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Notions of denseness

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Abstract

The notion of a completely saturated packing [4] is a sharper version of maximum density, and the analogous notion of a completely reduced covering is a sharper version of minimum density. We define two related notions: uniformly recurrent and weakly recurrent dense packings, and diffusively dominant packings. Every compact domain in Euclidean space has a uniformly recurrent dense packing. If the domain self-nests, such a packing is limit-equivalent to a completely saturated one. Diffusive dominance is yet sharper than complete saturation and leads to a better understanding of n–saturation.

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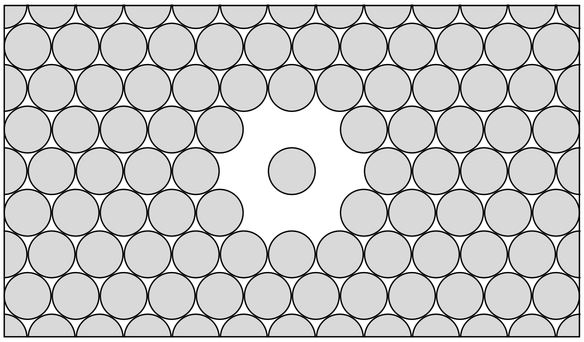
What is the best way to pack congruent copies of some compact domain in a geometric space? Is there a best way? The first notion of optimality is *maximum density*. While this notion is useful, it is also incomplete. For example, the packing $\mathcal{A}$ in Figure 1 has the maximum density, but there is clearly something missing. Intuitively, the packing $\mathcal{B}$ is uniquely optimal.

$\mathcal{A}$



$\mathcal{B}$

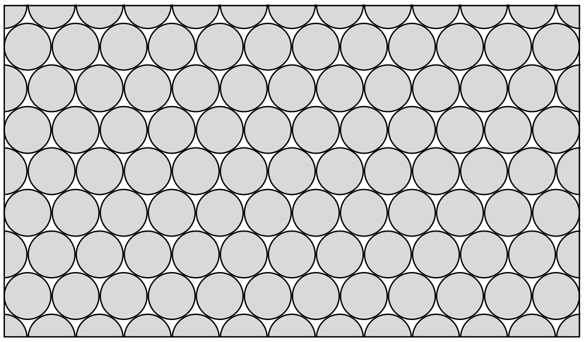


Figure 1: Two dense circle packings

The problem of defining optimality for packings and coverings has been considered previously by Fejes Tóth , Conway and Sloane and probably by others. Fejes Tóth [5] defines a packing (or covering) to be *solid* if no finite rearrangement of its elements yields a non-congruent packing (or covering). This is a very strict condition which is only sometimes achievable. Conway and Sloane [3] propose a definition of optimality for sphere packings only which, as we will discuss in Section 4, is too strict to be useful. In this article we will introduce two other notions: uniformly and weakly recurrent dense packings, and diffusively dominant packings. We will show that these notions have some favorable properties, and in particular we will compare them to the notion of complete saturation [4].

A packing $\mathcal{P}$ of some body (ie, a compact domain) K is *n–saturated* if it is not possible to replace any $n-1$ elements of $\mathcal{P}$ by n and still have a packing. It is *completely saturated* if it is saturated for all n. We can express saturation by a weak partial ordering on packings: $\mathcal{P}_1 \succeq_{S,n} \mathcal{P}_2$ if the packing $\mathcal{P}_1$ is obtained from $\mathcal{P}_2$ by repeatedly replacing $n-1$ elements of $\mathcal{P}_2$ by n or more (either a larger finite number or an infinite number). Dropping the restriction on n, we say that $\mathcal{P}_1 \succeq_S \mathcal{P}_2$. Thus $\mathcal{P}$ is completely saturated if and only if it is maximal with respect to the partial ordering $\succeq_S$. For example, the packing $\mathcal{A}$ in Figure 1 is not 1–saturated. On the other hand, any packing which is both periodic and has maximum density, such as the packing $\mathcal{B}$ in Figure 1, is completely saturated.

Dually, a covering $\mathcal{P}$ is *n–reduced* if it is not possible to replace n elements by $n-1$ and still have a covering. It is *completely reduced* if it is n–reduced for all n. The main open problem about complete saturation and complete reduction is the following:

Question 1 *Does every body K in Euclidean space $\mathbb{R}^d$ admit a completely saturated packing and a completely reduced covering? What about bodies in hyperbolic space $\mathbb{H}^d$ or other homogeneous spaces?*

In this article we will present a stronger version of a previous result [4] that strictly self-nesting domains have completely saturated packings and completely reduced coverings. By Theorem 3, every dense packing of a self-nesting K has arbitrarily large regions that are arbitrarily close to some completely saturated packing. The technical argument is actually the same; only the conclusion is different.

A packing $\mathcal{A}$ is a *limit* of a packing $\mathcal{B}$, or $\mathcal{A} \succeq_L \mathcal{B}$, if a sequence of translates of $\mathcal{B}$ converges to $\mathcal{A}$ in the Hausdorff topology. It is *uniformly recurrent* if it is maximal with respect to this weak partial ordering in the space of packings. It is *weakly recurrent* if it is a limit of (generally different) uniformly recurrent packings. Given a finite measure μ in $\mathbb{R}^d$, a packing $\mathcal{A}$ *μ–dominates* a packing $\mathcal{B}$, or $\mathcal{A} \succeq_\mu \mathcal{B}$, if the inequality

$$\mu * \chi_{\mathcal{A}} \geq \mu * \chi_{\mathcal{B}}$$

holds everywhere, where $\chi_{\mathcal{P}}$ is the characteristic measure of $\mathcal{P}$ and $\mu * \nu$ is the convolution of the measures μ and ν. We say that $\mathcal{A}$ *diffusively dominates* $\mathcal{B}$, or $\mathcal{A} \succeq_D \mathcal{B}$, if $\mathcal{A} \succeq_\mu \mathcal{B}$ for some μ. Diffusive domination is also a weak partial ordering, since

$$\mathcal{A} \succeq_\mu \mathcal{B} \succeq_\nu \mathcal{C} \implies \mathcal{A} \succeq_{\mu * \nu} \mathcal{C}.$$

The packing $\mathcal{A}$ is *μ–dominant* or *diffusively dominant* if it is maximal with respect to the corresponding partial orderings. Diffusive dominance implies μ–dominance for all μ as follows: If $\mathcal{A}$ were the former but not the latter, then

$$\mu * \chi_{\mathcal{A}} \leq \mu * \chi_{\mathcal{B}}$$

for some μ without equality holding everywhere, which would preclude the possibility that $\mathcal{A} \succeq_{\mu * \nu} \mathcal{B}$ for any ν. Yet there would be a ν such that $\mathcal{A} \succeq_{\nu} \mathcal{B}$.

Unfortunately there exists a sequence of completely saturated packings of a square that converges to a packing which is not even simply saturated [4]. In the present context we can conclude that the saturation partial orderings $\succeq_{S,n}$ and $\succeq_S$ are irreparably incompatible with the Hausdorff topology on the space of packings. By contrast, limits and uniform recurrence *are* compatible, in the sense that the set of pairs of packings $\mathcal{A} \succeq_L \mathcal{B}$ is closed. The μ–dominant partial orderings have an intermediate property: For any fixed μ, the set of pairs $\mathcal{A} \succeq_{\mu} \mathcal{B}$ is closed. Nonetheless the set of pairs $\mathcal{A} \succeq_D \mathcal{C}$ is not closed.

Our arguments below apply to the dual case of thin coverings. Indeed, they apply more generally to k–packings or k–coverings of finite collections of bodies (or even compact families of bodies), which could have weighted density or other variations. They also apply to restricted congruences, such as packings by translates, provided that the set of allowed isometries includes all translations. Theorem 2 also applies to packings and coverings in arbitrary homogeneous Riemannian geometries, such as hyperbolic geometry and complex hyperbolic geometry, while Theorem 9 applies to packings and coverings in sub-exponential geometries such as nil-geometry and solve-geometry [12]. We will assume these generalizations implicitly.

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1 Basic notions

A *domain* is a set which is the closure of its interior and a *body* is a compact domain. Let $\mathcal{P}$ be a collection of congruent copies of a body K in Euclidean space $\mathbb{R}^d$. The *density* $\delta(\mathcal{P})$ of $\mathcal{P}$ is defined as

$$\delta(\mathcal{P}) = \lim_{r\to\infty} \frac{\sum_{D\in\mathcal{P}} \mathrm{Vol}\ \ B(p,r)\cap D}{\mathrm{Vol}\ \ B(p,r)}. \tag{1}$$

In this expression p is an arbitrary (fixed) point in $\mathbb{R}^d$ and $B(p,r)$ is the round ball of radius r centered at p. The collection $\mathcal{P}$ has *uniform density* if the convergence in equation (1) is uniform in p.

The collection $\mathcal{P}$ is a *packing* if its elements have disjoint interiors, and that it is a *covering* if the union of its elements is all of $\mathbb{R}^d$. A packing is *dense* if it has maximum density, and a covering is *thin* if it has minimum density. *Uniformly dense* means that density is both maximum and uniform.

These definitions have the following desirable properties [7]:

(1) If the limit exists for some p, it exists for all p.

(2) The supremum $\delta(K)$ (the packing density of K) of the density $\delta(\mathcal{P})$ is achieved by a uniformly dense packing.

(3) $\delta(K)$ is greater than or equal to the upper density of any packing, in which the limit in equation (1) is replaced by a lim sup.

(4) If $\mathcal{P}$ has uniform density, then

$$\delta(\mathcal{P}) = \lim_{r\to\infty} \frac{\sum_{D\in\mathcal{P}} \mathrm{Vol}\ (rR+p)\cap D}{\mathrm{Vol}\ \ rR}$$

uniformly in $p\in\mathbb{R}^d$, where R is an arbitrary body and $rR+p$ denotes R dilated by a factor of r and translated by p.

We define a modified Hausdorff distance between collections of domains and we will use the induced Hausdorff topology. By definition, $d(\mathcal{P},\mathcal{Q}) \le \epsilon$ means that there exists a binary relation $K\sim L$ for $K\in\mathcal{P}$ and $L\in\mathcal{Q}$ such that $K\sim L$ implies that $d(K,L)\le\epsilon$. For each $K\in\mathcal{P}$ in the ball $B(0,1/\epsilon)$, there must exist a unique $L\in\mathcal{Q}$ such that $K\sim L$ and vice versa. (The relation need not satisfy any other axioms.)

A *weak partial ordering* $\succeq$ on a set S is defined as any binary relation such that $x\succeq x$ and $x\succeq y\succeq z$ implies $x\succeq z$. Any weak partial ordering induces an equivalence relation: x and y are equivalent when $x\succeq y\succeq x$.

2 Recurrence

All of the results of this section have implications for weakly recurrent packings, but it is more natural to state them in terms of uniformly recurrent packings.

Theorem 2 *Every packing $\mathcal{P}$ of a body K admits a uniformly recurrent packing $\mathcal{Q}$ as a limit. If $\mathcal{P}$ is uniformly dense, so is $\mathcal{Q}$.*

Proof The translation group $\mathbb{R}^d$ acts on the space of packings in $\mathbb{R}^d$, and the action is continuous relative to our Hausdorff topology. This group action may be interpreted as a dynamical system with a d–dimensional time variable. The notions of limits and uniform recurrence are then familiar from the dynamical systems point of view.

To say that $\mathcal{A}$ is a limit of $\mathcal{B}$ is to say that $\mathcal{A}$ lies in the closure of the orbit of $\mathcal{B}$. A limit equivalence class is an invariant set (not necessarily closed) in which each orbit is dense. If such an invariant set is closed, it is called a *minimal set.* A uniformly recurrent packing is a point in a minimal set in this dynamical system. Since the space of packings is compact, minimal sets exists in the closure of every orbit.

The set of uniformly dense packings is unfortunately not compact, but it is exhausted by invariant compacta in the following way: If $\mathcal{P}$ is uniformly dense, the proportion of a large disk covered by $\mathcal{P}$ must approach the packing density $\delta(K)$ at a certain rate. Ie, there is some function $\epsilon(r)$ which converges to 0 as r goes to infinity and such that the density in any ball $B(p,r)$ is within $\epsilon(r)$ of $\delta(K)$. The set of packings that are uniform using any fixed $\epsilon(r)$ is compact, and it is also invariant under the action. $\square$

Note that the density of a uniformly recurrent packing $\mathcal{P}$ necessarily exists uniformly. In particular if $\mathcal{P}$ is dense, it is uniformly dense.

Theorem 2 can also be argued from the point of view of general topology: If X is a compact Hausdorff space, and if $\succeq$ is a weak partial ordering which is closed as a subset of $X \times X$, then it has a maximal element. (Proof: By compactness every ascending chain C has a convergent cofinal subchain. By closedness, the limit is an upper bound for C. Therefore Zorn's Lemma produces a maximal element of $\succeq$ on X.)

Given a body K in Euclidean space and an allowed isometry group of G for collections of copies K (eg, the group of translations or the group of all isometries), the domain K (strictly) *self-nests* if for every $\epsilon > 0$ there exists $g \in G$ such that gK lies in the interior of $(1+\epsilon)K$. For collections of translates, the condition holds if K is strictly star-shaped (star-shaped with a continuous radial function). For general arrangements of congruent copies, it is a restatement of the strict nested similarity property of Reference [4]. For example, the spiral-like body in Figure 2 (the 1–neighborhood of a part of a logarithmic spiral) has this property.

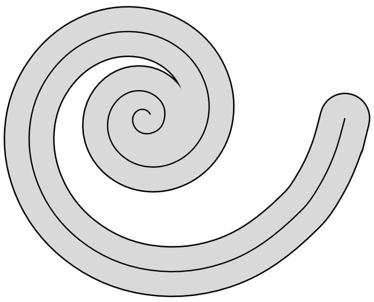

Figure 2: A self-nesting spiral-like body

Theorem 3 *If $K \subset \mathbb{R}^d$ is a self-nesting body, then every uniformly recurrent, dense packing $\mathcal{P}$ is limit-equivalent to a completely saturated one.*

We follow the argument in Reference [4].

Lemma 4 *Let $\mathcal{P}$ be a dense packing of a self-nesting body $K \subset \mathbb{R}^d$. For every radius r and every $\epsilon > 0$, there is an $\alpha > 0$ with the following property: Consider points p such that the restriction of the packing $\mathcal{P}$ to the ball $B(p, r)$ is within α in modified Hausdorff distance of a packing in this disk that is not completely saturated. The set S of such points has density at most ϵ.*

Proof Since K is self-nesting, we can *loosen* any packing $\mathcal{P}$ by homothetic expansion: We expand the packing and each copy of K by a factor of $1+\gamma$, and then we replace each copy of $(1+\gamma)K$ by a copy of K contained in its interior. For every γ and K, there is an α such that in the loosened packing $\mathcal{P}_\gamma$, no two copies of K are within 2α of each other.

Intuitively, if we loosen $\mathcal{P}$, we decrease the density by expansion, but we may then increase it by re-saturation. We choose the constant γ so that the latter would outweigh the former if $\mathcal{P}$ failed to satisfy the conclusion of the theorem. More precisely, we choose γ so that

$$d\gamma\delta(K) < \frac{\epsilon(\mathrm{Vol}\ K)}{\mathrm{Vol}\ B(p, 2r)}.$$

Let α be the constant in the previous paragraph. Consider a maximal packing of balls $\{B(p_i, r)\}$ such that $\mathcal{P}$ is within α of an unsaturated packing in each ball. By maximality, the corresponding collection $\{B(p_i, 2r)\}$ covers S. In the loosened packing $\mathcal{P}_\gamma$, we can cram in at least one extra copy of K in each expanded ball $(1+\gamma)B(p_i,r)$. If S had upper density greater than ϵ, the new packing $\mathcal{P}'_\gamma$ would have greater density than that of $\mathcal{P}$, contradicting the assumption that $\mathcal{P}$ is dense. □

Proof of Theorem 3 Let $\epsilon_k = 1/2^{k+1}$ and $r_k = 2^k$, and choose the corresponding α_k according to Lemma 4. For each n, there is a non-zero measure of points $p \in \mathbb{R}^d$ such that $\mathcal{P}$ is simultaneously at least α_k away from unsaturated in $B(p,r_k)$ for all $1 \le k \le n$. Let p_n be one such point. The sequence of translated packings $\{\mathcal{P} - p_n\}$ must have a convergent subsequence. The limit $\mathcal{Q}$ of this subsequence has the same property for all k; in particular it is completely saturated.

By construction $\mathcal{Q}$ is a limit of $\mathcal{P}$. Since $\mathcal{P}$ is uniformly recurrent, the two packings are limit-equivalent. □

Question 5 *Is it possible that in an equivalence class of uniformly recurrent packings of a self-nesting domain K, some representatives are completely saturated and others are not?*

Theorem 6 *The only uniformly recurrent dense packing of a circle (a round disk) is the hexagonal packing.*

Proof The result follows from the fact that the unique smallest possible Voronoi region in a circle packing is a regular hexagon [6]. For every $\epsilon > 0$, the union of the Voronoi regions that are more than ϵ away from the regular circumscribed hexagon must have density 0 in the plane in a dense packing. Consequently there are arbitrarily large disks in which every Voronoi region is within ϵ of the optimal shape. Taking the limit $\epsilon \to 0$, the conclusion is that there are arbitrarily large regions that converge to the hexagonal packing. Thus the hexagonal packing is a limit of every dense packing.

Conversely, any periodic packing is the only limit of itself. In particular, the hexagonal circle packing has this property. □

Note that even if there are distinct uniformly dense packings they may all form a single minimal set, or limit equivalence class. In this case the equivalence class as a whole may be considered the unique uniformly recurrent solution to the packing problem.

Example 7 The Penrose tilings of unit rhombuses form a limit equivalence class. Non-Penrose tilings of the rhombuses are usually eliminated by certain matching rules, but one may equally well use notches and teeth for this purpose. The notches and teeth may be chosen so that the tiles are self-nesting. Indeed there is a set of three convex polygons due to Robert Ammann that only tile in Penrose fashion [8]. Thus Theorem 3 applies to families of bodies whose densest packings fall into uncountable limit equivalence classes.

Example 8 Bi-infinite sequences of symbols from a finite alphabet are equivalent to "packings" of unit intervals colored by the same alphabet. Every periodic sequence is uniformly recurrent, but there are also others, such as the Thue–Morse sequence [10] in the two letter alphabet $\{0,1\} = \mathbb{Z}/2$. The nth term of the sequence, for $n \geq 0$, is the sum in $\mathbb{Z}/2$ of the binary digits of n. The sequence is only infinite in one direction, but any bi-infinite limit of translates of the Thue–Morse sequence is also uniformly recurrent and aperiodic. By contrast, every sequence over a finite alphabet is weakly recurrent, since every finite sequence can be extended to a periodic one. Binary sequences are also a model for Barlow packings of spheres in $\mathbb{R}^3$ [1]. It seems likely that a sphere packing is weakly recurrent among dense packings if and only if it is Barlow [9, 11], but not all Barlow packings are uniformly recurrent.

3 Diffusion

We first state some of the notions of diffusion and measure more precisely. We assume that all measures are defined on Borel sets in $\mathbb{R}^d$. If a measure μ has a density function $f(x)$,

$$\mu(A) = \int_A f(x)dx,$$

we will write $\mu(dx)$ for the density $f(x)dx$. The characteristic measure χ_S of a set S is defined by

$$\chi_S(A) = \text{Vol } A \cap S.$$

If $\mathcal{P}$ is a collection of sets, its characteristic measure $\chi_\mathcal{P}$ is defined as the sum:

$$\chi_\mathcal{P} = \sum_{A \in \mathcal{P}} \chi_A.$$

If μ and ν are measures in $\mathbb{R}^d$ and the total measure of μ is finite, then their convolution $\mu * \nu$ is defined by linear extension of addition of points:

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(p,q) | p + q \in A\}),$$

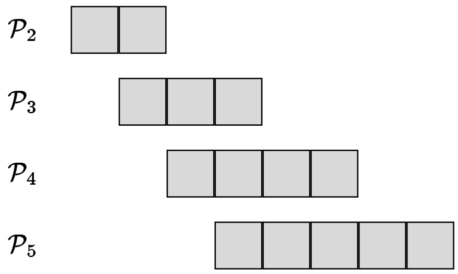

Figure 3: Anomalously replacing one square by two

where $\mu \times \nu$ is the product measure on $\mathbb{R}^{2d}$.

Even though the saturation partial orderings $\succeq_{S,n}$ are not closed in the Hausdorff topology, we can still ask what happens if we have an increasing sequence of packings. The idea is that the packings in the sequence are improving, so perhaps they must converge to a packing which is better still. Unfortunately this is not the case. For example, we can let $\mathcal{P}_k$ be a string of k unit squares whose centers start at $(k, 0)$ and ending at $(2k-1, 0)$ (Figure 3). Although $\mathcal{P}_{k+1}$ is obtained from $\mathcal{P}_k$ by replacing one square by two, the limit is the empty packing.

This disease may be cured by considering another partial ordering $\succeq_{C,n}$ on packings of a body K. We say that $\mathcal{A} \succeq_{C,n} \mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by replacing $k < n$ copies of the body K by $k+1$ in such a way that the union of the $2k+1$ copies of K is a connected set. Clearly $\succeq_{C,n}$ and $\succeq_{S,n}$ have the same maximal elements.

Theorem 9 *If a sequence of packings $\{\mathcal{P}_i\}$ of a body K in Euclidean space $\mathbb{R}^d$ increases under the connected n–saturation partial ordering,*

$$\mathcal{P}_1 \preceq_{C,n} \mathcal{P}_2 \preceq_{C,n} \mathcal{P}_3 \preceq_{C,n} \cdots,$$

it is eventually constant in every bounded region in $\mathbb{R}^d$.

Theorem 9 follows from a favorable comparison between connected saturation and μ–diffuse domination.

Lemma 10 *For every n and K, there exists a measure μ such that if $\mathcal{A} \succeq_{C,n} \mathcal{B}$ are packings of K, then $(\chi_{\mathcal{A}} * \mu) > (\chi_{\mathcal{B}} * \mu)$ everywhere.*

Proof Assume that K has diameter 1. Then the maximum diameter of a connected union of $2n-1$ copies of K is $2n-1$. Let μ be the measure with density function

$$\mu(dx) = \left(\frac{n}{n+1}\right)^{|x|/(2n-1)} dx.$$

If B is the union of $k < n$ interior-disjoint copies of K, A is the union of $k+1$ such copies, and $A \cup B$ is connected, then we claim that

$$\mu(A) \geq \mu(B).$$

The reason is that if we express $\mu(A)$ and $\mu(B)$ as integrals, the ratio of volumes of A to B is

$$\frac{k+1}{k} > \frac{n}{n+1},$$

but the ratio of the integrands is at least $n/(n+1)$.

Suppose that $\mathcal{A}$ is formed from $\mathcal{B}$ by replacing the components of B with the components of A. Then

$$(\chi_{\mathcal{A}} * \mu)(p) - (\chi_{\mathcal{B}} * \mu)(p) = \mu(A-p) - \mu(B-p) > 0,$$

as desired. □

Note that Lemma 10 implies that a μ–dominant packing is necessarily n–saturated. Thus a diffusively dominant packing is completely saturated. By the argument of the lemma, if a packing $\mathcal{A}$ maximizes $\mu * \chi_{\mathcal{A}}$ at a single point, then $\mathcal{A}$ is already n–saturated. This produces n–saturated packings which may be far from periodic. If μ is determined by Lemma 10, then μ–dominance is slightly stronger than n–saturation, but one can find such a μ–dominant packing by maximizing $\nu * \mu * \chi_c A$ at any single point, where ν is any measure with full support on $\mathbb{R}^d$. (For example, $\nu = \mu$.) Alternatively, one can find μ–dominant packings by using the fact that the relation $\succeq_\mu$ is closed in the space of pairs of packings and appealing to Zorn's Lemma as in Section 2.

Proof of Theorem 9 The argument of Lemma 10 actually shows that its conclusion is "true by a margin", in the following sense: Let R be a bounded domain and consider the measure μ from the lemma. Then there is a constant $\epsilon > 0$ such that if $\mathcal{A}$ is obtained from $\mathcal{B}$ by a connected replacement of $k < n$ copies of K by $k+1$, and if this replacement intersects R, then the diffused measure in $B(p,r)$ increases by at least ϵ:

$$(\chi_{\mathcal{A}} * \mu)(R) \geq (\chi_{\mathcal{B}} * \mu)(R) + \epsilon$$

If the connected replacement does not intersect R, the diffused measure in R at least does not decrease. At the same time, the total measure in R is bounded above by $|\mu|(\mathrm{Vol}\ R)$. Thus only finitely many of the replacements in the sequence

$$\mathcal{P}_1 \preceq_{C,n} \mathcal{P}_2 \preceq_{C,n} \mathcal{P}_3 \preceq_{C,n} \cdots$$

may meet R. □

Corollary 11 *For every integer n, every Euclidean body K has an n-saturated packing which is also uniformly dense.*

Proof Apply the process of Theorem 9 to a uniformly dense packing. The usual notion of uniform density is that the limit

$$\lim_{r\to\infty} \frac{\chi_{\mathcal{P}}(B(p,r))}{\mathrm{Vol}\ B(p,r)}$$

converges uniformly in p. If μ is a finite measure, then it is equivalent to demand that

$$|\mu|\delta(\mathcal{P}) = \lim_{r\to\infty} \frac{(\mu * \chi_{\mathcal{P}})(B(p,r))}{\mathrm{Vol}\ B(p,r)} \tag{2}$$

converges uniformly in p. Since the diffused measure $\mu * \chi_{\mathcal{P}}$ strictly increases under connected replacements of $k < n$ copies, and since the density is already maximized, the limit in equation (2) does not change and continues to converge uniformly. □

Example 12 Consider a square rim with an extra mass in one corner, as in Figure 4. The figure shows a pair of packings $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}$ is uniformly recurrent, $\mathcal{A}$ is not, and $\mathcal{B}$ is a limit of $\mathcal{A}$. At the same time, we conjecture that $\mathcal{A}$ is diffusively dominant, while $\mathcal{B}$ is certainly not, since $\mathcal{A}$ strictly diffusively dominates $\mathcal{B}$.

The example of a square with a mass in one corner illustrates an artificial aspect of diffusive dominance: It depends on the mass distribution of the body K, which is not naturally determined by the geometry of K. This shortcoming is absent for packings by translates. Let C be the set of centers of some packing of translates of some body K. Let χ_C denote the measure with a unit atom at each point in C. An arbitrary mass distribution for K may be represented by a finite measure μ, and the corresponding mass distribution of the packing is $\mu * \chi_C$. If ν is another such measure, then $\mu * \chi_C$ is diffusively dominant if and only if $\nu * \chi_C$ is, since they become equal if we convolve the former with ν and the latter with μ.

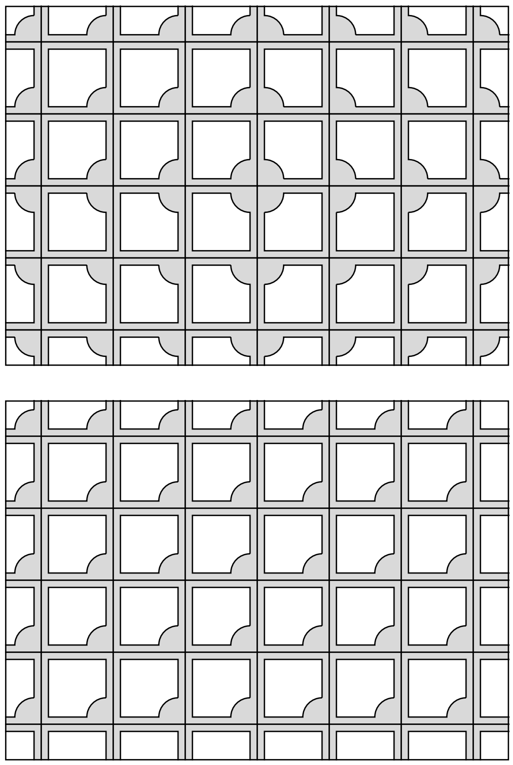

Figure 4: Packing a square with a mass in one corner

Conjecture 13 *The hexagonal packing is the unique diffusively dominant circle packing.*

More generally, we conjecture that among packings by translates of a centrally symmetric convex body in $\mathbb{R}^2$, the densest lattice packings are diffusively dominant, and no other packings are.

4 Others' notions

In this section we relate our notions to Fejes Tóth's solid packings and coverings [5] and Conway and Sloane's definition of tight sphere packings [3].

The shape packed in Figure 4 does not admit a solid packing, while the tiling of the three shapes in Figure 5 is solid but not even weakly recurrent. Our examples in two dimensions suggest that solidity diverges from uniform or weak

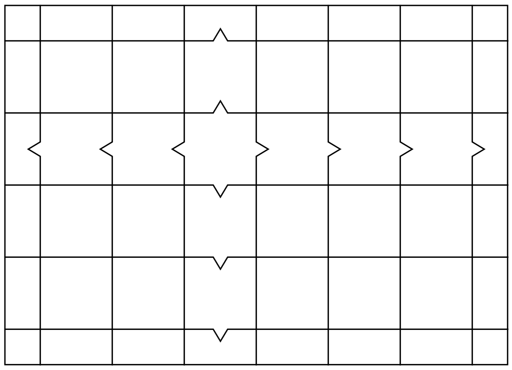

Figure 5: A solid tiling which is not weakly recurrent

recurrence for sphere packings in high dimensions. The relationship with diffusive dominance is less clear. Certain sphere packings, in particular the hexagonal circle packing, the E_8 lattice, and the Leech lattice [2], might have special properties that imply that they are simultaneously solid, uniquely uniformly recurrent, and diffusively dominant.

Conway and Sloane say the following:

> Suppose we can dissect the space of a packing into finitely many polyhedral pieces in such a way that each sphere center lies in the interior of some piece, and there are also some *empty* pieces containing no centers. Then if we can rearrange the *nonempty* pieces into another dissection in which the centers are at least as far apart as they were originally, we call the packing loose.
> Provisionally, we may call a packing tight if it is not loose. Packings that are tight in this sense certainly have the highest possible density. However, we are not sure that this particular definition is the right one, and perhaps some other meaning for "tight" should be used in the Postulates below.

We argue that this definition is indeed not the right one. We provisionally call the Conway–Sloane operation a *tightening.* We claim that any periodic sphere packing in two or more dimensions can be tightened to introduce a hole. Figure 6 shows the construction for the hexagonal circle packing in $\mathbb{R}^2$. We slide B and D past A, C, E, and F to create space for two circle centers. Then we slide C and D past A and B, discarding F and using E to plug the gap between A and B. In the end one circle center has disappeared.

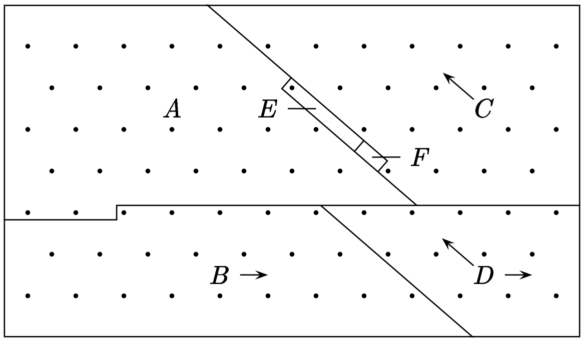

Figure 6: Tightening the hexagonal circle packing

We can generalize the argument of Theorem 6 to produce a sensible criterion which is similar to the Conway–Sloane proposal. Let K be a domain with unit volume and with packing density δ. Suppose that for each packing $\mathcal{P}$ of K, we can form a packing $\mathcal{Q}$ of domains (in general non-congruent) with volume $1/\delta$ whose elements are in bijection with those of $\mathcal{P}$. Suppose further that $\mathcal{Q}$ satisfies the following conditions.

(1) The assignment $\mathcal{P} \mapsto \mathcal{Q}$ is translation-invariant on the space of packings.

(2) The assignment $\mathcal{P} \mapsto \mathcal{Q}$ is continuous in the Hausdorff topology on the space of packings.

(3) Each element of $\mathcal{Q}$ lies a bounded distance from the associated element of $\mathcal{P}$, with the bound independent of $\mathcal{P}$.

Then:

Theorem 14 *If $\mathcal{P}$ and $\mathcal{Q}$ are defined as above, and if $\mathcal{P}$ is weakly recurrent and dense, then $\mathcal{Q}$ is a tiling.*

The complicated hypothesis of Theorem 14 is designed to fit the argument of Theorem 6. Theorem 6 is a special case because we can form the packing $\mathcal{Q}$ from the Voronoi tiling of a circle packing. For example, we can truncate each Voronoi region so that it has the same area as the circumscribed hexagon.

In conclusion, Theorem 14 provides a sense, as Conway and Sloane desired, in which weakly recurrent dense packings have no gaps [11]. So we may adopt their postulate as a conjecture:

Conjecture 15 *Every weakly recurrent, dense sphere packing in dimension $2 \leq n \leq 8$ fibers over one in dimension 2^k, where 2^k is the largest power of 2 strictly less than n.*

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