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Flag Manifolds and the Landweber–Novikov Algebra

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Abstract

We investigate geometrical interpretations of various structure maps associated with the Landweber–Novikov algebra S^* and its integral dual S_*. In particular, we study the coproduct and antipode in S_*, together with the left and right actions of S^* on S_* which underly the construction of the quantum (or Drinfeld) double $\mathcal{D}(S^*)$. We set our realizations in the context of double complex cobordism, utilizing certain manifolds of bounded flags which generalize complex projective space and may be canonically expressed as toric varieties. We discuss their cell structure by analogy with the classical Schubert decomposition, and detail the implications for Poincaré duality with respect to double cobordism theory; these lead directly to our main results for the Landweber–Novikov algebra.

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1 Introduction

The Landweber–Novikov algebra S^* was introduced in the 1960s as an algebra of cohomology operations in complex cobordism theory, and was subsequently described by Buchstaber and Shokurov [6] in terms of differential operators on a certain algebraic group. More recently, both S^* and its integral dual S_* have been studied from alternative viewpoints [15], [18], [22], reflecting the growth in popularity of Hopf algebras throughout mathematics. Nevertheless, the interpretations have remained predominately algebraic, although the underlying motivations have ranged from theoretical physics to combinatorics.

Our purpose here is to provide a purely geometric description of S_*, incorporating its structure maps and certain left and right actions by S^*; the importance of the latter is their contribution to the adjoint action, which figures prominently in the construction of the quantum (or Drinfeld) double $\mathcal{D}(S^*)$. We work in the context of double complex cobordism, whose properties we have developed in a preliminary article [8]. So far as we are aware, double cobordism theories first appeared in [20], and in the associated work [23]. To emphasize our geometric intent we return to the notation of the 60s, and write bordism and cobordism functors as $\Omega_*(\ )$ and $\Omega^*(\ )$ throughout.

The realizations we seek are provided by a family of bounded flag manifolds with various double U–structures. These manifolds were originally constructed by Bott and Samelson [4] without reference to flags or U–structures, and were introduced into complex cobordism theory in [21]. We consider their algebraic topology in detail, describing computations in bordism and cobordism theory which provide the essential link with the Landweber–Novikov algebra, and are related to the generalized Schubert calculus of Bressler and Evens [5]. These results appear to be of independent interest and extend to the topological study of other toric manifolds [3], [9], as well as being related to Magyar's program [17] for the description of arbitrary Bott–Samelson varieties in combinatorial terms. We hope to record such extensions in a future work.

For readers who seek background information in algebra, combinatorics, and geometry, we suggest the classic books by Kassel [14], Aigner [2], and Griffiths and Harris [12] respectively.

We begin in section 2 by summarizing prerequisites and notation connected with double complex cobordism, recalling the coefficient ring Ω_*^{DU} and its subalgebra G_*, together with the canonical isomorphism which identifies them with the Hopf algebroid A_*^U and its sub-Hopf algebra S_* respectively. In section 3 we study the geometry and topology of the bounded flag manifolds $B(Z_{n+1})$, describing their toric structure and introducing the posets of subvarieties X_Q which serve to desingularize their cells. In section 4 we define the basic U– and double U–structures on X_Q which underlie the geometrical realization of G_*, and use them to compute $\Omega_U^*(X_Q)$ and $\Omega_*^U(X_Q)$; the methods extend to double cobordism, although several aspects of duality demand extra care. We apply this material in section 5 to calculate Ω_*^{DU}–theory characteristic numbers of the X_Q, interpreting the results by means of the calculus of section 3. Under the canonical isomorphism, realizations of the relevant structure maps for S^* and S_* follow immediately.

We use the following notation and conventions without further comment.

Given a complex m–plane bundle ξ over a finite CW complex, we let $\xi^\perp$ denote the complementary $(n-m)$–plane bundle in some suitably high-dimensional trivial bundle $\mathbb{C}^n$.

We write A_U^* for the algebra of complex cobordism operations, and A_*^U for its continuous dual $\mathrm{Hom}_{\Omega_*^U}(A_U^*, \Omega_*^U)$, forcing us in turn to write S^* for the graded Landweber–Novikov algebra, and S_* for its dual $\mathrm{Hom}_{\mathbb{Z}}(S^*, \mathbb{Z})$; neither of these notations is entirely standard.

Several of our algebras are polynomial in variables such as b_k of grading $2k$, where b_0 is the identity. An additive basis is therefore given by monomials of the form $b_1^{\omega_1} b_2^{\omega_2} \ldots b_n^{\omega_n}$, which we denote by b^ω, where ω is the sequence of nonnegative, eventually zero integers $(\omega_1, \omega_2, \ldots, \omega_n, 0 \ldots)$. The set of all such sequences forms an additive semigroup, and $b^\psi b^\omega = b^{\psi+\omega}$. Given any ω, we write $|\omega|$ for $2\sum i\omega_i$, which is the grading of b^ω. We distinguish the sequences $\epsilon(m)$, which have a single nonzero element 1 and are defined by $b^{\epsilon(m)} = b_m$ for each integer $m \geq 1$. It is often convenient to abbreviate the formal sum $\sum_{k\geq 0} b_k$ to b, and write $(b)_k^n$ for the component of the nth power of b in grading $2k$; negative values of n are permissible.

When dualizing, we choose dual basis elements of the form c_ω, defined by $\langle c_\omega, b^\psi \rangle = \delta_{\omega,\psi}$; this notation is designed to be consistent with our convention on gradings, and to emphasize that the elements c_ω are not necessarily monomials themselves.

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2 Double complex cobordism

In this section we summarize the appropriate parts of [8], concerning the notation and conventions of double complex cobordism, and operations and cooperations in the corresponding homology and cohomology theories.

Double complex cobordism is based on manifolds M whose stable normal bundles are equipped with a splitting of the form $\nu \cong \nu_\ell \oplus \nu_r$. We refer to an equivalence class of such splittings as a *double U–structure* (ν_ℓ, ν_r) on M, writing $(M; \nu_\ell, \nu_r)$ if the manifold requires emphasis. It is helpful to think of ν_ℓ and ν_r as the *left* and *right normal bundles* of the structure, respectively. We may follow Stong [24] and Quillen [19] in setting up the corresponding bordism and cobordism functors geometrically, taking necessary care with the double indexing inherent in the splitting. Cartesian product ensures that $\Omega^*_{DU}(X)$ is a graded ring for any space or spectrum X. Both functors admit an involution χ induced by interchanging the order of ν_ℓ and ν_r, and we find it convenient to write $\chi(M)$ for $(M; \nu_r, \nu_\ell)$. The coefficient ring Ω^{DU}_* is the *double complex cobordism ring.*

We may recombine the left and right normal bundles to obtain a forgetful homomorphism $\pi\colon \Omega^{DU}_*(X) \to \Omega^U_*(X)$; conversely, we may interpret any standard U–structure as either of the double U–structures $(\nu, 0)$ or $(0, \nu)$, thereby inducing multiplicative natural transformations ι_ℓ and $\iota_r\colon \Omega^U_*(X) \to \Omega^{DU}_*(X)$, which are interchanged by χ. All these transformations have cohomological counterparts, and the compositions $\pi \circ \iota_\ell$ and $\pi \circ \iota_r$ reduce to the identity. Given an element θ of $\Omega^U_*(X)$ or $\Omega^*_U(X)$, we write $\iota_\ell(\theta)$ and $\iota_r(\theta)$ as θ_ℓ and θ_r respectively.

From the homotopy theoretic viewpoint, it is convenient to work in any of the currently fashionable categories which admit well-behaved smash products; a coordinate-free approach suffices, as described in [10], for example. The Pontryagin–Thom construction then ensures that the double complex bordism and cobordism functors are represented by the Thom spectrum $MU \wedge MU$, which we label as DU, and the cobordism ring Ω^{DU}_* is identified with the homotopy ring $\pi_*(DU)$. The transformation π is induced by the product map on MU, whilst ι_ℓ and ι_r are induced by the left and right inclusion of MU in DU respectively, using the unit $S^0 \to MU$ on the opposite factor.

We may also identify the homotopy ring of $MU \wedge MU$ with the Ω^U_*–algebra $\Omega^U_*(MU)$, adopting the convention of [1] (and most subsequent authors) in taking the argument as the second factor. Of course, $\Omega^U_*(MU)$ is also the Hopf algebroid A^U_* of cooperations in complex bordism theory. The Thom isomorphism $\Omega_*^U(MU) \cong \Omega_*^U(BU_+)$ provides a further description, whose ring structure is induced by the Whitney sum map on the Grassmannian BU; it is commonly used to transfer the standard polynomial generator β_n in $\Omega_{2n}^U(BU)$ to the polynomial generator b_n in $\Omega_{2n}^U(MU)$, for each $n \geq 0$. Monomials β^ω in the β_n are dual to the universal Chern classes c_ω in $\Omega_U^*(BU)$, and monomials b^ω in the b_n are dual to the Landweber–Novikov operations s_ω in the algebra of complex cobordism operations A_U^*. The *Landweber–Novikov algebra* S^* is the sub-Hopf algebra generated additively by the s_ω, with coproduct induced by the Cartan formulae; its integral dual S_* is the polynomial subalgebra $\mathbb{Z}[b_1, b_2, \dots]$ of A_*^U, with coproduct δ induced from that of A_*^U by restriction. We combine our isomorphisms as

$$\Omega_*^{DU} \cong \Omega_*^U(MU) \cong \Omega_*^U(BU_+), \tag{2.1}$$

referring to the first as the *canonical isomorphism*, and to the composition as h. An analysis of the Pontryagin–Thom construction confirms that h maps the double cobordism class of any $(M; \nu_\ell, \nu_r)$ to the cobordism class of the singular U–manifold $\nu_r \colon M \to BU$.

There are two complex orientation classes x_ℓ and x_r in $\Omega_{DU}^2(CP^\infty)$, arising from the first Chern class x in $\Omega_U^2(CP^\infty)$; indeed, DU is the universal example of a *doubly complex oriented* spectrum. More generally, there are left and right Chern classes $c_{\psi,\ell}$ and $c_{\omega,r}$ in $\Omega_{DU}^*(BU)$, dual to monomials β_ℓ^ψ and β_r^ω in $\Omega_*^{DU}(BU)$. We obtain mutually inverse expansions

$$x_r = \sum_{n \geq 0} g_n x_\ell^{n+1} \qquad \text{and} \qquad x_\ell = \sum_{n \geq 0} \bar{g}_n x_r^{n+1} \tag{2.2}$$

in $\Omega_{DU}^2(CP^\infty)$, where g_n and $\bar{g}_n$ lie in Ω_{2n}^{DU} for all n and are interchanged by the involution χ. For $n > 0$ they are annihilated by the transformation π, whilst $g_0 = \bar{g}_0 = 1$. As documented in [8], the image of g_n under the canonical isomorphism is b_n, and the isomorphism h of (2.1) therefore satisfies $h(g_n) = \beta_n$ in $\Omega_{2n}^U(BU_+)$, for each $n \geq 0$.

These observations arise from minor manipulations with the definitions, and suggest that we introduce the polynomial subalgebra G_* of Ω_*^{DU}, generated by the elements g_n (or, equivalently, by the elements $\bar{g}_n$) for $n \geq 0$. We may then incorporate our previous remarks and formulate the geometric viewpoint; we also appeal to [21], recalling the construction of singular manifolds $\beta \colon B^n \to CP^\infty$ to represent β_n in $\Omega_{2n}^U(CP^\infty)$, where B^n is an iterated 2–sphere bundle which admits a bounding U–structure for each $n \geq 0$.

Proposition 2.3 *The canonical isomorphism identifies G_* with the dual of the Landweber–Novikov algebra S_* in A_*^U; a representative for the generator g_n is given by $(B^n; \nu \oplus \beta^\perp, \beta)$, for each $n \geq 0$.*

We shall apply Proposition 2.3 to realize the coproduct and antipode of S_*, given by

$$\delta(b_n) = \sum_{k \geq 0} (b)_{n-k}^{n+1} \otimes b_k \qquad \text{and} \qquad \chi(b_n) = (b)_n^{-(n+1)}, \tag{2.4}$$

and the left and right actions of S^* on S_*, given by

$$\langle y, s_\ell a \rangle = \langle \chi(s) y, a \rangle \qquad \text{and} \qquad \langle y, s_r a \rangle = \langle ys, a \rangle;$$

here s and y lie in S^*, and the actions on a in S_* extend naturally to A_*^U. Equivalently, we may write

$$s_\ell a = \sum \langle \chi(s), a_1 \rangle a_2 \qquad \text{and} \qquad s_r a = \sum \langle s, a_2 \rangle a_1 \tag{2.5}$$

where $\delta(a) = \sum a_1 \otimes a_2$, confirming that either of the left or right actions determines (and is determined by) the coproduct δ.

We consider the algebra $A_{DU}^* \cong \Omega_{DU}^*(DU)$ of operations in double complex cobordism theory, whose continuous Ω_*^{DU}–dual is the corresponding Hopf algebroid of cooperations $A_*^{DU} \cong \Omega_*^{DU}(DU)$. An element s of S^* yields operations $s_\ell \otimes 1$ and $1 \otimes s_r$ by action on the first or second factor MU of DU, leading to the *left and right Landweber–Novikov operations* $s_{\psi,\ell} \otimes 1$ and $1 \otimes s_{\omega,r}$, which commute in A_{DU}^* by construction. It follows that A_{DU}^* contains the subalgebra $S_\ell^* \otimes S_r^*$, and that A_*^{DU} contains the subalgebra $S_{*,\ell} \otimes S_{*,r} \cong \mathbb{Z}[b_{j,\ell} \otimes 1, 1 \otimes b_{k,r} : j,k \geq 0]$; these are integrally dual Hopf algebras. Of course $S_\ell^* \otimes S_r^*$ acts on the coefficient ring Ω_*^{DU}, and we need only unravel the definitions in order to express the result in terms of the canonical isomorphism.

Proposition 2.6 *The canonical isomorphism identifies the actions of the algebras $S_\ell^* \otimes 1$ and $1 \otimes S_r^*$ on Ω_*^{DU} with the left and right actions of S^* on A_*^U respectively; in particular G_* is closed under the action of $S_\ell^* \otimes S_r^*$.*

Since S^* is cocommutative, the image of the coproduct $\delta \colon S^* \to S_\ell^* \otimes S_r^*$ defines a third subalgebra S_d^* of A_{DU}^*. The canonical isomorphism identifies the resulting *diagonal action* of S_d^* on G_* with the *adjoint action* of S^* on S_*; this is fundamental to the formation of the quantum double $\mathcal{D}(S^*)$ [14], and underlies the description of $\mathcal{D}(S^*)$ as a subalgebra of A_{DU}^* [7], [8].

By analogy with standard cobordism theory the action of $S_\ell^* \otimes S_r^*$ on Ω_*^{DU} may be expressed in terms of characteristic numbers, since the operation $s_{\psi,\ell} \otimes$

$s_{\omega,r}$ corresponds to the Chern class $c_{\psi,\ell} \otimes c_{\omega,r}$ under the appropriate Thom isomorphism $A^*_{DU} \cong \Omega^*_{DU}(BU \times BU_+)$. So the action of $s_{\psi,\ell} \otimes s_{\omega,r}$ on the cobordism class of $(M;\nu_\ell,\nu_r)$ is given by the Kronecker product

$$\langle c_{\psi,\ell}(\nu_\ell)c_{\omega,r}(\nu_r),\sigma\rangle \tag{2.7}$$

in Ω^{DU}_*, where σ in $\Omega^{DU}_*(M)$ is the canonical orientation class represented by the identity map on M. The left and right actions of S^* are therefore given by restriction, yielding $\langle c_{\psi,\ell}(\nu_\ell),\sigma\rangle$ and $\langle c_{\omega,r}(\nu_r),\sigma\rangle$ respectively. Our procedure for computing the actions of S^*_ℓ and S^*_r on G_* in Theorem 5.4 is now revealed; we take the double U–cobordism class of $(M;\nu_\ell,\nu_r)$, form the Poincaré duals of $c_{\psi,\ell}(\nu_\ell)$ and $c_{\omega,r}(\nu_r)$ respectively, and record the double U–cobordism classes of the resulting source manifolds.

3 Bounded flag manifolds

In this section we introduce our family of bounded flag manifolds, and discuss their topology in terms of a cellular calculus which is intimately related to the Schubert calculus for classic flag manifolds. Our description is couched in terms of nonsingular subvarieties, anticipating applications to cobordism in the next section. We also invest the bounded flag manifolds with certain canonical U– and double U–structures, and so relate them to our earlier constructions in Ω^{DU}_*. Much of our notation differs considerably from that introduced in [21].

We shall follow combinatorial convention by writing $[n]$ for the set of natural numbers $\{1,2,\dots,n\}$, equipped with the standard linear ordering $<$. Every interval in the poset $[n]$ has the form $[a,b]$ for some $1 \leq a \leq b \leq n$, and consists of all m satisfying $a \leq m \leq b$; our convention therefore dictates that we abbreviate $[1,b]$ to $[b]$. It is occasionally convenient to interpret $[0]$ as the empty set, and $[\infty]$ as the natural numbers. We work in the context of the Boolean algebra $\mathcal{B}(n)$ of finite subsets of $[n]$, ordered by inclusion. We decompose each such subset $Q \subseteq [n]$ into maximal subintervals $I(1)\cup\cdots\cup I(s)$, where $I(j) = [a(j),b(j)]$ for $1 \leq j \leq s$, and assign to Q the monomial b^ω, where ω_i records the number of intervals $I(j)$ of cardinality i for each $1 \leq i \leq n$; we refer to ω as the *type* of Q, noting that it is independent of the choice of n. We display the elements of Q in increasing order as $\{q_i : 1 \leq i \leq d\}$, and abbreviate the complement $[n] \setminus Q$ to Q'. We also write $I(j)^+$ for the subinterval $[a(j),b(j)+1]$ of $[n+1]$, and $Q^\wedge$ for $Q \cup \{n+1\}$. It is occasionally convenient to set $b(0)$ to 0 and $a(s+1)$ to $n+1$.

We begin by recalling standard constructions of complex flag manifolds and some of their simple properties, for which a helpful reference is [13]. We work in an ambient complex inner product space Z_{n+1}, which we assume to be invested with a preferred orthonormal basis $z_1, \ldots, z_{n+1}$, and we write Z_E for the subspace spanned by the vectors $\{z_e : e \in E\}$, where $E \subseteq [n+1]$. We abbreviate $Z_{[a,b]}$ to $Z_{a,b}$ (and $Z_{[b]}$ to Z_b) for each $1 \leq a < b \leq n+1$, and write $CP(Z_E)$ for the projective space of lines in Z_E. We let $V - U$ denote the orthogonal complement of U in V for any subspaces $U < V$ of Z_{n+1}, and we regularly abuse notation by writing 0 for the subspace which consists only of the zero vector. A complete flag V in Z_{n+1} is a sequence of proper subspaces

$$0 = V_0 < V_1 < \cdots < V_i < \cdots < V_n < V_{n+1} = Z_{n+1},$$

of which the *standard* flag $Z_0 < \cdots < Z_i < \cdots < Z_{n+1}$ is a specific example. The flag manifold $F(Z_{n+1})$ is the set of all flags in Z_{n+1}, topologized as the quotient $U(n+1)/T$ of the unitary group $U(n+1)$ by its maximal torus.

The flag manifold is a nonsingular complex projective algebraic variety of dimension $\binom{n+1}{2}$, whose cells e_α are even dimensional, indexed by elements of the symmetric group $\mathfrak{S}_{n+1}$, and partially ordered by the decomposition of α into a product of transpositions. The closure of every e_α is an algebraic subvariety, generally singular, known as the *Schubert variety* X_α. Whether considered as cells or subvarieties, the e_α define a basis for the integral homology and cohomology groups $H_*(F(Z_{n+1}))$ and $H^*(F(Z_{n+1}))$, which are integrally dual. The manipulation of cup and cap products and Poincaré duality in these terms is known as the *Schubert calculus* for $F(Z_{n+1})$.

An alternative description of $H^*(F(Z_{n+1}))$ is provided by Borel's computations with the *characteristic homomorphism* $H^*(BT) \to H^*(U(n+1)/T)$, induced by the canonical torus bundle $U(n+1)/T \to BT$. Noting that $H^*(BT)$ is a polynomial algebra on two dimensional generators x_i for $1 \leq i \leq n$, Borel identifies $H^*(F(Z_{n+1}))$ with the ring of *coinvariants* under the action of $\mathfrak{S}_{n+1}$. In this context, x_i is the first Chern class of the line bundle over $F(Z_{n+1})$ obtained by associating $V_i - V_{i-1}$ to each flag V.

The interaction between the Schubert and Borel descriptions of the cohomology of $F(Z_{n+1})$ is a fascinating area of combinatorial algebra and has led to a burgeoning literature on the subject of *Schubert polynomials*, beautifully introduced in MacDonald's book [16].

We call a flag U in Z_{n+1} *bounded* if each i–dimensional component U_i contains the first $i-1$ basis vectors $z_1, \ldots, z_{i-1}$, or equivalently, if $Z_{i-1} < U_i$ for every $1 \leq i \leq n+1$. We define the *bounded flag manifold* $B(Z_{n+1})$ to be the set of all bounded flags in Z_{n+1}, topologized as a subvariety of $F(Z_{n+1})$; it is straightforward to check that $B(Z_{n+1})$ is nonsingular, and has dimension

n. Clearly $B(Z_2)$ is isomorphic to the projective line $CP(Z_2)$ with the standard complex structure, whilst $B(Z_1)$ consists solely of the trivial flag. We occasionally abbreviate $B(Z_{n+1})$ to B_n, in recognition of its dimension.

The algebraic torus $(C^*)^n$ is contained in Z_n, and each of its points t determines a line $L_t < Z_{n+1}$ with basis vector $t + z_{n+1}$. We may therefore embed $(C^*)^n$ in $B(Z_{n+1})$ as an open dense subset, by assigning the bounded flag

$$0 < L_t < L_t \oplus Z_1 < \cdots < L_t \oplus Z_i < \cdots < L_t \oplus Z_{n-1} < Z_{n+1}$$

to each t. The standard action of $(C^*)^n$ on this torus extends to the whole of $B(Z_{n+1})$ by coordinatewise multiplication on Z_n (fixing z_{n+1}), and therefore imposes a canonical toric variety structure [11].

There is a map $p_h\colon B(Z_{n+1}) \to B(Z_{h+1,n+1})$ for each $1 \leq h \leq n$, defined by factoring out Z_h. Thus $p_h(U)$ is given by

$$0 < U_{h+1} - Z_h < \cdots < U_i - Z_h < \cdots < U_n - Z_h < Z_{h+1,n+1}$$

for each bounded flag U. Since $Z_{i-1} < U_i$ for all $1 \leq i \leq n+1$, we deduce that $Z_{h+1,i-1} < U_i - Z_h$ for all $i > h+1$, ensuring that $p_h(U)$ is indeed bounded. We may readily check that p_h is the projection of a fiber bundle, with fiber $B(Z_{h+1})$. In particular, p_1 has fiber $CP(Z_2)$, and after $n-1$ applications we may exhibit $B(Z_{n+1})$ as an iterated bundle

$$B(Z_{n+1}) \to \cdots \to B(Z_{h,n+1}) \to \cdots \to B(Z_{n,n+1}), \tag{3.1}$$

where the fiber of each map is isomorphic to CP^1. This construction was introduced in [21].

We define maps q_h and $r_h\colon B(Z_{n+1}) \to CP(Z_{h,n+1})$ by letting $q_h(U)$ and $r_h(U)$ be the respective lines $U_h - Z_{h-1}$ and $U_{h+1} - U_h$, for each $1 \leq h \leq n$. We remark that $q_h = q_1 \cdot p_{h-1}$ and $r_h = r_1 \cdot p_{h-1}$ for all h, and that the appropriate q_h and r_h may be assembled into maps q_Q and $r_Q\colon B(Z_{n+1}) \to \times_Q CP(Z_{h,n+1})$, where h varies over an arbitrary subset Q of $[n]$. In particular, $q_{[n]}$ is an embedding which associates to each flag U the n–tuple $(U_1, \ldots, U_h - Z_{h-1}, \ldots, U_n - Z_{n-1})$, and describes $B(Z_{n+1})$ as a projective algebraic variety.

We proceed by analogy with the Schubert calculus for $F(Z_{n+1})$. To every flag U in $B(Z_{n+1})$ we assign the *support* $S(U)$, given by $\{j \in [n] : U_j \neq Z_j\}$, and consider the subspace

$$e_Q = \{U \in B(Z_{n+1}) : S(U) = Q\}$$

for each Q in the Boolean algebra $\mathcal{B}(n)$. For example, $e_\emptyset$ is the singleton consisting of the standard flag.

Lemma 3.2 *The subspace $e_Q \subset B(Z_{n+1})$ is an open cell of dimension $2|Q|$, whose closure X_Q is the union of all e_R for which $R \subseteq Q$ in $\mathcal{B}(n)$.*

Proof If $Q = \cup_j I(j)$, then e_Q is homeomorphic to the cartesian product $\times_j e_{I(j)}$, so it suffices to assume that Q is an interval $[a,b]$. If U lies in $e_{[a,b]}$ then $U_{a-1} = Z_{a-1}$ and $U_{b+1} = Z_{b+1}$ certainly both hold; thus $e_{[a,b]}$ consists of those flags U for which $q_j(U)$ is a fixed line L in $CP(Z_{a,b+1}) \setminus CP(Z_{a,b})$ for all $a \leq j \leq b$. Therefore $e_{[a,b]}$ is a $2(b-a+1)$–cell, as sought. Obviously $e_R \subset X_{[a,b]}$ for each $R \subseteq Q$, so it remains only to observe that the limit of a sequence of flags in $e_{[a,b]}$ cannot have fewer components satisfying $U_j = Z_j$, and must therefore lie in e_R for some $R \subseteq [a,b]$. □

Clearly $X_{[n]}$ is $B(Z_{n+1})$, so that Lemma 3.2 provides a CW decomposition for B_n with 2^n cells.

We now prove that all the subvarieties X_Q are nonsingular, in contrast to the situation for $F(Z_{n+1})$.

Proposition 3.3 *For any $Q \subseteq [n]$, the subvariety X_Q is diffeomorphic to the cartesian product $\times_j B(Z_{I(j)^+})$.*

Proof We may define a smooth embedding $i_Q \colon \times_j B(Z_{I(j)^+}) \to B(Z_{n+1})$ by choosing the components of $i_Q(U(1), \dots, U(s))$ to be

$$T_k = \begin{cases} Z_{a(j)-1} \oplus U(j)_i & \text{if } k = a(j)+i-1 \text{ in } I(j) \\ Z_k & \text{if } k \in [n+1] \setminus Q, \end{cases} \tag{3.4}$$

where $U(j)_i < Z_{I(j)^+}$ for each $1 \leq i \leq b(j) - a(j) + 1$; the resulting flag is indeed bounded, since $Z_{a(j),a(j)+i-1} < U(j)_i$ holds for all such i and $1 \leq j \leq s$. Any flag T in $B(Z_{n+1})$ for which $S(T) \subseteq Q$ must be of the form (3.4), so that i_Q has image X_Q, as required. □

We may therefore interpret the set

$$\mathcal{X}(n) = \{X_Q : Q \in \mathcal{B}(n)\}$$

as a Boolean algebra of nonsingular subvarieties of $B(Z_{n+1})$, ordered by inclusion, on which the support function $S \colon \mathcal{X}(n) \to \mathcal{B}(n)$ induces an isomorphism of Boolean algebras. Moreover, whenever Q has type ω then X_Q is isomorphic to the cartesian product $B_1^{\omega_1} B_2^{\omega_2} \dots B_n^{\omega_n}$, and so may be abbreviated to B^ω. In this important sense, S preserves types. We note that the complex dimension $|Q|$ of X_Q may be written as $|\omega|$.

The following quartet of lemmas is central to our computations in section 4.

Lemma 3.5 *The map* $r_{Q'}\colon B(Z_{n+1}) \to \times_{Q'} CP(Z_{h,n+1})$ *is transverse to the subvariety* $\times_{Q'} CP(Z_{h+1,n+1})$*, whose inverse image is* X_Q.

Proof Let T be a flag in $B(Z_{n+1})$. Then $r_h(T)$ lies $CP(Z_{h+1,n+1})$ if and only if $T_{h+1} = T_h \oplus L_h$ for some line L_h in $Z_{h+1,n+1}$. Since $Z_h < T_{h+1}$, this condition is equivalent to requiring that $T_h = Z_h$, and the proof is completed by allowing h to range over Q'. □

Lemma 3.6 *The map* $q_{Q'}\colon B(Z_{n+1}) \to \times_{Q'} CP(Z_{h,n+1})$ *is transverse to the subvariety* $\times_{Q'} CP(Z_{h+1,n+1})$*, whose inverse image is diffeomorphic to* $B(Z_{Q^\wedge})$.

Proof Let T be a flag in $B(Z_{n+1})$ such that $q_h(T)$ lies $CP(Z_{h+1,n+1})$, which occurs if and only if $T_h = Z_{h-1} \oplus L_h$ for some line L_h in $Z_{h+1,n+1}$. Whenever this equation holds for all h in some interval $[a,b]$, we deduce that L_h actually lies in $Z_{b+1,n+1}$. Thus we may describe T globally by

$$T_k = Z_{[k-1]\setminus Q} \oplus U_i,$$

where U_i lies in $Z_{Q^\wedge}$, and i is $k - |[k-1] \setminus Q|$. Clearly $U_{i-1} < U_i$ and $Z_{\{q_1,\dots,q_{i-1}\}} < U_i$ for all appropriate i, so that U lies in $B(Z_{Q^\wedge})$. We may now identify the required inverse image with the image of the natural smooth embedding $j_Q\colon B(Z_{Q^\wedge}) \to B(Z_{n+1})$, as sought. □

We therefore define Y_Q to consist of all flags T for which the line $T_h - Z_{h-1}$ lies in $Z_{Q^\wedge}$ for every h in Q'. It follows that Y_Q is isomorphic to B_k whenever Q has cardinality k; for example, $Y_{[n]}$ is $B(Z_{n+1})$ itself and $Y_\emptyset$ consists of the single flag determined by $T_1 = Z_{n+1}$. The set

$$\mathcal{Y}(n) = \{Y_Q : Q \in \mathcal{B}(n)\}$$

is also a Boolean algebra of nonsingular subvarieties.

Lemma 3.7 *For any* $1 \le m \le n-h$*, the map* $q_h\colon B(Z_{n+1}) \to CP(Z_{h,n+1})$ *is transverse to the subvariety* $CP(Z_{h+m,n+1})$*, whose inverse image is diffeomorphic to* $Y_{[h,h+m-1]'}$.

Proof Let T be a flag in $B(Z_{n+1})$ such that $q_h(T)$ lies $CP(Z_{h+m,n+1})$, which occurs if and only if $T_h = Z_{h-1} \oplus L_h$ for some line L_h in $Z_{h+m,n+1}$. Following the proof of Lemma 3.6 we immediately identify the required inverse image with $Y_{[h-1]\cup[h+m,n]}$, as sought. □

Lemma 3.8 *The following intersections in $B(Z_{n+1})$ are transverse:*

$$X_Q \cap X_R = X_{Q\cap R} \quad \text{and} \quad Y_Q \cap Y_R = Y_{Q\cap R} \quad \text{whenever} \quad Q \cup R = [n],$$

$$\text{and} \quad X_Q \cap Y_R = \begin{cases} X_{Q,R} & \text{if } Q \cup R = [n] \\ \emptyset & \text{otherwise,} \end{cases}$$

where $X_{Q,R}$ denotes the submanifold $X_{Q\cap R} \subseteq B(Z_{R^\wedge})$. Moreover, m copies of $Y_{\{h\}'}$ may be made self-transverse so that

$$Y_{\{h\}'} \cap \cdots \cap Y_{\{h\}'} = Y_{[h,h+m-1]'}$$

for each $1 \leq h \leq n$ and $1 \leq m \leq n-h$.

Proof The first three formulae follow directly from the definitions, and dimensional considerations ensure that the intersections are transverse. The manifold $X_{Q,R}$ is diffeomorphic to $\times_j Y_{R(j)}$ as a submanifold of $B(Z_{n+1})$, where $Q = \cup_j I(j)$ and $R(j) = I(j) \cap R$ for each $1 \leq j \leq s$.

Since $Y_{\{h\}'}$ is defined by the single constraint $U_h = Z_{h-1} \oplus L_h$, where L_h is a line in $Z_{h+1,n+1}$, we may deform the embedding $j_{\{h\}'}$ (through smooth embeddings, in fact) to $m-1$ further embeddings in which the L_h is constrained to lie in $Z_{[h,n+1]\setminus\{h+i-1\}}$, for each $2 \leq i \leq m$. The intersection of the m resulting images is determined by the single constraint $L_h < Z_{h+m,n+1}$, and the result follows by applying Lemma 3.7. □

It is illuminating to consider the toric structure of $B(Z_{n+1})$ in these terms.

Proposition 3.9

(1) *For each $1 \leq h \leq n$, the projection $q_h \colon B(Z_{n+1}) \to CP(Z_{h,n+1})$ is equivariant with respect to an action of the torus $(C^*)^{n-h+1}$, and the equivariant filtration*

$$CP(Z_h) \subset \cdots \subset CP(Z_{h,i}) \subset \cdots \subset CP(Z_{h,n+1})$$

lifts to an equivariant filtration of the irregular values of q_h.

(2) *The quotient of $B(Z_{n+1})$ by the action of the compact torus T^n is homeomorphic to the n–cube I^n.*

Proof For (1), we choose the subtorus of $(C^*)^n$ in which the first $h-1$ coordinates are 1; in particular, when $h=1$ the result refers to toric structures on $B(Z_{n+1})$ and CP^n. For (2), we proceed inductively from the observation that the invariant submanifolds of the action of T^n are the subvarieties $X_{Q\setminus R,Q}$

for all pairs $R \subseteq Q \subseteq [n]$; in particular, the fixed points are standard flags in the subvarieties $B(Z_{Q^\wedge})$, and so display the vertices of the quotient in bijective correspondence with the subsets Q. $\square$

The second part of Proposition 3.9 refers to the structure of $B(Z_{n+1})$ as a toric *manifold* [9], and may be extended by algebraic geometers to a more detailed description of the associated fan [11].

4 Normal structures and duality

In this section we describe the basic $U-$ and double U–structures on the varieties X_Q, and compute their cobordism rings. We pay special attention to Poincaré duality, which makes delicate use of the normal structures and is of central importance to our subsequent applications.

We consider complex line bundles γ_i and ρ_i over $B(Z_{n+1})$, classified respectively by the maps q_i and r_i for each $1 \leq i \leq n$. We set γ_0 to 0 and ρ_0 to γ_1, which are compatible with the choices above and enable us to write

$$\gamma_i \oplus \rho_i \oplus \rho_{i+1} \oplus \cdots \oplus \rho_n \cong \mathbb{C}^{n-i+2} \tag{4.1}$$

for every $0 \leq i \leq n$. We may follow [21] in using (3.1) to obtain an expression of the form $\tau \oplus \mathbb{R} \cong (\oplus_{i=2}^{n+1} \gamma_i) \oplus \mathbb{R}$ for the tangent bundle of $B(Z_{n+1})$, as prophesied by the toric structure; so (4.1) leads to an isomorphism $\nu \cong \oplus_{i=2}^{n}(i-1)\rho_i$. We refer to the resulting U–structure as the *basic* U–structure on $B(Z_{n+1})$. We emphasize that these isomorphisms are of real bundles only, and that the basic U–structure is not compatible with any complex structure on the underlying variety. On $B(Z_2)$, for example, the basic U–structure is that of a 2–sphere S^2, rather than CP^1. Indeed, the basic U–structure on $B(Z_{n+1})$ extends over the 3–disc bundle associated to $\gamma_1 \oplus \mathbb{R}$ for all values of n, so that $B(Z_{n+1})$ represents zero in Ω_{2n}^U.

By virtue of (4.1) we may introduce the double U–structure $(\bigoplus_{i=1}^n i\rho_i, \gamma_1)$, which we again label *basic*; equivalently, we rewrite ν_ℓ as $\gamma = -(\gamma_1 \oplus \cdots \oplus \gamma_n)$. The basic double U–structure does not bound, however, as we shall see in Proposition 4.2. Given any cartesian product of manifolds $B(Z_{n+1})$, we also refer to the product of basic structures as basic.

Proposition 4.2 *With the basic double U–structure, $B(Z_{n+1})$ represents g_n in Ω_*^{DU}; if ν_ℓ and ν_r are interchanged, it represents $\bar{g}_n$.*

Proof It suffices to apply Proposition 2.3 for g_n, because the bundle γ_1 over $B(Z_{n+1})$ coincides with the bundle β of [21] over B^n. The result for $\bar{g}_n$ follows by applying the involution χ. □

Corollary 4.3 *The cobordism classes of the basic double U–manifolds X_Q give an additive basis for G_* as Q ranges over finite subsets of $[\infty]$.*

Proof It suffice to combine Propositions 3.3 and 4.2, remarking that X_Q represents g^ω whenever Q has type ω. □

Henceforth we shall insist that B_n denotes $B(Z_{n+1})$ (or any isomorph) equipped exclusively with the basic double U–structure.

Proposition 4.4 *Both $\mathcal{X}(n)$ and $\mathcal{Y}(n)$ are Boolean algebras of basic U–submanifolds, in which the intersection formulae of Lemma 3.8 respect the basic U–structures.*

Proof It suffices to prove that the pullbacks in Lemmas 3.5, 3.6 and 3.7 are compatible with the basic U–structures. Beginning with Lemma 3.5, we note that whenever ρ_h over $B(Z_{n+1})$ is restricted by i_Q to a factor $B(Z_{I(j)^+})$, we obtain ρ_{k+1} if $h = a(j)+k$ lies in $I(j)$ and γ_1 if $h = a(j)-1$; for all other values of h, the restriction is trivial. Since the construction of Lemma 3.5 identifies $\nu(i_Q)$ with the restriction of $\oplus_h \rho_h$ as h ranges over Q', we infer an isomorphism $\nu(i_Q) \cong (\times_j \gamma_1) \oplus \mathbb{C}^{n-j-|Q|}$ over X_Q (unless $1 \in Q$, in which case the first γ_1 is trivial). Appealing to (4.1), we then verify that this is compatible with the basic structures in the isomorphism $\nu^{X_Q} \cong i_Q^*(\nu^{B(Z_{n+1})}) \oplus \nu(i_Q)$, as claimed. The proofs for Lemmas 3.6 and 3.7 are similar, noting that the restriction of ρ_h to Y_Q is ρ_k if $h = q_k$ lies in Q, and is trivial otherwise, and that the restriction of γ_h is γ_k if $h = q_k$ lies in Q, and is γ_{k+1} if q_k is the greatest element of Q for which $h > q_k$ (meaning γ_1 if $h < q_1$, and the trivial bundle if $h > q_k$ for all k). Since the construction of Lemma 3.6 identifies $\nu(j_Q)$ with the restriction of $\oplus_h \gamma_h$ as h ranges over Q', we infer an isomorphism

$$\nu(j_Q) \cong \bigoplus_{j=1}^{s+1} \Big(a(j) - b(j-1) - 1\Big)\gamma_{c(j)} \tag{4.5}$$

over Y_Q, where $c(j) = j + \sum_{i=0}^{j-1}(b(i) - a(i))$. This isomorphism is also compatible with the basic structures in $\nu^{Y_Q} \cong j_Q^*(\nu^{B(Z_{n+1})}) \oplus \nu(j_Q)$, once more by appeal to (4.1). □

The corresponding results for double U–structures are more subtle, since we are free to choose our splitting of $\nu(i_Q)$ and $\nu(j_Q)$ into left and right components.

Corollary 4.6 *The same results hold for double U–structures with respect to the splittings $\nu(i_Q)_\ell = 0$ and $\nu(i_Q)_r = \nu(i_Q)$, and $\nu(j_Q)_\ell = \nu(j_Q)$ and $\nu(j_Q)_r = 0$.*

Proof One extra fact is required in the calculation for i_Q, namely that γ_1 on $B(Z_{n+1})$ restricts trivially to X_Q (or to γ_1 if $1 \in Q$). □

At this juncture we may identify the inclusions of X_Q in $F(Z_{n+1})$ with certain of the desingularizations introduced by Bott and Samelson [4]. For example, $X_{[n]}$ is the desingularization of the Schubert variety $X_{(n+1,1,2,\ldots,n)}$, and the resolution map is actually an isomorphism in this case. Moreover, the corresponding U–cobordism classes form the cornerstone of Bressler and Evens's calculus for $\Omega_U^*(F(Z_{n+1}))$. In both of these applications, however, the underlying complex manifold structures suffice. The basic U–structures become vital when investigating the Landweber–Novikov algebra (and could also have been used in [5], although an alternative calculus would result). We leave the details to interested readers.

We now use the basic structures on X_Q to investigate Poincaré duality in bordism and cobordism, beginning with the CW decomposition for $B(Z_{n+1})$ which stems from Lemma 3.2. Since the cells e_Q occur only in even dimensions, the corresponding homology classes x_Q^H form a basis for the integral homology groups $H_*(B(Z_{n+1}))$ as Q ranges over $\mathcal{B}(n)$. Applying $\mathrm{Hom}_{\mathbb{Z}}$ determines a dual basis $Hd(x_Q^H)$ for the cohomology $H^*(B(Z_{n+1}))$; we delay clarifying the cup product structure until after Theorem 4.8 below, although it may also be deduced directly from the toric properties of $B(Z_{n+1})$.

We introduce the complex bordism classes x_Q and y_Q in $\Omega^U_{2|Q|}(B(Z_{n+1}))$, represented respectively by the inclusions i_Q and j_Q of the subvarieties X_Q and Y_Q with their basic U–structures. By construction, the fundamental class in $H_{2|Q|}(X_Q)$ maps to x_Q^H in $H_{2|Q|}(B(Z_{n+1}))$ under i_Q; thus x_Q maps to x_Q^H under the Thom homomorphism $\Omega_*^U(B(Z_{n+1})) \to H_*(B(Z_{n+1}))$. The Atiyah–Hirzebruch spectral sequence for $\Omega_*^U(B(Z_{n+1}))$ therefore collapses, and the classes x_Q form an Ω_*^U–basis as Q ranges over $\mathcal{B}(n)$. The classes $x_{[n]}$ and $y_{[n]}$ coincide, since they are both represented by the identity map. They constitute the *basic fundamental class* in $\Omega_{2n}^U(B(Z_{n+1}))$, with respect to which the Poincaré duality isomorphism is given by

$$Pd(w) = w \cap x_{[n]}$$

in $\Omega^U_{2(n-d)}(B(Z_{n+1}))$, for any w in $\Omega_U^{2d}(B(Z_{n+1}))$.

An alternative source of elements in $\Omega_U^2(B(Z_{n+1}))$ is provided by the Chern classes

$$x_i = c_1(\gamma_i) \qquad \text{and} \qquad y_i = c_1(\rho_i)$$

for each $1 \leq i \leq n$. It follows from (4.1) that

$$x_i = -y_i - y_{i+1} - \cdots - y_n \tag{4.7}$$

for every i. Given $Q \subseteq [n]$, we write $\prod_Q x_h$ as x^Q and $\prod_Q y_h$ as y^Q in $\Omega_U^{2|Q|}(B(Z_{n+1}))$, where h ranges over Q in both products.

We may now discuss the implications of our intersection results of Lemma 3.8 for the structure of $\Omega_U^*(B(Z_{n+1}))$. It is convenient (but by no means necessary) to use Quillen's geometrical interpretation of cobordism classes, which provides a particularly succinct description of cup and cap products and Poincaré duality, and is conveniently summarized in [5].

Theorem 4.8 *The complex bordism and cobordism of $B(Z_{n+1})$ satisfy*

(1) $Pd(x^{Q'}) = y_Q$ *and* $Pd(y^{Q'}) = x_Q$*;*

(2) *the elements* $\{y_Q : Q \subseteq [n]\}$ *form an* Ω_*^U*–basis for* $\Omega_*^U(B(Z_{n+1}))$*;*

(3) $Hd(x_Q) = x^Q$ *and* $Hd(y_Q) = y^Q$*;*

(4) *there is an isomorphism of rings*

$$\Omega_U^*(B(Z_{n+1})) \cong \Omega_*^U[x_1, \ldots, x_n]/(x_i^2 = x_i x_{i+1}),$$

where i ranges over $[n]$ and x_{n+1} is interpreted as 0.

Proof For (1), we apply Lemma 3.6 and Proposition 4.4 to deduce that $x^{Q'}$ in $\Omega_U^{2|Q'|}(B(Z_{n+1}))$ is the pullback of the Thom class under the collapse map onto $M(\nu(j_Q))$. Hence $x^{Q'}$ is represented geometrically by the inclusion $j_Q \colon Y_Q \to B(Z_{n+1})$, and therefore $Pd(x^{Q'})$ is represented by the same singular U–manifold in $\Omega_{2|Q|}(B(Z_{n+1}))$. Thus $Pd(x^{Q'}) = y_Q$. An identical method works for $Pd(y^{Q'})$, by applying Lemma 3.5. For (2), we have already shown that the x_Q form an Ω_*^U–basis for $\Omega_*^U(B(Z_{n+1}))$. Thus by (1) the y^Q form a basis for $\Omega_U^*(B(Z_{n+1}))$, and therefore so do the x^Q by (4.7); the proof is concluded by appealing to (1) once more. To establish (3), we remark that the cap product $x^Q \cap x_R$ is represented geometrically by the fiber product of $j_{Q'}$ and i_R, and is therefore computed by the intersection theory of Lemma 3.8.

Bearing in mind the crucial fact that each basic U–structure bounds (except in dimension zero!), we obtain

$$\langle x^Q, x_R\rangle = \delta_{Q,R} \tag{4.9}$$

and therefore that $Hd(x_Q) = x^Q$, as sought. The result for $Hd(y_Q)$ follows similarly. To prove (4) we note that it suffices to obtain the product formula $x_i^2 = x_i x_{i+1}$, since we have already demonstrated that the monomials x^Q form a basis in (2). Now x_i and x_{i+1} are represented geometrically by $Y_{\{i\}'}$ and $Y_{\{i+1\}'}$ respectively, and products are represented by intersections; according to Lemma 3.8 (with $m=2$), both x_i^2 and $x_i x_{i+1}$ are therefore represented by the same subvariety $Y_{\{i,i+1\}'}$, so long as $1 \le i < n$. When $i = n$ we note that x_n pulls back from CP^1, so that $x_n^2 = 0$, as required. □

For any $Q \subseteq [n]$, we obtain the corresponding structures for the complex bordism and cobordism of X_Q by applying the Künneth formula to Theorem 4.8. Using the same notation as in $B(Z_{n+1})$ for any cohomology class which restricts along (or homology class which factors through) the inclusion i_Q, we deduce, for example, a ring isomorphism

$$\Omega_U^*(X_Q) \cong \Omega_*^U[x_i : i \in Q]/(x_i^2 = x_i x_{i+1}), \tag{4.10}$$

where x_i is interpreted as 0 for all $i \notin Q$.

The relationship between the classes x_i and y_i in $\Omega_U^*(B(Z_{n+1}))$ is described by (4.7), but may be established directly by appeal to the third formula of Lemma 3.8, as in the proof of Theorem 4.8; for example, we deduce immediately that $x_i y_i = 0$ for all $1 \le i \le n$. When applied with arbitrary m, the fourth formula of Lemma 3.8 simply iterates the quadratic relations, and produces nothing new.

The results of Theorem 4.8 extend to any complex oriented cohomology theory as usual; in particular, we may substitute double complex cobordism, so long as we choose left or right Chern classes consistently throughout. To understand duality, however, we must also attend to the choice of splittings provided by Corollary 4.6, and the failure of formulae such as (4.9) because the manifolds B_n are no longer double U–boundaries. Since, by (2.7), duality lies at the heart of our applications to the Landweber–Novikov algebra, we treat these issues with care below.

We are particularly interested in the left and right Chern classes x_ℓ^Q, y_ℓ^Q, x_r^Q and y_r^Q in $\Omega_{DU}^{2|Q|}(B_n)$, and we seek economical geometric descriptions of their Poincaré duals. We continue to write x_R and y_R in $\Omega^{DU}_{2|R|}(B_n)$ for the homology classes represented by the respective inclusions of X_R and Y_R with their basic double U–structures.

Proposition 4.11 *In $\Omega^{DU}_{2(n-|Q|)}(B_n)$, we have that*

$$Pd(x^{Q'}_\ell)=y_Q \qquad \text{and} \qquad Pd(y^{Q'}_r)=x_Q,$$

whilst $Pd(x^{Q'}_r)$ and $Pd(y^{Q'}_\ell)$ are represented by the inclusion of Y_Q and X_Q with the respective double U–structures

$$\big(\nu^{Y_Q}-(\nu(j_Q)\oplus\gamma_1),\ \nu(j_Q)\oplus\gamma_1\big) \quad \text{and} \quad (\nu^{X_Q}-i^*_Q\gamma_1,\ i^*_Q\gamma_1),$$

for all $n\geq 0$.

Proof The first two formulae follow at once from Corollary 4.6, by analogy with (1) of Theorem 4.8. The second require the interchange of the left and right components of the normal bundles of j_Q and i_Q respectively, plus the observation that $j^*_Q(\gamma_1)$ is always γ_1, whatever Q. □

Proposition 4.11 extends to X_Q by the Künneth formula, which we express in terms of restriction along i_Q in our applications below; it also extends to general doubly complex oriented cohomology theories in the obvious fashion. It inspires many interesting cobordism calculations, of which we offer a single example.

Proposition 4.12 *The map $q_h\colon B_n\to CP^{n-h+1}$ represents either of the expressions*

$$\sum_{m=0}^{n+1-h} g_{n-m}\beta_{m,\ell} \qquad \text{or} \qquad \sum_{j\geq m=0}^{n+1-h} g_{n-j}(g)^m_{j-m}\beta_{m,r}$$

in $\Omega^{DU}_{2n}(CP^{n-h+1})$, for each $1\leq h\leq n$.

Proof The coefficient of $\beta_{m,\ell}$ in the first expression is given by $\langle x^m_{h,\ell}, x_{[n]}\rangle$; by Proposition 4.11, this is g_{n-m} when $1\leq m\leq n-h+1$, and zero otherwise, as required. To convert the result into the second expression, we dualize the expansion (2.2). □

5 Applications

In our final section, we apply the duality calculations to realize the left and right actions of the Landweber–Novikov algebra on its dual; some preliminary combinatorics is helpful.

Fixing the subset $Q = \cup_{j=1}^{s} I(j)$ of $[n]$, we consider the additive semigroup $H(Q)$ of nonnegative integer sequences h of the form $(h_1, \ldots, h_n)$, where $h_i = 0$ for all $i \notin Q$; for any such h, we set $|h| = 2\sum_i h_i$. Whenever h satisfies $\sum_{i=l}^{b(j)} h_i \leq b(j) - l + 1$ for all $a(j) \leq l \leq b(j)$, we define the subset $hQ \subseteq Q$ by

$$\{m : \sum_{i=l}^{m} h_i < m - l + 1 \text{ for all } a(j) \leq l \leq m \leq b(j)\};$$

otherwise, we set $hQ = Q$. It follows that $hQ = Q \cap h[n]$ for all h in $H(Q)$, and we introduce the subset $S(h) \subseteq [s]$ of indices j for which $I(j) \cap hQ \neq \emptyset$. We also identify the subsemigroup $K(Q) \subseteq H(Q)$ of sequences k for which k_i is nonzero only if $i = a(j)$ for some $1 \leq j \leq s$.

For each h in $H(Q)$ and k in $K(Q)$, our applications require us to invest the manifold $X_{Q,(h+k)[n]}$ of Lemma 3.8 with a non-basic double–U structure. In terms of the decomposition $\times_{S(h+k)} Y_{I(j)\cap(h+k)[n]}$, this is given by

$$\Big(\mathop{\times}_{S(h+k)} (\gamma - k_{a(j)}\gamma_1),\ \mathop{\times}_{S(h+k)} (k_{a(j)} + 1)\gamma_1 \Big), \tag{5.1}$$

and we denote the resulting double–U manifold by $X^k_{Q,(h+k)[n]}$. For example, when h is 0 and k has a single nonzero element $k_{a(j)} = m$ for some $1 \leq j \leq s$ and $m \leq b(j) - a(j)$, then $X^k_{Q,(h+k)[n]}$ reduces to the manifold $X_{Q\setminus[a(j),a(j)+m-1]}$ with double U–structure

$$\big(\gamma^{\times j-1} \times (\gamma - m\gamma_1) \times \gamma^{\times s-j},\ \gamma_1^{\times j-1} \times (m+1)\gamma_1 \times \gamma_1^{\times s-j}\big). \tag{5.2}$$

This case is important enough to motivate the notation $X_P^{m:j}$ (omitting the $:j$ if $s = 1$) for any X_P whose basic double U–structure is similarly amended on its jth factor $Y_{I(j)}$; in particular, (5.2) describes $X^{m:j}_{Q\setminus[a(j),a(j)+m-1]}$.

We may now apply Proposition 4.11 to compute the effect of the left and right actions of S^* on S_* under the canonical isomorphism. To ease computations with the left action we consider the monomial basis of *tangential* Landweber–Novikov operations $\bar{s}_\psi$ for A^U_*; under the universal Thom isomorphism, these correspond to the Chern classes $\perp^* c_\psi$ induced by the involution $\perp$ of complementation on BU. There are therefore expressions

$$\bar{s}_\psi = \sum_\omega \lambda_{\psi,\omega} s_\omega, \tag{5.3}$$

where the $\lambda_{\psi,\omega}$ are integers and the summation ranges over sequences ω for which $|\omega| = |\psi|$ and $\sum \omega_i \geq \sum \psi_i$. For each $Q \subseteq [n]$, it is also helpful to partition $K(Q)$ and $H(Q)$ into compatible blocks $K(Q,\psi)$ and $H(Q,\psi)$ for every indexing sequence ψ; each block consists of those sequences k or h which have ψ_i entries i for each $i \geq 1$, and all other entries zero. Thus, for example, $|h| = |\psi|$ for all h in $H(Q,\psi)$. Any such block will be empty whenever ψ is incompatible with Q in the appropriate sense.

Theorem 5.4 *Up to double U–cobordism, the actions of S^*_ℓ and S^*_r on additive generators of G_* are induced by*

$$\bar{s}_{\psi,\ell}(X_Q) = \sum_{H(Q,\psi)} X_{Q,h[n]} \qquad \text{and} \qquad s_{\omega,r}(X_Q) = \sum_{K(Q,\omega)} X^k_{Q,k[n]}$$

respectively.

Proof We combine (2.7) with Proposition 4.11, recalling that c_θ is evaluated on any sum of line bundles $\oplus_{i=1}^r \lambda_i$ by forming the symmetric sum of all monomials $c_1(\lambda_1)^{i_1} \ldots c_r(\lambda_r)^{i_r}$, where θ_i of the exponents take the value i for each $1 \leq i \leq r$. We note that the product structure in $\Omega^*_{DU}(B_n)$ allows us to replace any x_i^m (either left or right) by $x^{[i,i+m-1]}$ when $[i, i+m-1] \subseteq Q$, and zero otherwise; indeed, the definitions of $H(Q)$ and $K(Q)$ are tailored exactly to these relations. For $s_{\psi,\ell}(X_Q)$ we set $k = 0$, and observe that $\bar{c}_{\psi,\ell}(\nu_\ell) = i^*_Q c_{\psi,\ell}(\oplus_Q \gamma_i)$. For $s_{\omega,r}(X_Q)$ we set $h = 0$, and observe in turn that $c_{\omega,r}(\nu_r) = i^*_Q c_{\omega,r}(\gamma_{a(1)} \oplus \cdots \oplus \gamma_{a(s)})$. The computations are then straightforward, although the bookkeeping demands caution. ◻

Recalling (5.1), we may combine the left and right actions by

$$\bar{s}_{\psi,\ell} \otimes s_{\omega,r}(X_Q) = \sum_{H(Q,\psi),\ K(Q,\omega)} X^k_{Q,(h+k)[n]},$$

from which the diagonal action follows immediately. If we prefer to express the action of S^*_ℓ in terms of the standard basis s_ω, we need only incorporate the integral relations (5.3).

Readers may observe that our expression in section 3 for ν_ℓ as the sum of line bundles $\bigoplus_{i=1}^n i\rho_i$ appears to circumvent the need to introduce the tangential operations $\bar{s}_\psi$. However, it contains $n(n+1)/2$ summands rather than n, and their Chern classes y_i are algebraically more complicated than the x_i used above, by virtue of (4.7). These two factors conspire to make the alternative calculations less palatable, and it is an instructive exercise to reconcile the two approaches in simple special cases. The apparent dependence of Theorem 5.4 on n is illusory (and solely for notational convenience), since k_i and h_i are zero whenever i lies in Q'.

We may specialize Theorem 5.4 to the cases when ψ and ω are of the form $\epsilon(m)$ for some integer $0 \le m \le |Q|$, or when $Q = [n]$ (so that we are dealing with polynomial generators of G_*), or both. We obtain

$$\bar{s}_{\epsilon(m),\ell}(X_Q) = \sum_j \sum_{i=a(j)}^{b(j)-m+1} X_{Q\setminus I(j)} \times Y_{I(j)\setminus[i,i+m-1]}$$
$$\text{and} \quad s_{\epsilon(m),r}(X_Q) = \sum_j X^{m:j}_{Q\setminus[a(j),a(j)+m-1]}, \tag{5.5}$$

where the summations range over all j with $b(j) - a(j) \ge m - 1$, and

$$\bar{s}_{\psi,\ell}(X_{[n]}) = \sum_{H([n],\psi)} Y_{h[n]}$$
$$\text{and} \quad s_{\omega,r}(X_{[n]}) = \begin{cases} X^m_{[m+1,n]} & \text{when } \omega = \epsilon(m) \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

These follow from (5.1), and the facts that $K(Q, \epsilon(m))$ consists solely of sequences containing a single nonzero entry m in some position $a(j)$, and $K([n], \omega)$ is empty unless $\omega = \epsilon(m)$ for some $0 \le m \le n$.

We might expect Theorem 5.4 to provide geometrical confirmation that G_* is closed under the action of $S^*_\ell \otimes S^*_r$ on Ω^{DU}_*, as noted in Proposition 2.6; however, it remains to show that X^{k+1}_{kQ} lies in G_*! Currently, we have no direct geometrical proof of this fact.

We now turn to the structure maps of S_*, continuing to utilize the canonical isomorphism to identify G_* and $G_* \otimes G_*$ with S_* and $S_* \otimes S_*$ respectively. We express monomial generators of $G_* \otimes G_*$ as double U–cobordism classes of *pairs* of basic double U–manifolds (X_Q, X_R), where Q and R range over independently chosen subsets of $[n]$.

Proposition 5.7 *Up to double U–cobordism, the coproduct δ and the antipode χ of the dual of the Landweber–Novikov algebra are induced by*

$$X_Q \mapsto \sum_{K(Q)} (X^k_{Q,k[n]}, X_{Q\setminus kQ}) \qquad \text{and} \qquad X_Q \mapsto \chi(X_Q)$$

respectively.

Proof For δ, we combine the right action of Theorem 5.4 with (2.5), and the observation that $X_{Q\backslash kQ}$ is isomorphic to B^{ω} for each k in $K(Q,\omega)$. For χ, we refer to Proposition 4.2. □

Corollary 5.8 *When equipped with the double U–structure*

$$\Big(\mathop{\times}_{j=1}^{s}(\gamma - m(j)\gamma_1),\ \mathop{\times}_{j=1}^{s}(m(j)+1)\gamma_1\Big),$$

the manifold X_Q represents $\prod_j (g)^{m(j)+1}_{b(j)-a(j)+1}$ in $\Omega^{DU}_{2|Q|}$ for any sequence of natural numbers $m(1)$, $m(2)$, … , $m(s)$.

Proof If we consider the coproduct for $Q=[n]$ in Proposition 5.7, we deduce that $X^{m}_{[m+1,n]}$ represents $(g)^{m+1}_{n-m}$ by appeal to (2.4). The result for general X_Q follows by applying this case to each factor $Y_{I(j)}$. □

Corollary 5.8 is particularly fascinating because it describes how to represent an intricate (but important) polynomial in the cobordism classes of the basic B_n by perturbing the double U–structure on a single manifold X_Q.

For a final comment on Proposition 5.7, we note that the elements of $\Omega^{DU}_* \otimes_{\Omega^U_*} \Omega^{DU}_*$ may be represented by *threefold* U–manifolds. Under the canonical isomorphism, the coproduct on the Hopf algebroid A^{DU}_* is then induced by mapping the double U–cobordism class of each $(M;\nu_\ell,\nu_r)$ to the threefold cobordism class of $(M;\nu_\ell,0,\nu_r)$, and the diagonal on G_* follows by restriction. Theories of multi U–cobordism are remarkably rich, and have applications to the study of iterated doubles and Adams–Novikov resolutions; we reserve our development of these ideas for the future.

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