, bounded by parts of ∂D and of arcs in $D \cap \Sigma$, that is a lower compressing disc for one of $S_1, \dots, S_n$. □

Lemma 9 *Let $S \subset M$ be a surface transversal to $\mathcal{T}^2$ with upper and lower compressing discs D_1, D_2 such that $\partial(D_1 \cap D_2) \subset \partial D_2 \cap S$. Assume either that $(\partial D_1) \cap D_2 \subset \mathcal{T}^1$ or that there is a splitting disc D_m for S such that $D_1 \cap D_m = \partial D_1 \cap \partial D_m = \{x\}$ is a single point and $D_2 \cap D_m = \emptyset$. Then S has a pair of independent or nested upper and lower compressing discs.*

Proof If $D_1 \cap D_2 \cap \mathcal{T}^1$ comprises more than a single point then the string of D_2 is contained in the string of D_1. Thus $D_1 \cap S$ contains an arc different from the base of D_1, bounding in D_1 a lower compressing disc, that forms with D_1 a pair of nested upper and lower compressing discs for S.

Assume that a component γ of $D_1 \cap D_2$ is a circle. Then there are discs $D_1' \subset D_1$ and $D_2' \subset D_2$ with $\partial D_1' = \partial D_2' = \gamma$. Since $\partial(D_1 \cap D_2) \subset \partial D_2$, D_2' does not contain arcs of $D_1 \cap D_2$. Thus if we choose γ innermost in D_2, then $D_1 \cap D_2' = \gamma$. By cut-and-paste of D_1 along D_2', one reduces the number of circle components in $D_1 \cap D_2$. Therefore we assume by now that $D_1 \cap D_2$ consists of isolated points in $\partial D_1 \cap \partial D_2$ and of arcs that do not meet ∂D_1.

Assume that there is a point $y \in (\partial D_1 \cap \partial D_2) \setminus \mathcal{T}^1$. Then there is an arc $\gamma \subset \partial D_1$ with $\partial\gamma = \{x, y\}$. Without assumption, let $\gamma \cap D_2 = \{y\}$. Let A be the closure of the component of $U(\gamma) \setminus (D_1 \cup D_2 \cup D_m)$ whose boundary contains arcs in both D_2 and D_m. Define $D_2^* = ((D_2 \cup D_m) \setminus U(\gamma)) \cup A$, that is to say, D_2^* is the connected sum of D_2 and D_m along γ. By construction, $(D_1 \cap D_2^*) \setminus \partial D_1 = (D_1 \cap D_2) \setminus \partial D_1$, and $\#(D_1 \cap D_2^*) < \#(D_1 \cap D_2)$. In that way, we remove all points of intersection of $(\partial D_1 \cap D_2) \setminus \mathcal{T}^1$. Thus by now we can assume that $D_1 \cap D_2$ consists of arcs in D_1 that do not meet ∂D_1, and possibly of a single point in $\mathcal{T}^1$.

Let $\gamma \subset D_1 \cap D_2$ be an outermost arc in D_2, that is to say, $\gamma \cup \partial D_2$ bounds a disc $D' \subset D_2 \setminus \mathcal{T}^1$ with $D_1 \cap D' = \gamma$. We move D_1 away from D' by an isotopy mod $\mathcal{T}^1$ and obtain a compressing disc D_1^* for S with $D_1^* \cap D_2 = (D_1 \cap D_2) \setminus \gamma$. In that way, we remove all arcs of $D_1 \cap D_2$ and finally get a pair of independent upper and lower compressing discs for S. □

Lemma 10 *Let $S \subset M$ be an almost 1–normal surface. If S has a compressing disc, then S is isotopic mod $\mathcal{T}^1$ to an almost 1–normal surface with* *a compressing disc contained in a single tetrahedron. In particular, S is not 1–normal.*

Proof Let D be a compressing disc for S. Choose S and D up to isotopy of $S \cup D$ mod $\mathcal{T}^1$ so that S is almost 1–normal and $\#(D \cap \mathcal{T}^2)$ is minimal. Choose an innermost component $\gamma \subset (D \cap \mathcal{T}^2)$, which is possible as $D \cap \mathcal{T}^2 \neq \emptyset$. There is a closed tetrahedron t of $\mathcal{T}$ and a component C of $D \cap t$ that is a disc, such that $\gamma = C \cap \partial t$. Let σ be the closed 2–simplex of $\mathcal{T}$ that contains γ. We obtain three cases.

(1) Let γ be a circle, thus $\partial C = \gamma$. Then there is a disc $D' \subset \sigma$ with $\partial D' = \gamma$ and a ball $B \subset t$ with $\partial B = C \cup D'$. By an isotopy mod $\mathcal{T}^1$ with support in $U(B)$, we move $S \cup D$ away from B, obtaining a surface S^* with a compressing disc D^*. If S^* is almost 1–normal, then we obtain a contradiction to our choice as $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$.

(2) Let γ be an arc with endpoints in a single component c of $S \cap \sigma$. Since S has no returns, γ is not the string of D. We apply to $S \cup D$ an isotopy mod $\mathcal{T}^1$ with support in $U(C)$ that moves C into $U(C) \setminus t$, and obtain a surface S^* with a compressing disc D^*. If S^* is almost 1–normal, then we obtain a contradiction to our choice as $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$.

(3) Let γ be an arc with endpoints in two different components c_1, c_2 of $S \cap \sigma$. If both c_1 and c_2 are normal arcs, then set $C' = C$, $c_1' = c_1$ and $c_2' = c_2$. If, say, c_1 is a circle, then we move $S \cup D$ away from C by an isotopy mod $\mathcal{T}^1$ with support in $U(C)$. If the resulting surface S^* is still almost 1–normal, then we obtain a contradiction to the choice of D.

In either case, S^* is not almost 1-normal, i.e., the isotopy introduces a return. Therefore there is a component of $C \setminus S$ with closure C' such that $\partial C' \cap S$ connects two normal arcs c_1', c_2' of $S \cap \sigma$.

Let $\gamma' = C' \cap \sigma$. Up to isotopy of C' mod $\mathcal{T}^2$ that is fixed on $\partial C' \cap S$, we assume that $\gamma' \cap (c_1' \cup c_2') \subset \partial \gamma'$. There is an arc α contained in an edge of σ with $\partial \alpha \subset c_1' \cup c_2'$. For $i \in \{1, 2\}$, there is an arc $\beta_i \subset c_i'$ that connects $\alpha \cap c_i'$ with $\gamma' \cap c_i'$. The circle $\alpha \cup \beta_1 \cup \beta_2 \cup \gamma'$ bounds a closed disc $D' \subset \sigma$. Eventually $D' \cup C'$ is a compressing disc for S contained in a single tetrahedron. □

Lemma 11 *Let $S \subset M$ be a 1–normal surface and let D be a splitting disc for S. Then, $(D, \partial D)$ is isotopic in $(M \setminus \mathcal{T}^1, S \setminus \mathcal{T}^1)$ to a disc embedded in S.*

Proof We choose D up to isotopy of $(D, \partial D)$ in $(M \setminus \mathcal{T}^1, S \setminus \mathcal{T}^1)$ so that $(\#((\partial D) \cap \mathcal{T}^2), \#(D \cap \mathcal{T}^2))$ is minimal in lexicographic order. Assume that

$\partial D \cap \mathcal{T}^2 \neq \emptyset$. Then, there is a tetrahedron t, a 2–simplex $\sigma \subset \partial t$, a component K of $S \cap t$, and a component γ of $\partial D \cap K$ with $\partial\gamma \subset \sigma$. Since S is 1–normal, the closure D' of one component of $K \setminus \gamma$ is a disc with $\partial D' \subset \gamma \cup \sigma$. By choosing γ innermost in D, we can assume that $D' \cap \partial D = \gamma$. An isotopy of $(D, \partial D)$ in $(M \setminus \mathcal{T}^1, S \setminus \mathcal{T}^1)$ with support in $U(D')$, moving ∂D away from D', reduces $\#(\partial D \cap \mathcal{T}^2)$, in contradiction to our choice. Thus $\partial D \cap \mathcal{T}^2 = \emptyset$.

Now, assume that $D \cap \mathcal{T}^2 \neq \emptyset$. Then, there is a tetrahedron t, a 2–simplex $\sigma \subset \partial t$, and a disc component C of $D \cap t$, such that $C \cap \sigma = \partial C$ is a single circle. There is a ball $B \subset t$ bounded by C and a disc in σ. An isotopy of D with support in $U(B)$, moving C away from t, reduces $\#(D \cap \mathcal{T}^2)$, in contradiction to our choice. Thus D is contained in a single tetrahedron t. Since S is 1–normal, ∂D bounds a disc D' in $S \cap t$. An isotopy with support in t that is constant on ∂D moves D to D', which yields the lemma. □

Corollary 1 *Let $S_0 \subset M$ be a 1–normal sphere that splits M, and let $S \subset B^-(S_0)$ be an almost 1–normal sphere disjoint to S_0 that is split equivalent to S_0. Then there is a $\mathcal{T}^1$–Morse embedding $J\colon S^2 \times I \to M$ with $J(S^2 \times I) = B^+(S) \cap B^-(S_0)$ and $c(J, \mathcal{T}^1) = 0$.*

Proof Let X be a graph isomorphic to $S_0 \cap \mathcal{T}^2$. Since $S^\times$ is a copy of S_0, there is an embedding $\varphi\colon X \times I \to B^+(S) \cap B^-(S_0)$ with $\varphi(X^0 \times I) = \varphi(X \times I) \cap \mathcal{T}^1$, $\varphi(X \times 0) = S_0 \cap \mathcal{T}^2 = S_0 \cap \varphi(X \times I)$, and $\varphi(X \times 1)$ is the union of the normal arcs in S.

Let $\gamma \subset S \cap \varphi(X \times I)$ be a circle that does not meet $\mathcal{T}^1$. Then, γ bounds a disc $D \subset \varphi(X \times I) \setminus \mathcal{T}^1$. The two components of $S \setminus \gamma$ are discs. One of them is disjoint to $\mathcal{T}^1$, since otherwise the disc D would give rise to a splitting disc for $S^\times = S_0$ that is not isotopic mod $\mathcal{T}^1$ to a sub-disc of S_0, in contradiction to the preceding lemma. Thus by cut-and-paste along sub-discs of $S \setminus \mathcal{T}^1$, we can assume that additionally $S \cap \varphi(X \times I) = \varphi(X \times 1)$.

Let $\gamma \subset X$ be a circle so that $\varphi(\gamma \times 0)$ is contained in the boundary of a tetrahedron of $\mathcal{T}$. Since S_0 is 1–normal, $\varphi(\gamma \times 0)$ bounds an open disc in $S_0 \setminus \mathcal{T}^2$. By the same argument as in the preceding paragraph, $\varphi(\gamma \times 1)$ bounds an open disc in $S \setminus \mathcal{T}^1$. One easily verifies that these two discs together with $\varphi(\gamma \times I)$ bound a ball in $B^+(S) \cap B^-(S_0)$ disjoint to $\mathcal{T}^1$. Hence $(B^+(S) \cap B^-(S_0)) \setminus U(\varphi(X \times I))$ is a disjoint union of balls in $M \setminus \mathcal{T}^1$, and this implies the existence of J. □

7 Reduction of surfaces

Let M be a closed connected orientable 3–manifold with a triangulation $\mathcal{T}$. In this section, we show how to get isotopies of embedded surfaces under which the number of intersections with $\mathcal{T}^1$ is monotonely non-increasing.

Definition 15 Let $S \subset M$ be a connected embedded surface that is transversal to $\mathcal{T}^2$ and splits M. Let D be an upper (resp. lower) bond of S, set $D_1 = U(D) \cap S$, and set $D_2 = B^+(S) \cap \partial U(D)$ (resp. $D_2 = B^-(S) \cap \partial U(D)$). An *elementary reduction* along D transforms S to the surface $(S \setminus D_1) \cup D_2$. *Upper* (resp. *lower*) *reductions* of S are the surfaces that are obtained from S by a sequence of elementary reductions along upper (resp. lower) bonds.

If S' is an upper or lower reduction of S, then $\|S'\| \leq \|S\|$ with equality if and only if $S = S'$. Obviously S is isotopic to S', such that $\| \cdot \|$ is monotonely non-increasing under the isotopy. If $\alpha \subset \mathcal{T}^1 \setminus \mathcal{T}^0$ is an arc with $\partial\alpha \subset S'$, then also $\partial\alpha \subset S$. It is easy to see that if S' has a lower compressing disc and is an upper reduction of S, then also S has a lower compressing disc.

We will construct surfaces with almost 1–normal upper or lower reductions. Let $N \subset M$ be a 3–dimensional sub–manifold, such that ∂N is pre-normal. Let $S \subset N$ be an embedded surface transversal to $\mathcal{T}^2$ that splits M and has no lower compressing disc.

Lemma 12 *Suppose that there is a system $\Gamma \subset N \setminus \mathcal{T}^1$ of arcs such that $S^\Gamma \subset N$ is connected, $\Gamma \subset B^-(S^\Gamma)$, and $\partial N \cap B^+(S^\Gamma)$ is 1–normal.*

If, moreover, Γ and an upper reduction $S' \subset N$ of S^Γ are chosen so that $\|S'\|$ is minimal, then S' is almost 1–normal.

Proof By hypothesis, $\Gamma \subset B^-(S^\Gamma)$, and S has no lower compressing discs. Thus by Lemma 8, S^Γ has no lower compressing discs. Therefore its upper reduction S' has no lower compressing discs.

Assume that S' is not almost 1–normal. Then S' has a compressing disc D' that is contained in a single tetrahedron t (see [16]), with string α' and base β'. Since S' has no lower compressing discs, D' is upper and does not contain proper compressing sub-discs. Thus $\alpha' \cap S' = \partial\alpha'$, i.e., all components of $(D' \cap S') \setminus \beta'$ are circles. Since ∂N is pre-normal, $\partial N \setminus \mathcal{T}^2$ is a disjoint union of discs. Therefore, since D' is contained in a single tetrahedron, we can assume by isotopy of D' mod $\mathcal{T}^2$ that $D' \cap \partial N$ consists of arcs. We have

$\alpha' \subset B^+(S') \subset B^+(S^\Gamma)$. It follows $\partial N \cap \alpha' = \emptyset$, since otherwise a sub-disc of D' is a compressing disc for $\partial N \cap B^+(S^\Gamma)$, which is impossible as $\partial N \cap B^+(S^\Gamma)$ is 1–normal by hypothesis. Thus $\partial N \cap \alpha' = \emptyset$ and $D' \subset N$.

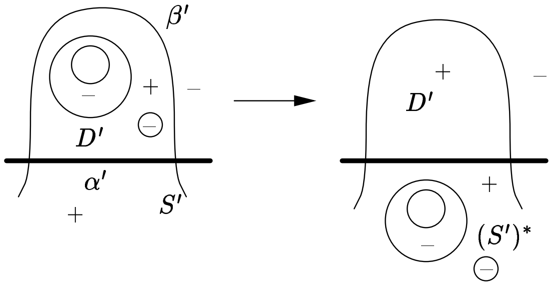

Figure 3: How to produce a bond

By an isotopy with support in $U(D')$ that is constant on β', we move $(D' \cap S') \setminus \beta'$ to $U(D') \setminus t$, and obtain from S' a surface $(S')^* \subset N$ that has D' as upper bond. This is shown in Figure 3, where $B^+(S')$ is indicated by plus signs and $\mathcal{T}^1$ is bold. The isotopy moves Γ to a system of arcs $\Gamma^* \subset N$ and moves S^Γ to S^{Γ^*} with $\Gamma^* \subset B^-(S^{\Gamma^*})$. Since $\alpha' \subset B^+(S')$, there is a homeomorphism $\varphi\colon B^-(S') \to B^-((S')^*)$ that is constant on $\mathcal{T}^1$ with $\varphi(B^-(S^\Gamma)) = B^-(S^{\Gamma^*})$. One obtains S' by a sequence of elementary reductions along bonds of S^Γ that are contained in $B^-(S')$. These bonds are carried by φ to bonds of S^{Γ^*}. Thus $(S')^*$ is an upper reduction of S^{Γ^*}. Since $(S')^*$ admits an elementary reduction along its upper bond D', we obtain a contradiction to the minimality of $\|S'\|$. Thus S' is almost 1–normal. □

Lemma 13 *Let Γ and S' be as in the previous lemma, and let G_1, G_2 be two connected components of $(S')^\times$ that both split M. Then there is no arc in $(\mathcal{T}^1 \setminus \mathcal{T}^0) \cap B^+(S') \cap N$ joining G_1 with G_2.*

Proof By the previous lemma, S' is almost 1–normal. Recall that one obtains $(S')^\times$ up to isotopy mod $\mathcal{T}^1$ by splitting S' along splitting discs that do not meet $\mathcal{T}^2$. Assume that there is an arc $\alpha \subset (\mathcal{T}^1 \setminus \mathcal{T}^0) \cap B^+(S') \cap N$ joining G_1 with G_2. Let Y be the component of $M \setminus (G_1 \cup G_2)$ that contains α.

By hypothesis, S^Γ is connected. Thus S' is connected, and there is an arc $\beta \subset S'$ with $\partial\beta = \partial\alpha$. Since G_1, G_2 split M, the set Y is the only component of $M \setminus (G_1 \cup G_2)$ with boundary $G_1 \cup G_2$. Thus there is a component β' of $\beta \cap Y$ connecting G_1 with G_2. There is a splitting disc $D \subset Y$ of S' contained in a single tetrahedron with $\beta' \cap D \neq \emptyset$. By choosing D innermost, we assume that

$\beta \cap D$ is a single point in ∂D. Since ∂N is pre-normal and D is contained in a single tetrahedron, we can assume by isotopy of D mod $\mathcal{T}^2$ that $D \cap \partial N = \emptyset$, thus $D \subset N$.

Choose a disc $D' \subset U(\alpha \cup \beta) \cap B^+(S')$ so that $D' \cap \mathcal{T}^1 = \alpha$ and $D' \cap S' = \beta \setminus U(\partial D)$. Then $D' \cap \partial N = \emptyset$, since $U(\alpha \cup \beta) \cap \partial N = \emptyset$. We split S' along D, pull the two components of $(S' \cap \partial U(D)) \setminus D$ along $(\partial D') \setminus (\alpha \cup \beta)$, and reglue. We obtain a surface $(S')^*$ with D' as an upper bond.

Since a small collar of ∂D in D is in $B^-(S')$, there is a homeomorphism $\varphi\colon B^-(S') \to B^-((S')^*)$ that is constant on $\mathcal{T}^1$. Set $\Gamma^* = \varphi(\Gamma)$. Then $\varphi(S^\Gamma) = S^{\Gamma^*}$ with $\Gamma^* \subset B^-(S^{\Gamma^*})$. As in the proof of the previous lemma, $(S')^*$ is an upper reduction of S^{Γ^*}, and $(S')^*$ admits an elementary reduction along D'. This contradiction to the minimality of $\|S'\|$ yields the lemma. □

8 Proof of Theorem 2

Let $\mathcal{T}$ be a triangulation of S^3 with a vertex $x_0 \in \mathcal{T}^0$. Let $\Sigma \subset S^3$ be a maximal system of disjoint 1–normal spheres with $\|\Sigma\| < 2^{185\, t(\mathcal{T})^2}$, as given by Construction 1. Construction 2 extends Σ to a system $\tilde{\Sigma} \subset S^3$ of disjoint 2–normal spheres that are pairwise non-isotopic mod $\mathcal{T}^2$, such that

(1) any component of $\tilde{\Sigma}$ has at most one octagon,

(2) any component of $S^3 \setminus \tilde{\Sigma}$ has at most one boundary component that is not 1–normal,

(3) if the boundary of a component N of $S^3 \setminus \tilde{\Sigma}$ is 1–normal, then N does not contain 2–normal spheres with exactly one octagon, and

(4) $\|\tilde{\Sigma}\| < 2^{190\, t(\mathcal{T})^2}$.

Let N be a component of $S^3 \setminus \tilde{\Sigma}$ that is not a regular neighbourhood of a vertex of $\mathcal{T}$. Let S_0 be the component of ∂N with $N \subset B^-(S_0)$, and let $S_1, \ldots, S_k$ be the other components of ∂N. Since Σ is maximal, any almost 1-normal sphere in N is a tube sum of copies of $S_0, S_1, \ldots, S_k$.

Lemma 14 $N \cap \mathcal{T}^0 = \emptyset$.

Proof If $x \in N \cap \mathcal{T}^0$, then the sphere $\partial U(x) \subset N$ is 1–normal. It is not isotopic mod $\mathcal{T}^1$ to a component of ∂N, since $N \neq U(x)$. This contradicts the maximality of Σ. □

Lemma 15 *If ∂N is 1–normal, then there is an arc in $\mathcal{T}^1 \cap \overline{N}$ that connects two different components of $\partial N \setminus S_0$.*

Proof Let $\partial N = S_0 \cup S_1 \cup \cdots \cup S_k$ be 1–normal. We first consider the case where there is an almost 1–normal sphere $S \in S_1 \circ \cdots \circ S_k$ in $\overline{N}$ that has a compressing disc D, with string α and base β. We choose D innermost, so that $\alpha \cap S = \partial\alpha$. In particular, $\alpha \cap \partial N = \partial\alpha$. Assume that $\alpha \not\subset \overline{N}$. Since $\partial D \setminus \alpha \subset \overline{N}$, there is an arc $\beta' \subset D \cap \partial N$ that connects the endpoints of α. The sub-disc $D' \subset D$ bounded by $\alpha \cup \beta'$ is a compressing disc for the 1–normal surface ∂N, in contradiction to Lemma 10. By consequence, $\alpha \subset \overline{N}$. Assume that $\partial\alpha$ is contained in a single component of $\partial N \setminus S_0$, say, in S_1. By Lemma 10, D is not a compressing disc for S_1, hence $\beta \not\subset S_1$. Thus there is a closed line in $S_1 \setminus \beta$ that separates $\partial\alpha$ on S_1, but not on S. This is impossible as S is a sphere. We conclude that if S has a compressing disc, then there is an arc $\alpha \subset \mathcal{T}^1 \cap N$ that connects different components of $\partial N \setminus S_0$.

It remains to consider the case where no sphere in $S_1 \circ \cdots \circ S_k$ contained in $\overline{N}$ has a compressing disc. We will show the existence of an almost 2–normal sphere in N with exactly one octagon, using the technique of thin position. This contradicts property (3) of $\tilde{\Sigma}$ (see the begin of this section), and therefore finishes the proof of the lemma. Let $J\colon S^2 \times I \to B^-(S_0)$ be a $\mathcal{T}^1$–Morse embedding, such that

(1) $J(S^2 \times 0) = S_0$,

(2) $J(S^2 \times \frac{1}{2}) \in S_1 \circ \cdots \circ S_k$ (or $\|J(S^2 \times \frac{1}{2})\| = 0$, in the case $\partial N = S_0$),

(3) $B^-\left(J(S^2 \times 1)\right) \cap \mathcal{T}^1 = \emptyset$, and

(4) $\kappa(J)$ is minimal.

Define $S = J(S^2 \times \frac{1}{2})$. Assume that for some $\xi \in I$ there is a pair $D_1, D_2 \subset M$ of nested or independent upper and lower compressing discs for $J_\xi = J(S^2 \times \xi)$. We show that we can assume $D_1, D_2 \subset B^-(S_0)$. Since S_0 is 1–normal, it has no compressing discs by Lemma 10. Thus $(D_1 \cup D_2) \cap S_0$ consists of circles. Any such circle bounds a disc in $S_0 \setminus \mathcal{T}^1$ by Lemma 11. By cut-and-paste of $D_1 \cup D_2$, we obtain $D_1, D_2 \subset B^-(S_0)$, as claimed. Now, one obtains from J an embedding $J'\colon S^2 \times I \to B^-(S_0)$ with $\kappa(J') < \kappa(J)$ by isotopy along $D_1 \cup D_2$, see [16], [22]. The embedding J' meets conditions (1) and (3) in the definition of J. Since $S \in S_1 \circ \cdots \circ S_k$ has no compressing discs by assumption, $S \cap D_i$ consists of circles. Thus S is split equivalent to $J'(S^2 \times \frac{1}{2})$. So J' meets also condition (2), $J'(S^2 \times \frac{1}{2}) \in S_1 \circ \cdots \circ S_k$, in contradiction to the choice of J. This disproves the existence of D_1, D_2. In conclusion, if J_ξ has upper and lower bonds, then it is impermeable.

Let ξ_{max} be the greatest critical parameter of J with respect to $\mathcal{T}^1$ in the interval $\left]0, \frac{1}{2}\right[$. We have $N \cap \mathcal{T}^0 = \emptyset$ by Lemma 14. Hence the critical point corresponding to ξ_{max} is a point of tangency of $J_{\xi_{max}}$ to some edge of $\mathcal{T}$. By assumption, S has no upper bonds, thus $\|S\| < \|J_{\xi_{max}-\epsilon}\|$ for sufficiently small $\epsilon > 0$. Let $\xi_{min} \in I$ be the smallest critical parameter of J with respect to $\mathcal{T}^1$. By Lemma 10, S_0 has no bonds, thus $\|S_0\| < \|J_{\xi_{min}+\epsilon}\|$. Therefore there are consecutive critical parameters $\xi_1, \xi_2 \in \left]0, \frac{1}{2}\right[$ such that

$$\|J_{\xi_1-\epsilon}\| < \|J_{\xi_1+\epsilon}\| > \|J_{\xi_2+\epsilon}\|.$$

Thus $J_{\xi_1+\epsilon}$ has both upper and lower bonds, and is therefore impermeable by the preceding paragraph. One component of $J^\times_{\xi_1+\epsilon}$ is a 2–normal sphere in N with exactly one octagon, by Proposition 3. The existence of that 2–normal sphere is a contradiction to the properties of $\tilde{\Sigma}$, which proves the lemma. □

We show that some tube sum $S \in S_1 \circ \cdots \circ S_k$ is isotopic to S_0 such that $\|\cdot\|$ is monotone under the isotopy. We consider three cases. In the first case, let ∂N be 1–normal.

Lemma 16 *If ∂N is 1–normal, then there is a sphere $S \in S_1 \circ \cdots \circ S_k$ in N with an upper reduction $S' \subset N$ so that there is a $\mathcal{T}^1$–Morse embedding $J\colon S^2 \times I \to S^3$ with $J(S^2 \times I) = B^+(S') \cap B^-(S_0)$ and $c(J, \mathcal{T}^1) = 0$.*

Proof By Lemma 15, there is an arc $\alpha \subset \mathcal{T}^1 \cap N$ that connects two components of $\partial N \setminus S_0$, say, S_1 with S_2. By Lemma 14, α is contained in an edge of $\mathcal{T}$. By Lemma 10, the 1–normal surfaces $S_1, \ldots, S_k$ have no lower compressing discs. Let $\Gamma \subset N$ be a system of $k-1$ arcs, such that the tube sum S of $S_1, \ldots, S_k$ along Γ is a sphere and an upper reduction $S' \subset N$ of S minimizes $\|S'\|$. We have $\|S'\| < \|S\|$, since it is possible to choose Γ so that S has an upper bond with string α. Since $\Gamma \subset B^-(S)$ and by Lemma 12, S' is almost 1–normal.

By the maximality of Σ, it follows $S' \in n_0 S_0 \circ \cdots \circ n_k S_k$ with non-negative integers $n_0, n_1, \ldots, n_k$. Moreover, $n_i \leq 2$ for $i = 0, \ldots, k$ by Lemma 13. Since S separates S_0 from $S_1, \ldots, S_k$, so does S'. Thus any path connecting S_0 with S_j for some $j \in \{1, \ldots, k\}$ intersects S' in an odd number of points. So alternatively $n_0 \in \{0, 2\}$ and $n_i = 1$ for all $i \in \{1, \ldots, k\}$, or $n_0 = 1$ and $n_i \in \{0, 2\}$ for all $i \in \{1, \ldots, k\}$. Since $\|S'\| < \|S^*\|$, it follows $n_0 = 1$ and $n_i = 0$ for $i \in \{1, \ldots, k\}$, thus $(S')^\times = S_0$. The existence of a $\mathcal{T}^1$–Morse embedding J with the claimed properties follows then by Corollary 1. □

The second case is that S_0 is 1–normal, and exactly one of $S_1, \ldots, S_k$ contains exactly one octagon, say, S_1. The octagon gives rise to an upper bond D of S_1

contained in a single tetrahedron. Since $\partial N \setminus S_1$ is 1–normal, $D \subset N$. Thus an elementary reduction of S_1 along D transforms S_1 to a sphere $F \subset N$. Since S_1 is impermeable by Proposition 4, F has no lower compressing disc (such a disc would give rise to a lower compressing disc for S_1 that is independent from D).

Lemma 17 *If $\partial N \setminus S_0$ is not 1–normal, then there is a sphere $S \in S_1 \circ \cdots \circ S_k$ in N with an upper reduction $S' \subset N$ so that there is a $\mathcal{T}^1$–Morse embedding $J\colon S^2 \times I \to S^3$ with $J(S^2 \times I) = B^+(S') \cap B^-(S_0)$ and $c(J, \mathcal{T}^1) = 0$.*

Proof We apply the Lemma 12 to $F, S_2, \ldots, S_k$, and together with the elementary reduction along D we obtain a sphere $S \in S_1 \circ S_2 \circ \cdots \circ S_k$ with an almost 1–normal upper reduction $S' \subset N$. One concludes $(S')^\times = S_0$ and the existence of J as in the proof of the previous lemma. □

We come to the third and last case, namely S_0 has exactly one octagon and $\partial N \setminus S_0$ is 1–normal. The octagon gives rise to a lower bond D of S_0, that is contained in N since $\partial N \setminus S_0$ is 1–normal. Thus an elementary reduction of S_0 along D yields a sphere $F \subset N$. Since S_0 is impermeable by Proposition 4, F has no upper compressing disc, similar to the previous case.

Lemma 18 *If S_0 is not 1–normal, then there is a lower reduction $S' \in S_1 \circ \cdots \circ S_k$ of S_0, with $S' \subset N$.*

Proof We apply Lemma 12 with $\Gamma = \emptyset$ to *lower* reductions of F, which is possible by symmetry. Thus, together with the elementary reduction along D, there is a lower reduction $S' \in n_0 S_0 \circ \cdots \circ n_k S_k$ of S_0, and $n_0, \ldots, n_k \leq 2$ by Lemma 13. Since $S' \subset B^-(F)$ and $S_0 \subset B^+(F)$, it follows $n_0 = 0$. Since S' separates $\partial N \cap B^+(F)$ from $\partial N \cap B^-(F)$, it follows $n_1, \ldots, n_k$ odd, thus $n_1 = \cdots = n_k = 1$. □

We are now ready to construct the $\mathcal{T}^1$–Morse embedding $H\colon S^2 \times I \to S^3$ with $c(H, \mathcal{T}^1)$ bounded in terms of $t(\mathcal{T})$, thus to finish the proof of Theorems 1 and 2. Let $x_0 \in \mathcal{T}^0$ be the vertex involved in the definition of $B^+(\cdot)$. We construct H inductively as follows.

Choose $\xi_1 \in]0,1[$ and choose $H|[0,\xi_1]$ so that $H_0 \cap \mathcal{T}^2 = \emptyset$, $H_{\xi_1} = \partial U(x_0) \subset \tilde{\Sigma}$, and x_0 is the only critical point of $H|[0,\xi_1]$.

For $i \geq 1$, let $H|[0,\xi_i]$ be already constructed. Our induction hypothesis is that $H_{\xi_i} \in S_0 \circ S^*$ for some component S_0 of $\tilde{\Sigma}$, and moreover for any choice of S_0 we have $H_{\xi_i} \subset B^+(S_0)$. Choose $\xi_{i+1} \in]\xi_i, 1[$.

Assume that S_0 is not of the form $S_0 = \partial U(x)$ for a vertex $x \in \mathcal{T}^0 \backslash \{x_0\}$. Then, let N_i be the component of $S^3 \backslash \tilde{\Sigma}$ with $N_i \subset B^-(S_0)$ and $\partial N_i = S_0 \cup S_1 \cup \cdots \cup S_k$ for $S_1, \ldots, S_k \subset \tilde{\Sigma}$. If S_0 is 1–normal, then let $S \in S_1 \circ \cdots \circ S_k$, S' and J be as in Lemmas 16 and 17. Then, we extend $H|[0, \xi_i]$ to $H|[0, \xi_{i+1}]$ induced by the embedding J, relating S_0 with S', and by the *inverses* of the elementary upper reductions, relating S' with S. If S_0 is not 1–normal, then let $S \in S_1 \circ \cdots \circ S_k$ be as in Lemma 18. We extend $H|[0, \xi_i]$ to $H|[0, \xi_{i+1}]$ along the elementary lower reductions, relating S_0 with S. In either case, $H_{\xi_{i+1}} \in S_1 \circ \cdots \circ S_k \circ S^*$. The critical points of $H|[\xi_i, \xi_{i+1}]$ are contained in N_i, given by elementary reductions. Thus the number of these critical points is $\leq \frac{1}{2} \max\{\|S_0\|, \|S\|\} \leq \frac{1}{2} \|\tilde{\Sigma}\| < 2^{190\, t(\mathcal{T})^2}$, by Construction 2. Since $H_{\xi_{i+1}} \subset B^+(S_m)$ for any $m = 1, \ldots, k$, we can proceed with our induction.

After at most $\#(\tilde{\Sigma})$ steps, we have $H^\times_{\xi_i} = \partial U(\mathcal{T}^0 \backslash \{x_0\})$. Then, choose $H|[\xi_i, 1]$ so that $H_1 \cap \mathcal{T}^2 = \emptyset$ and the set of its critical points is $\mathcal{T}^0 \setminus \{x_0\}$. By Proposition 2 holds $\#(\tilde{\Sigma}) \leq 10\, t(\mathcal{T})$. Thus finally

$$c(H, \mathcal{T}^1) < \#(\mathcal{T}^0) + 10\, t(\mathcal{T}) \cdot 2^{190\, t(\mathcal{T})^2} < 2^{196\, t(\mathcal{T})^2}. \quad \square$$

9 Proof of Propositions 3 and 4

Let M be a closed connected 3–manifold with a triangulation $\mathcal{T}$. We prove Proposition 3, that states that any impermeable surface in M is isotopic mod $\mathcal{T}^1$ to an almost 2–normal surface with exactly one octagon. The proof consists of the following three lemmas.

Lemma 19 *Any impermeable surface in M is almost 2–normal, up to isotopy mod $\mathcal{T}^1$.*

Proof We give here just an outline. A complete proof can be found in [16]. Let $S \subset M$ be an impermeable surface. By definition, it has upper and lower bonds with strings α_1, α_2. By isotopies mod $\mathcal{T}^1$, one obtains from S two surfaces $S_1, S_2 \subset M$, such that S_i has a return $\beta_i \subset \mathcal{T}^2$ with $\partial \beta_i = \partial \alpha_i$, for $i \in \{1, 2\}$. A surface that has both upper and lower returns admits an independent pair of upper and lower compressing discs, thus is not impermeable. By consequence, under the isotopy mod $\mathcal{T}^1$ that relates S_1 and S_2 occurs a surface S' that has no returns at all, thus is almost k–normal for some natural number k.

If there is a boundary component ζ of a component of $S' \setminus \mathcal{T}^2$ and an edge e of $\mathcal{T}$ with $\#(\zeta \cap e) > 2$, then there is an independent pair of upper and lower compressing discs. Thus $k = 2$. $\square$

Lemma 20 *Let $S \subset M$ be an almost 2–normal impermeable surface. Then S contains at most one octagon.*

Proof Two octagons in different tetrahedra of $\mathcal{T}$ give rise to a pair of independent upper and lower compressing discs for S. Two octagons in one tetrahedron of $\mathcal{T}$ give rise to a pair of nested upper and lower compressing discs for S. Both is a contradiction to the impermeability of S. □

Lemma 21 *Let $S \subset M$ be an almost 2–normal impermeable surface. Then S contains at least one octagon.*

Proof By hypothesis, S has both upper and lower bonds. Assume that S does not contain octagons, i.e., it is almost 1–normal. We will obtain a contradiction to the impermeability of S by showing that S has a pair of independent or nested compressing discs.

According to Lemma 10, we can assume that S has a compressing disc D_1 with string α_1 that is contained in a single closed tetrahedron t_1. Choose D_1 innermost, i.e., $\alpha_1 \cap S = \partial\alpha_1$. Without assumption, let D_1 be *upper*. Since S has no octagon by assumption, α_1 connects two different components ζ_1, η_1 of $S \cap \partial t_1$. Let D be a lower bond of S. Choose S, D_1 and D so that, in addition, $\#(D \cap \mathcal{T}^2)$ is minimal.

Let C be the closure of an innermost component of $D \setminus \mathcal{T}^2$, which is a disc. There is a closed tetrahedron t_2 of $\mathcal{T}$ and a closed 2–simplex $\sigma_2 \subset \partial t_2$ of $\mathcal{T}$ such that $\partial C \cap \partial t_2$ is a single component $\gamma \subset \sigma_2$. We have to consider three cases.

(1) Let γ be a circle, thus $\partial C = \gamma$. There is a disc $D' \subset \sigma_2$ with $\partial D' = \gamma$ and a ball $B \subset t_2$ with $\partial B = C \cup D'$. We move $S \cup D$ away from B by an isotopy mod $\mathcal{T}^1$ with support in $U(B)$, and obtain a surface S^* with a lower bond D^*. As D is a bond, $S \cap D'$ consists of circles. Therefore the normal arcs of $S \cap \mathcal{T}^2$ are not changed under the isotopy, and the isotopy does not introduce returns, thus S^* is almost 1–normal. Since $\xi_1 \cap D' = \eta_1 \cap D' = \emptyset$ and $C \cap S = \emptyset$, it follows $B \cap \partial D_1 = \emptyset$. Thus D_1 is an upper compressing disc for S^*, and $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$ in contradiction to our choice.

(2) Let γ be an arc with endpoints in a single component c of $S \cap \sigma$. By an isotopy mod $\mathcal{T}^1$ with support in $U(C)$ that moves C into $U(C) \setminus t_2$, we obtain from S and D a surface S^* with a lower bond D^*. Since D is a bond, the isotopy does not introduce returns, thus S^* is almost 1–normal.

One component of $S^* \cap t_1$ is isotopic mod $\mathcal{T}^2$ to the component of $S \cap t_1$ that contains $\partial D_1 \cap S$. Thus up to isotopy mod $\mathcal{T}^2$, D_1 is an upper compressing disc for S^*, and $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$ in contradiction to our choice.

(3) Let γ be an arc with endpoints in two different components c_1, c_2 of $S \cap \sigma$. Assume that, say, c_1 is a circle. By an isotopy mod $\mathcal{T}^1$ with support in $U(C)$ that moves C into $U(C) \setminus t_2$, we obtain from S and D a surface S^* with a lower bond D^*. Since D is a bond, the isotopy does not introduce returns, thus S^* is almost 1–normal. There is a disc $D' \subset \sigma$ with $\partial D' = c_1$. Let K be the component of $S \cap t_1$ that contains $\partial D_1 \cap S$. One component of $S^* \cap t_1$ is isotopic mod $\mathcal{T}^2$ either to K or, if $\partial D' \cap \partial K \neq \emptyset$, to $K \cup D'$. In either case, D_1 is an upper compressing disc for S^*, up to isotopy mod $\mathcal{T}^2$. But $\#(D^* \cap \mathcal{T}^2) < \#(D \cap \mathcal{T}^2)$ in contradiction to our choice. Thus, c_1 and c_2 are normal arcs.

Since S is almost 1–normal, c_1, c_2 are contained in different components ζ_2, η_2 of $S \cap \partial t_2$. Since D is a lower bond, $\partial(C \cap D_1) \subset \partial C \cap S$. There is a sub-arc α_2 of an edge of t_2 and a disc $D' \subset \sigma$ with $\partial D' \subset \alpha_2 \cup \gamma \cup \zeta_2 \cup \eta_2$ and $\alpha_2 \cap S = \partial \alpha_2$. The disc $D_2 = C \cup D' \subset t_2$ is a lower compressing disc for S with string α_2, and $\partial(D_1 \cap D_2) \subset \partial D_2 \cap S$. At least one component of $\partial t_1 \setminus (\zeta_1 \cup \eta_1)$ is a disc that is disjoint to D_2. Let D_m be the closure of a copy of such a disc in the interior of t_1, with $\partial D_m \subset S$. By construction, $D_1 \cap D_m = \partial D_1 \cap \partial D_m$ is a single point and $D_2 \cap D_m = \emptyset$. Thus by Lemma 9, S has a pair of independent or nested upper and lower compressing discs and is therefore not impermeable. □

Proof of Proposition 4 Let $S \subset M$ be a connected 2–normal surface that splits M, and assume that exactly one component O of $S \setminus \mathcal{T}^2$ is an octagon. The octagon gives rise to upper and lower bonds of S.

Let D_1, D_2 be any upper and lower compressing discs for S. We have to show that D_1 and D_2 are neither impermeable nor nested. It suffices to show that $\partial D_1 \cap \partial D_2 \not\subset \mathcal{T}^1$. To obtain a contradiction, assume that $\partial D_1 \cap \partial D_2 \subset \mathcal{T}^1$. Choose D_1, D_2 so that $\#(\partial D_1 \setminus \mathcal{T}^2) + \#(\partial D_2 \setminus \mathcal{T}^2)$ is minimal.

Let t be a tetrahedron of $\mathcal{T}$ with a closed 2–simplex $\sigma \subset \partial t$, and let β be a component of $\partial D_1 \cap t$ (resp. $\partial D_2 \cap t$) such that $\partial \beta$ is contained in a single component of $S \cap \sigma$. Since S is 2–normal, there is a disc $D \subset S \cap t$ and an arc $\gamma \subset S \cap \sigma$ with $\partial D = \beta \cup \gamma$. By choosing β innermost in D, we can assume that $D \cap (\partial D_1 \cup \partial D_2) = \beta$. An isotopy of $(D_1, \partial D_1)$ (resp. $(D_2, \partial D_2)$) in (M, S) with support in $U(D)$ that moves β to $U(D) \setminus t$ reduces $\#(\partial D_1 \setminus \mathcal{T}^2)$ (resp.

$\#(\partial D_2 \setminus \mathcal{T}^2))$, leaving $\partial D_1 \cap \partial D_2$ unchanged. This is a contradiction to the minimality of D_1, D_2.

For $i = 1, 2$, there are arcs $\beta_i \subset \partial D_i \setminus \mathcal{T}^1$ and $\gamma_i \subset D_i \cap \mathcal{T}^2$ such that $\beta_i \cup \gamma_i$ bounds a component of $D_i \setminus \mathcal{T}^2$, by an innermost arc argument. Let t_i be the tetrahedron of $\mathcal{T}$ that contains β_i, and let $\sigma_i \subset \partial t_i$ be the close 2–simplex that contains γ_i. We have seen above that $\partial \beta_i$ is not contained in a single component of $S \cap \sigma_i$. Since S is 2–normal, i.e., has no tubes, it follows that $\beta_i \subset O$. Since collars of β_1 in D_1 and of β_2 in D_2 are in different components of $t \setminus O$, it follows $\beta_1 \cap \beta_2 \neq \emptyset$. Thus $\partial D_1 \cap \partial D_2 \not\subset \mathcal{T}^1$, which yields Proposition 4. □

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