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An elementary approach to the mapping class group of a surface

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Abstract

We consider an oriented surface S and a cellular complex X of curves on S, defined by Hatcher and Thurston in 1980. We prove by elementary means, without Cerf theory, that the complex X is connected and simply connected. From this we derive an explicit simple presentation of the mapping class group of S, following the ideas of Hatcher–Thurston and Harer.

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1 Introduction

Let S be a compact oriented surface of genus $g \geq 0$ with $n \geq 0$ boundary components and k distinguished points. The mapping class group $\mathcal{M}_{g,n,k}$ of S is the group of the isotopy classes of orientation preserving homeomorphisms of S which keep the boundary of S and all the distinguished points pointwise fixed. In this paper we study the problem of finding a finite presentation for $\mathcal{M}_{g,n} = \mathcal{M}_{g,n,0}$. We restrict our attention to the case $n = 1$. The case $n = 0$ is easily obtained from the case $n = 1$. In principle the case of $n > 1$ (and the case of $k > 0$) can be obtained from the case $n = 1$ via standard exact sequences, but this method does not produce a global formula for the case of several boundary components and the presentation (in contrast to the ones we shall describe for the case $n = 0$ and $n = 1$) becomes rather ugly. On the other hand Gervais in [6] succeeded recently to produce a finite presentation of $\mathcal{M}_{g,n}$ starting from the results in [20] and using a new approach.

A presentation for $g = 1$ has been known for a long time. A quite simple presentation for $g = 2$ was established in 1973 in [1], but the method did not generalize to higher genus. In 1975 McCool proved in [19], by purely algebraic methods, that $\mathcal{M}_{g,1}$ is finitely presented for any genus g. It seems that extracting an explicit finite presentation from his proof is very difficult. In 1980 appeared the groundbreaking paper of Hatcher and Thurston [9] in which they gave an algorithm for constructing a finite presentation for the group $\mathcal{M}_{g,1}$ for an arbitrary g. In 1981 Harer applied their algorithm in [7] to obtain a finite (but very unwieldy) explicit presentation of $\mathcal{M}_{g,1}$. His presentation was simplified by Wajnryb in 1983 in [20]. A subsequent Errata [3] corrected small errors in the latter. The importance of the full circle of ideas in these papers can be jugded from a small sample of subsequent work which relied on the presentation in [20], eg [14], [16], [17], [18]. The proof of Hatcher and Thurston was deeply original, and solved an outstanding open problem using novel techniques. These included arguments based upon Morse and Cerf Theory, as presented by Cerf in [4].

In this paper we shall give, in one place, a complete hands-on proof of a simple presentation for the groups $\mathcal{M}_{g,0}$ and $\mathcal{M}_{g,1}$. Our approach will follow the lines set in [9], but we will be able to use elementary methods in the proof of the connectivity and simple connectivity of the cut system complex. In particular, our work does not rely on Cerf theory. At the same time we will gather all of the computational details in one place, making the result accesible for independent checks. Our work yields a slightly different set of generators and relations from the ones used in [20] and in [3]. The new presentation makes the computations in section 4 a little simpler. We shall give both presentations and prove that they are equivalent.

A consequence of this paper is that, using Lu [16] or Matveev and Polyak [18], the fundamental theorem of the Kirby calculus [13] now has a completely elementary proof (ie, one which makes no appeal to Cerf theory or high dimensional arguments).

This paper is organised as follows. In section 2 we give a new proof of the main theorem of Hatcher and Thurston. In section 3 we derive a presentation of the mapping class group following (and explaining) the procedure described in [9] and in [7]. In section 4 we reduce the presentation to the simple form of Theorem 1 repeating the argument from [20] with changes required by a slightly different setup. In section 5 we deduce the case of a closed surface and in section 6 we translate the presentation into the form given in [20], see Remark 1 below.

We start with a definition of a basic element of the mapping class group.

Definition 1 A (positive) Dehn twist with respect to a simple closed curve α on an oriented surface S is an isotopy class of a homeomorphism h of S, supported in a regular neighbourhood N of α (an annulus), obtained as follows: we cut S open along α, we rotate one side of the cut by 360 degrees to the right (this makes sense on an oriented surface) and then glue the surface back together, damping out the rotation to the identity at the boundary of N. The Dehn twist (or simply twist) with respect to α will be denoted by T_α. If curves α and β intersect only at one point and are transverse then $T_\alpha(\beta)$, up to an isotopy, is obtained from the union $\alpha \cup \beta$ by splitting the union at the intersection point.

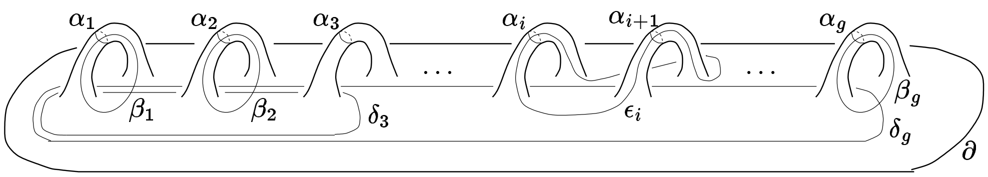

Figure 1: Surface $S_{g,1}$

We shall say that two elements a, b of a group are braided (or satisfy the braid relation) if $aba = bab$.

Theorem 1 *Let $S_{g,1}$ be a compact, orientable surface of genus $g \geq 3$ with one boundary component. Let a_i, b_i, e_i denote Dehn twists about the curves*

$\alpha_i, \beta_i, \epsilon_i$ on $S_{g,1}$ depicted on Figure 1. The mapping class group $\mathcal{M}_{g,1}$ of $S_{g,1}$ is generated by elements $b_2, b_1, a_1, e_1, a_2, e_2, \ldots, a_{g-1}, e_{g-1}, a_g$ and has defining relations:

(M1) *Elements b_2 and a_2 are braided and b_2 commutes with b_1. Every other pair of consecutive elements (in the above order) is braided and every other pair of non-consecutive elements commute.*

(M2) $\quad (b_1a_1e_1a_2)^5 = b_2a_2e_1a_1b_1^2a_1e_1a_2b_2$

(M3) $\quad d_3a_1a_2a_3 = d_{1,2}d_{1,3}d_{2,3}$, *where*

$$d_{1,2} = (a_2e_1a_1b_1)^{-1}b_2(a_2e_1a_1b_1),\quad d_{1,3} = t_2d_{1,2}t_2^{-1},\quad d_{2,3} = t_1d_{1,3}t_1^{-1},$$
$$t_1 = e_1a_1a_2e_1,\quad t_2 = e_2a_2a_3e_2,\quad d_3 = b_2a_2e_1b_1^{-1}d_{1,3}b_1e_1^{-1}a_2^{-1}b_2^{-1}.$$

Theorem 2 *The mapping class group $\mathcal{M}_{2,1}$ of an orientable surface $S_{2,1}$ of genus $g = 2$ with one boundary component is generated by elements b_2, b_1, a_1, e_1, a_2 and has defining relations* (M1) *and* (M2)*.*

Theorem 3 *The mapping class group $\mathcal{M}_{g,0}$ of a compact, closed, orientable surface of genus $g > 1$ can be obtained from the above presentation of $\mathcal{M}_{g,1}$ by adding one more relation:*

(M4) $\quad [b_1a_1e_1a_2\ldots a_{g-1}e_{g-1}a_ga_ge_{g-1}a_{g-1}\ldots e_1a_1b_1, d_g] = 1$, *where*

$$t_i = e_ia_ia_{i+1}e_i, \text{ for } i = 1, 2, \ldots, g-1,\quad d_2 = d_{1,2},$$
$$d_i = (b_2a_2e_1b_1^{-1}t_2t_3\ldots t_{i-1})d_{i-1}(b_2a_2e_1b_1^{-1}t_2t_3\ldots t_{i-1})^{-1}, \text{ for } i = 3, 4, \ldots, g$$

The presentations in Theorems 1 and 3 are equivalent to but not the same as those in [20] and [3]. We now give alternative presentations of Theorems 1 and 3, with the goal of correlating the work in this paper with that in [20] and [3]. See Remark 1, below, for a dictionary which allows one to move between Theorems 1′ and 3′ and the results in [20] and [3]. See Section 6 of this paper for a proof that the presentations in Theorems 1 and 1′, and also Theorems 3 and 3′, are equivalent.

Theorem 1′ *The mapping class group $\mathcal{M}_{g,1}$ admits a presentation with generators $b_2, b_1, a_1, e_1, a_2, e_2, \ldots, a_{g-1}, e_{g-1}, a_g$ and with defining relations:*

(A) *Elements b_2 and a_2 are braided and b_2 commutes with b_1. Every other pair of consecutive elements (in the above order) is braided and every other pair of non-consecutive elements commute.*

(B) $\quad (b_1a_1e_1)^4 = (a_2e_1a_1b_1^2a_1e_1a_2)b_2(a_2e_1a_1b_1^2a_1e_1a_2)^{-1}b_2$

(C) $\quad e_2e_1b_1\tilde{d}_3 = \tilde{t}_1^{-1}\tilde{t}_2^{-1}b_2\tilde{t}_2\tilde{t}_1\tilde{t}_2^{-1}b_2\tilde{t}_2b_2$ *where*

$\tilde{t}_1 = a_1b_1e_1a_1\,,\ \ \tilde{t}_2 = a_2e_1e_2a_2\,,\quad \tilde{d}_3 = (a_3e_2a_2e_1a_1u)v(a_3e_2a_2e_1a_1u)^{-1}\,,$
$u = e_2^{-1}a_3^{-1}\tilde{t}_2^{-1}b_2\tilde{t}_2a_3e_2\,,\ \ v = (a_2e_1a_1b_1)^{-1}b_2(a_2e_1a_1b_1)\,.$

Theorem 3′ *The mapping class group $\mathcal{M}_{g,0}$ of a compact, closed, orientable surface of genus $g > 1$ can be obtained from the above presentation of $\mathcal{M}_{g,1}$ by adding one more relation:*

(D) $\quad [a_ge_{g-1}a_{g-1}\dots e_1a_1b_1^2a_1e_1\dots a_{g-1}e_{g-1}a_g, \tilde{d}_g] = 1$, *where*

$\tilde{d}_g = u_{g-1}u_{g-2}\dots u_1b_1(u_{g-1}u_{g-2}\dots u_1)^{-1}, \quad u_1 = (b_1a_1e_1a_2)^{-1}v_1a_2e_1a_1\,,$
$u_i = (e_{i-1}a_ie_ia_{i+1})^{-1}v_ia_{i+1}e_ia_i \quad$ *for* $\quad i = 2,\dots,g-1\,,$
$v_1 = (a_2e_1a_1b_1^2a_1e_1a_2)b_2(a_2e_1a_1b_1^2a_1e_1a_2)^{-1}\,,$
$v_i = \tilde{t}_{i-1}^{-1}\tilde{t}_i^{-1}v_{i-1}\tilde{t}_i\tilde{t}_{i-1} \quad$ *for* $\quad i = 2,\dots,g-1\,,$
$\tilde{t}_1 = a_1e_1b_1a_1\,,\quad \tilde{t}_i = a_ie_ie_{i-1}a_i \quad$ *for* $\quad i = 2,\dots,g-1\,.$

Remark 1 We now explain how to move back and forth between the results in this paper and those in [20] and [3].

(i) The surface and the curves in [20] look different from the surface and the curves on Figure 1. However if we compare the Dehn twist generators in Theorem 1′ with those in Theorem 1 of [20] and [3] we see that corresponding curves have the same intersection pattern. Thus there exists a homeomorphism of one surface onto the other which takes the curves of one family onto the corresponding curves of the other family. The precise correspondence is given by:

$$\begin{gathered}(b_2, b_1, a_1, e_1, a_2, e_2, \dots, a_{g-1}, e_{g-1}, a_g)\\ \longleftrightarrow \quad (d, a_1, b_1, a_2, b_2, a_3, \dots, b_{n-1}, a_n, b_n)\,,\end{gathered}$$

where the top sequence refers to Dehn twists about the curves in Figure 1 of this paper and the bottom sequence refers to Dehn twists about the curves in Figure 1 on page 158 of [20].

(ii) Composition of homeomorphisms in [20] was performed from left to right, while in the present paper we use the standard composition from right to left.

(iii) The element d_g in this paper represents a Dehn twist about the curve δ_g in Figure 1 of this paper. The element $\tilde{d}_g$ in relation (D) of Theorem 3′ represents a Dehn twist about the curve β_g in Figure 1. We wrote d_g as a particular product of the generators in $\mathcal{M}_{g,1}$. It follows from the argument in the last section that any other such product representing d_g will also do.

(iv) In the case of genus $g = 2$ we should omit relation (C) in Theorems $1'$ and $3'$.

Here is the plan of the proof of the theorems. Following Hatcher and Thurston we define a 2–dimensional cell complex X on which the mapping class group $\mathcal{M}_{g,1}$ acts by cellular transformations and the action is transitive on the vertices of X. We give a new elementary proof of the fact that X is connected and simply connected. We then describe the stabilizer H of one vertex of X under the action of $\mathcal{M}_{g,1}$ and we determine an explicit presentation of H. Following the algorithm of Hatcher and Thurston we get from it a presentation of $\mathcal{M}_{g,1}$. Finally we reduce the presentation to the form in Theorem 1 and as a corollary get Theorem 3.

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2 Cut-system complex

We denote by S a compact, connected oriented surface of genus $g > 0$ with $n \geq 0$ boundary components. We denote by $\bar{S}$ a closed surface obtained from S by capping each boundary component with a disk. By a *curve* we shall mean a simple closed curve on S. We are mainly interested in the isotopy classes of curves on S. The main goal in the proofs will be to decrease the number of intersection points between different curves. If the Euler characteristic of S is negative we can put a hyperbolic metric on S for which the boundary curves are geodesics. Then the isotopy class of any non-separating curve on S contains a unique simple closed geodesic, which is the shortest curve in its isotopy class. If we replace each non-separating curve by the unique geodesic isotopic to it we shall minimize the number of intersection points between every two non-isotopic curves, by Corollary 5.1. So we can think of curves as geodesics. In the proof we may construct new curves, which are not geodesics and which have small intersection number with some other curves. When we replace the new curve by the corresponding geodesic we further decrease the intersection number. If S is a closed torus we can choose a flat metric on S. Now geodesics are not unique but still any choice of geodesics will minimize the intersection number of any pair of non-isotopic curves. If two curves are isotopic on S then they correspond to the same geodesic, but we shall call them disjoint because we can isotop one off the other. Geodesics are never tangent.

If α and β are curves then $|\alpha \cap \beta|$ denotes their geometric intersection number, ie, the number of intersection points of α and β. If α is a curve we denote by $[\alpha]$ the homology class represented by α in $H_1(\bar{S}, \mathbb{Z})$, up to a sign. We denote by $i(\alpha, \beta)$ the absolute value of the algebraic intersection number of α and β. It depends only on the classes $[\alpha]$ and $[\beta]$.

We shall describe now the cut-system complex X of S. To construct X we consider collections of g disjoint curves $\gamma_1, \gamma_2, \ldots, \gamma_g$ in S such that when we cut S open along these curves we get a connected surface (a sphere with $2g+n$ holes). An isotopy class of such a collection we call a cut system $\langle \gamma_1, \ldots, \gamma_g \rangle$. We can say that a cut system is a collection of geodesics. A curve is contained in a cut-system if it is one of the curves of the collection. If γ_i' is a curve in S, which meets γ_i at one point and is disjoint from other curves γ_k of the cut system $\langle \gamma_1, \ldots, \gamma_g \rangle$, then $\langle \gamma_1, \ldots, \gamma_{i-1}, \gamma_i', \gamma_{i+1}, \ldots, \gamma_g \rangle$ forms another cut system. In such a situation the replacement $\langle \gamma_1, \ldots, \gamma_i, \ldots, \gamma_g \rangle \to \langle \gamma_1, \ldots, \gamma_i', \ldots, \gamma_g \rangle$ is called a simple move. For brewity we shall often drop the symbols for unchanging circles and shall write $\langle \gamma_i \rangle \to \langle \gamma_i' \rangle$. The cut systems on S form the 0–skeleton (the vertices) of the complex X. We join two vertices by an edge if and only if the corresponding cut systems are related by a simple move. We get the 1–skeleton X^1. By a *path* we mean an edge-path in X^1. It consists of a sequence of vertices $\mathbf{p} = (v_1, v_2, \ldots, v_k)$ where two consecutive vertices are related by a simple move. A path is closed if $v_1 = v_k$. We distinguish three types of closed paths:

If three vertices (cut-systems) have $g-1$ curves $\gamma_1, \ldots, \gamma_{g-1}$ in common and if the remaining three curves $\gamma_g, \gamma_g', \gamma_g''$ intersect each other once, as on Figure 2, C3, then the vertices form a closed triangular path:

(C3) a triangle $\langle \gamma_g \rangle \to \langle \gamma_g' \rangle \to \langle \gamma_g'' \rangle \to \langle \gamma_g \rangle$.

If four vertices have $g-2$ curves $\gamma_1, \ldots, \gamma_{g-2}$ in common and the remaining pairs of curves consist of (γ_{g-1}, γ_g), $(\gamma_{g-1}', \gamma_g)$, $(\gamma_{g-1}', \gamma_g')$, $(\gamma_{g-1}, \gamma_g')$ where the curves intersect as on Figure 2, C4, then the vertices form a closed square path:

(C4) a square $\langle \gamma_{g-1}, \gamma_g \rangle \to \langle \gamma_{g-1}', \gamma_g \rangle \to \langle \gamma_{g-1}', \gamma_g' \rangle \to \langle \gamma_{g-1}, \gamma_g' \rangle \to \langle \gamma_{g-1}, \gamma_g \rangle$.

If five vertices have $g-2$ curves $\gamma_1, \ldots, \gamma_{g-2}$ in common and the remaining pairs of curves consist of (γ_{g-1}, γ_g), $(\gamma_{g-1}, \gamma_g')$, $(\gamma_{g-1}', \gamma_g')$, $(\gamma_{g-1}', \gamma_g'')$ and (γ_g, γ_g'') where the curves intersect as on Figure 2, C5, then the vertices form a closed pentagon path:

(C5) a pentagon $\langle\gamma_{g-1},\gamma_g\rangle \to \langle\gamma_{g-1},\gamma'_g\rangle \to \langle\gamma'_{g-1},\gamma'_g\rangle \to \langle\gamma'_{g-1},\gamma''_g\rangle \to \langle\gamma''_g,\gamma_g\rangle \to \langle\gamma_{g-1},\gamma_g\rangle$.

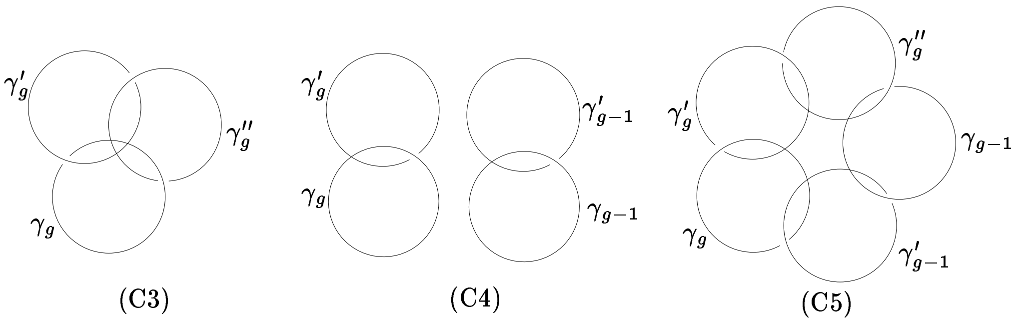

Figure 2: Configurations of curves in paths (C3), (C4) and (C5)

X is a 2–dimensional cell complex obtained from X^1 by attaching a 2–cell to every closed edge-path of type (C3), (C4) or (C5). The mapping class group of S acts on S by homeomorphisms so it takes cut systems to cut systems. Since the edges and the faces of X are determined by the intersections of pairs of curves, which are clearly preserved by homeomorphisms, the action on X^0 extends to a cellular action on X.

In this section we shall prove the main result of [9]:

Theorem 4 *X is connected and simply connected.*

We want to prove that every closed path $\mathbf{p}$ is null-homotopic. If $\mathbf{p}$ is null-homotopic we shall write $\mathbf{p} \sim \mathbf{o}$. We start with a closed path $\mathbf{p} = (v_1, \ldots, v_k)$ and try to simplify it. If $\mathbf{q}$ is a short-cut, an edge path connecting a vertex v_i of $\mathbf{p}$ with v_j, $j > i$, we can split $\mathbf{p}$ into two closed edge-paths:
$\mathbf{p}_1 = (v_j, v_{j+1}, \ldots, v_k, v_2, \ldots, v_{i-1}, \mathbf{q})$ and $\mathbf{p}_2 = (\mathbf{q}^{-1}, v_{i+1}, v_{i+2}, \ldots, v_j)$.
If both paths are null-homotopic in X then $\mathbf{p} \sim \mathbf{o}$. We want to prove Theorem 4 by splitting path $\mathbf{p}$ into simpler paths according to a notion of complexity which is described in the next definition.

Definition 2 Let $\mathbf{p} = (v_1, v_2, \ldots, v_k)$ be a path in X. Let α be a fixed curve of some fixed v_j. We define *distance* from α to a vertex v_i to be $d(\alpha, v_i) = min\{|\alpha \cap \beta| : \beta \in v_i\}$. The *radius of* $\mathbf{p}$ *around* α is equal to the maximum distance from α to the vertices of $\mathbf{p}$. The path $\mathbf{p}$ is called a *segment* if every vertex of $\mathbf{p}$ contains a fixed curve α. We shall write α–segment if we want to stress the fact that α is the common curve of the segment. If the segment has several fixed curves we can write $(\alpha, \beta, \ldots, \gamma)$–segment.

Theorem 4 will be proven by induction on the genus of S, for the given genus it will be proven by induction on the radius of a path $\mathbf{p}$ and for a given radius m around a curve α we shall induct on the number of segments of $\mathbf{p}$ which have a common curve γ with $|\gamma \cap \alpha| = m$. The main tool in the proof of Theorem 4 is a reduction of the number of intersection points of curves.

Definition 3 Curves α and β on S have an *excess intersection* if there exist curves α' and β', isotopic to α and β respectively, and such that $|\alpha \cap \beta| > |\alpha' \cap \beta'|$. Curves α and β form a 2–*gon* if there are arcs a of α and b of β, which meet only at their common end points and do not meet other points of α or β and such that $a \cup b$ bound a disk, possibly with holes, on S. The disk is called a 2–gon. We can *cut off* the 2–gon from α by replacing the arc a of α by the arc b of β. We get a new curve α' (see Figure 3).

Lemma 5 (Hass–Scott, see [8], Lemma 3.1) *If α, β are two curves on a surface S having an excess intersection then they form a 2–gon (without holes) on S.*

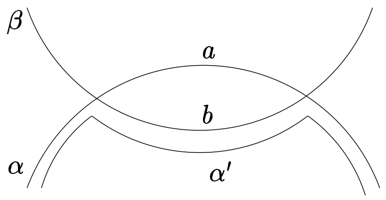

Figure 3: Curves α and β form a 2–gon.

Corollary 5.1 *Two simple closed geodesics on S have no excess intersection. In particular if we replace two curves by geodesics in their isotopy class then the number of intersection points between the curves does not increase.*

Proof If there is a 2–gon we can shorten one geodesic in its homotopy class, by first cutting off the 2–gon and then smoothing corners. □

Lemma 6 *Consider a finite collection of simple closed geodesics on S. Suppose that curves α and β of the collection form a minimal 2–gon (which does not contain another 2–gon). Let α' be the curve obtained by cutting off the minimal 2–gon from α and passing to the isotopic geodesic. Then $|\alpha \cap \alpha'| = 0$, $|\beta \cap \alpha'| < |\beta \cap \alpha|$ and $|\gamma \cap \alpha'| \leq |\gamma \cap \alpha|$ for any other curve γ of the collection. In particular if $|\gamma \cap \alpha| = 1$ then $|\gamma \cap \alpha'| = 1$. Also $[\alpha] = [\alpha']$.*

Proof Since the 2–gon formed by arcs a and b of α and β is minimal every other curve γ of the collection intersects the 2–gon along arcs which meet a and b once. Thus cutting off the 2–gon will not change $|\alpha \cap \gamma|$ and it may only decrease after passing to the isotopic geodesic. Clearly α and α' are disjoint and homologous on $\bar{S}$ and by passing to α' we remove at least two intersection points of α with β. □

2.1 The case of genus 1

In this section we shall assume that S is a surface of genus one, possibly with boundary. By $\bar{S}$ we shall denote the closed torus obtained by glueing a disk to each boundary component of S. We want to prove:

Proposition 7 *If S has genus one then the cut system complex X of S is connected and simply connected.*

On a closed torus $\bar{S}$ the homology class $[\alpha]$ of a curve α is defined by a pair of relatively prime integers, up to a sign, after a fixed choice of a basis of $H_1(\bar{S}, \mathbb{Z})$. If $[\alpha] = (a_1, a_2)$ and $[\beta] = (b_1, b_2)$ then the absolute value of their algebraic intersection number is equal $i(\alpha, \beta) = |a_1 b_2 - a_2 b_1|$. If α and β are geodesics on $\bar{S}$ then $|\alpha \cap \beta| = i(\alpha, \beta)$, therefore it is also true for curves on S which form no 2–gons, because then they have no excess intersection on $\bar{S}$.

Lemma 8 *Let α, β be nonseparating curves on S and suppose that $|\alpha \cap \beta| = k \neq 1$. Then there exists a nonseparating curve δ such that if $k = 0$ then $|\delta \cap \alpha| = |\delta \cap \beta| = 1$ and if $k > 1$ then $|\delta \cap \alpha| < k$ and $|\delta \cap \beta| < k$.*

Proof If $|\alpha \cap \beta| = 0$ then the curves are isotopic on $\bar{S}$ and they split S into two connected components S_1 and S_2. We can choose points P and Q on α and β respectively and connect them by simple arcs in S_1 and in S_2. The union of the arcs forms the required curve δ. If $k > 1$ and if the curves have an excess intersection on $\bar{S}$ then they form a 2–gon on S. We can cut off the 2–gon decreasing the intersection, by Lemma 6. If there are no 2–gons then the algebraic and the geometric intersection numbers are equal. In particular all intersections have the same sign. Consider two intersection points consecutive along β. Choose δ as on Figure 4. Then $|\delta \cap \alpha| = 1$ so it is nonseparating and $|\delta \cap \beta| < k$. □

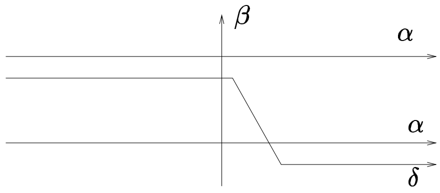

Figure 4: Curve δ has smaller intersection with α and β.

Corollary 8.1 *If surface S has genus 1 then the cut system complex of S is connected.*

Proof A cut system on S is an isotopy class of a single curve. If two curves intersect once they are connected by an edge. It follows from the last lemma by induction that any two curves can be connected by an edge-path in X. □

We now pass to a proof that closed paths are null-homotopic.

Lemma 9 *A closed path $\mathbf{p} = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_1)$ such that $|\delta_2 \cap \delta_4| = 0$ is null-homotopic in X.*

Proof Let $\beta = T^{\pm 1}_{\delta_2}(\delta_3)$ be the image of δ_3 by the Dehn Twist along δ_2. Recall that as a set $\beta = \delta_2 \cup \delta_3$ outside a small neighbourhood of $\delta_2 \cap \delta_3$. From this we get that $|\beta \cap \delta_2| = 1$, $|\beta \cap \delta_3| = 1$ and $|\beta \cap \delta_4| = 1$. Thus $\mathbf{p}' = (\delta_1, \delta_2, \beta, \delta_4, \delta_1)$ is a closed path which is homotopic to $\mathbf{p}$ because $\mathbf{p} - \mathbf{p}'$ splits into a sum of two triangles $\mathbf{t_1} = (\beta, \delta_2, \delta_3, \beta)$ and $\mathbf{t_2} = (\beta, \delta_3, \delta_4, \beta)$. We also have $i(\beta, \delta_1) = |i(\delta_3, \delta_1) \pm i(\delta_2, \delta_1)|$, so for a suitable choice of the sign of the twist we have $i(\delta_3, \delta_1) > i(\beta, \delta_1)$ unless $i(\delta_3, \delta_1) = 0$. We may assume by induction that $i(\delta_3, \delta_1) = 0$. If $|\delta_1 \cap \delta_3| > 0$ then δ_1 and δ_3 have an excess intersection on $\bar{S}$ and form a 2–gon on S. We can cut off the 2–gon from δ_3 getting a new curve β such that $[\beta] = [\delta_3]$, $|\beta \cap \delta_3| = 0$, $|\beta \cap \delta_1| < |\delta_3 \cap \delta_1|$ and $|\beta \cap \delta_i| = |\delta_3 \cap \delta_i|$ for $i \neq 1$, by Lemma 6. We get a new closed path $\mathbf{p}' = (\delta_1, \delta_2, \beta, \delta_4, \delta_1)$ and the difference between it and the old path is a closed path $\mathbf{q} = (\beta, \delta_2, \delta_3, \delta_4, \beta)$ with $|\beta \cap \delta_3| = 0$ and $|\delta_2 \cap \delta_4| = 0$. So by induction it suffices to assume that $|\delta_1 \cap \delta_3| = 0$. If we now let $\beta = T_{\delta_2}(\delta_3)$ then β intersects each of the four curves once so our path is a sum of four triangles and thus is null-homotopic in X. □

Lemma 10 *If $\mathbf{p} = (\alpha_1, \dots, \alpha_k)$ is a closed path then there exists a closed path $\mathbf{p}' = (\alpha'_1, \dots, \alpha'_k)$ such that $\mathbf{p}'$ is homotopic to $\mathbf{p}$ in X, $[\alpha_i] = [\alpha'_i]$ for all i and the collection of curves $\alpha'_1, \dots, \alpha'_{k-1}$ forms no 2–gons.*

Proof Suppose that there exists a 2–gon bounded by arcs of two curves in $\mathbf{p}$. Then there also exists a minimal 2–gon bounded by arcs of curves α_i and α_j. If we cut off the 2–gon from α_i we get a curve α_i' such that $[\alpha_i] = [\alpha_i']$, α_i' is disjoint from α_i, $|\alpha_j \cap \alpha_i| > |\alpha_j \cap \alpha_i'|$ and $|\alpha_m \cap \alpha_i'| \leq |\alpha_m \cap \alpha_i|$ for any other curve α_m, by Lemma 6. It follows that if we replace α_i by α_i' we get a new closed path $\mathbf{p}'$ with a smaller number of intersection points between its curves. The difference of the two paths is a closed path $\mathbf{q} = (\alpha_{i-1}, \alpha_i, \alpha_{i+1}, \alpha_i', \alpha_{i-1})$ which is null-homotopic by Lemma 9. Thus $\mathbf{p}'$ is homotopic to $\mathbf{p}$ in X. Lemma 10 follows by induction on the total number of intersection points between pairs of curves of $\mathbf{p}$. □

Proof of Proposition 7 Let $\mathbf{p} = (\alpha_1, \ldots, \alpha_k, \alpha_1)$ be a closed path in X^1. We may assume that the path has no 2–gons. By Lemma 5 there is no excess intersection on the closed torus $\bar{S}$. It means that the geometric intersection number of two curves of $\mathbf{p}$ is equal to their algebraic intersection number. Let $m = max\{i(\alpha_1, \alpha_j) | j = 1, \ldots, k\}$ be the *radius* of $\mathbf{p}$ around α_1. Suppose first that $m = 1$. Two disjoint curves on $\bar{S}$ have the same homology class, and two curves representing the same class have algebraic intersection equal to 0. It follows that two consecutive curves in a path cannot be both disjoint from α_1. If $k > 4$ then either $|\alpha_1 \cap \alpha_3| = 1$ or $|\alpha_1 \cap \alpha_4| = 1$. We get an edge which splits $\mathbf{p}$ into two shorter closed paths with radius 1. If $k = 4$ and $|\alpha_1 \cap \alpha_3| = 0$ then $\mathbf{p} \sim \mathbf{o}$ by Lemma 9. If $k = 3$ then $\mathbf{p}$ is a triangle. Suppose now that $m > 1$. We may assume by induction that every path of radius less than m is null-homotopic and every path of radius m which has less curves α_j with $|\alpha_1 \cap \alpha_j| = m$ is also null-homotopic. Choose the smallest i such that $i(\alpha_1, \alpha_i) = m$. Then $i(\alpha_1, \alpha_{i-1}) < m$ and $i(\alpha_1, \alpha_{i+1}) \leq m$. Choose a basis of the homology group $H_1(\bar{S})$ which contains the curve α_1. A homology class of a curve is then represented by a pair of integers (a, b). We consider the homology classes and their intersection numbers up to a sign. We have $[\alpha_1] = (1, 0)$, $[\alpha_{i-1}] = (a, b)$, $[\alpha_i] = (p, m)$ and $[\alpha_{i+1}] = (c, d)$. The intersection form is defined by $i((a, b), (c, d)) = |ad - bc|$. Thus $am - bp = \pm 1, cm - dp = \pm 1, |b| < m, |d| \leq m$. We get $m(ad - bc) = (\pm d \pm b)$. Since $2m > |b| + |d|$ we must have $|ad - bc| = 1$ or $ad - bc = 0$. In the first case $i(\alpha_{i-1}, \alpha_{i+1}) = |\alpha_{i-1} \cap \alpha_{i+1}| = 1$. We can "cut off" the triangle $\mathbf{q} = (\alpha_{i-1}, \alpha_i, \alpha_{i+1}, \alpha_{i-1})$ getting a path which is null-homotopic by the induction hypothesis. If $ad - bc = 0$ then $i(\alpha_{i-1}, \alpha_{i+1}) = |\alpha_{i-1} \cap \alpha_{i+1}| = 0$. Let $\beta = T^{\pm 1}_{\alpha_{i-1}}(\alpha_i)$. Then $|\beta \cap \alpha_{i-1}| = 1$ and $|\beta \cap \alpha_{i+1}| = 1$. We can replace α_i by β getting a new closed path. Their difference is the closed path $\mathbf{q} = (\alpha_{i-1}, \alpha_i, \alpha_{i+1}, \beta, \alpha_{i-1})$ which is null homotopic by lemma 9. Thus the new path is homotopic to the old path. For a suitable choice of the sign of the Dehn twist we have $i(\beta, \alpha_1) < m$. It may happen that $|\beta \cap \alpha_1| \geq m$. We can get rid of 2–gons by Lemma 10 and thus get rid of the excess intersection. We get a homotopic path which is null-homotopic by the induction hypothesis. This concludes the proof of Proposition 7 □

2.2 Paths of radius 0

From now until the end of section 2, we assume that S is a surface of genus $g > 1$ with a finite number of boundary components. We denote by X the cut-system complex of S. We assume:

Induction Hypothesis 1 The cut-system complex of a surface of genus less than g is connected and simply-connected.

We want to prove that every closed path in X is null-homotopic. We shall start with paths of radius zero. The simplest paths of radius zero are closed segments.

Lemma 11 *A closed segment is null-homotopic in X.*

Proof When we cut S open along the common curve α the remaining curves of each vertex form a vertex of a closed path in the cut-system complex of a surface of a smaller genus. By Induction Hypothesis 1 it is a sum of paths of type (C3), (C4) and (C5) there. When we adjoin α to every vertex we get a splitting of the original paths into null-homotopic paths. □

In a similar way we prove:

Lemma 12 *If two vertices of X have one or more curves in common we can connect them by a path all of whose vertices contain the common curves.*

Proof If we cut S open along the common curves the remaining collection of curves form two vertices of the cut-system complex on the new surface of smaller genus. They can be connected by a path. If we adjoin all the common curves to each vertex of this path we get a path in X with the required properties. □

Remark 2 If α and β are two disjoint non-separating curves on S then $\alpha \cup \beta$ does not separate S if and only if $[\alpha] \neq [\beta]$. In this case the pair α, β can be completed to a cut-system on S.

We shall now construct two simple types of null-homotopic paths in X.

Lemma 13 *Let α_1, α_2, α_3 be disjoint curves such that the union of any two of them does not separate S but the union of all three separates S. Then there exist disjoint curves β_1, β_2, β_3 and a closed path*

$$\text{(C6)} \qquad \langle\alpha_1,\alpha_2\rangle \to \langle\alpha_1,\beta_2\rangle \to \langle\alpha_1,\alpha_3\rangle \to \langle\beta_1,\alpha_3\rangle \to \langle\alpha_2,\alpha_3\rangle \\ \to \langle\alpha_2,\beta_3\rangle \to \langle\alpha_2,\alpha_1\rangle,$$

which is null-homotopic in X.

Proof Let $\gamma_3,\ldots,\gamma_g$ be a cut system on a surface $S-(\alpha_1\cup\alpha_2\cup\alpha_3)$ (not connected), ie, a collection of curves which does not separate the surface any further. Then $\langle\alpha_1,\alpha_2,\gamma_3,\ldots,\gamma_g\rangle$ is a cut system on S. Let S_1 and S_2 be the components of a surface obtained by cutting S open along all α_i's and γ_j's. An arc connecting different components of the boundary does not separate the surface so we can find (consecutively) disjoint arcs b_1 connecting α_1 with α_2, b_2 connecting α_2 with α_3 and b_3 connecting α_3 with α_1 in S_1, and similar arcs b_1', b_2' and b_3' in S_2 with the corresponding ends coinciding in S. The pairs of corresponding arcs form the required curves β_1, β_2 and β_3 and we get a closed path $\mathbf{p}$ described in (C6). Moreover the curves β_2 and β_3 are disjoint and $[\beta_2]\neq[\beta_3]$. To prove that the path is null-homotopic in X we choose a curve $\delta=T_{\alpha_2}(\beta_1)$. Then $|\delta\cap\alpha_1|=|\delta\cap\alpha_2|=|\delta\cap\beta_1|=|\delta\cap\beta_2|=1$ and $|\delta\cap\alpha_3|=|\delta\cap\beta_3|=0$. Figure 5 shows how $\mathbf{p}$ splits into a sum of triangles (C3), squares (C4) and pentagons (C5) and therefore is null-homotopic in X. □

$$\begin{tikzcd}
\langle\alpha_1,\beta_2\rangle \arrow[rrrrrr] \arrow[dr] & & & & & & \langle\alpha_1,\alpha_3\rangle \arrow[dl] \arrow[dd] \\
& \langle\beta_2,\beta_3\rangle \arrow[rr] \arrow[dr] & & \langle\delta,\beta_3\rangle \arrow[rr] \arrow[dl] & & \langle\delta,\alpha_3\rangle \arrow[dl] \arrow[dr] & \\
\langle\alpha_1,\alpha_2\rangle \arrow[uu] & & \langle\alpha_2,\beta_3\rangle \arrow[ll] & & \langle\alpha_2,\alpha_3\rangle \arrow[ll] & & \langle\beta_1,\alpha_3\rangle \arrow[ll]
\end{tikzcd}$$

Figure 5: Hexagonal path

Proposition 14 *If $\mathbf{p}$ is a path of radius 0 around a curve α then $\mathbf{p}\sim\mathbf{o}$.*

Proof Let v_0 be a vertex of $\mathbf{p}$ containing α. We shall prove the proposition by induction on the number of segments of $\mathbf{p}$ having a fixed curve disjoint from α. Consider the maximal α–segment of $\mathbf{p}$ which contains the vertex v_0. We shall call it the first segment. Let v_1 be the last vertex of the first segment. The next vertex contains a curve β disjoint from α such that β is the common curve of the next segment of $\mathbf{p}$. Since $|\alpha \cap \beta| = 0$ the simple move from v_1 to the next vertex does not involve β hence v_1 also contains β. Let v_2 be the last vertex of the second segment. If there are only two segments then v_2 also contains both α and β. By Lemma 12 there is an (α,β)–segment connecting v_1 and v_2. Then $\mathbf{p}$ is a sum of a closed α–segment and a closed β–segment. So we may assume that there is a third segment. The vertex $\tilde{v}$ of $\mathbf{p}$ following v_2 contains a curve γ disjoint from α and γ is the common curve of the third segment. Let v_3 be the last vertex of the third segment. We shall reduce the number of segments. There are three cases.

Case 1 Vertex v_2 does not contain γ. Since $\tilde{v}$ contains γ and does not contain β we have $|\beta \cap \gamma| = 1$. Let S_1 be a surface of genus $g-1$ obtained by cutting S open along $\beta \cup \gamma$. Vertices v_2 and $\tilde{v}$ have $g-1$ curves in common and the common curves form a cut system u on S_1. The union $\beta \cup \gamma$ cannot separate $S - \alpha$ hence α does not separate S_1 and it belongs to a cut-system u' on S_1. Vertices u and u' can be connected by a path $\mathbf{q}$ in the cut-system complex of S_1. If we adjoin β (respectively γ) to each vertex of $\mathbf{q}$ we get a path $\mathbf{q_2}$ (respectively $\mathbf{q_1}$). Path $\mathbf{q_2}$ connects v_2 to a vertex u_2 containing α and β and path $\mathbf{q_1}$ connects $\tilde{v}$ to a vertex u_1 containing α and γ. The corresponding vertices of $\mathbf{q_1}$ and $\mathbf{q_2}$ are connected by an edge so the middle rectangle on Figure 6 splits into a sum of squares of type (C4) and is null-homotopic. We can connect v_1 to u_2 by an (α,β)–segment so the triangle on Figure 6 is a closed β–segment and is also null-homotopic. The part of $\mathbf{p}$ between v_1 and $\tilde{v}$ can be replaced by the lower path on Figure 6. We get a new path $\mathbf{p}'$, which has a smaller number of segments (no β–segment) and is homotopic to $\mathbf{p}$ in X.

Case 2 Vertex v_2 contains γ and $\alpha \cup \gamma$ does not separate S. If there exists a vertex v which contains α and β and γ we can connect it to v_1 and v_2 using Lemma 12. We get a closed segment and the remaining path has one segment less (Figure 7). Otherwise $\alpha \cup \beta \cup \gamma$ separate S and we can apply Lemma 13. There exist vertices w_1 containing α and β, w_2 containing β and γ and w_3 containing α and γ and a β–segment from w_1 to w_2, a γ–segment from w_2 to w_3 and an α–segment from w_3 to w_1. The sum of the segments is null-homotopic. We now connect v_1 to w_1 by an (α,β)–segment and v_2 to w_2 by

$$\begin{tikzcd}
v_0 \arrow[r, "\alpha"] & v_1 \arrow[r, "\beta"] \arrow[dr, "{(\alpha,\beta)}"'] & v_2 \arrow[r] & \tilde{v} \arrow[r, "\gamma"] & v_3 \\
& & u_2 \arrow[u, "\beta"] \arrow[r] & u_1 \arrow[u, "\gamma"'] &
\end{tikzcd}$$

Figure 6: Path of radius 0, Case 1

a (β,γ)–segment. Thus the second segment of $\mathbf{p}$ can be replaced by a sum of an α–segment and a γ–segment, and the difference is a closed β–segment plus a null-homotopic hexagonal path of Lemma 13 (see the right side of Figure 7).

$$\begin{tikzcd}
v_1 \arrow[r, "\beta"] \arrow[dr, "{(\alpha,\beta)}"'] & v_2 \arrow[r, "\gamma"] & v_3 \\
& v \arrow[u, "{(\beta,\gamma)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \arrow[r, "\beta"] \arrow[dd, "{(\alpha,\beta)}"] & v_2 \arrow[r, "\gamma"] & v_3 \\
& w_2 \arrow[u, "{(\beta,\gamma)}"'] & \\
w_1 \arrow[ur, "\beta"] \arrow[r, "\alpha"] & w_3 \arrow[u, "\gamma"'] &
\end{tikzcd}$$

Figure 7: Path of radius 0, Case 2

Case 3 Vertex v_2 contains γ and $\alpha \cup \gamma$ separates S into two surfaces S_1 and S_2. If there were only three segments then, as at the vertex v_1, the first vertex of the first segment would contain both α and γ and their union would not separate S. This contradicts our assumptions. It follows in particular that every closed path of radius zero with at most three segments (where the common curve of each segment is disjoint from a fixed curve of the first segment) is null-homotopic. We may assume that the path $\mathbf{p}$ has a fourth segment with a fixed curve δ disjoint from α. Since $[\gamma] = [\alpha]$ we cannot have $|\gamma \cap \delta| = 1$. Therefore δ is not involved in the simple move from v_3 to the next vertex and v_3 contains δ. In particular $[\gamma] \neq [\delta]$ and $[\alpha] \neq [\delta]$. We may assume that β lies in S_1. If δ lies in S_2 then there is a vertex w which contains α and β and δ. We can connect w to v_1 by an (α,β)–segment, to v_2 by a β–segment and to v_3 by a δ–segment. We get a new path, homotopic to $\mathbf{p}$, which does not contain β–segment nor γ–segment (see Figure 8, left part.)

Suppose now that δ lies in S_1. Consider the cut-system complex X_1 of S_1 and choose a vertex s of X_1 which contains δ and a vertex s' of X_1 which contains β. Let $\mathbf{q}$ be a path in X_1 which connects s to s'. Let t be a fixed vertex of the cut-system complex X_2 of S_2 (if X_2 is not empty.) We add α and all curves of t to each vertex of the path $\mathbf{q}$ and get an α–segment in X connecting a vertex w_2, containing δ, to a vertex w_2', containing β. Then we add γ and all curves of t to each vertex of the path $\mathbf{q}$ and get a γ–segment in X connecting a vertex w_3, containing δ, to a vertex w_3', containing β. We now connect v_1 to w_2 by an α–segment, v_2 to w_2' by a β–segment, v_3 to w_3' by a γ–segment and v_4 to w_3 by a δ–segment (see Figure 8, the right side.) Corresponding vertices of the two vertical segments on Figure 8, the right side, have a common curve δ_i, a curve of a vertex of the path $\mathbf{q}$ disjoint from α and from γ, and can be connected by a δ_i–segment. We get a "ladder"such that each small rectangle in this ladder has radius zero around γ and consists of only three segments. Therefore it is null-homotopic. Every other closed path on Figure 8, the right side, has a similar property. We get a new path, homotopic to $\mathbf{p}$, which does not contain β–segment nor γ–segment. □

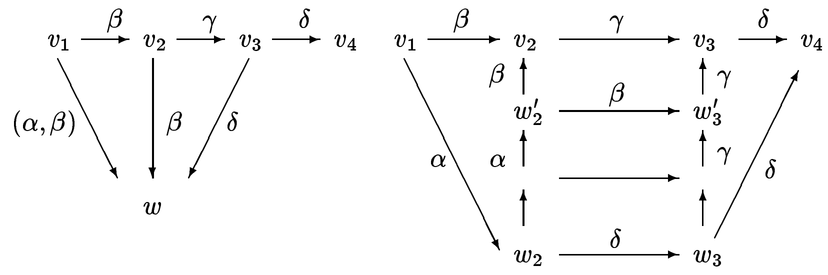

Figure 8: Path of radius 0, Case 3

2.3 The general case

We now pass to the general case and we want to prove it by induction on the radius of a closed path.

Induction hypothesis 2 A closed path of radius less than m is null-homotopic.

We want to prove that a closed path $\mathbf{p}$ of radius m around a curve α is null-homotopic. The general idea is to construct a *short-cut*, an edge-path which splits $\mathbf{p}$ and is close to α and to a fixed β–segment. The first step is to construct one intermediate curve.

Lemma 15 *Let γ_1 and γ_2 be non-separating curves on S such that $|\gamma_1 \cap \gamma_2| = n > 1$. Then there exists a non-separating curve δ such that $|\gamma_1 \cap \delta| < n$ and $|\gamma_2 \cap \delta| < n$. Suppose that we are also given non-separating curves α, β and an integer $m > 0$ such that $|\alpha \cap \beta| \leq m$, $|\gamma_1 \cap \alpha| < m$, $|\gamma_2 \cap \alpha| \leq m$, and $|\beta \cap \gamma_1| = |\beta \cap \gamma_2| = 0$. Then we can find a curve δ as above which also satisfies $|\delta \cap \alpha| < m$ and $|\delta \cap \beta| = 0$.*

Proof We orient the curves γ_1 and γ_2 and split the union $\gamma_1 \cup \gamma_2$ into a different union of oriented simple closed curves as follows. We start near an intersection point, say P_1, on the side of γ_2 after γ_1 crosses it and on the side of γ_1 before γ_2 crosses it. Now we move parallel to γ_1 to the next intersection point with γ_2, say P_2. We do not cross γ_2 at P_2 and move parallel to γ_2, in the positive direction, back to P_1. We get a curve δ_1. Now we start near P_2 and move parallel to γ_1 until we meet an intersection point, say P_3, which is either equal to P_1 or was not met before. We do not cross γ_2 at P_3 and move parallel to γ_2, in the positive direction, back to P_2. We get a curve δ_2. And so on. Curve δ_i meets γ_1 near some points of $\gamma_1 \cap \gamma_2$, but not near P_i and it meets γ_2 near some points of $\gamma_1 \cap \gamma_2$, but not near P_{i+1}. So δ_i meets both curves less than n times. Let $\bar{\gamma}$ denote the (oriented) homology class represented by an oriented curve γ in $H_1(\bar{S}, \mathbb{Z})$. We have $\bar{\gamma}_1 + \bar{\gamma}_2 = \bar{\delta}_1 + \ldots + \bar{\delta}_k$. Now we repeat a similar construction for the opposite orientation of γ_2 starting near the same point P_1. We get new curves $\epsilon_1, \ldots, \epsilon_r$ and $\bar{\gamma}_1 - \bar{\gamma}_2 = \bar{\epsilon}_1 + \ldots + \bar{\epsilon}_r$. Also $\bar{\delta}_1 - \bar{\epsilon}_1 = \bar{\gamma}_2$. Combining these equalities in $H_1(\bar{S}, \mathbb{Z})$ we get $\bar{\epsilon}_1 + \sum_{i \neq 1} \bar{\delta}_i = \bar{\gamma}_1$, $\bar{\delta}_1 + \sum_{i \neq 1} \bar{\epsilon}_i = \bar{\gamma}_1$, $\sum_{i \neq 1} \bar{\delta}_i - \sum_{i \neq 1} \bar{\epsilon}_i = \bar{\gamma}_2$. A simple closed curve separates S if and only if it represents 0 in $H_1(\bar{S}, \mathbb{Z})$. Since γ_1 and γ_2 are non-separating it follows that either δ_1 and some δ_i, $i \neq 1$, are not separating or ϵ_1 and some ϵ_i, $i \neq 1$, are not separating. And each of them meets γ_1 and γ_2 less than n times, so it can be chosen for δ. If we are also given curves α and β and integer $m > 0$ which satisfy the assumptions of the Lemma then $|\gamma_1 \cap \alpha| + |\gamma_2 \cap \alpha| = \Sigma|\delta_i \cap \alpha| = \Sigma|\epsilon_i \cap \alpha| \leq 2m - 1$ therefore one of the constructed nonseparating curves intersects α less than m times and is disjoint from β. □

Lemma 16 *Let γ_1 and γ_2 be disjoint non-separating curves on S such that $\gamma_1 \cup \gamma_2$ separates S. Then there exists a non-separating curve δ such that $|\gamma_1 \cap \delta| = 1$ and $|\gamma_2 \cap \delta| = 1$. Suppose that we are also given non-separating curves α, β and an integer $m > 0$ such that $|\alpha \cap \beta| \leq m$, $|\alpha \cap \beta| = 1$ if $m = 1$,*

$|\gamma_1 \cap \alpha| < m$, $|\gamma_2 \cap \alpha| \leq m$, and $|\beta \cap \gamma_1| = |\beta \cap \gamma_2| = 0$. Then we can find a curve δ as above which also satisfies $|\delta \cap \alpha| < m$ and $|\delta \cap \beta| < m$.

Proof By our assumptions $\gamma_1 \cup \gamma_2$ separates S into two components S_1 and S_2. We can choose a simple arc d_1 in S_1 which connects γ_1 to γ_2 and a simple arc d_2 in S_2 which connects γ_1 to γ_2. Then we can slide the end-points of d_1 and d_2 along γ_1 and γ_2 to make the end-points meet. We get a nonseparating curve δ which intersects γ_1 and γ_2 once. Suppose that we are also given curves α and β and an integer $m > 0$. We need to alter the arcs d_1 and d_2, if necessary, in order to decrease the intersection of δ with α and β. We may assume that β lies in S_1. We have to consider several cases.

Case 1 Curve α lies in S_1. Then d_2 is disjoint from α. If $m = 1$ then $|\alpha \cap \beta| = 1$ so the union $\alpha \cup \beta$ does not separate its regular neighbourhood and does not separate S_1. We can choose d_1 disjoint from α and β and then δ is also disjoint from α and β.

Suppose that $m > 1$. If α separates S_1 (but does not separate S) then it separates γ_1 from γ_2 in S_1. There exists an arc d in S_1 which connects γ_1 with γ_2 and is disjoint from α, if α does not separate S_1, or meets α once, if α separates S_1. We choose such an arc d which has minimal number of intersections with β. If $|\beta \cap d| > m$ then there exist two points P and Q of $\beta \cap d$, consecutive along β, and not separated by a point of $\beta \cap \alpha$. We can move along d to P then along β, without crossing β, to Q, and then continue along d to its end. This produces an arc which meets α at most once and has smaller number of intersections with β. So we may assume that $|\beta \cap d| = m$ and that every pair of points of $\beta \cap d$ consecutive along β is separated by a point of $\beta \cap \alpha$. We now alter d as follows. Consider the intersection $d \cap (\alpha \cup \beta)$. If the first or the last point along d of this intersection belongs to α we start from this end of d. Otherwise we start from any end. We move along d to the first point, say P, of intersection with $\alpha \cup \beta$. If $P \in \alpha$ we continue along α, without crossing it, to the next point of $\alpha \cap \beta$. Then along β, without crossing it, to the last point, say Q, of $\beta \cap d$ on d, and then along d to its end. The new arc crosses β at most once, near Q, and crosses α less than m times. If $P \in \beta$ we continue along β, without crossing β, to Q and then along d to its end, which produces a similar result. We can choose such an arc for d_1 and then the curve δ satisfies the Lemma.

Case 2 Curve α meets γ_1 or γ_2. Then $m > 1$, because if $m = 1$ and $|\gamma_1 \cap \alpha| = 0$ and α crosses γ_2 into S_1 then it must cross it again in order to exit S_1, and this contradicts $|\gamma_2 \cap \alpha| \leq m$. The arcs of α split S_1 (and S_2) into connected components. One of the components must meet both γ_1 and γ_2 (Otherwise the union of all components meeting γ_1 has α for a boundary component and then α is disjoint from γ_1 and γ_2.) Choosing d_1 (respectively d_2) in such a component we can make them disjoint from α. Now we want to modify d_1 in such a way that $|d_1 \cap \alpha| = 0$ and $|d_1 \cap \beta| < m$. There are three subcases.

Case 2a There exists an arc a_1 of α in S_1 which connects γ_1 and γ_2. Choose d_1 parallel to this arc. It may happen that d_1 meets β m times. Then a_1 is the only arc of α which meets β. We then modify d_1 as follows. We move from γ_1 along d_1 until it meets β. Then we turn along β, away from a_1, to the next point of a_1. We turn before crossing a_1 and move parallel to a_1 to γ_2. The new arc does not meet α and meets β less than m times.

Case 2b There exists an arc of α in S_1 which connects γ_1 and β and there exists an arc of α which connects γ_2 and β. Then there exist points P and Q of $\alpha \cap \beta$, consecutive along β, and arcs a_1 and a_2 of α such that a_1 connects γ_1 to P and a_2 connects Q to γ_2. We move along a_1 to P then along β, without crossing β, to Q, and then along a_2 to γ_2. The new arc does not meet α and meets β less than m times.

Case 2c If an arc of α in S_1 meets β then it meets only γ_1. (The case of γ_2 is similar.) We consider an arc d in S_1 which is disjoint from α and connects γ_1 and γ_2. We start at γ_2 and move along d to the first point of intersection with β. Then we move along β, without crossing it, to the first point of intersection with α. Then we move along α, away from β, to γ_1. The new arc does not meet α and meets β less than m times. If β is disjoint from α then β is either disjoint from a component of $S_1 - \alpha$ which connects γ_1 to γ_2 or is contained in it. We can find an arc in the component (disjoint from α) which connects γ_1 with γ_2 and meets β at most once.

So in each case we have an arc d_1 which is disjoint from α and meets β less than m times. We now slide the end-points of d_1 along γ_1 and γ_2 to meet the end-points of d_2. Each slide can be done along one of two arcs of γ_i. Choosing suitably we may assume that d_1 meets at most $m/2$ points of α sliding along γ_2 and at most $(m-1)/2$ points of α sliding along γ_1. The curve δ obtained from d_1 and d_2 meets α and β less than m times.

Case 3 The curve α lies in S_2. Then $|\alpha \cap \beta| = 0$ so we must have $m > 1$. We can choose d_2 which is disjoint from β and meets α at most once and we can choose d_1 which is disjoint from α and meets β at most once. The curve δ obtained from d_1 and d_2 meets α and β less than m times. □

Lemma 17 *Let δ_1 and δ_2 be non-separating curves on S and let w_1 be a vertex of X containing δ_1 and let w_2 be a vertex of X containing δ_2. Then there exists an edge-path $\mathbf{q} = (w_1 = z_1, z_2, \ldots, z_k = w_2)$ connecting w_1 and w_2. Suppose that we are also given non-separating curves α, β and an integer $m > 0$ such that $|\alpha \cap \beta| \leq m$, $|\alpha \cap \beta| = 1$ if $m = 1$, $|\delta_1 \cap \alpha| < m$, $|\delta_2 \cap \alpha| \leq m$, and $|\beta \cap \delta_1| = |\beta \cap \delta_2| = 0$. Then there exists a path $\mathbf{q}$ as above and an integer j, $1 \leq j < k$, such that $d(z_i, \beta) < m$ for all i, $d(z_i, \alpha) < m$ for $1 \leq i \leq j < k$ and z_i contains δ_2 for $j < i \leq k$.*

Proof We shall prove the lemma by induction on $|\delta_1 \cap \delta_2| = n$.

If $\delta_1 = \delta_2$ we can connect w_1 and w_2 by a δ_1–segment, by Lemma 12.

If $n = 1$ there exist vertices u_1, u_2 in X which are connected by an edge and such that $\delta_1 \in u_1$, $\delta_2 \in u_2$. Now we can connect u_1 to w_1 and w_2 to u_2 as in the previous case.

If $n = 0$ and $\delta_2 \cup \delta_1$ does not separate S then there exists a vertex v containing both curves δ_2 and δ_1. We can connect v to w_1 and w_2 as in the first case.

Suppose now that $n = 0$ and that $\delta_2 \cup \delta_1$ separates S. Then, by Lemma 16, there exists a curve δ such that $|\delta_2 \cap \delta| = |\delta_1 \cap \delta| = 1$. We can find a vertex v containing δ and we can connected v to w_1 and w_2 as in the second case. If we are also given curves α, and β and an integer m we can choose δ which also satisfies $|\alpha \cap \delta| < m$ and $|\beta \cap \delta| < m$. Then the path obtained by connecting v to w_1 and w_2 have all vertices in a distance less than m from β and in a distance less than m from α, except for the final δ_2–segment which ends at w_2 (Curve δ_2 may have distance m from α.)

If $n > 1$ then by Lemma 15 there exists a curve δ such that $|\delta_1 \cap \delta| < n$ and $|\delta_2 \cap \delta| < n$. We choose a vertex v containing δ. By induction on n we can connect v to w_1 and w_2. If we are also given curves α and β and an integer m then we can find δ which also satisfies $|\delta \cap \alpha| < m$ and $|\delta \cap \beta| = 0$. By induction on n we can connect w_1 to v and v to w_2 by a path the vertices of which are closer to β than m, and closer to α than m except for a final δ_2–segment which ends at w_2. □

As an immediate corollary we get:

Corollary 17.1 *Complex X is connected.*

We need one more lemma before we prove that every closed path is null-homotopic in X.

Lemma 18 *Let α, β, γ be non-separating curves on S such that $|\alpha\cap\beta| = m$, $|\alpha \cap \gamma| \leq m$, $|\beta \cap \gamma| = 1$. There exists a non-separating curve δ such that $|\delta \cap \alpha| < m$, $|\delta \cap \beta| = 0$ and $|\delta \cap \gamma| \leq 1$. If $m = 1$ then $|\delta \cap \gamma| = 0$ and $[\delta]$ is different from $[\alpha]$, $[\beta]$ and $[\gamma]$.*

Proof When we split S along $\gamma \cup \beta$ we get a surface S_1 with a "rectangular" boundary component ∂ consisting of two β–edges (vertical) and two γ–edges (horizontal on pictures of Figure 9). We can think of S_1 as a rectangle with holes and with some handles attached to it. Curve α intersects S_1 along some arcs a_i with end-points P_i and Q_i on ∂. If, for some i, points P_i and Q_i lie on the same β–edge then $m > 1$ and we can construct a curve δ consisting of an arc parallel to a_i and an arc parallel to the arc of β which connects P_i and Q_i passing through the point $\gamma \cap \beta$. Then $|\delta \cap \beta| = 0$, $|\delta \cap \alpha| < m$ and $|\delta \cap \gamma| = 1$. Recall that if two curves intersect exactly at one point then they are both non-separating on S. Therefore δ satisfies the conditions of the Lemma. If for some i points P_i and Q_i lie on different γ–edges of ∂ then we can modify the arc a_i sliding its end-point P_i along the γ edge to the point corresponding to Q_i. We get a closed curve δ satisfying the conditions of the Lemma. So we may assume that there are no arcs a_i of the above types.

Suppose that for every pair i, j the pairs of end-points P_i, Q_i and P_j, Q_j do not separate each other on ∂. Then we can connect the corresponding end points by nonintersecting intervals inside a rectangle. In the other words a regular neighbourhood of $\alpha \cup \partial$ in S_1 is a planar surface homeomorphic to a rectangle with holes. Since S_1 has positive genus there exists a subsurface of S_1 of a positive genus attached to one hole or a subsurface of S_1 which connects two holes of the rectangle. Such a subsurface contains a curve δ which is non-separating on S and is disjoint from α, β and γ and the homology class $[\delta]$ is different from $[\alpha]$, $[\beta]$ and $[\gamma]$. This happens in particular when $m = 1$ because then there is at most one point on every edge and the pairs of end points of arcs do not separate each other.

So we may assume that $m > 1$ and that there exists a pair of arcs, say a_1 and a_2, such that the pair P_1, Q_1 separates the pair P_2, Q_2 in ∂. Since an arc a_i does not connect different γ–edges we must have two points, say P_1 and P_2, on the same edge. Suppose that they lie on a β–edge, say the left edge. Choosing an intermediate point, if there is one, we may assume that P_1 and P_2 are consecutive points of α along β. We have different possible configurations of pairs of points. For each of them we construct curves δ_i, as on Figure 9. Each δ_i is disjoint from β and intersects γ at most once, and if it is disjoint from γ it intersects some other curve once. So δ_i is not-separating. We shall prove that we can always choose a suitable δ_i with $|\delta_i \cap \alpha| < m$. Observe that δ_i may meet α only along the boundary ∂, and not along the arc connecting P_1 to P_2.

Case 1 Points Q_1 and Q_2 lie on the same γ–edge, say lower edge. If there is no point of α on γ to the left of Q_1 then $|\delta_1 \cap \alpha| < m$. If there is a point of α on γ to the left of Q_1 then $|\delta_2 \cap \alpha| < m$.

Case 2 Points Q_1 and Q_2 lie on different γ–edges. Then $|\delta_3 \cap \alpha| < m$.

Case 3 Points Q_1 and Q_2 lie on the right edge. Then $|\delta_4 \cap \alpha| < m$.

Case 4 One of the points Q_i, say Q_1, lies on a γ–edge and the other lies on a β–edge. Let u_i, $i = 1, \ldots, 6$ denote the number of intersection points of α with the corresponding piece of ∂ on Figure 9. Then $u_3 + u_4 = u_5 + u_6 = |\alpha \cap \beta| = m$ and $u_1 + u_2 \leq m$. Also $|\delta_1 \cap \alpha| = u_1 + u_4$, $|\delta_5 \cap \alpha| = u_1 + u_3$, $|\delta_6 \cap \alpha| = u_2 + u_5$ and $|\delta_7 \cap \alpha| = u_2 + u_6$. Moreover, since P_2 and Q_2 are connected by an arc of α, they represent different points on S (otherwise it would be the only arc of α) and $u_4 \neq u_6$. It follows that $|\delta_i \cap \alpha| < m$ for $i = 1, 5, 6$ or 7.

We may assume now that for every pair of arcs (i, j) whose end-points separate each other no two end-points lie on the same β–edge. If P_i and Q_i lie on a γ–edge and P_j lie in between then a_i together with the interval of γ between P_i and Q_i form a nonseparating curve which meets α less than m times. So we may assume that P_1 and P_2 lie on different β–edges, say P_1 on the left edge and P_2 on the right edge, and Q_1 and Q_2 lie on the lower edge. Replacing Q_1 or Q_2 by an intermediate point, if necessary, we may also assume that for every point Q_i between Q_1 and Q_2 the corresponding point P_i also lies between Q_1 and Q_2. Now if there is no point of α on the left edge below P_1 then $|\delta_1 \cap \alpha| < m$. If there is such point of α consider the one closest to P_1 and call it P_3. Then, by our assumptions, point Q_3 lies on the lower edge to the left of Q_2 and $|\delta_8 \cap \alpha| < m$.

This concludes the proof of the Lemma. □

Proposition 19 *A path* $\mathbf{p}$ *of radius* m *around* α *is null-homotopic.*

Proof Let v_0 be a vertex of $\mathbf{p}$ containing α. We say that $\mathbf{p}$ begins at v_0. Let v_1 be the first vertex of $\mathbf{p}$ which has distance m from α. Let $\mathbf{q}$ be the

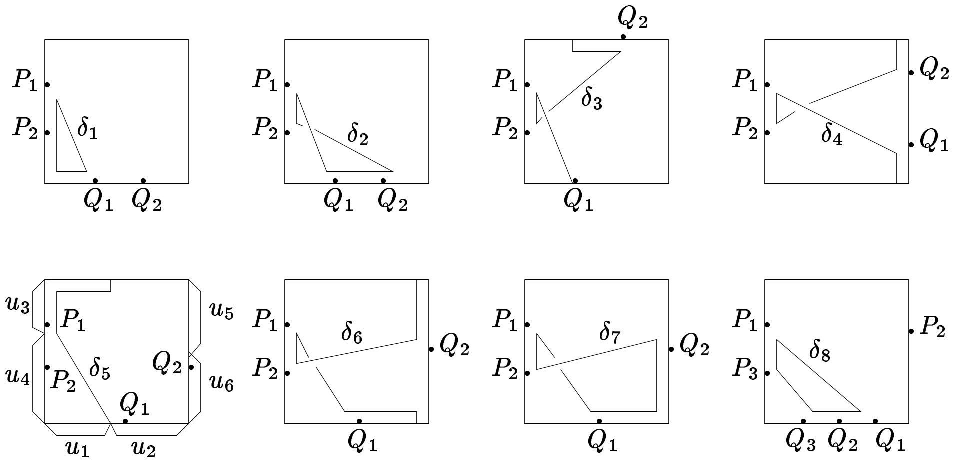

Figure 9: Constructing curve δ

maximal segment of $\mathbf{p}$, which starts at v_1 and contains some fixed curve β satisfying $|\beta \cap \alpha| = m$ and such that no vertex of $\mathbf{q}$ contains a curve β' satisfying $|\beta' \cap \alpha| < m$. Let v_2 be the last vertex of $\mathbf{q}$. Let u_1 be the vertex of $\mathbf{p}$ preceding v_1 and let u_2 be the vertex of $\mathbf{p}$ following v_2. Vertex u_1 contains a curve γ_1 such that $|\gamma_1 \cap \alpha| < m$. Vertex u_2 is the first vertex of the second segment which has a fixed curve γ_2 such that $|\gamma_2 \cap \alpha| \le m$. If u_1 contains β then $|\gamma_1 \cap \beta| = 0$. Otherwise, since v_1 does not contain γ_1, the move from u_1 to v_1 involves γ_1 and β, so $|\gamma_1 \cap \beta| = 1$. If v_2 contains γ_2 then $|\gamma_2 \cap \beta| = 0$. It may also happen that $|\gamma_2 \cap \alpha| < m$ and that $\beta \in u_2$. Then also $|\gamma_2 \cap \beta| = 0$. Otherwise $|\gamma_2 \cap \beta| = 1$. We want to construct vertices w_1, w_1', w_2 and w_2' and edge paths connecting them, as on Figure 10, so that the rectangles are null homotopic. Then we can replace the part of $\mathbf{p}$ between u_1 and u_2 by the path connecting consecutively u_1 to w_1', w_1' to w_1, w_1 to w_2, w_2 to w_2' and w_2' to u_2. We denote the new path by $\mathbf{p}'$.

In our construction vertex w_i contains a nonseparating curve δ_i disjoint from β. If $|\gamma_i \cap \beta| = 0$ we let $\delta_i = \gamma_i$, $w_i = w_i' = u_i$ and the corresponding rectangle degenerates to an edge. If $|\gamma_i \cap \beta| = 1$ we proceed as follows. By Lemma 18 there exists a nonseparating curve δ_i such that $|\delta_i \cap \beta| = 0$, $|\delta_i \cap \alpha| < m$, and $|\delta_i \cap \gamma_i| \le 1$. If $|\delta_i \cap \gamma_i| = 0$ (this is always the case if $m = 1$) then $[\delta_i] \ne [\gamma_i]$ and $[\delta_i] \ne [\beta]$ because they have different intersections. There exists a vertex w_i' containing δ_i and γ_i and a vertex w_i containing δ_i and β. We can connect u_i to w_i' by a γ_i–segment, w_i' to w_i by a δ_i–segment and w_i to v_i by a β–segment. The corresponding rectangle has radius zero around δ_i, so it is null-homotopic by the Induction Hypothesis 2.

If $|\delta_i \cap \gamma_i| = 1$ there exist vertices w_i' and w_i which are connected by an edge and contain γ_i and δ_i respectively. We connect w_i' to u_i by a γ_i–segment. We now apply Lemma 17 to vertices w_i and v_i with $\delta_1, \delta_2, \alpha, \beta$ replaced by $\delta_i, \beta, \gamma_i, \beta$ respectively and $m > 1$. There exists a path connecting w_i to v_i such that all vertices of the path have distance less than m from γ_i and β. The corresponding rectangle has radius less than m around γ_i so it is null-homotopic, by the Induction Hypothesis 2.

We now apply Lemma 17 to vertices w_1 and w_2. There exists a path $\mathbf{q} = (w_1 = z_1, z_2, \ldots, z_k = w_2)$ connecting w_1 and w_2 such that $d(z_i, \beta) < m$ for all i, $d(z_i, \alpha) < m$ for $1 \leq i \leq j < k$ and z_i contains δ_2 for $j < i \leq k$. In particular the middle rectangle on Figure 10 has radius less than m around β so it is null-homotopic by the Induction Hypothesis 2. All vertices of the new part of path $\mathbf{p}'$ have distance less than m from α except for the final γ_2–segment from w_2' to u_2, if $|\gamma_2 \cap \beta| = 1$, or final $\delta_2 = \gamma_2$–segment of $\mathbf{q}$, if $|\gamma_2 \cap \beta| = 0$ and the right rectangle degenerates. Thus $\mathbf{p}'$ has smaller number of segments at the distance m from α, it has no β–segment, and it is null-homotopic by induction on the number of segments of $\mathbf{p}$ at the distance m from some curve α. □

$$
\begin{tikzcd}
{} \arrow[r] & u_1 \arrow[r] \arrow[d, "\gamma_1"'] & v_1 \arrow[r, "\beta"] & v_2 \arrow[r] & u_2 \arrow[r, "\gamma_2"] & {} \\
 & w_1' \arrow[r, "\delta_1"] & w_1 \arrow[u] \arrow[r] & w_2 \arrow[u] \arrow[r, "\delta_2"] & w_2' \arrow[u, "\gamma_2"'] &
\end{tikzcd}
$$

Figure 10: Reducing a path of radius m

This concludes the proof of Theorem 4.

3 A presentation of $M_{g,1}$

In this section we shall consider a surface S of genus $g > 1$ with one boundary component ∂. Let $\mathcal{M}_{g,1}$ be the mapping class group of S. Let X be the cut-system complex of S described in the previous section. We shall establish a presentation of $\mathcal{M}_{g,1}$ via its action on X. The action of $\mathcal{M}_{g,1}$ on X is defined by its action on vertices of X. If $v = \langle C_1, \dots, C_g \rangle$ is a vertex of X and $g \in \mathcal{M}_{g,1}$ then $g(v) = \langle g(C_1), \dots, g(C_g) \rangle$.

We start with some properties of homeomorphisms of a surface. Then we describe stabilizers of vertices and edges of the action of $\mathcal{M}_{g,1}$ on X. Finally we consider the orbits of faces of X and determine a presentation of X.

In order to shorten some long formulas we shall adopt the following notation for conjugation: $a * b = aba^{-1}$. As usually $[a,b] = aba^{-1}b^{-1}$.

Remark 3 Some proofs of relations between homeomorphisms of a surface are left to the reader. The general idea of a proof is as follows. We split the surface into a union of disks by a finite number of curves (and arcs with the end-points on the boundary if the surface has a boundary). We prove that the given product of homeomorphisms takes each curve (respectively arc) onto an isotopic curve (arc), preserving some fixed orientation of the curve (arc). Then the product is isotopic to a homeomorphism pointwise fixed on each curve and arc. But a homeomorphism of a disk fixed on its boundary is isotopic to the identity homeomorphism, relative to the boundary, by Lemma of Alexander. Thus the given product of homeomorphisms is isotopic to the identity.

Dehn proved in [5] that every homeomorphism of S is isotopic to a product of twists. We start with some properties of twists.

Lemma 20 *Let α be a curve on S, let h be a homeomorphism and let $\alpha' = h(\alpha)$. Then $T_{\alpha'} = hT_\alpha h^{-1}$.*

Proof Since h maps α to α' we may assume that (up to isotopy) it also maps a neighborhood N of α to a neighborhood N' of α'. The homeomorphism h^{-1} takes N' to N, then T_α maps N to N, twisting along α, and h takes N back to N'. Since T_α is supported in N, the composite map is supported in N' and is a Dehn twist about α'. □

Lemma 21 *Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be a chain of curves, ie, the consecutive curves intersect once and non-consecutive curves are disjoint. Let N denote the regular neighbourhood of the union of these curves. Let c_i denote the twist along γ_i. Then the following relations hold:*

(i) *The "commutativity relation": $c_i c_j = c_j c_i$ if $|i - j| > 1$.*

(ii) *The "braid relation": $c_i c_j(\gamma_i) = \gamma_j$, and $c_i c_j c_i = c_j c_i c_j$ if $|i - j| = 1$.*

(iii) *The "chain relation": If k is odd then N has two boundary components, ∂_1 and ∂_2, and $(c_1c_2\dots c_k)^{k+1} = T_{\partial_1}T_{\partial_2}$. If k is even then N has one boundary component ∂_1 and $(c_1c_2\dots c_k)^{2k+2} = T_{\partial_1}$.*

(iv) $(c_2c_1c_3c_2)(c_4c_3c_5c_4)(c_2c_1c_3c_2) = (c_4c_5c_3c_4)(c_2c_1c_3c_2)(c_4c_3c_5c_4)$.

(v) $(c_1c_2\dots c_k)^{k+1} = (c_1c_2\dots c_{k-1})^k(c_kc_{k-1}\dots c_2c_1^2c_2\dots c_{k-1}c_k) = (c_kc_{k-1}\dots c_2c_1^2c_2\dots c_{k-1}c_k)(c_1c_2\dots c_{k-1})^k$.

Proof Relation (i) is obvious. It follows immediately from the definition of Dehn twist that $c_2(\gamma_1) = c_1^{-1}(\gamma_2)$. Both statements of (ii) follow from this and from Lemma 20. Relation (iii) is a little more complicated. It can be proven by the method explained in Remark 3. Relations (iv) and (v) follow from the braid relations (i) and (ii) by purely algebraic operations. We shall prove (iv).

$$
\begin{aligned}
&(c_2c_1c_3c_2)(c_4c_3c_5c_4)(c_2c_1c_3c_2) = c_2c_3c_1c_2c_4c_5c_3c_4c_2c_3c_1c_2 = \\
&c_2c_3c_4c_1c_5c_2c_3c_2c_4c_3c_1c_2 = c_2c_3c_4c_3c_1c_5c_2c_3c_4c_3c_1c_2 = \\
&c_4c_2c_3c_4c_1c_5c_2c_4c_3c_1c_2c_4 = c_4c_2c_3c_1c_4c_5c_4c_2c_3c_1c_2c_4 = \\
&c_4c_5c_2c_3c_1c_4c_2c_3c_1c_2c_5c_4 = c_4c_5c_2c_3c_1c_2c_1c_4c_3c_2c_5c_4 = \\
&c_4c_5c_3c_2c_3c_4c_1c_2c_3c_2c_5c_4 = c_4c_5c_3c_2c_3c_4c_3c_1c_2c_3c_5c_4 = \\
&c_4c_5c_3c_4c_2c_3c_4c_1c_2c_3c_5c_4 = (c_4c_5c_3c_4)(c_2c_1c_3c_2)(c_4c_3c_5c_4).
\end{aligned}
$$

We now prove (v). We prove by induction that for $s \le k$ we have $(c_1c_2\dots c_k)^s = (c_1c_2\dots c_{k-1})^s(c_kc_{k-1}\dots c_{k-s+1})$. Using (i) and (ii) one checks easily that for $i > 1$ we have $c_i(c_1c_2\dots c_k) = (c_1c_2\dots c_k)c_{i-1}$. Now

$$
\begin{aligned}
&(c_1c_2\dots c_k)^{s+1} = (c_1c_2\dots c_{k-1})^s(c_kc_{k-1}\dots c_{k-s+1})(c_1c_2\dots c_k) = \\
&(c_1c_2\dots c_{k-1})^s c_1c_2\dots c_kc_{k-1}\dots c_{k-s} = (c_1c_2\dots c_{k-1})^{s+1}c_kc_{k-1}\dots c_{k-s}.
\end{aligned}
$$

For $s = k+1$ we get $(c_1c_2\dots c_k)^{k+1} = (c_1c_2\dots c_{k-1})^k(c_kc_{k-1}\dots c_1c_1c_2\dots c_k)$.

This proves the first equality in (v). By (i) and (ii) $(c_kc_{k-1}\dots c_2c_1^2c_2\dots c_{k-1}c_k)$ commutes with c_i for $i < k$, which implies the second equality. □

The next lemma was observed by Dennis Johnson in [12] and was called a *lantern relation.*

Lemma 22 *Let U be a disk with the outer boundary ∂ and with 3 inner holes bounded by curves $\partial_1, \partial_2, \partial_3$ which form vertices of a triangle in the clockwise order. For $1 \le i < j \le 3$ let $\alpha_{i,j}$ be the simple closed curve in U which bounds a neighbourhood of the "edge" (∂_i, ∂_j) of the triangle (see Figure 11). Let d be the twist along ∂, d_i the twist along ∂_i and $a_{i,j}$ the twist along $\alpha_{i,j}$.*

Then $dd_1d_2d_3 = a_{1,2}a_{1,3}a_{2,3}$.

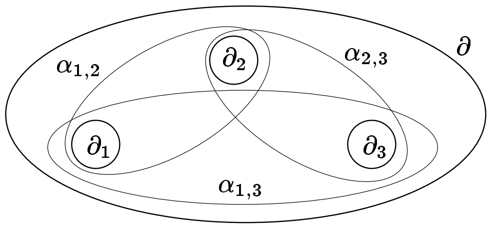

Figure 11: Lantern relation

We now describe a presentation of the mapping class group of a disk with holes.

Lemma 23 *Let U be a disk with the outer boundary ∂ and with n inner holes bounded by curves $\partial_1, \partial_2, \ldots, \partial_n$. For $1 \le i < j \le n$ let $\alpha_{i,j}$ be the simple closed curve in U shown on Figure 12, separating two holes ∂_i and ∂_j from the other holes. Let d be the twist along ∂, d_i the twist along ∂_i and $a_{i,j}$ the twist along $\alpha_{i,j}$. Then the mapping class group of U has a presentation with generators d_i and $a_{i,j}$ and with relations*

(Q1) $[d_i, d_j] = 1$ *and* $[d_i, a_{j,k}] = 1$ *for all* i, j, k.

(Q2) *pure braid relations*

(a) $a_{r,s}^{-1} * a_{i,j} = a_{i,j}$ *if* $r < s < i < j$ *or* $i < r < s < j$,

(b) $a_{r,s}^{-1} * a_{s,j} = a_{r,j} * a_{s,j}$ *if* $r < s < j$,

(c) $a_{r,j}^{-1} * a_{r,s} = a_{s,j} * a_{r,s}$ *if* $r < s < j$,

(d) $[a_{i,j}, a_{r,j}^{-1} * a_{r,s}] = 1$ *if* $r < i < s < j$.

Proof Relations (Q2) come in place of standard relations in the pure braid group on n strings and we shall first prove the equivalence of (Q2) to the standard presentation of the pure braid group. The standard presentation has generators $a_{i,j}$ and relations (this is a corrected version of the relations in [2]):

(i) $a_{r,s}^{-1} * a_{i,j} = a_{i,j}$ if $r < s < i < j$ or $i < r < s < j$,

(ii) $a_{r,s}^{-1} * a_{s,j} = a_{r,j} * a_{s,j}$ if $r < s < j$,

(iii) $[a_{r,j}, a_{s,j}] = [a_{r,s}^{-1}, a_{r,j}^{-1}]$ if $r < s < j$,

(iv) $a_{r,s}^{-1} * a_{i,j} = [a_{r,j},\ a_{s,j}] * a_{i,j}$ if $r < i < s < j$.

So relations (a) and (b) are the same as (i) and (ii) respectively. We can substitute relation (iii) in (iv) and get (d), after cancellation of $a_{r,s}$. When we

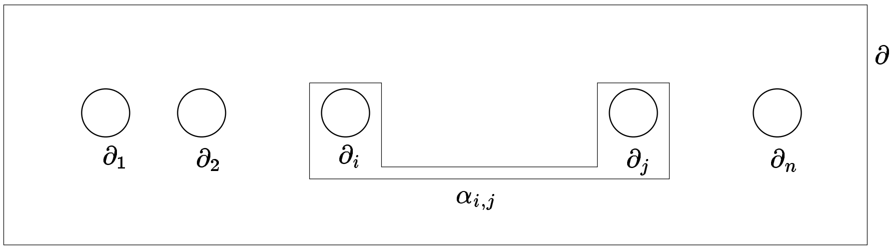

Figure 12: Generators of $\mathcal{M}_{0,n}$

substitute (ii) for the first three terms of (iii) we get (c), after cancellation of $a_{r,s}^{-1}$.

We now consider the disk U with holes. When we glue a disk with a distinguished center to each curve ∂_i we get a disk with n distinquished points. Its mapping class group is isomorphic to the pure braid group P_n with generators $a_{i,j}$ and with relations (Q2). In the passage from the mapping class group of U to the mapping class group of the punctured disk we kill exactly the twists d_i, which commute with everything. One can check that the removal of the disks does not affect the relations (Q2) so the mapping class group of U has a presentation with relations (Q1) and (Q2). $\square$

We now consider the surface S. We shall fix some curves on S. The surface S consists of a disk with g handles attached to it. For $i = 1,\dots,g$ and $j = 1,\dots,g-1$ we fix curves α_i, β_i, ϵ_j (see Figure 1). Curve α_i is a meridian curve across the i-th handle, β_i is a curve along the i-th handle and ϵ_i runs along i-th handle and $(1+i)$-th handle. Curves $\alpha_1,\alpha_2,\dots,\alpha_g$ form a cut-system.

We denote by I_0 the set of indices $I_0 = \{-g, 1-g, 2-g, \dots, -1, 1, 2, \dots, g-1, g\}$. When we cut S open along the curves $\alpha_1,\dots,\alpha_g$ we get a disk S_0 with $2g$ holes bounded by curves ∂_i, $i \in I_0$, where curves ∂_i and ∂_{-i} correspond to the same curve α_i on S (see Figure 13). The glueing back map identifies ∂_i with ∂_{-i} according to the reflection with respect to the x–axis. Curves on S can be represented on S_0. If a curve on S meets some curves α_i then it is represented on S_0 by a disjoint union of several arcs. In particular ϵ_i is represented by two arcs joining ∂_{-i} to ∂_{-i-1} and ∂_i to ∂_{i+1}. We denote by $\delta_{i,j}$, $i < j \in I_0$, curves on S represented on Figure 13. Curve $\delta_{i,j}$ separates holes ∂_i and ∂_j from the other holes on S_0. We now fix some elements of $\mathcal{M}_{g,1}$.

Definition 4 We denote by a_i, b_i, e_j the Dehn twists along the curves $\alpha_i, \beta_i, \epsilon_j$ respectively. We fix the following elements of $\mathcal{M}_{g,1}$:

$s = b_1 a_1 a_1 b_1$.
$t_i = e_i a_i a_{i+1} e_i$ for $i = 1, \dots, g-1$.
$d_{1,2} = (b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1}) * b_2$.

For $i < j \in I_0$ we let

$d_{i,j} = (t_{i-1} t_{i-2} \dots t_1 t_{j-1} t_{j-2} \dots t_2) * d_{1,2}$ if $i > 0$,
$d_{i,j} = (t_{-i-1}^{-1} t_{-i-2}^{-1} \dots t_1^{-1} s^{-1} t_{j-1} t_{j-2} \dots t_2) * d_{1,2}$ if $i < 0$ and $i + j > 0$,
$d_{i,j} = (t_{-i-1}^{-1} t_{-i-2}^{-1} \dots t_1^{-1} s^{-1} t_j t_{j-1} \dots t_2) * d_{1,2}$ if $i < 0$, $j > 0$ and $i + j < 0$,
$d_{i,j} = (t_{-j-1}^{-1} t_{-j-2}^{-1} \dots t_1^{-1} t_{-i-1}^{-1} t_{-i-2}^{-1} \dots t_2^{-1} s^{-1} t_1^{-1} s^{-1}) * d_{1,2}$ if $j < 0$,
$d_{i,j} = (t_{j-1}^{-1} d_{j-1,j} t_{j-2}^{-1} d_{j-2,j-1} \dots t_1^{-1} d_{1,2}) * (s^2 a_1^4)$ if $i + j = 0$.

The products described in the above definition represent very simple elements of $\mathcal{M}_{g,1}$ and we shall explain their meaning now. We shall first define special homeomorphisms of S_0.

Definition 5 A *half-twist* $h_{i,j}$ along a curve $\delta_{i,j}$ is an isotopy class (on S_0 relative to its boundary) of a homeomorphism of S_0 which is fixed outside $\delta_{i,j}$ and is equal to a counterclockwise "rotation" by 180 degrees inside $\delta_{i,j}$. In particular $h_{i,j}$ switches the two holes ∂_i and ∂_j inside $\delta_{i,j}$ so it is not fixed on the boundary of S_0, but $h_{i,j}^2$ is fixed on the boundary of S_0 and is isotopic to the full Dehn twist along $\delta_{i,j}$.

Lemma 24 *The result of the action of t_k (respectively s) on a curve $\delta_{i,j}$ is the same as the result of the action of the product of half-twists $h_{k,k+1} h_{-k-1,-k}$ (respectively the result of the action of $h_{-1,1}$) on $\delta_{i,j}$. So t_k rotates $\delta_{i,j}$ around $\delta_{k,k+1}$ counterclockwise and around $\delta_{-k-1,-k}$ conterclockwise and switches the corresponding holes. If the pair (i,j) is disjoint from $\{k, k+1, -k, -k-1\}$ than t_k leaves $\delta_{i,j}$ fixed. It also leaves curves $\delta_{k,k+1}$ and $\delta_{-k-1,-k}$ fixed. In a similar way s rotates $\delta_{i,j}$ counterclockwise around $\delta_{-1,1}$ and switches the holes. If (i,j) is disjoint from $(-1,1)$ than s leaves $\delta_{i,j}$ fixed. Also $\delta_{-1,1}$ is fixed by s.*

Proof The result of the action can be checked directly. □

Lemma 25 *The element $d_{i,j}$ of $\mathcal{M}_{g,1}$ is equal to the twist along the curve $\delta_{i,j}$ for all i, j.*

Proof We start with the curve β_2 and apply to it the product of twists $b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1}$. We get the curve $\delta_{1,2}$. Therefore, by Lemma 20, $d_{1,2}$ is equal to the twist along $\delta_{1,2}$. For $i \neq -j$ we start with $\delta_{1,2}$ and apply consecutive factors t_i and s, one at a time. We check that the result is $\delta_{i,j}$ so $d_{i,j}$ is the twist along $\delta_{i,j}$, by Lemma 20. For $d_{-1,1}$ we use (iii) of Lemma 21. The curve $\delta_{-1,1}$ is the boundary of a regular neighbourhood of $\alpha_1 \cup \beta_1$ and $(a_1 b_1)^6 = s^2 a_1^4$ by (i) and (ii) of Lemma 21, therefore $d_{-1,1} = (a_1 b_1)^6$ is the twist along $\delta_{-1,1}$ by (iii) of Lemma 21. Now we apply a suitable product of t_i's and $d_{i,i+1}$'s to $\delta_{-1,1}$ and get $\delta_{-j,j}$. Therefore $d_{-j,j}$ is equal to the twist along $\delta_{-j,j}$ by Lemma 20. □

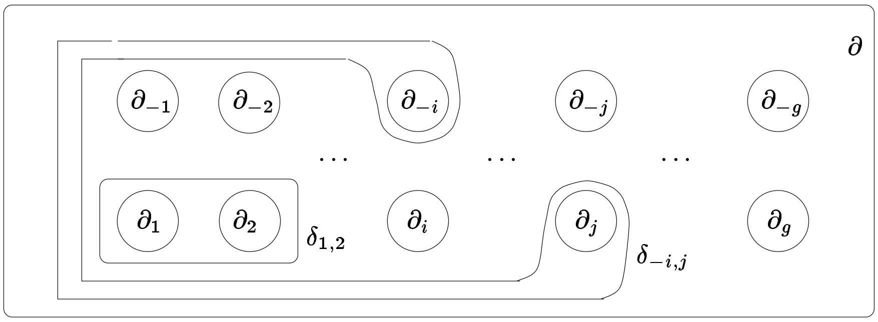

Figure 13: Curves $\delta_{i,j}$ on S_0

We can explain now the relations in Theorem 1.

Lemma 26 *The relations* (M1), (M2) *and* (M3) *from Theorem 1 are satisfied in* $\mathcal{M}_{g,1}$.

Proof Relations (M1) follow from Lemma 21 (i) and (ii). Curves $\beta_1, \alpha_1, \epsilon_1$ form a chain. One boundary component of a regular neighbourhood of $\alpha_1 \cup \beta_1 \cup \epsilon_1$ is equal to β_2. It is easy to check that $a_2 e_1 a_1 b_1^2 a_1 e_1 a_2(\beta_2)$ is equal to the other boundary component. By Lemma 21 (iii) and by Lemma 20 we have $(b_1 a_1 e_1)^4 = b_2 a_2 e_1 a_1 b_1^2 a_1 e_1 a_2 b_2 (a_2 e_1 a_1 b_1^2 a_1 e_1 a_2)^{-1}$. This is equivalent to relation (M2) by Lemma 21 (v). Consider now relation (M3). Applying consecutive twists one can check that $(b_2 a_2 e_1 b_1^{-1})(\delta_{1,3}) = \delta_3$, where δ_3 is the curve on Figure 1. Thus d_3 represents the twist along δ_3. When we cut S along curves α_1, α_2, α_3 and δ_3 we split off a sphere with four holes from surface S. Since elements $d_{i,j}$ represent twists along curves $\delta_{i,j}$, relation (M3) follows from lantern relation, Lemma 22. □

Our first big task is to establish a presentation of a stabilizer of one vertex of X. Let v_0 be a fixed vertex of X corresponding to the cut system $\langle \alpha_1, \alpha_2, \ldots, \alpha_g \rangle$. Let H be the stabilizer of v_0 in $\mathcal{M}_{g,1}$.

Proposition 27 *The stabilizer H of vertex v_0 admits the following presentation:*

The set of generators consists of $a_1, a_2, \ldots, a_g$, s, t_1, t_2, $\ldots$, t_{g-1} and the $d_{i,j}$'s for $i<j$, $i,j \in I_0$.

The set of defining relations consists of:

(P1) $[a_i, a_j] = 1$ *and* $[a_i, d_{j,k}] = 1$ *for all* $i,j,k \in I_0$.

(P2) *pure braid relations*

(a) $d_{r,s}^{-1} * d_{i,j} = d_{i,j}$ *if* $r<s<i<j$ or $i<r<s<j$,

(b) $d_{r,s}^{-1} * d_{s,j} = d_{r,j} * d_{s,j}$ *if* $r<s<j$,

(c) $d_{r,j}^{-1} * d_{r,s} = d_{s,j} * d_{r,s}$ *if* $r<s<j$,

(d) $[d_{i,j}, d_{r,j}^{-1} * d_{r,s}] = 1$ *if* $r<i<s<j$.

(P3) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ *for* $i = 1, \ldots, g-2$ *and* $[t_i, t_j] = 1$ *if* $1 \le i < j-1 < g-1$.

(P4) $s^2 = d_{-1,1} a_1^{-4}$ *and* $t_i^2 = d_{i,i+1} d_{-i-1,-i} a_i^{-2} a_{i+1}^{-2}$ *for* $i = 1, \ldots, g-1$.

(P5) $[t_i, s] = 1$ *for* $i = 2, \ldots, g-1$.

(P6) $s t_1 s t_1 = t_1 s t_1 s$.

(P7) $[s, a_i] = 1$ *for* $1 \le i \le g$, $t_i * a_i = a_{i+1}$ *for* $1 \le i \le g-1$,
$[a_i, t_j] = 1$ *for* $1 \le i \le g$, $j \ne i, i-1$.

(P8) $s * d_{i,j} = d_{i,j}$ *if* $i \ne \pm 1$ *and* $j \ne \pm 1$ *or if* $i = -1$ *and* $j = 1$,
$s * d_{-1,j} = d_{1,j}$ *for* $2 \le j \le g$, $s * d_{i,-1} = d_{i,1}$ *for* $-g \le i \le -2$,
$t_k * d_{i,j} = d_{i,j}$ *if* $1 \le k \le g-1$ *and* ($j = i+1 = k+1$, *or* $j = i+1 = -k$ *or* $i,j \notin \{\pm k, \pm(k+1)\}$),
$t_k * d_{k,j} = d_{k+1,j}$ *for* $1 \le k \le g-1$ *and* $k+2 \le j \le g$,
$t_k * d_{i,-k-1} = d_{i,-k}$ *for* $1 \le k \le g-1$ *and* $-g \le i \le -k-2$,
$t_k * d_{-k-1,k} = d_{-k,k+1}$, $t_k * d_{-k-1,k+1} = d_{k,k+1} * d_{-k,k}$, *for* $1 \le k \le g-1$
$t_k * d_{-k-1,j} = d_{-k,j}$ *for* $1 \le k \le g-1$ *and* $j > -k$, $j \ne k, k+1$,
$t_k * d_{i,k} = d_{i,k+1}$ *for* $1 \le k \le g-1$ *and* $i < k$, $i \ne -k, -k-1$.

Proof An element of H leaves the cut-system v_0 invariant but it may permute the curves α_i and may reverse their orientation. Clearly a_i belongs to H. One can easily check that $t_i(\alpha_i) = \alpha_{i+1}$, $t_i(\alpha_{i+1}) = \alpha_i$, $t_i(\alpha_k) = \alpha_k$ for $k \ne i, i+1$.

$s(\alpha_1)$ is equal to α_1 with the opposite orientation and s is fixed on other α_i's. Thus t_i's and s belong to H. By Lemma 25 we know that $d_{i,j}$ is a twist along the curve $\delta_{i,j}$ so it also belongs to H. We shall prove in the next section that the relations (P1) – (P8) follow from the relations (M1) – (M3), so they are satisfied in $\mathcal{M}_{g,1}$ and thus also in H. The group H can be defined by two exact sequences.

$$1 \to \mathbb{Z}_2^g \to \pm\Sigma_g \to \Sigma_g \to 1. \tag{1}$$

$$1 \to H_0 \to H \to \pm\Sigma_g \to 1. \tag{2}$$

Before defining the objects and the homomorphisms in these sequences we shall recall the following fact from group theory.

Lemma 28 *Let $1 \to A \to B \to C \to 1$ be an exact sequence of groups with known presentations $A = \langle a_i | Q_j \rangle$ and $C = \langle c_i | W_j \rangle$. A presentation of B can be obtained as follows: Let b_i be a lifting of c_i to B. Let R_j be a word obtained from W_j by substitution of b_i for each c_i. Then R_j represents an element d_j of A which we write as a product of generators a_i of A. Finally for every a_i and b_j the conjugate $b_j * a_i$ represents an element $a_{i,j}$ of A, which we write as a product of the generators a_i.*

Then $B = \langle a_i, b_j | Q_j, R_j = d_j, b_j * a_i = a_{i,j} \rangle$.

We now describe the sequence (1) and the group $\pm\Sigma_g$. This is the group of permutations of the set $I_0 = \{-g, 1-g, \ldots, -1, 1, 2, \ldots, g\}$ such that $\sigma(-i) = -\sigma(i)$. The homomorphism $\pm\Sigma_g \to \Sigma_g$ forgets the signs. A generator of the kernel changes the sign of one letter. The sequence splits, Σ_g can be considered as the subgroup of the permutations which take positive numbers to positive numbers. Let $\tau_i = (i, i+1)$ be a transposition in Σ_g for $i = 1, 2, \ldots, g-1$. Then

(S1) $[\tau_i, \tau_j] = 1$ for $|i-j| > 1$,

(S2) $\tau_i * \tau_{i+1} = \tau_{i+1}^{-1} * \tau_i$ for $i = 1, \ldots, g-2$,

(S3) $\tau_i^2 = 1$ for $i = 1, \ldots, g-1$.

This defines a presentation of Σ_g. Further let σ_i for $i = 1, \ldots, g$ denote the change of sign of the i-th letter in a signed permutation. Then $\sigma_i^2 = 1$ and $[\sigma_i, \sigma_j] = 1$ for all i, j. Finally the conjugation gives $\tau_i * \sigma_i = \sigma_{i+1}$, $\tau_i * \sigma_{i+1} = \sigma_i$ and $[\sigma_j, \tau_i] = 1$ for $j \neq i$ and $j \neq i+1$. In fact it suffices to use one generator $\sigma = \sigma_1$ and the relations $\sigma_i = (\tau_{i-1}\tau_{i-2} \ldots \tau_1) * \sigma$. We get the relations

(S4) $\sigma^2 = 1$,

(S5) $[\sigma, \tau_j] = 1$ and $[(\tau_i\tau_{i-1}\ldots\tau_1)*\sigma, \tau_j] = 1$ for $1 \le i \le g-1$ and $j \ne i$ and $j \ne i+1$,

(S6) $[(\tau_i\tau_{i-1}\ldots\tau_1)*\sigma, \sigma] = 1$ and $[(\tau_i\tau_{i-1}\ldots\tau_1)*\sigma, (\tau_j\tau_{j-1}\ldots\tau_1)*\sigma] = 1$ for $1 \le i, j \le g-1$.

Group $\pm\Sigma_g$ has a presentation with generators $\sigma, \tau_1, \ldots, \tau_{g-1}$ and with defining relations (S1) – (S6).

We shall describe now the sequence (2). A homeomorphism in H may permute the curves α_i and may change their orientations. We fix an orientation of each curve α_i and define a homomorphism $\phi_1\colon H \to \pm\Sigma_g$ as follows: a homeomorphism h is mapped onto a permutation $i \mapsto \pm j$ if $h(\alpha_i) = \alpha_j$ and the sign is "$+$" if the orientations of $h(\alpha_i)$ and of α_j agree, and "$-$" otherwise. If h preserves the isotopy class of α_i and preserves its orientation then it is isotopic to a homeomorphism fixed on α_i. The kernel of ϕ_1 is the subgroup H_0 of the elements of H represented by the homeomorphisms which keep the curves $\alpha_1, \alpha_2, \ldots, \alpha_g$ pointwise fixed. We want to find a presentation of H from the sequence (2). We start with a presentation of H_0. An element of H_0 induces a homeomorphism of S_0. When we glue back the corresponding pairs of boundary components of S_0 we get the surface S. This glueing map induces a homomorphism from the mapping class group of S_0 onto H_0.

We shall prove that the kernel of this homomorphism is generated by products $d_i d_{-i}^{-1}$, where d_i is the twist along curve ∂_i, so both twists are identified with a_i in H_0. It suffices to assume, by induction, that we glue only one pair of boundaries ∂_i and ∂_{-i} on S_0 and get a nonseparating curve α_i on S. Homeomorphism $d_i d_{-i}^{-1}$ induces a spin map s of S along α_i (see [2], Theorem 4.3 and Fig. 14). Let γ be a curve on S which meets α_i at one point P. Let h_0 be a homeomorphism of S_0 which induces a homeomorphism h of S isotopic to the identity and fixed on α_i. We shall prove that for a suitable power k the map $s^k h$ is fixed on γ (after an isotopy of S liftable to S_0). If $h(\gamma)$ forms a 2–gon with γ we can get rid of the 2–gon by an isotopy liftable to S_0 (fixed on α_i). If $h(\gamma)$ and γ form no 2–gons then they are tangent at P. Let us move $h(\gamma)$ off γ, near the point P, to a curve γ'. If γ' and γ intersect then they form a 2–gon. But then $h(\gamma)$ and γ form a self-touching 2–gon (see Figure 14). Clearly the spin map s or s^{-1} removes the 2–gon. If γ' and γ are disjoint then they bound an annulus. If the annulus contains the short arc of α_i between γ' and γ then $h(\gamma)$ is isotopic to γ by an isotopy liftable to S_0. If the annulus contains the long arc of α_i between γ' and γ then the situation is similar to Figure 14, but $h(\gamma)$ does not meet γ outside P. The spin map s or s^{-1} takes $h(\gamma)$ onto a curve isotopic to γ by an isotopy liftable to S_0. So we may assume, by induction on $|h(\gamma)\cap\gamma|$, that $h(\gamma)=\gamma$. We proceed as in Remark 3. We spilt S into disks by curves α_i, γ, and other curves disjoint from α_i. Homeomorphism h takes each curve to an isotopic curve and the 2–gons which appear do not contain α_i. Thus all isotopies are liftable to S_0.

It follows that the kernel is generated by $d_id_{-i}^{-1}$'s. By Lemma 23 the map-

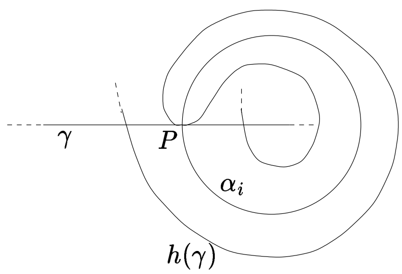

Figure 14: Self-touching 2–gon

ping class group of S_0 has a presentation with generators d_k and $d_{i,j}$ and with relations (Q1) and (Q2), where $a_{i,j}=d_{i,j}$, $i,j\in I_0$. Therefore H_0 has a presentation with generators $a_1,\dots,a_g$ and $d_{i,j}$ for $i<j\in I_0$ and with relations (P1), (P2). In these relations $d_{i,j}$ is represented by a Dehn twist along $\delta_{i,j}$.

We come back to sequence (2). We see from the action of t_i and s on α_j that we can lift τ_i to t_i and σ to s. Relations (S1) and (S2) lift to (P3). Relations (S3) and (S4) lift to (P4). Relation (S5) lifts to $[(t_it_{i-1}\dots t_1)*s,t_j]=1$ for $j\neq i$ and $j\neq i+1$ and it follows from (P3) and (P5). We shall deal with (S6) a little later. We now pass to the conjugation of the generators of H_0 by s and t_k. Since $s^2\in H_0$ and $t_k^2\in H_0$, by (P4), it suffices to know the result of the conjugation of each generator of H_0 by either s or by s^{-1}, and by either t_k or by t_k^{-1}, the other follows. The result of the conjugation is described in relations (P7) and (P8). Finally we lift the relation (S6). We start with the case $[\tau_1*\sigma,\sigma]$. It lifts to $t_1st_1^{-1}st_1s^{-1}t_1^{-1}s^{-1}=$ (by (P6)) $t_1st_1^{-1}ss^{-1}t_1^{-1}s^{-1}t_1=t_1st_1^{-2}s^{-1}t_1^{-1}t_1^2$. We have $t_1^2\in H_0$, by (P4), and the conjugation of an element of H_0 by s and t_i is already determined by (P7) and (P8). So we know how to lift $[\tau_1*\sigma,\sigma]=1$. In the general case we have a commutator $[(t_it_{i-1}\dots t_1)*s,(t_jt_{j-1}\dots t_1)*s]$. If $i>j$ then, by (P3) and (P5), this commutator is equal to the conjugate of $[t_1*s,s]$ by $t_jt_{j-1}\dots t_1t_it_{i-1}\dots t_2$. This is a conjugation of an element of H_0 by t_k's, so the result is determined by (P7) and (P8).

This concludes the proof of Proposition 27. □

We shall prove now that $\mathcal{M}_{g,1}$ acts transitively on vertices of X, so there is only one vertex orbit, and that H acts transitively on edges incident to v_0, so there is only one edge orbit. When we cut S open along the curves $\alpha_1, \dots, \alpha_g$ we get a disk with $2g$ holes. It is homeomorphic to any other such disk and we can prescribe the homeomorphism on the boundary components, hence every two cut systems can be transformed into each other by a homeomorphism and $\mathcal{M}_{g,1}$ acts transitively on X^0. Let w be a vertex connected to v_0 by an edge. Then w contains a curve β which intersects some curve α_i at one point and is disjoint from the other curves of v_0. We cut S open along $\alpha_1, \alpha_2, \dots, \alpha_g, \beta$ and get a disk with $2g-1$ holes, including one "big" hole bounded by arcs of both α_i and β. If we cut S along curves belonging to another edge incident to v_0 we get a similar situation. There exists a homeomorphism of one disk with $2g-1$ holes onto the other which preserves the identification of curves of the boundary and takes the set $\alpha_1, \alpha_2, \dots, \alpha_g$ onto itself. It induces a homeomorphism of S which leaves v_0 invariant and takes one edge onto the other. Thus H acts transitively on the edges incident to v_0.

We now fix one such edge and describe its stabilizer. If we replace curve α_1 of cut-system v_0 by curve β_1 we get a new cut-system connected to v_0 by an edge. We denote the cut-system by v_0' and the edge by $\mathbf{e}_0$. Let H' be the stabilizer of $\mathbf{e}_0$ in H.

Lemma 29 *The stabilizer H' of the edge $\mathbf{e}_0$ is generated by $a_1^2 s$, $t_1 s t_1$, a_2, $d_{2,3}$, $d_{-2,2}$, $d_{-1,1} d_{-1,2} d_{1,2} a_1^{-2} a_2^{-1}$, and $t_2, \dots, t_{g-1}$.*

Proof An element h of H' may permute curves $\alpha_2, \dots, \alpha_g$ and may reverse their orientations. It may also reverse simultaneously orientations of both curves α_1 and β_1 (it preserves the orientation of S so it must preserve algebraic intersection of curves). We check that $a_1^2 s$ reverses orientations of β_1 and α_1, $t_1 s t_1$ reverses orientation of α_2 and leaves β_1 and α_1 invariant. Elements t_i permute the curves $\alpha_2, \dots, \alpha_g$. Modulo these homeomorphisms we may assume that h is fixed on all curves α_i and β_1. It induces a homeomorphism of S cut open along all these curves. We get a disk with holes and, by Lemma 23, its mapping class group is generated by twists around holes and twists along suitable curves surrounding two holes at a time. Element $d_{-1,1} = (a_1^2 s)^2$ represents twist around the big hole (the cut along $\alpha_1 \cup \beta_1$). Conjugates of a_2 by t_j's produce twists around other holes. The conjugate of $d_{2,3}$ by $(t_1 s t_1)^{-1}$ is equal to $d_{-2,3}$ and the conjugate of $d_{-2,3}$ by $(t_2 t_1 s t_1)^{-1}$ is equal to $d_{-3,-2}$. Every $d_{k,k+1}$ can be obtained from these by conjugation by t_i's, $i > 1$. It is now clear that every curve $\delta_{i,j}$ with $i, j \neq \pm 1$ can be obtained from $d_{2,3}$ and $d_{-2,2}$ by conjugation by the elements chosen in the Lemma. So the corresponding twists are products of the above generators. We also need twists along curves which surround the "big" hole and another hole. Consider such a curve on S_0. It must contain inside both holes ∂_{-1} and ∂_1, including the arc between them corresponding to β_1, and one other hole. One such curve, call it γ, contains $\partial_{-1}, \partial_1, \partial_2$. By Lemma 22 the twist along γ is equal to $d_{-1,1}d_{-1,2}d_{1,2}a_1^{-2}a_2^{-1}$. Any other curve which contains $\partial_{-1}, \partial_1, \partial_i$ with $i > 1$ is obtained from γ by application of t_i's. The curve which contains $\partial_{-1}, \partial_1, \partial_{-2}$ is obtained from γ by application of $(t_1 s t_1)^{-1}$ and a curve which contains $\partial_{-1}, \partial_1, \partial_i$ with $i < -2$ is obtained from the last curve by application of t_i^{-1}'s. Therefore all generators of H' are products of the generators in the Lemma. $\square$

We now distinguish one more element of $\mathcal{M}_{g,1}$, which does not belong to H. Let $r = a_1 b_1 a_1$. The element r is a "quarter-twist". It switches curves α_1 and β_1, $r^2 = s a_1^2 = h_{-1,1}$ is a half-twist around $\delta_{-1,1}$ and $r^4 = d_{-1,1}$. Also r leaves the other curves α_i fixed, so it switches the vertices of the edge $\mathbf{e}_0$, $r(v_0) = v_0'$ and $r(v_0') = v_0$.

We now describe precisely a construction from [15] and [9] which will let us determine a presentation of $\mathcal{M}_{g,1}$. This construction was very clearly explained in [10].

Let us consider a free product $(H * \mathbb{Z})$ where the group $\mathbb{Z}$ is generated by r. An *h-product* is an element of $(H * \mathbb{Z})$ with positive powers of r, so it has a form $h_1 r h_2 r \dots h_k r h_{k+1}$. We have an obvious map η: $(H * \mathbb{Z}) \to \mathcal{M}_{g,1}$ through which the h–products act on X. We shall prove that η is onto and we shall find the h products which normally genarate the kernel of η.

To every edge-path $\mathbf{p} = (v_0, v_1, \dots, v_k)$ which begins at v_0 we assign an h–product $g = h_1 r h_2 r \dots h_k r h_{k+1}$ such that $h_1 r \dots h_m r(v_0) = v_m$ for $m = 1, \dots, k$. We construct it as follows. There exists $h_1 \in H$ such that $h_1(v_0) = v_0$ and $h_1(v_0') = v_1$. Then $h_1 r(v_0) = v_1$. Next we transport the second edge to v_0. $(h_1 r)^{-1}(v_1) = v_0$ and $(h_1 r)^{-1}(v_2) = v_1'$. There exists $h_2 \in H$ such that $h_2 r(v_0) = v_1'$ and $h_1 r h_2 r(v_0) = v_2$. And so on. Observe that the elements h_i in the h–product corresponding to an edge-path $\mathbf{p}$ are not uniquely determined. In particular h_{k+1} is an arbitrary element of H. The construction implies:

Lemma 30 *The generators of H together with the element r generate the group $\mathcal{M}_{g,1}$.*

Proof Let g be an element of $\mathcal{M}_{g,1}$. Then $g(v_0)$ is a vertex of X and can be connected to v_0 by an edge-path $\mathbf{p} = (v_0, v_1, \dots, v_k = g(v_0))$. Let $g_1 = h_1 r h_2 r \dots h_k r$ be an h–product corresponding to $\mathbf{p}$. Then $g_1(v_0) = v_k = g(v_0)$, therefore $\eta(g_1^{-1})g = h_{k+1}$ leaves v_0 fixed and belongs to the stabilizer H of v_0. It follows that $g = \eta(h_1 r h_2 r \dots h_k r h_{k+1})$ in $\mathcal{M}_{g,1}$. $\square$

By the inverse process we define an edge-path induced by the h–product $g = h_1 r h_2 r \dots h_k r h_{k+1}$. The edge-path starts at v_0 then $v_1 = h_1 r(v_0)$, $v_2 = h_1 r h_2 r(v_0)$ and so on. The last vertex of the path is $v_k = g(v_0)$.

Remark 4 An h–product g represents an element in H if and only if $v_k = v_0$. This happens if and only if the corresponding edge-path is closed. We can multiply such g by a suitable element of H on the right and get an h–product which represents the identity in $\mathcal{M}_{g,1}$ and induces the same edge-path as g.

We now describe a presentation of $\mathcal{M}_{g,1}$.

Theorem 31 *The mapping class group $\mathcal{M}_{g,1}$ admits the following presentation:*

The set of generators consists of a_1, $a_2, \dots, a_g$, s, t_1, t_2, $\dots$, t_{g-1}, r and the $d_{i,j}$'s for $i < j$, $i, j \in I_0$.

The set of defining relations consists of relations (P1) – (P8) *of Proposition 27 and of the following relations:*

(P9) *r commutes with $a_1^2 s$, $t_1 s t_1$, a_2, $d_{2,3}$, $d_{-2,2}$, $d_{-1,1} d_{-1,2} d_{1,2} a_1^{-2} a_2^{-1}$, and $t_2, \dots, t_{g-1}$.*

(P10) $r^2 = s a_1^2$.

(P11) $(k_i r)^3 = (k_i s a_1)^2$ *for $i = 1, 2, 3, 4$, where*
$k_1 = a_1$, $k_2 = d_{1,2}$, $k_3 = a_1^{-1} a_2^{-2} d_{1,2} d_{-2,1} d_{-2,2}$, $k_4 = a_1^{-1} a_2^{-1} a_3^{-1} d_{1,2} d_{1,3} d_{2,3}$,
$(r k_5 r k_5^{-1})^2 = s a_1^2 k_5 s a_1^2 k_5^{-1}$, *where* $k_5 = a_2 t_1 d_{1,2}^{-1}$, $(r a_1 t_1)^5 = (s a_1^2 t_1)^4$.

Relations (P9) – (P11) say that some elements of $H * \mathbb{Z}$ belong to $ker(\eta)$, so Theorem 31 claims that $(H * \mathbb{Z})$ modulo relations (P9) – (P11) is isomorphic to $\mathcal{M}_{g,1}$.

Relations (P9) tell us that r commutes with the generators of the stabilizer H' of the edge $\mathbf{e}_0$. We shall prove this claim now. An element h of H' leaves the edge $\mathbf{e}_0$ fixed but it may reverse the orientations of α_1 and β_1. The element $r^2 = s a_1^2$ does exactly this and commutes with r. Modulo this element we may assume that h leaves α_1 and β_1 pointwise fixed. But then we may also assume that it leaves some neighbourhood of α_1 and β_1 pointwise fixed. On the other hand r is equal to the identity outside a neighbourhood of α_1 and β_1 so it commutes with h.

From relations (P9) and (P10) we get information about h–products.

Claim 1 If two h–products represent the same element in $\mathcal{M}_{g,1}$ and induce the same edge-path then they are equal in $(H * \mathbb{Z})/(P9)$.

Proof If two h–products $g_1 = h_1 r \dots r h_{k+1}$ and $g_2 = f_1 r \dots r f_{k+1}$ induce the same edge-path $\mathbf{p} = (v_0, v_1, \dots, v_k)$ then $h_1 r(v_0) = f_1 r(v_0)$. Therefore $h_1^{-1} f_1 r(v_0) = r(v_0)$ and $h_1^{-1} f_1 \in H'$ commutes with r in $(H * \mathbb{Z})/(P9)$. Now $f_1 r f_2 = h_1 h_1^{-1} f_1 r f_2 = h_1 r f_2'$ in $(H * \mathbb{Z})/(P9)$. Therefore g_2 and a new h–product $h_1 r f_2' r f_3 r \dots r f_{k+1}$ are equal in $(H * \mathbb{Z})/(P9)$ and induce the same edge-path $\mathbf{p}$. If we apply $r^{-1} h_1^{-1}$ to the vertices $(v_1, v_2, \dots, v_k)$ of $\mathbf{p}$ we get a shorter edge-path which starts at v_0 and is induced by both shorter h–products $h_2 r \dots r h_{k+1}$ and $f_2' r \dots r f_{k+1}$. Claim 1 follows by induction on k. □

Two different edge-paths may be homotopic in the 1–skeleton X^1. This means that there is a backtracking v_i, v_{i+1}, v_i along the edge-path.

Claim 2 If two h–products represent the same element in $\mathcal{M}_{g,1}$ and induce edge-paths which are equal modulo back-tracking then the h–products are equal in $(H * \mathbb{Z})/((P9), (P10))$.

Proof Consider an h–product $g = g_i h_{i+1} r h_{i+2} r$, where g_i is an h–product inducing a shorter edge-path $\mathbf{p}$ and the edge-path induced by g has a back-tracking at the end: $g_i(v_0) = v_i$, $g_i h_{i+1} r(v_0) = v_{i+1}$, and $g_i h_{i+1} r h_{i+2} r(v_0) = v_i$. Clearly the h–product $g_i h_{i+1} r r$ induces the same edge-path. In particular $g_i h_{i+1} r^2(v_0) = v_i$ hence there exists $h' \in H$ such that $\eta(g_i h_{i+1} r^2 h') = \eta(g_i h_{i+1} r h_{i+2} r)$. Now by Claim 1 the h–products are equal in $(H * \mathbb{Z})/(P9)$. But $g_i h_{i+1} r^2 h'$ is equal in $(H * \mathbb{Z})/(P10)$ to a shorter h–product which induces the edge-path $\mathbf{p}$. Claim 2 follows by induction on the number of back-trackings. □

Relations (P11) correspond to edge-paths of type (C3), (C4) and (C5). Six relations correspond to six particular edge-paths.

As in (P11) we let $k_1 = a_1$, $k_2 = d_{1,2}$, $k_3 = d_{1,2} d_{-2,1} d_{-2,2} a_2^{-2} a_1^{-1}$, $k_4 = a_1^{-1} a_2^{-1} a_3^{-1} d_{1,2} d_{1,3} d_{2,3} = d_3$, $k_5 = a_2 d_{1,2}^{-1} t_1$, $k_6 = a_1 t_1$.

We now choose six h–products. For $i = 1,2,3,4$ let $g_i = (k_i r)^3$, $g_5 = (r k_5 r k_5^{-1})^2$ and $g_6 = (r k_6)^5$. Product g_i appears in the corresponding relation in (P11).

For $i = 1, \dots, 6$ let γ_i be the corresponding curve on Figure 15, represented on surface S_0 ($\gamma_5 = \beta_2$).

For $i = 1,2,3,4$ homeomorphism k_i leaves all curves α_i invariant. Also $k_i r(\alpha_1) = \gamma_i$, $k_i r(\gamma_i) = \beta_1$, $k_i r(\beta_1) = \alpha_1$. It follows that the h–product g_i represents the edge path $\mathbf{p}_i = (\langle\alpha_1\rangle \to \langle\gamma_i\rangle \to \langle\beta_1\rangle \to \langle\alpha_1\rangle)$.

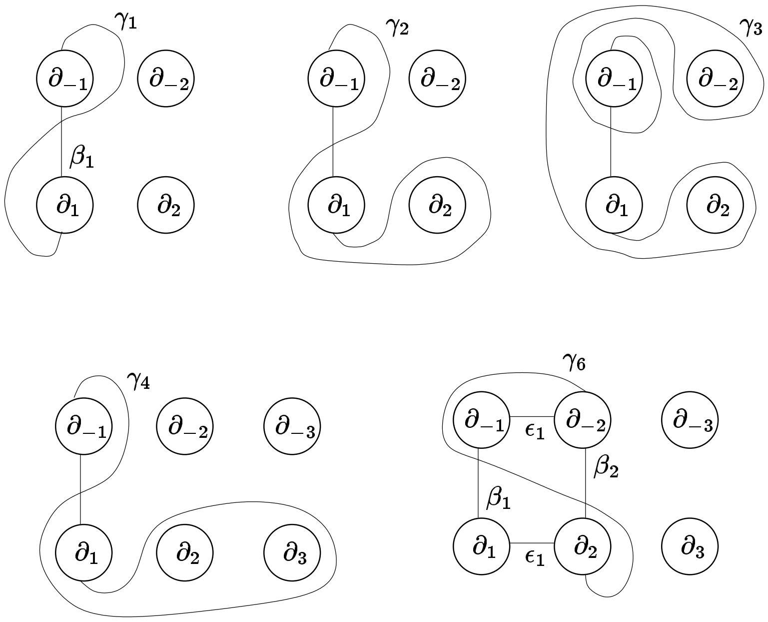

Figure 15: Curves γ_i

It is not hard to check that for $i = 5, 6$ the h–product g_i represents the edge-path $\mathbf{p}_i$, where

$$\mathbf{p}_5 = (\langle\alpha_1, \alpha_2\rangle \to \langle\beta_1, \alpha_2\rangle \to \langle\beta_1, \gamma_5\rangle \to \langle\alpha_1, \gamma_5\rangle \to \langle\alpha_1, \alpha_2\rangle).$$

$$\mathbf{p}_6 = (\langle\alpha_1, \alpha_2\rangle \to \langle\beta_1, \alpha_2\rangle \to \langle\beta_1, \epsilon_1\rangle \to \langle\gamma_6, \epsilon_1\rangle \to \langle\gamma_6, \alpha_1\rangle \to \langle\alpha_2, \alpha_1\rangle).$$

Since g_i represents a closed edge-path it is equal in $\mathcal{M}_{g,1}$ to some element $h_i \in H$. Then $V_i = g_i h_i^{-1}$ also represents $\mathbf{p}_i$ and is equal to the identity in $\mathcal{M}_{g,1}$. We shall prove in the next section that h_i is equal in H to the right hand side of the corresponding relation in (P11) and thus $V_i = 1$ in $(H * \mathbb{Z})/((P9), (P10), (P11))$.

We already know, by Theorem 4, that every closed path in X is a sum of paths of type (C3), (C4) and (C5). Some of these paths are represented by the paths $\mathbf{p}_1 - \mathbf{p}_6$. We say that a closed path is *conjugate* to a path $\mathbf{p}_i$ if it has a form $\mathbf{q}_1\mathbf{q}_2\mathbf{q}_1^{-1}$, where $\mathbf{q}_1$ starts at v_0 and $\mathbf{q}_2$ is the image of $\mathbf{p}_i$ under the action of some element of $\mathcal{M}_{g,1}$. We shall prove that every closed path is a sum of paths conjugate to one of the paths $\mathbf{p}_1 - \mathbf{p}_6$ or their inverses. This will imply Theorem 31.

We shall start with Harer's reduction for paths of type (C3) (see [7]). We fix curves $\alpha_2, \dots, \alpha_g$ and consider cut-systems containing one additional curve. Consider a triangular path $(\langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle)$. Give orientations to curves α, β, γ so that the algebraic intersection numbers satisfy $(\alpha, \beta) = (\alpha, \gamma) = 1$. Switching β and γ, if necessary, we may also assume $(\beta, \gamma) = -1$. We cut S open along $\alpha_2, \dots, \alpha_g$ and get a torus $S_{1,2g-1}$ with $2g-1$ holes. Each square on Figure 16 represents a part of the universal cover of a closed torus, punctured above holes of $S_{1,2g-1}$. We show more than one preimage of some curves in the universal cover. A fundamental region is a square bounded by α and β with $2g-1$ holes, one of them bounded by the boundary ∂ of S. We may assume that γ crosses α and β in two distinct points. Then γ splits the square into three regions: F_1 to the right of γ after γ crosses α, F_2 to the left of γ after γ crosses β, and F_0 (see Figure 16 (a)). Reversing the orientations of all curves we can switch the regions F_1 and F_2. Let l_i be the number of holes in F_i for $i = 0, 1, 2$. We want to prove that every triangular path is a sum of triangular paths with $l_1 \leq 2$, $l_2 = 0$, and with the hole ∂ not in the region F_1. We shall prove it by induction on $l_1 + l_2$. A thin circle on Figure 16 denotes a single hole and a thick circle denotes all remaining holes of the region.

Suppose first that $l_1 > 1$. Move curves α, β, γ off itself (on a closed torus) in such a way that the region bounded by α and α' contains only one hole, belonging to F_1, the region bounded by β and β' contains another hole of F_1 and the region bounded by γ and γ' contains both of these holes. Up to an isotopy (translating holes and straightening curves) the situation looks like on Figure 16 (c).

Now consider Figure 16 (b) — an octahedron projected onto one of its faces. All faces of the octahedron are "triangles" in X. In order to understand regions F_0, F_1, F_2 corresponding to each face we translate the fundamental domain to a suitable square in the universal cover. For every face different from (α, β, γ) the region F_1 has at least one hole less than l_1 and the region F_0 has all of its original holes and possibly some more. Clearly the boundary of the face (α, β, γ) is a sum of conjugates of the boundaries of the other faces. So by induction we may assume that $l_1 \leq 1$. By symmetry we may assume that $l_2 \leq 1$ and $l_1 > 2$. We chose new curves α', β' and γ' whose liftings are shown on Figure 16 (d). Consider again the octahedron on Figure 16 (b). Now region F_2 is fixed for all faces of the octahedron and region F_0 has all of its original holes and some additional holes for all faces different from (α, β, γ). So by induction we may assume that $l_1 \leq 2$ and $l_2 \leq 1$. Suppose $l_1 = 2$ and $l_2 = 1$ and choose new curves α', β' and γ' as on Figure 16 (e). Again consider the octahedron. Now for each face different from (α, β, γ) at least one of the

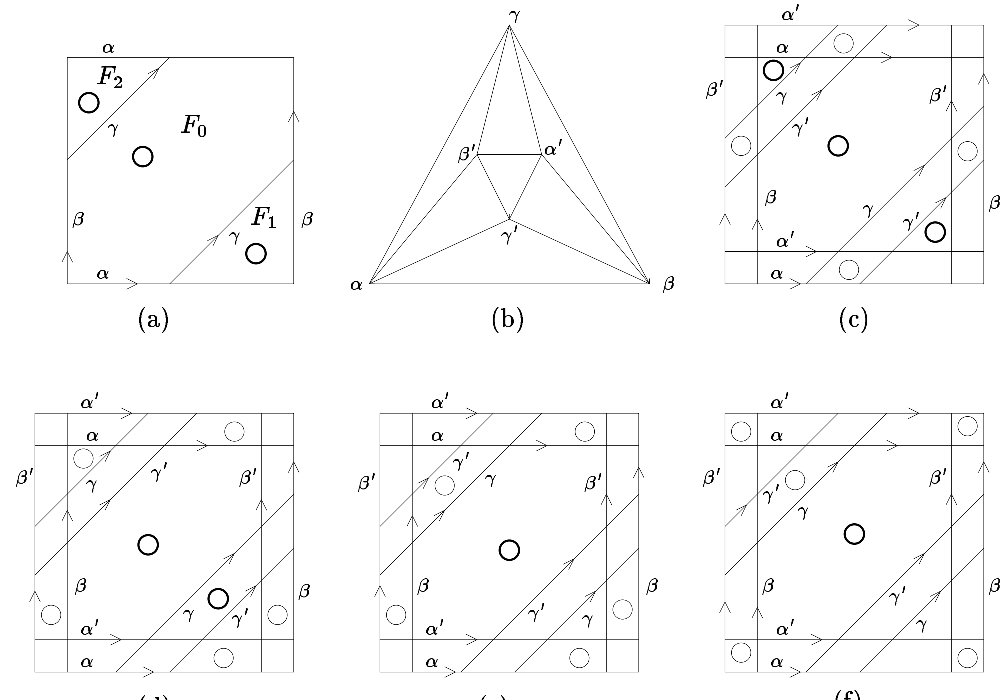

Figure 16: Reduction for paths of type (C3)

regions F_1 and F_2 looses at least one of its holes. Suppose now that each region has exactly one hole. Consider curves α', β' and γ' on Figure 16 (f). Again for each face of the corresponding octahedron different from the face (α, β, γ) at least one of the regions F_1 and F_2 looses at least one of its holes. So we may assume that $l_2 = 0$ and $l_1 \leq 2$. Finally it may happen that a hole in F_1 is bounded by ∂. We can isotop γ (on the torus with holes) over F_2 to the other side of the intersection of α and β. Clearly F_1 becomes now F_0. We can repeat the previous reduction and the hole ∂ will remain in F_0. Thus we are left with four cases:

$l_1 = l_2 = 0$,

$l_1 = 1$, $l_2 = 0$,

$l_1 = 2$, $l_2 = 0$ and the two holes in F_1 correspond to the same curve α_i,

$l_1 = 2$, $l_2 = 0$ and the two holes in F_1 correspond to different curves α_i and α_j.

Clearly each case is uniquely determined up to homeomorphism (of one square with holes onto another, preserving γ). Paths $\mathbf{p}_i$, $i = 1, 2, 3, 4$ represent triangle paths of these four types with α corresponding to α_1, β corresponding to β_1 and γ corresponding to γ_i. Given any path of type (C3) we can map it onto a path which starts at v_0. Applying an element of H we may assume that the second vertex is v_0'. Then the path is of the type considered in the above reduction and it is a sum of conjugates of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and their inverses (we may need to switch β and γ in the proof).

Consider now the path $\mathbf{p}_5$. When we cut S open along $\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1$ and γ_5 we get a disk with two "big" holes and $2g-4$ "small" holes. Any other path $\mathbf{q}$ of type (C4) produces a similar configuration. There exists $g \in G$ which takes $\mathbf{p}_5$ on $\mathbf{q}$. Therefore every path of type (C4) is a conjugate of $\mathbf{p}_5$.

Consider now $\mathbf{p}_6$ and another path $\mathbf{q}$ of type (C5). When we cut S open along the first four changing curves of $\mathbf{p}_6$: $\alpha_1, \alpha_2, \beta_1, \epsilon_1$ and along all fixed curves of the cut systems we get a disk with one "big" hole and $2g-4$ "small" holes. When we do the same for the curves in $\mathbf{q}$ we get a similar configuration. We may map $\mathbf{q}$ onto a path

$$(\langle \alpha_1, \alpha_2 \rangle \to \langle \beta_1, \alpha_2 \rangle \to \langle \beta_1, \epsilon_1 \rangle \to \langle \gamma, \epsilon_1 \rangle \to \langle \gamma, \alpha_1 \rangle \to \langle \alpha_2, \alpha_1 \rangle)$$

in which only the curve γ is different from the corresponding curve γ_6 in $\mathbf{p}_6$ and all other curves are as in $\mathbf{p}_6$. Consider curve $\gamma_5 = \beta_2$ on Figure 15. Curve γ_5 can be homotop onto the union of β_1, ϵ_1 and a part of α_1. Since γ intersects β_1 once and does not intersect α_1 nor ϵ_1 it must intersect γ_5 once. Therefore we may form a subgraph of X as on Figure 17.

$$\begin{tikzcd}
\langle \alpha_2, \alpha_1 \rangle \arrow[rr] \arrow[dr] & & \langle \alpha_2, \beta_1 \rangle \arrow[rr] & & \langle \epsilon_1, \beta_1 \rangle \arrow[dl] \\
& \langle \alpha_1, \gamma_6 \rangle & & \langle \epsilon_1, \gamma_6 \rangle \arrow[ll] & \\
& \langle \alpha_1, \gamma_5 \rangle \arrow[u] & & \langle \epsilon_1, \gamma_5 \rangle \arrow[ll] \arrow[u] & \\
\langle \alpha_1, \gamma \rangle \arrow[uuu] \arrow[ur] & & & & \langle \epsilon_1, \gamma \rangle \arrow[uuu] \arrow[ul] \arrow[llll]
\end{tikzcd}$$

Figure 17: Reduction for paths of type (C5)

Here bottom and middle square edge-paths are of type (C4). In the square edge-path on the left side (and on the right side) only one curve changes so the path is a sum of triangles by Proposition 7. The top pentagon edge-path is equal to $\mathbf{p}_6$. The new edge-path coincides with the outside pentagon. It is a sum of conjugates of the other edge paths.

We can now complete the proof of Theorem 31. Let $W \in H * \mathbb{Z}$ be such that $\eta(W) = 1$. We want to prove that $W = 1$ in $(H * \mathbb{Z})/((P9), (P10), (P11))$. Modulo (P10) we can write W as an h–product g which represents a closed edge path $\mathbf{p}$. By Theorem 4 the edge-path $\mathbf{p}$ is a sum of conjugates of path of type (C3), (C4) and (C5). By the above discussion it is a sum of conjugates of the special paths $\mathbf{p}_1 - \mathbf{p}_6$ and their inverses, modulo backtracking. So, modulo backtracking, $\mathbf{p} = \Pi \mathbf{q}_i f_i(\mathbf{p}_{j_i}^{\pm 1})\mathbf{q}_i^{-1}$ for some $f_i \in \mathcal{M}_{g,1}$. Each path $\mathbf{q}_i f_i(\mathbf{p}_{j_i}^{\pm 1})\mathbf{q}_i^{-1}$ can be represented by an h–product g_i. Since the path is closed we have $\eta(g_i) \in H$, so we can correct the h–product g_i and assume that $\eta(g_i) = 1$. The product Πg_i represents the path $\Pi \mathbf{q}_i f_i(\mathbf{p}_{j_i}^{\pm 1})\mathbf{q}_i^{-1}$, so, by Claims 1 and 2, W is equal to Πg_i in $(H * \mathbb{Z})/((P9), (P10))$. It suffices to prove that $g_i = 1$ in $(H * \mathbb{Z})/((P9), (P10), (P11))$. Let $k = j_i$ and suppose that g_i represents a path $\mathbf{q}_i f_i(\mathbf{p}_k)\mathbf{q}_i^{-1}$. Let u_i be an h–product representing $\mathbf{q}_i$. Then $u_i(v_0) = f_i(v_0)$ is the first vertex of $f_i(\mathbf{p}_k)$, hence $\eta(u_i)^{-1} f_i = h_i \in H$. Recall that $\mathbf{p}_k$ can be represented by an h–product V_k which is equal to the identity in $(H * \mathbb{Z})/((P9), (P10), (P11))$. The h–product $u_i h_i V_k$ represents $\mathbf{q}_i f_i(\mathbf{p}_k)$ and there exists an h–product $u_i h_i V_k w_i$ such that $\eta(u_i h_i V_i w_i) = 1$ and $u_i h_i V_k w_i$ represents the edge-path $\mathbf{q}_i f_i(\mathbf{p}_k)\mathbf{q}_i^{-1}$. Clearly $u_i h_i w_i$ represents the edge-path $\mathbf{q}_1 \mathbf{q}_1^{-1}$ which is null-homotopic by backtracking. By Claims 1 and 2 the h–product g_i is equal to $u_i h_i V_k w_i$ in $(H * \mathbb{Z})/((P9), (P10))$, $u_i h_i V_k w_i$ is equal to $u_i h_i w_i$ in $(H*\mathbb{Z})/((P9), (P10), (P11))$ and $u_i h_i w_i = 1$ in $(H*\mathbb{Z})/((P9), (P10))$.

The path inverse to $\mathbf{p}_k$ is represented by some other h–product V_k' but then $V_k V_k'$ represents a path contractible by back-tracking. Thus $V_k V_k' = 1$ in $(H * \mathbb{Z})/((P9), (P10))$ and $V_k = 1$ in $(H*\mathbb{Z})/((P9), (P10), (P11))$ hence also $V_k' = 1$ in $(H * \mathbb{Z})/((P9), (P10), (P11))$.

This concludes the proof of Theorem 31. □

4 Reduction to a simple presentation

Let us recall the following obvious direction of Tietze's Theorem.

Lemma 32 *Consider a presentation of G with generators $g_1, \dots, g_k$ and relations $R_1, \dots, R_s$. If we add another relation R_{s+1}, which is valid in G, we get another presentation of G. If we express some element g_{k+1} of G as a product P of the generators $g_1, \dots, g_k$ we get a new presentation of G with generators $g_1, \dots, g_{k+1}$ and with defining relations $R_1, \dots, R_s, g_{k+1}^{-1}P$. Conversely suppose that we can express some generator, say g_k, as a product P of $g_1, \dots, g_{k-1}$. Let us replace every appearence of g_k in each relation R_i, $i = 1, \dots, s$ by P, getting a new relation P_i. Then G has a presentation with generators $g_1, \dots, g_{k-1}$ and with relations $P_1, \dots, P_s$.*

We start with the presentation of the mapping class group $\mathcal{M}_{g,1}$ established in Theorem 31. The generators represent the mapping classes of corresponding homeomorphisms of the surface $S = S_{g,1}$ represented on Figure 1. We adjoin additional generators

$$\begin{aligned} &b_1 = a_1^{-1} r a_1^{-1}, \quad b_2 = (t_1 a_1 b_1) * d_{1,2}, \quad e_1 = (r d_{1,2} a_2^{-1}) * b_2 \\ &e_{i+1} = (t_i t_{i+1}) * e_i \quad \text{for} \quad i = 1, \dots, g-2. \end{aligned} \tag{3}$$

These generators also represent the corresponding twists in $\mathcal{M}_{g,1}$. We adjoin the relations (M1) – (M3). Now, by Theorem 31, generators $s, t_i, d_{i,j}, r$ can be expressed in $\mathcal{M}_{g,1}$ by the formulas from Definition 4. We substitute for each of these generators the corresponding product of $b_2, b_1, a_1, e_1, a_2, \dots, a_{g-1}, e_{g-1}, a_g$ in all relations (P1) – (P11) and in (3). We check easily that the relations (3) become trivial modulo the relations (M1) – (M3) (the last one will be proven in (17).) We want to prove that all relations (P1) – (P11) follow from relations (M1) – (M3).

Remark 5 We shall establish many auxiliary relations of increasing complexity which follow from the relations (M1) – (M3). We shall explain some standard technique which one can use (some proofs will be left to the reader). From the braid relation $aba = bab$ one can derive several other useful relations, like: $a * b = b^{-1} * a$, $a * (b^2) = b^{-1} * (a^2)$, $(ab) * a = b$. When we want to prove that $[a, b] = 0$ we shall usually try to prove that $a * b = b$. The relation $aba = bab$ tells us that a can "jump" over ba to the right becoming b. By consecutive jumping to the right we can prove that $a_1(e_1 a_1 a_2 e_1 e_2 a_2) = (e_1 a_1 a_2 e_1 e_2 a_2) e_2$. We also get $e_1 a_1 a_2 e_1 e_2 a_2 = e_1 a_2 e_2 a_1 e_1 a_2$ and $(b_1 a_1 e_1 a_2) * b_2 = (b_2^{-1} a_2^{-1} e_1^{-1} a_1^{-1}) * b_1$. We shall say that some relations follow by (J) – jumping, if they follow *easily* from (M1) by the above technique.

We start the list of the auxiliary relations.

$$t_i * a_i = a_{i+1}, \; t_i * a_{i+1} = a_i, \; t_i * a_k = a_k \quad \text{for} \quad k \neq i, i+1, \tag{4}$$

$s * a_i = a_i$ for $i = 1, \ldots, g$ by (J).

Let $w_0 = a_g e_{g-1} a_{g-1} e_{g-2} \ldots e_1 a_1 b_1$.

(5) $\quad w_0^{-1} * b_2 = d_{1,2},\ w_0^{-1} * b_1 = a_1,\ w_0^{-1} * a_i = e_i,\ w_0^{-1} * e_i = a_{i+1},$

$d_{1,2} * b_1 = b_1^{-1} * d_{1,2},\ d_{1,2} * e_2 = e_2^{-1} * d_{1,2},\ [d_{1,2}, a_i] = 1,\ [d_{1,2}, e_j] = 1,$ for $j \neq 2$, $[d_{1,2}, t_j] = 1$ for $j \neq 2$.

Proof of (5) We have $w_0^{-1} * b_2 =$ (by (M1)) $(b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1}) * b_2 = d_{1,2}$. Other results of conjugation by w_0 follow by (J). Now
$d_{1,2} * b_1 = (b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1} b_2 a_2 e_1 a_1 b_1) * b_1 =$ (by J)
$(b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1} b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1}) * b_2 =$ (by jumping from left side to the right)
$(b_1^{-1} b_1^{-1} a_1^{-1} e_1^{-1} a_2^{-1}) * b_2 = b_1^{-1} * d_{1,2}$.

Other relations follow from (M1) by conjugation by w_0^{-1}. □

(6) $\quad a_k * d_{i,j} = d_{i,j}$ for all i, j, k, by (J) and (4) and (5).

(7) $\quad t_i * t_{i+1} = t_{i+1}^{-1} * t_i$ *for* $i = 1, 2, \ldots, g-2$,

(by the calculations similar to the proof of Lemma 21 (iv)),
$[t_i, t_k] = 1$ for $|i-k| > 1$, $[t_i, s] = 1$ for $i > 1$, by (M1).

Using relations (5) and (7) we can write the elements $d_{i,j}$ in a different way.

(8) $\quad d_{i,i+1} = (t_{i-1} t_i t_{i-2} t_{i-1} \ldots t_1 t_2) * d_{1,2}$ for $i > 0$,

$d_{-i-1,-i} = (t_{i-1}^{-1} t_i^{-1} t_{i-2}^{-1} t_{i-1}^{-1} \ldots t_1^{-1} t_2^{-1}) * d_{-2,-1}$ for $i > 0$,
$d_{i,i+1} = (t_{i-1} t_i) * d_{i-1,i},\ \ t_k * d_{i,i+1} = d_{i,i+1}$ for $|k-i| \neq 1$.

Let $w_1 = a_2 e_1 a_1 b_1^2 a_1 e_1 a_2$.

(9) $\quad (b_1 a_1 e_1)^4 = b_2 w_1 b_2 w_1^{-1},\ w_1 * b_2 = w_1^{-1} * b_2,\ [w_1 * b_2, b_2] = 1$

Proof of (9) We have $b_2 w_1 b_2 =$ (by (M2)) $(b_1 a_1 e_1 a_2)^5 =$ (as in the proof of Lemma 21 (v)) $(b_1 a_1 e_1)^4 w_1 =$ (by (J)) $w_1 (b_1 a_1 e_1)^4$.
Also $b_2 (b_1 a_1 e_1)^4 = (b_1 a_1 e_1)^4 b_2$, by (M1).

Therefore $w_1 b_2 w_1^{-1} = b_2^{-1} (b_1 a_1 e_1)^4 = (b_1 a_1 e_1)^4 b_2^{-1} = w_1^{-1} b_2 w_1$ commutes with b_2. □

(10) $\quad s t_1 s = b_1 a_1 e_1 a_2^2 e_1 a_1 b_1 t_1 = t_1 b_1 a_1 e_1 a_2^2 e_1 a_1 b_1$ hence $s t_1 s t_1 = t_1 s t_1 s$

Proof of (10) We have a sequence of transformations by (J).
$st_1s =$
$b_1a_1a_1b_1e_1a_1a_2e_1b_1a_1a_1b_1 = b_1a_1a_1e_1b_1a_1b_1a_2e_1a_1a_1b_1 =$
$b_1a_1a_1e_1a_1b_1a_2a_1e_1a_1a_1b_1 = b_1a_1e_1a_1e_1b_1a_2e_1e_1a_1e_1b_1 =$
$b_1a_1e_1a_1b_1a_2e_1a_2e_1a_1b_1e_1 = b_1a_1e_1a_2a_1b_1a_2e_1a_1b_1a_2e_1 =$
$b_1a_1e_1a_2a_2e_1a_1b_1t_1$.

The second equality follows immediately by symmetry. □

Let $w_2 = e_2a_2e_1a_1^2e_1a_2e_2$.

(11) $$(a_1e_1a_2)^4 = t_1^2a_1^2a_2^2 = d_{1,2}d_{-2,-1}, \quad [d_{1,2}, d_{-2,-1}] = 1.$$

$d_{-2,-1} = w_2 * d_{1,2} = w_2^{-1} * d_{1,2} = (b_1a_1e_1a_2) * b_2$

Proof of (11) We have $d_{-2,-1} = (s^{-1}t_1^{-1}s^{-1}) * d_{1,2} =$ (by (10))
$((b_1a_1e_1a_2a_2e_1a_1b_1)^{-1}t_1^{-1}b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1}) * b_2 =$ (by (J))
$((b_1a_1e_1a_2a_2e_1a_1b_1)^{-1}b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1}) * b_2 =$ (by (9)) $(b_1a_1e_1a_2) * b_2$.

Conjugating (9) by $(b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1})$ we get $(a_1e_1a_2)^4 = d_{1,2}d_{-2,-1}$. Also, by (M1),

$t_1^2a_1^2a_2^2 = (a_1e_1a_2)^4$. This proves the first relation. The second relation follows from it by (5). Conjugating (9) by w_0^{-1} we get, by (5), $(a_1e_1a_2)^4 = d_{1,2}w_2d_{1,2}w_2^{-1}$ and $w_2 * d_{1,2} = w_2^{-1} * d_{1,2}$. Therefore, from the first relation, $d_{-2,-1} = w_2 * d_{1,2} = w_2^{-1} * d_{1,2} = (b_1a_1e_1a_2) * b_2$. □

Definition 6 If A is a product of the generators we denote by A' the element obtained from A by replacing each generator by its inverse. We call A' the element *symmetric to* A.

Remark 6 Relations (M1) and (M2) are symmetric. They remain valid when we replace each generator by its inverse. Therefore every relation between some elements of $\mathcal{M}_{g,1}$ (products of generators) which follows from (M1) and (M2) remains valid if we replace each element by the element symmetric to it.

(12) For $i + j \neq 0$ $d_{i,j}$ is symmetric to $d_{-j,-i}^{-1}$.

Proof of (12) Element $d_{1,2}$ is symmetric to $d_{-2,-1}^{-1}$, by (11). Also $t_1' = t_1^{-1}$ and $s' = s^{-1}$. We see immediately that $d_{i,j}' = d_{-j,-i}^{-1}$ for $i > 0$. If $i < 0$ and $i + j > 0$ then

$d_{i,j}=(t_{-i-1}^{-1}\dots t_1^{-1}s^{-1}t_{j-1}\dots t_2)*d_{1,2}$
and $d_{-j,-i}^{-1}=(t_{j-1}^{-1}\dots t_1^{-1}s^{-1}t_{-i}\dots t_2)*d_{1,2}^{-1}$.

Jumping with the positive powers of t_k to the left we get

$d_{-j,-i}^{-1}=(t_{-i-1}\dots t_1st_{j-1}^{-1}\dots t_2^{-1}s^{-1}t_1^{-1}s^{-1})*d_{1,2}^{-1}=$
$(t_{-i-1}\dots t_1st_{j-1}^{-1}\dots t_2^{-1})*d'_{1,2}=d'_{i,j}$. □

(13) $d_{i+1,i+2}=(t_i^{-1}t_{i+1}^{-1})*d_{i,i+1}=(t_it_{i+1})*d_{i,i+1}$ for $i=1,\dots,g-2$.

Proof of (13) For $i=1$ we have $(t_2t_1^2t_2)*d_{1,2}=$ (by 11)
$(e_2a_2a_3e_2e_1a_1a_2e_1e_1a_1a_2e_1e_2a_2a_3e_2e_2^{-1}a_2^{-1}e_1^{-1}a_1^{-2}e_1^{-1}a_2^{-1}e_2^{-1})*d_{-2,-1}$
$=$ (by (J)) $(e_2a_2a_3e_2e_1a_1a_2e_1e_1a_2e_2a_3a_1^{-1}e_1^{-1}a_2^{-1}e_2^{-1})*d_{-2,-1}=$ (by (J))
$(e_2a_2a_3e_2e_1a_1a_2e_1e_1a_1^{-1}a_2e_1^{-1}e_2a_2^{-1}a_3e_2^{-1})*d_{-2,-1}=$ (by (J))
$(e_2a_2a_3e_2e_1a_2e_1^{-1}a_1a_1e_1a_2e_1^{-1}e_2a_2^{-1}a_3e_2^{-1})*d_{-2,-1}=$ (by (J))
$(a_3^{-1}e_2a_2e_1a_1^2e_1a_2e_2a_3)*d_{-2,-1}=$ (by (6) and (11)) $a_3^{-1}*d_{1,2}=$ (by (6)) $d_{1,2}$.

So (13) is true for $i=1$. We continue by induction. Conjugating relation (13) by $t_it_{i+1}t_{i+2}$ we get, by (8) and (7), relation (13) for index $i+1$. □

(14) $$t_i^2=d_{i,i+1}d_{-i-1,-i}a_i^{-2}a_{i+1}^{-2}.$$

Proof of (14) The relation is true for $i=1$, by (11). We proceed by induction.
$t_{i+1}^2=(t_i^{-1}t_{i+1}^{-1})*t_i^2=(t_i^{-1}t_{i+1}^{-1})*(d_{i,i+1}d_{-i-1,-i}a_i^{-2}a_{i+1}^{-2})$
$=$ (by (7), (13), (8) and (4)) $d_{i+1,i+2}d_{-i-2,-i-1}a_{i+1}^{-2}a_{i+2}^{-2}$. □

(15) $$d_{-i,i} \text{ is symmetric to } d_{-i,i}^{-1}.$$

Proof of (15) The relation is true for $i=1$. The general case follows from (14), (4), (5), (12) and the definitions. □

(16) $$[b_1,d_{-2,2}]=1$$

Proof of (16) By the definition $d_{-2,2}=(t_1^{-1}d_{1,2})*(s^2a_1^4)=$ (by (4))
$a_2^4((d_{1,2}t_1^{-1})*s^2)$.
Now $t_1^{-1}*s=t_1^{-1}st_1ss^{-1}=$ (by (10)) $b_1a_1e_1a_2^2e_1a_1^{-1}b_1^{-1}$.

Taking squares we get $t_1^{-1} * s^2 = b_1a_1e_1a_2^2e_1^2a_2^2e_1a_1^{-1}b_1^{-1}$ and $d_{-2,2} = a_2^4d_{1,2}b_1a_1e_1a_2^2e_1^2a_2^2e_1a_1^{-1}b_1^{-1}d_{1,2}^{-1}$. Now b_1 commutes with $d_{-2,2}$, by (4), (5) and (J). □

$$(17) \qquad (t_it_{i+1}) * e_i = e_{i+1} \quad \text{for} \quad i = 1, \dots, g-2.$$

Proof of (17) We have, by (J), $(t_it_{i+1}) * e_i =$ $(e_ia_ia_{i+1}e_ie_{i+1}a_{i+1}a_{i+2}e_{i+1}) * e_i = (e_ia_ia_{i+1}e_{i+1}e_ia_{i+1}) * e_i = e_{i+1}$. □

$$(18) \qquad [b_1, d_{i,i+1}] = 1 \ if \ i > 1,$$

$$[e_k, d_{i,i+1}] = 1 \ \text{ if } \ |k-i| \neq 1, i > 0, \quad d_{i,i+1} * e_k = e_k^{-1} * d_{i,i+1} \ \text{ if } \ |k-i| = 1.$$

Proof of (18) By (13) and (J) $d_{2,3} = (e_1^{-1}a_2^{-1}e_2^{-1}a_3^{-1}a_1^{-1}e_1^{-1}a_2^{-1}e_2^{-1}b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1}) * b_2 =$ $(e_1^{-1}a_2^{-1}e_2^{-1}a_3^{-1}a_1^{-1}b_1^{-1}e_1^{-1}a_1^{-1}a_2^{-1}e_1^{-1}e_2^{-1}a_2^{-1}) * b_2$.

Now b_1 commutes with $d_{2,3}$, by (J). For $i > 2$ we have b_1 commutes with $d_{i,i+1}$ by (M1) and (8). Conjugating by $a_1b_1t_1$ we get, by (7) and (J), $[e_1, d_{i,i+1}] = 1$, for $i > 2$. We also have $d_{1,2} * b_1 = b_1^{-1} * d_{1,2}$, by (5). We conjugate this equality by $u = a_1b_1t_1t_2$ and get $d_{2,3} * e_1 = e_1^{-1} * d_{2,3}$, by (8) and the first part of the proof. Conjugating relations (5) and the above relations by suitable products t_it_{i+1} we get all remaining relations, by (17). □

$$(19) \qquad [b_1, d_{1,2}sd_{1,2}] = 1, \quad \text{hence} \quad [s, d_{1,2}sd_{1,2}] = 1,$$

$$[e_j, d_{i,i+1}t_jd_{i,i+1}] = 1, \quad \text{hence} \quad [t_j, d_{i,i+1}t_jd_{i,i+1}] = 1, \quad \text{if} \quad |i-j| = 1.$$

Proof of (19) By (5) we have $(d_{1,2}sd_{1,2}) * b_1 = (d_{1,2}b_1a_1a_1b_1b_1^{-1}) * d_{1,2} =$ $(d_{1,2}b_1) * d_{1,2} = b_1$. The other case is similar, but we use (18) instead of (5). □

Remark 7 Relation $uvuv = vuvu$ implies $(uv) * u = v^{-1} * u$ and $(u^{-1}v^{-1}) * u = v * u$. Relations (19) will be often used in this form.

Observe that relations (4) – (14) imply in particular that relations (P1) and relations (P3) – (P7) follow from (M1) and (M2). We shall prove now that relations (P8) follow from (M1) and (M2).

Definition 7 We say that homeomorphism t_k (respectively $t_k^{-1}d_{k+1,k+2}$ or s) moves a curve $\delta_{i,j}$ properly if it takes it to some curve $\delta_{p,q}$.

Lemma 33 *If t_k (respectively s) moves curve $\delta_{i,j}$ to some curve $\delta_{p,q}$ then $t_k * d_{i,j} = d_{p,q}$ (respectively $s * d_{i,j} = d_{p,q}$).*

Remark 8 Since the action of t_k and s on a curve $\delta_{i,j}$ is described by Lemma 24 and is easily determined, Lemma 33 helps us to understand the result of the conjugation. In fact the action corresponds exactly to relations (P8), so we have to prove that relations (P8) follow from (M1) and (M2).

Proof of Lemma 33 We know that $[t_i, d_{i,i+1}] = 1$, by (8), (7), and (5).

If $i < 0$ and $i + j = 1$ then $t_{j-1}^{-1} * d_{i,j} = (t_{j-1}^{-1}t_{j-2}^{-1}\dots t_1^{-1}s^{-1}t_{j-1}\dots t_2) * d_{1,2} = d_{i-1,j-1}$.

For $i > 0$ or $i < 0$, $i + j > 0$ all other cases of conjugation by t_k follow from (5), (7) and the definitions. The other cases of $i \neq -j$ follow by symmetry.

Consider conjugation by s for $i \neq -j$. Again it suffices to consider $i > 0$ or $i < 0$, $i+j > 0$. The other cases follow by symmetry. We have $s^{-1}*d_{1,j} = d_{-1,j}$.

If $i > 1$ then $d_{i,j} =$ (by (7)) $(t_{i-1}\dots t_2t_{j-1}\dots t_3) * d_{2,3}$ and $s * d_{i,j} = d_{i,j}$, by (7), (6) and (18).

If $i < 1$ then $d_{i,j} =$ (by 7) $(t_{-i-1}^{-1}\dots t_2^{-1}t_{j-1}\dots t_3) * d_{-2,3}$. Also $s^{-1} * d_{-2,3} = (s^{-1}t_1^{-1}s^{-1}t_2) * d_{1,2} = (s^{-1}t_1^{-1}s^{-1}t_1^{-1}) * d_{2,3} =$ (by (10)) $(t_1^{-1}s^{-1}t_1^{-1}s^{-1}) * d_{2,3} = d_{-2,3}$ (by (6) and (18)). So $s^{-1} * d_{i,j} = d_{i,j}$, by (7).

We now consider conjugation of $d_{-j,j}$. Clearly $s * d_{-1,1} = d_{-1,1}$, by (M1). Also $s * d_{-2,2} = d_{-2,2}$, by (6) and (16). For $i > 1$ we have $[s, t_i] = 1$, by (M1), hence $s * d_{-j,j} = d_{-j,j}$ for all j, by the first part of the proof.

Consider conjugation by t_k.

For $k > j$ we have $t_k * d_{-j,j} = d_{-j,j}$, by (7) and the first part.

Curves $t_j(\delta_{-j,j})$ and $t_{j-1}(\delta_{-j,j})$ are not of the form $\delta_{p,q}$.

Consider $k = j - 2$ (the other cases follow by conjugation and by the first part of the proof). We have

$d_{-j,j} = (d_{j-1,j}t_{j-1}^{-1}t_{j-2}^{-1}d_{j-2,j-1}) * d_{2-j,j-2} =$ (by (8) and (13))
$(t_{j-2}^{-1}t_{j-1}^{-1}d_{j-2,j-1}t_{j-1}t_{j-2}t_{j-1}^{-1}t_{j-2}^{-1}d_{j-2,j-1}) * d_{2-j,j-2} =$ (by (7) and (8))
$(t_{j-2}^{-1}t_{j-1}^{-1}t_{j-2}^{-1}d_{j-2,j-1}t_{j-1}d_{j-2,j-1}) * d_{2-j,j-2}$. Now, by (7) and (19),
$t_{j-2}^{-1} * d_{-j,j} = (t_{j-2}^{-1}t_{j-1}^{-1}t_{j-2}^{-1}d_{j-2,j-1}t_{j-1}d_{j-2,j-1}t_{j-1}^{-1}) * d_{2-j,j-2} = d_{-j,j}$ (by the previous case). □

We now pass to the biggest task of this section: the relations (P2).

(20) $\qquad d_{i,j}$ commutes with $d_{-1,1}$ if $i,j \neq \pm 1$,

$d_{i,j}$ commutes with $d_{k,k+1}$ if all indices are distinct.

Proof of (20) We know by Lemma 33 that $d_{i,j}$ commutes with s, hence also with $d_{-1,1}$. Consider the other cases. We assume first that $k>0$. Consider curves $\delta_{i,j}$ and $\delta_{k,k+1}$. We want to move the curves properly to some standard position by application of products of s and t_m's. This moves holes (see definition 5) and corresponds to conjugation of $d_{i,j}$ and $d_{k,k+1}$, by the previous lemma.

Case 1 $\;i \neq -j\;$ Observe that for every k either $t_{k+1}t_k$ or $t_{k+1}^{-1}t_k^{-1}$ moves $\delta_{i,j}$ properly. Both products take $\delta_{k+1,k+2}$ onto $\delta_{k,k+1}$ and conjugation of $d_{k+1,k+2}$ by either product produces $d_{k,k+1}$. If either $|i|$ or $|j|$ is bigger than $k+1$ we may assume, moving $\delta_{i,j}$ properly, and not moving $\delta_{k,k+1}$, that either $|i|$ or $|j|$ is equal to $k+2$. Then either t_kt_{k+1} or $t_k^{-1}t_{k+1}^{-1}$ moves $\delta_{i,j}$ properly and moves $\delta_{k,k+1}$ to $\delta_{k+1,k+2}$ and leaves at most one index $|i|$ or $|j|$ bigger than $k+1$. Applying this procedure again, if necessary, we may assume $j<k$. If $j>0$ we can move $\delta_{i,j}$ properly, without moving $\delta_{k,k+1}$, and get a curve $\delta_{i,j}$ with $j<0$. Now applying consecutive products $t_{m+1}t_m$ or $t_{m+1}^{-1}t_m^{-1}$ we reach $k=1$. Further moves will produce one of the following three cases:

Case 1a $\;d_{1,2}$ commutes with $d_{-2,-1}$. True, by (11).

Case 1b $\;d_{1,2}$ commutes with $d_{-3,-1}$ or $d_{-3,-2}$. We can conjugate $d_{-3,-2}$ by t_1 and get $d_{-3,-1}$. Now $d_{-3,-1}=t_2^{-1}*d_{-2,-1}=$ (by (11)) $(e_2^{-1}a_3^{-1}a_2^{-1}e_2^{-1}w_2)*$ $d_{1,2}=$ (by (J) and (5)) $(e_2^{-1}a_3^{-1}e_1a_1d_{1,2}^{-1}e_2^{-1}a_2^{-1}e_1^{-1})*a_1=$ (by (J) and (5)) $(e_2^{-1}d_{1,2}^{-1}a_3^{-1}e_2^{-1}e_1a_1a_2^{-1}e_1^{-1})*a_1$.

The last expression commutes with $d_{1,2}$, by (J) and (5).

Case 1c $\;d_{-2,-1}$ commutes with $d_{3,4}$. The proof is rather long. We consider relation (M3): $d_3=a_3^{-1}a_2^{-1}a_1^{-1}d_{1,2}d_{1,3}d_{2,3}$, where $d_3=(b_1^{-1}b_2a_2e_1e_2a_2a_3e_2b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1})*b_2=$ $(b_1^{-1}b_2a_2e_1e_2a_2a_3e_2b_2a_2e_1a_1)*b_1$. It follows, by (J), that d_3 commutes with $a_2e_1e_2a_2a_3e_2$, hence $d_3=(b_1^{-1}((a_2e_1e_2a_2a_3e_2)^{-1}*b_2)b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1})*b_2$. We now conjugate relation (M3) by $u=(a_4e_3a_3e_2a_2e_1a_1b_1)^{-1}a_3e_2a_2e_1a_1b_1$. When we write $d_{1,2}=(b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1})*b_2$ we see that $d_{1,2}$ commutes with u, by (J).

All other factors on the RHS commute with u by (J), so we may replace d_3 in the relation (M3) by $u * d_3 =$
$(a_4e_3a_3e_2a_2e_1a_1b_1)^{-1} * ((a_3e_2a_2e_1a_1((a_2e_1e_2a_2a_3e_2)^{-1} * b_2)b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1}) * b_2)$.

We conjugate each term by $(a_4e_3a_3e_2a_2e_1a_1b_1)^{-1}$ and get
$u * d_3 = (e_3a_3e_2a_2e_1((e_2a_2a_3e_2e_3a_3)^{-1} * d_{1,2})a_1^{-1}e_1^{-1}a_2^{-1}e_2^{-1}) * d_{1,2}$.

We now conjugate each term of relation (M3) by $t_2^{-1}t_3^{-1} =$
$e_2^{-1}a_3^{-1}e_3^{-1}a_4^{-1}a_2^{-1}e_2^{-1}a_3^{-1}e_3^{-1}$.

The RHS becomes $a_4^{-1}a_3^{-1}a_1^{-1}(t_2^{-1} * d_{1,2})((t_2^{-1}t_3^{-1}t_2) * d_{1,2})d_{3,4}$.

The LHS becomes
$(e_2^{-1}a_3^{-1}e_3^{-1}a_4^{-1}) * ((e_1((e_2a_2a_3e_2e_3a_3)^{-1} * d_{1,2})(a_1^{-1}e_1^{-1}a_2^{-1}e_2^{-1})) * d_{1,2})$.

We conjugate each bracket.
$(e_2^{-1}a_3^{-1}e_3^{-1}a_4^{-1}) * e_1 = e_1$.
$(e_2^{-1}a_3^{-1}e_3^{-1}a_4^{-1}(e_2a_2a_3e_2e_3a_3)^{-1}) * d_{1,2} =$ (by (J) and (5)) $(t_3^{-1}t_2^{-1}) * d_{1,2}$.
$(e_2^{-1}a_3^{-1}e_3^{-1}a_4^{-1}a_1^{-1}e_1^{-1}a_2^{-1}e_2^{-1}) * d_{1,2} =$ (by (J) and (5)) $(a_1^{-1}e_1^{-1}t_2^{-1}) * d_{1,2}$.

We shall prove that all terms of the obtained equation commute with $d_{-2,-1}$, except possibly for $d_{3,4}$. Therefore $d_{3,4}$ also commute with $d_{-2,-1}$. Clearly $a_1, e_1, a_2, a_3, a_4, t_1, t_3$ commute with $d_{-2,-1}$ by (5) and symmetry. Also $d_{1,2}$ commutes with $d_{-2,-1}$ by Case 1a and $d_{1,3}$ and $d_{2,3}$ commute with $d_{-2,-1}$ by Case 1b. Therefore $t_2^{-1} * d_{1,2} = t_1 * d_{2,3}$ commute with $d_{-2,-1}$ and $(t_2^{-1}t_3^{-1}t_2) * d_{1,2} =$ (by (7) and (5)) $(t_3t_2^{-1}) * d_{1,2}$ commutes with $d_{-2,-1}$. It follows that $d_{3,4}$ commutes with $d_{-2,-1}$

Case 2 $i = -j$ We have to prove that $d_{k,k+1}$ commutes with $d_{-j,j}$ if $j \neq k, k+1$. If $j < k$ then the result follows by Lemma 33 and by Case 1. If $j > k+2$ we can properly move $\delta_{k,k+1}$ to $\delta_{j-2,j-1}$, without moving $\delta_{-j,j}$. So we may assume $j = k+2$. We have
$d_{-j,j} = (d_{k+1,k+2}t_{k+1}^{-1}t_k^{-1}d_{k,k+1}) * d_{-k,k} =$ (by (6) and (14))
$(d_{-k-2,-k-1}^{-1}t_{k+1}t_kd_{-k-1,-k}^{-1}) *\ d_{-k,k}$.

We conjugate $d_{k,k+1}$ by $(d_{-k-2,-k-1}^{-1}t_{k+1}t_kd_{-k-1,-k}^{-1})^{-1}$ and get, by Case 1 and (13), $(d_{-1-k,-k}t_k^{-1}t_{k+1}^{-1}d_{-k-2,-k-1}) * d_{k,k+1} = d_{-1-k,-k} * d_{k+1,k+2} = d_{k+1,k+2}$, which commutes with $d_{-k,k}$ by the first part of Case 2.

Suppose now that $k < 0$. Cases 1a and 1b follow by symmetry. In the Case 1c we arrive by symmetry at the situation where we have to prove that $d_{1,2}$ commutes with $d_{-4,-3}$. Conjugating by $t_2t_1t_3t_2$ we get a pair $d_{3,4}$, $d_{-2,-1}$, which commutes by Case 1c. Now Case 2 follows by symmetry. □

Lemma 34 *If $t_k^{-1}d_{k,k+1}$ takes a curve $\delta_{i,j}$ to $\delta_{p,q}$ then $t_k^{-1}d_{k,k+1} * d_{i,j} = d_{p,q}$.*

Proof We shall list the relevant cases. If i or j is equal to k it becomes $k+1$, if i or j is equal to $-k$ it becomes $-k-1$. In particular $t_k^{-1}d_{k,k+1}(\delta_{-k,k}) = \delta_{-k-1,k+1}$. Indices $k+1$ and $-k-1$ are forbidden, they do not move properly, except for $\delta_{k,k+1}$ and $\delta_{-k-1,-k}$ which are fixed by $t_k^{-1}d_{k,k+1}$. Other indices i, j do not change.

We now pass to the proof of the Lemma. If $i = -j$ the result follows from the definitions and from (20) and Lemma 33. Suppose $i \neq -j$. If i and j are different from k then $d_{i,j}$ commutes with $d_{k,k+1}$, by (20), and t_k^{-1} moves $\delta_{i,j}$ properly, so we are done by Lemma 33. If i and j are different from $-k$ we can replace $t_k^{-1}d_{k,k+1}$ by $t_k d_{-k-1,-k}^{-1} a_k^2 a_{k+1}^2$, using (14), and we are done by a similar argument. □

Lemmas 33 and 34 allow us to reduce relations (P2) to relatively small number of cases. We can apply product of half-twists $h_{k,k+1}$ and $h_{-k-1,-k}$ either in the same direction, conjugating by t_k, or in opposite directions, conjugating by $t_k^{-1}d_{k,k+1}$, and move properly curves corresponding to elements $d_{p,q}$ in the relations (P2) into a small number of standard configurations.

(21) $\quad d_{i,j}$ commutes with $d_{r,s}$ if $r < s < i < j$ or $i < r < s < j$.

Proof of (21) Moving curves $\delta_{i,j}$ and $\delta_{r,s}$ properly we can arrive at a situation $s = r+1$ or $j = i+1$ or $-r = s = 1$ or $-i = j = 1$. In particular if $i < r < s < j$ and $r = -s$ then conjugating by $t_2 d_{2,3}^{-1} \dots t_{s-1} d_{1-s,s-1}^{-1}$ we get a pair $d_{-1,1}$, $d_{i,j}$. Then (21) follows from (20). □

(22) $$d_{r,i}^{-1} * d_{i,j} = d_{r,j} * d_{i,j} \quad \text{if} \quad r < i < j.$$

Proof of (22) Indices $(i,-i)$ move together (either remain fixed or move to $(i-1, 1-i)$ or to $(i+1, -i-1)$) when we conjugate by t_k or s or $t_k^{-1}d_{k,k+1}$. Moving curves properly we can arrive at one of the following four cases depending on the pairs of opposite indices.

Case 1 There is no pair of opposite numbers among r, i, j. We may assume that $(r, i, j) = (1, 2, 3)$.

$d_{1,2}^{-1} * d_{2,3} =$ (by (13)) $(d_{1,2}^{-1}t_1^{-1}t_2^{-1}) * d_{1,2} =$ (by (5) and (19)) $(t_1^{-1}t_2) * d_{1,2}$.
$d_{1,3} * d_{2,3} = (t_2 d_{1,2} t_2^{-1} t_1 t_2) * d_{1,2} =$ (by (5) and (7)) $(t_2 t_1 d_{1,2} t_2) * d_{1,2} =$ (by (19)) $(t_2 t_1 t_2^{-1}) * d_{1,2} =$ (by (5) and (7)) $(t_1^{-1}t_2) * d_{1,2}$.

Case 2 $i = -r$ We may assume that $(r,i,j) = (-1,1,2)$.

$d_{-1,1}^{-1} * d_{1,2} = (s^{-2}a_1^{-4}) * d_{1,2} =$ (by (19)) $(s^{-1}d_{1,2}s) * d_{1,2} = d_{-1,2} * d_{1,2}$.

Case 3 $r = -j$ We may assume that $(r,i,j) = (-2,1,2)$. We have to prove that $d_{-2,1}^{-1} * d_{1,2} = d_{-2,2} * d_{1,2}$. We conjugte by $d_{1,2}^{-1}t_1$. The right hand side becomes $s^2 * d_{1,2}$ and the left hand side becomes $(d_{1,2}^{-1}t_1 t_1^{-1} s^{-1} d_{1,2}^{-1} s t_1) * d_{1,2} =$ (by (5) and (19)) $(s d_{1,2}^{-1} s^{-1} d_{1,2}^{-1}) * d_{1,2} =$ (by (19)) $s^2 * d_{1,2}$.

Case 4 $i = -j$ We may assume that $(r,i,j) = (-2,-1,1)$.

$d_{-2,-1}^{-1} * d_{-1,1} = a_1^4(d_{-2,-1}^{-1} * s^2) =$ (by (19) and symmetry) $a_1^4((s d_{-2,-1} s^{-1}) * s^2) = d_{-2,1} * d_{-1,1}$. □

$$d_{r,j}^{-1} * d_{r,i} = d_{i,j} * d_{r,i} \quad \text{if} \quad r < i < j. \tag{23}$$

Proof of (23) Let us apply symmetry to relation (22). We get $d_{-i,-r} * d_{-j,-i}^{-1} = d_{-j,-r}^{-1} * d_{-j,-i}^{-1}$ if $-j < -i < -r$. This is relation (23) after a suitable change of indices. □

$$[d_{i,j}, d_{r,j}^{-1} * d_{r,s}] = 1 \quad \text{if} \quad r < i < s < j. \tag{24}$$

Proof of (24) Again we have to consider different cases depending on pairs of opposite indices. For each of them we move curves properly to some standard position. If we apply symmetry we get $[d_{-j,-i}, d_{-j,-r} * d_{-s,-r}] = 1$. Now conjugate by $d_{-j,-r}^{-1}$ and get $[d_{-s,-r}, d_{-j,-r}^{-1} * d_{-j,-i}] = 1$ if $-j < -s < -i < -r$. This is again relation (24) with different pairs of opposite indices. So $i = -j$ is equivalent to $r = -s$ and $s = -j$ is equivalent to $r = -i$. We are left with the following five cases.

Case 1 There is no pair of opposite numbers among r, i, s, j. We may assume that $(r,i,s,j) = (1,2,3,4)$. Conjugate by t_3^{-1}.
$t_3^{-1} * d_{2,4} = d_{2,3}$.
$(t_3^{-1}d_{1,4}^{-1}) * d_{1,3} = (t_2 d_{1,2}^{-1} t_2^{-1} t_3^{-1} t_2) * d_{1,2} =$ (by (7)) $(t_2 d_{1,2}^{-1} t_3 t_2^{-1} t_3^{-1}) * d_{1,2} =$ (by

(5)) $(t_2 t_3 d_{1,2}^{-1} t_2^{-1}) * d_{1,2} =$ (by (19)) $(t_2 t_3 t_2) * d_{1,2} = d_{1,4}$, and it commutes with $d_{2,3}$, by (21).

Case 2 $r = -i$ We may assume that $(r,i,s,j) = (-1,1,2,3)$. We conjugate by t_2^{-1}.
$t_2^{-1} * d_{1,3} = d_{1,2}$.
$(t_2^{-1} d_{-1,3}^{-1}) * d_{-1,2} = (t_2^{-1} s^{-1} t_2 d_{1,2}^{-1} t_2^{-1}) * d_{1,2} =$ (by (7) and (19)) $(s^{-1} t_2) * d_{1,2} = d_{-1,3}$, and it commutes with $d_{1,2}$, by (21).

Case 3 $r = -s$ We may assume that $(r,i,s,j) = (-2,1,2,3)$. By (23) we have $d_{-2,3}^{-1} * d_{-2,2} = d_{2,3} * d_{-2,2}$, so we have to prove that $d_{1,3}$ commutes with $d_{2,3} * d_{-2,2}$. We conjugate by t_2^{-1} and get
$t_2^{-1} * d_{1,3} = d_{1,2}$.
$(t_2^{-1} d_{2,3}) * d_{-2,2} = d_{-3,3}$, and it commutes with $d_{1,2}$, by (21).

Case 4 $r = -j$ and $i \neq -s$ We may assume that $(r,i,s,j) = (-3,1,2,3)$. By (23) we have $d_{-3,3}^{-1} * d_{-3,2} = d_{2,3} * d_{-3,2}$. After conjugation by $t_1^{-1} d_{2,3}^{-1}$ we have to prove that $t_1^{-1} * d_{-3,2} = d_{-3,1}$ commutes with $(t_1^{-1} d_{2,3}^{-1}) * d_{1,3} = (t_1^{-1} d_{2,3}^{-1} t_1^{-1}) * d_{2,3} =$ (by (19)) $d_{2,3}$. This is true by (21).

Case 5 $i = -s$ We may assume that $(r,i,s,j) = (-2,-1,1,m)$, where $m = 2$ or $m = 3$. By (23) we have $d_{-2,m}^{-1} * d_{-2,1} = d_{1,m} * d_{-2,1}$. After conjugation by $s^{-1} d_{1,m}^{-1}$ we have to consider $s^{-1} * d_{-2,1} = d_{-2,-1}$ and $(s^{-1} d_{1,m}^{-1}) * d_{-1,m}$. For $m = 2$ the last expression is equal $(s^{-1} d_{1,2}^{-1} s^{-1}) * d_{1,2} = d_{1,2}$, by (19). For $m = 3$ we get $(s^{-1} d_{1,3}^{-1} s^{-1}) * d_{1,3} = (s^{-1} t_2 d_{1,2}^{-1} t_2^{-1} t_2 s^{-1}) * d_{1,2} =$ (by (19)) $d_{1,3}$. Both elements commute with $d_{-2,-1}$, by (21). □

This concludes the proof of the fact that relations (P2) follow from (M1) – (M3).

We now pass to the relations (P9) – (P11).

Consider first relations (P9). Clearly a_1 commutes with all the elements in (P9), by (4) and (6), so it suffices to prove that b_1 commutes with these elements. It commutes with $a_1^2 s$, $t_1 s t_1$, a_2, and t_i, for $i > 1$, by (J). Also b_1 commutes with $d_{-2,2}$, by (16), and commutes with $d_{2,3}$ by (18). Finally it commutes with $d_{1,2} s d_{1,2}$, by (19), hence also commutes with $d_{-1,1} d_{-1,2} d_{1,2} a_1^{-2} a_2^{-1} = a_1^4 s^2 s^{-1} d_{1,2} s d_{1,2} a_1^{-2} a_2^{-1} = a_1^2 s d_{1,2} s d_{1,2} a_2^{-1}$. This proves relations (P9). Relation (P10) follows from the definitions and (M1).

We now pass to relations (P11).

For $i=1,2,3,4$ we have $g_i=(k_ir)^3$. By (6) $k_i*a_1=a_1$ therefore it suffices to prove that $k_i*b_1=b_1^{-1}*k_i$. Then
$g_i=(k_ia_1b_1a_1)^3=$ (by (6)) $k_ia_1b_1a_1^2k_ib_1k_ia_1^2b_1a_1=k_ia_1sk_isa_1$ and this is exactly relation (P11) for $i=1,2,3,4$.

$$k_1*b_1=b_1^{-1}*k_1. \tag{25}$$

Proof of (25) Since $k_1=a_1$ the result follows from (M1). □

$$k_2*b_1=b_1^{-1}*k_2. \tag{26}$$

Proof of (26) Since $k_2=d_{1,2}$ the result follows from (5). □

$$k_3*b_1=b_1^{-1}*k_3. \tag{27}$$

Proof of (27) We have
$k_3=a_1^{-1}a_2^{-2}d_{1,2}d_{-2,1}d_{-2,2}=a_1^{-1}a_2^{-2}d_{1,2}t_1^{-1}s^{-1}d_{1,2}st_1t_1^{-1}d_{1,2}s^2a_1^4d_{1,2}^{-1}t_1=$
(by (5) and (J)) $a_1^{-1}t_1^{-1}d_{1,2}s^{-1}d_{1,2}sd_{1,2}s^2a_1^2d_{1,2}^{-1}t_1=$ (by (19))
$a_1^{-1}t_1^{-1}d_{1,2}sd_{1,2}sa_1^2t_1=$ (by (5) and (J)) $a_1^{-1}t_1^{-1}(d_{1,2}b_1a_1)^4t_1$.

Let $u=t_1b_1d_{1,2}a_1b_1$. It follows from (5) and (J) that $u*a_1=d_{1,2}$, $u*e_1=b_1$, $u*a_2=a_1$, $u*d_{1,2}=a_2$. Conjugating (11) by u we get $(d_{1,2}b_1a_1)^4=a_2(u*d_{-2,-1})$, hence $k_3=$ (by (4)) $(t_1^{-1}u)*d_{-2,-1}=(b_1d_{1,2}a_1b_1)*d_{-2,-1}$. We want to prove that b_1 and k_3 are braided. It suffices to prove it for their common conjugates. We conjugate by b_1^{-1} and get b_1 and $(d_{1,2}a_1b_1)*d_{-2,-1}$. Now we conjugate by $a_1^{-1}b_1^{-1}d_{1,2}^{-1}$ and get, by (5), (6) and symmetry, $d_{1,2}$ and $b_1*d_{-2,-1}=d_{-2,-1}^{-1}*b_1$. Now we conjugate by $d_{-2,-1}$ and get, by (20), $d_{1,2}$ and b_1, which are braided. □

$$k_4*b_1=b_1^{-1}*k_4 \tag{28}$$

Proof of (28) By the definition and by (M3) we have $k_4=d_3=(b_2a_2e_1b_1^{-1})*d_{1,3}$. Conjugating k_4 and b_1 by $t_2^{-1}b_1e_1^{-1}a_2^{-1}b_2^{-1}$ we get $d_{1,2}$ and b_1, which are braided, by (5). □

$$g_5=sa_1^2k_5sa_1^2k_5^{-1} \tag{29}$$

Proof of (29) By the definition $g_5 = (rk_5rk_5^{-1})^2$ where $k_5 = a_2d_{1,2}^{-1}t_1$. We shall prove that r commutes with $k_5rk_5^{-1}$. Then $g_5 = r^2k_5r^2k_5^{-1} = sa_1^2k_5sa_1^2k_5^{-1}$, as required.

$k_5rk_5^{-1} = a_2d_{1,2}^{-1}e_1a_1a_2e_1a_1b_1a_1e_1^{-1}a_1^{-1}a_2^{-1}e_1^{-1}d_{1,2}a_2^{-1} =$ (by (J) and (5))
$a_2^2d_{1,2}^{-1}e_1a_1b_1a_1^{-1}e_1^{-1}d_{1,2} =$ (by (J))
$a_2^2d_{1,2}^{-1}b_1^{-1}a_1^{-1}e_1a_1b_1d_{1,2}$.

The last expression commutes with a_1 and b_1, by (J) and (5)). □

(30) $$g_6 = (sa_1^2t_1)^4$$

Proof of (30) By the definition $g_6 = (ra_1t_1)^5$. The required relation is proved by a rather long computation, using (J). Observe first that $sa_1^2 = (b_1a_1)^3$, and that $ra_1 = (b_1a_1)^2$. We also have

$t_1(b_1a_1)^2t_1 = e_1a_1a_2e_1b_1a_1b_1a_1e_1a_1a_2e_1 =$
$e_1a_1b_1a_2e_1a_1b_1a_1e_1a_1a_2e_1 = b_1e_1a_1b_1a_2e_1a_1b_1a_1e_1a_2e_1 =$
$b_1a_1e_1a_1b_1a_2e_1a_1b_1a_1e_1a_2 = b_1a_1e_1a_1a_2e_1b_1a_1b_1a_1e_1a_2 =$
$(b_1a_1)t_1(b_1a_1)^2e_1a_2$

and

$e_1a_2(b_1a_1)^2t_1 = b_1e_1a_1a_2b_1a_1e_1a_1a_2e_1 = b_1e_1a_1a_2e_1b_1a_1e_1a_2e_1 =$
$b_1e_1a_1a_2e_1b_1a_2a_1e_1a_2 = b_1a_1e_1a_1a_2e_1b_1a_1e_1a_2 = (b_1a_1)t_1(b_1a_1)e_1a_2 =$
$b_1a_1e_1a_1b_1a_2e_1a_1e_1a_2 = b_1a_1b_1e_1a_1b_1a_2e_1a_1a_2 = b_1a_1b_1a_1e_1a_1b_1a_2e_1a_1 =$
$(b_1a_1)^2t_1(b_1a_1)$.

The required result follows from the above relations. □

This concludes the proof of Theorem 1.

5 Mapping class group of a closed surface

We shall consider in this section the mapping class group $\mathcal{M}_g$ of a closed surface $S_{g,0}$ of genus $g > 1$. We shall keep the notation from the previous section. In particular $\mathcal{M}_{g,1}$ is the mapping class group of $S = S_{g,1}$ and $S_{g,0}$ is obtained from S by capping the boundary ∂ of S by a disk D with a distinguished center p, and then forgetting p. We have two exact sequences

$$1 \to \mathbb{Z} \xrightarrow{\psi} \mathcal{M}_{g,1} \xrightarrow{\phi} \mathcal{M}_{g,0,1} \to 1$$

$$1 \to \pi_1(S_{g,0,1}, p) \xrightarrow{\sigma} \mathcal{M}_{g,0,1} \xrightarrow{e_*} \mathcal{M}_{g,0,0} \to 1$$

In the first sequence the Dehn twist $\Delta = T_\partial$ belongs to the kernel of ϕ. We shall prove now that it generates the kernel. When we split the surface $S_{g,1}$ open along the curves $\beta_1, \alpha_1, \epsilon_1, \alpha_2, \ldots, \epsilon_{g-1}, \alpha_g$ we get an annulus N, and one boundary of N is equal to ∂. If $h \in ker(\phi)$ than h takes each curve $\gamma \in \{\beta_1, \alpha_1, \epsilon_1, \alpha_2, \ldots, \epsilon_{g-1}, \alpha_g\}$ onto a curve $h(\gamma)$ which is isotopic to γ in $S_{g,0}$ by an isotopy fixed on p. Therefore γ and $h(\gamma)$ form 2–gons which are disjoint from p and hence from D. It follows that h is isotopic in $S_{g,1}$ to a homeomorphism equal to the identity on all curves $\beta_1, \alpha_1, \epsilon_1, \alpha_2, \ldots, \epsilon_{g-1}, \alpha_g$. But then it is a homeomorphism of the annulus N so it is isotopic to a power of Δ.

The second sequence is described in [2], Theorem 4.3. The kernel of e_* is generated by spin-maps $T_{\gamma'} T_\gamma^{-1}$, where γ and γ' are simple, nonseparating curves which bound an annulus on $S_{g,0,1}$ containing the distinguished point p. The composition $e_*\phi$ is an epimorphism from the group $\mathcal{M}_{g,1}$ onto the group $\mathcal{M}_g = \mathcal{M}_{g,0,0}$ and its kernel is generated by Δ and by the spin maps $T_{\gamma'} T_\gamma^{-1}$, where γ and γ' are simple, nonseparating curves on S which bound an annulus with a hole bounded by ∂. Clearly all such annuli are equivalent by a homeomorphism of S, hence all spin maps are conjugate in $\mathcal{M}_{g,1}$. It suffices to consider one spin map $T_{\delta'_g} T_{\delta_g}^{-1}$, where δ_g and δ'_g are curves on Figure 18. T_{δ_g} is equal to the element d_g in the relation (M4).

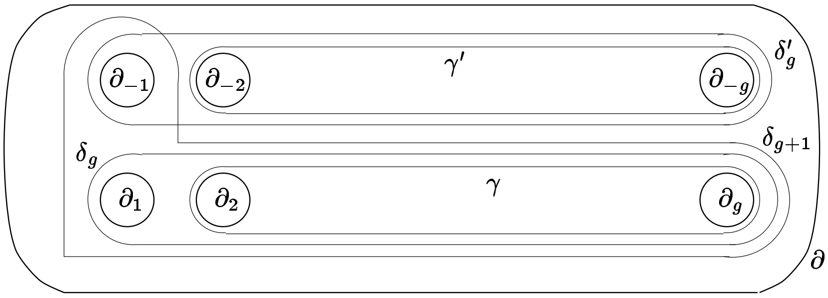

Figure 18: Spin maps in the proof of Theorem 3

Let $w = b_1 a_1 e_1 a_2 \ldots a_{g-1} e_{g-1} a_g^2 e_{g-1} a_{g-1} \ldots a_2 e_1 a_1 b_1$. It is easy to check, drawing pictures, that $w(\delta_g) = \delta'_g$. Therefore, by Lemma 20, relation (M4) is equivalent, modulo relations in $\mathcal{M}_{g,1}$, to $T_{\delta_g} = T_{\delta'_g}$. By the above argument $\mathcal{M}_g$ has a presentation with relations (M1) – (M4) and relation $\Delta = 1$. We have to prove that the last relation follows from the others.

Let $\mathcal{M}' = (\mathcal{M}_{g,1})/(M4)$. Let d_g, d'_g, d_{g+1}, c, c' be twists along curves δ_g,

δ'_g, δ_{g+1}, γ, γ' respectively, depicted on Figure 18. Each element of $\mathcal{M}_{g,1}$ represents an element in $\mathcal{M}'$ which we denote by the same symbol.

From now on, till the end of this section, symbols denote elements in $\mathcal{M}'$. We want to prove that $\Delta = 1$.

All relations from Lemma 21 are true in $\mathcal{M}'$. We have $d_g * b_1 = b_1^{-1} * d_g$ and d_g commutes with every a_i and e_i. By Lemma 21, (iii) and (v) we have:

$$\Delta = (a_g e_{g-1} a_{g-1} \dots e_1 a_1 b_1 d_g)^{2g+2} =$$
$$(a_g e_{g-1} a_{g-1} \dots e_1 a_1)^{2g} (b_1 a_1 \dots a_g a_g \dots a_1 b_1)(d_g b_1 a_1 \dots a_g a_g \dots a_1 b_1 d_g),$$
$$d_g d'_g = (a_g e_{g-1} a_{g-1} \dots e_1 a_1)^{2g} =$$
$$(a_g e_{g-1} a_{g-1} \dots e_2 a_2)^{2g-2} (e_1 a_2 \dots a_g a_g \dots a_2 e_1)(a_1 e_1 a_2 \dots a_g a_g \dots a_2 e_1 a_1),$$
$$(a_g e_{g-1} a_{g-1} \dots e_2 a_2)^{2g-2} = cc',$$
$$(d_g b_1 a_1)^4 = c d_{g+1}.$$

We also see that $c' d_{g+1}^{-1}$ and $d'_g d_g^{-1}$ are spin maps, hence $c' = d_{g+1}$ and $d_g = d'_g$. Therefore

$$(a_g e_{g-1} a_{g-1} \dots e_2 a_2)^{2g-2} = (d_g b_1 a_1)^4,$$
$$d_g^2 = (a_g e_{g-1} a_{g-1} \dots e_2 a_2)^{2g-2} (e_1 a_2 \dots a_g a_g \dots a_2 e_1)(a_1 e_1 a_2 \dots a_g a_g \dots a_2 e_1 a_1),$$

hence

$$(a_1 e_1 a_2 \dots a_g a_g \dots a_2 e_1 a_1) = (e_1 a_2 \dots a_g a_g \dots a_2 e_1)^{-1} (d_g b_1 a_1)^{-4} d_g^2.$$

Further

$$\Delta = (a_g e_{g-1} a_{g-1} \dots e_1 a_1)^{2g} (b_1 a_1 \dots a_g a_g \dots a_1 b_1)(d_g b_1 a_1 \dots a_g a_g \dots a_1 b_1 d_g) =$$
$$d_g^2 b_1 (a_1 e_1 a_2 \dots a_g a_g \dots a_2 e_1 a_1) b_1 (d_g b_1 a_1 \dots a_g a_g \dots a_1 b_1 d_g) =$$
$$d_g^2 b_1 (e_1 a_2 \dots a_g a_g \dots a_2 e_1)^{-1} (d_g b_1 a_1)^{-4} d_g^2 b_1 (d_g b_1 a_1 \dots a_g a_g \dots a_1 b_1 d_g).$$

Now $(d_g b_1 a_1 \dots a_g a_g \dots a_1 b_1 d_g)$ commutes with d_g, by (M4), and commutes with b_1 and a_1, by (J), hence

$$\Delta =$$
$$d_g^2 b_1 (e_1 a_2 \dots a_g a_g \dots a_2 e_1)^{-1} (d_g b_1 a_1)^{-1} (d_g b_1 a_1 \dots a_g a_g \dots a_1 b_1 d_g)(d_g b_1 a_1)^{-3} d_g^2 b_1$$
$= d_g^2 b_1 a_1 b_1 d_g (a_1^{-1} b_1^{-1} d_g^{-1})^3 d_g^2 b_1 = 1$, by (J).

This concludes the proof of Theorem 3.

6 Equivalence of presentations

In this section we shall prove that the presentations of $\mathcal{M}_{g,1}$ in Theorems 1 and $1'$ are equivalent. The relations (M1) coincide with the relations (A). It follows from relations (A) that b_2 commutes with the left hand side of the relation (B). Thus (B) is equivalent to

$$(b_2a_2e_1a_1b_1^2a_1e_1a_2b_2)(a_2e_1a_1b_1^2a_1e_1a_2)^{-1} = (b_1a_1e_1)^4.$$

Multiplying by $(a_2e_1a_1b_1^2a_1e_1a_2)$ on the right we get

$(b_2a_2e_1a_1b_1^2a_1e_1a_2b_2) = (b_1a_1e_1)^4(a_2e_1a_1b_1^2a_1e_1a_2) = (b_1a_1e_1a_2)^5$, as in the proof of Lemma 21 (v), and we get a relation identical with (M2).

We now pass to relation (M3). We shall transform it using relations (M1) and (M2) and then we shall conjugate it by $w = a_3e_2a_2e_1a_1b_1$ to get the relation (C). Since (M1)=(A) and (M2) is equivalent to (B) in presence of (M1), it will prove that (M3) is equivalent to (C) in presence of (M1) and (M2). It follows from (M1) and the definitions that each factor on the right hand side of (M3) commutes with $a_1a_2a_3$, therefore d_3 also commutes with $a_1a_2a_3$. Recall the relations (5), (7) and (19) from section 4, which follow from the relations (M1) and (M2).

(5) $d_{1,2}t_1 = t_1d_{1,2}$,

(7) $t_1t_2t_1 = t_2t_1t_2$,

(19) $t_2d_{1,2}t_2d_{1,2} = d_{1,2}t_2d_{1,2}t_2$.

We now have

$$\begin{aligned}
&d_{1,2}d_{1,3}d_{2,3} = d_{1,2}t_2d_{1,2}t_2^{-1}t_1t_2d_{1,2}t_2^{-1}t_1^{-1} = \text{(by 7)}\\
&d_{1,2}t_2d_{1,2}t_1t_2t_1^{-1}d_{1,2}t_2^{-1}t_1^{-1} = \text{(by 5)}\ d_{1,2}t_2t_1d_{1,2}t_2d_{1,2}t_1^{-1}t_2^{-1}t_1^{-1} = \text{(by 7)}\\
&d_{1,2}t_2t_1d_{1,2}t_2d_{1,2}t_2^{-1}t_1^{-1}t_2^{-1} = \text{(by 19)}\ d_{1,2}t_2t_1t_2^{-1}d_{1,2}t_2d_{1,2}t_1^{-1}t_2^{-1} = \text{(by 7 and 5)}\\
&t_1^{-1}d_{1,2}t_2t_1d_{1,2}t_2d_{1,2}t_1^{-1}t_2^{-1} = \text{(by 5)}\ t_1^{-1}d_{1,2}t_2d_{1,2}t_1t_2t_1^{-1}d_{1,2}t_2^{-1} = \text{(by 7 and 19)}\\
&t_1^{-1}t_2^{-1}d_{1,2}t_2d_{1,2}t_1t_2d_{1,2}t_2^{-1} = \text{(by 5 and 19)}\ t_1^{-1}t_2^{-1}d_{1,2}t_2t_1t_2^{-1}d_{1,2}t_2d_{1,2}.
\end{aligned}$$

We now conjugate everything by w and, using (M1), we get

$$\begin{aligned}
&w * a_1 = b_1,\ \ w * e_1 = a_1,\ \ w * a_2 = e_1,\ \ w * e_2 = a_2,\ \ w * a_3 = e_2,\\
&w * t_1 = a_1b_1e_1a_1 = \tilde{t}_1,\ \ w * t_2 = a_2e_2e_1a_2 = \tilde{t}_2,\ \ w * d_{1,2} = b_2.
\end{aligned}$$

Therefore after conjugation by w the right hand side of (M3) becomes the right hand side of (C).

We have shown in the proof of relations (20), Case 1c, using only relations (M1), that

$$d_3 = (b_1^{-1}((a_2e_1e_2a_2a_3e_2)^{-1} * b_2)b_1^{-1}a_1^{-1}e_1^{-1}a_2^{-1}) * b_2.$$

When we conjugate the last expression by w we get exactly the expression for $\tilde{d}_3$ in Theorem $1'$.

This proves the equivalence of the presentations in Theorems 1 and $1'$.

In order to compare Theorems 3 and $3'$ we need another set of generators. Let us call the curves $\beta_2, \beta_1, \alpha_1, \epsilon_1, \alpha_2 \ldots, \epsilon_{g-1}, \alpha_g$ — *the generating curves.* Let β_g be the curve shown on Figure 1 and let β_2' be a curve which intersects ϵ_{g-2} once and intersects β_g once and is disjoint from the other *generating curves.* Then the curves $\beta_2', \alpha_g, \epsilon_{g-1}, \alpha_{g-1}, \ldots, \epsilon_1, \alpha_1, \beta_1$ have the same intersection pattern as the *generating curves* and the curve β_g plays the same role with respect to these curves as the curve δ_g with respect to the *generating curves.* Let b_g and b_2' be twists along the curves β_g and β_2' respectively. Then, by Theorem 1, we have a new presentation of $\mathcal{M}_{g,1}$ with generators $b_2', a_g, e_{g-1}, a_{g-1}, \ldots, a_1, b_1$ and with defining relations (M1$'$), (M2$'$), (M3$'$) corresponding to (M1), (M2), (M3). It is a presentation of the same group and therefore, when we express b_2' in terms of the generators from Theorem 1, it is equivalent to the presentation ((M1), (M2), (M3)) and to the presentation ((A), (B), (C)). By Theorem 3 the group $\mathcal{M}_{g,0}$ has a presentation with relations (M1$'$), (M2$'$), (M3$'$) and one more relation

(M4$'$) $\quad [a_ge_{g-1}a_{g-1}\ldots e_1a_1b_1^2a_1e_1\ldots a_{g-1}e_{g-1}a_g, b_g] = 1.$

Here b_g is some product of generators which represents the Dehn twist of $S_{g,1}$ along the curve β_g. All such products are equivalent modulo relations (M1$'$), (M2$'$), (M3$'$). Relation (D) of Theorem $3'$ has the same form with b_g replaced by $\tilde{d}_g$. Therefore in order to check that the presentations in Theorems 3 and $3'$ are equivalent it suffices to prove that the expression for $\tilde{d}_g$ in (D) also represents the Dehn twist with respect to the curve β_g. This task (of drawing very many pictures) is left to the reader.

References

[1] **J S Birman**, **H Hilden**, *On mapping class groups of closed surfaces as covering spaces*, from: "Advances in the theory of Riemann surfaces", Ann. Math. Stud. 66 (1971) 81–115

[2] **J S Birman**, *Braids, links and mapping class groups*, Annals of Math. Studies 82 (1974)

[3] **J S Birman**, **B Wajnryb**, *Errata: presentations of the mapping class group*, Israel J. Math. 88 (1994) 425–427

[4] **J Cerf**, *La stratification naturelle ...*, Publ. Math. I.H.E.S. 39 (1970) 5–173

[5] **M Dehn**, *Die Gruppe der Abbildungsklassen*, Acta Math. 69 (1938) 135–206

[6] **S Gervais**, *A finite presentation of the mapping class group of an oriented surface*, preprint

[7] **J Harer**, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. 72 (1983) 221–239

[8] **J Hass**, **P Scott**, *Intersections of curves on surfaces*, Israel J. Math. 51 (1985),90–120.

[9] **A Hatcher**, **W Thurston**, *A presentation for the mapping class group of a closed orientable surface*, Topology, 19 (1980) 221–237

[10] **M Heusner**, *Eine Präsentation der Abbildungsklassengruppe einer geschlossenen, orientierbaren Fläche*, Diplom-Arbeit, University of Frankfurt

[11] **S Humphries**, *Generators for the mapping class group*, from: "Topology of Low-dimensional Manifolds", LNM 722 (1979) 44–47

[12] **D Johnson**, *Homeomorphisms of a surface which act trivially on homology*, Proc. Amer. Math. Soc. 75 (1979) 119–125

[13] **R C Kirby**, *A calculus of framed links in S^3*, Invent. Math. 45 (1978) 35–56

[14] **T Kohno**, *Topological invariants for 3–manifolds using representations of mapping class groups I*, Topology, 31 (1992) 203–230

[15] **F Laudenbach**, *Presentation du groupe de diffeotopies d'une surface compacte orientable*, Travaux de Thurston sur les surfaces, Asterisque 66–67 (1979) 267–282

[16] **Ning Lu**, *A simple proof of the fundamental theorem of Kirby calculus on links*, Trans. Amer. Math. Soc. 331 (1992) 143–156

[17] **M Matsumoto**, *A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities*, preprint

[18] **S Matveev**, **M Polyak**, *On a tangle presentation of the mapping class groups of surfaces*, Contemporary Mathematics, 164 (1994) 219–229

[19] **J McCool**, *Some finitely presented subgroups of the automorphism group of a free group*, J. Algebra, 35 (1975) 205–213

[20] **B Wajnryb**, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. 45 (1983) 157–174