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Intrinsically linked graphs and even linking number

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Abstract We study intrinsically linked graphs where we require that every embedding of the graph contains not just a non-split link, but a link that satisfies some additional property. Examples of properties we address in this paper are: a two component link with $lk(A,L) = k2^r, k \neq 0$, a non-split n-component link where all linking numbers are even, or an n-component link with components L, A_i where $lk(L, A_i) = 3k, k \neq 0$. Links with other properties are considered as well.

For a given property, we prove that every embedding of a certain complete graph contains a link with that property. The size of the complete graph is determined by the property in question.

AMS Classification 57M15; 57M25,05C10

Keywords Intrinsically linked graph, spatial graph, graph embedding, linking number

1 Introduction

In the early 1980's, Sachs, and independently, Conway and Gordon proved that every embedding of K_6 contains two disjoint cycles that form a nontrivial link [11], [3]. Since that time there has been extensive study of *intrinsically linked graphs*, that is, graphs whose every embedding into three-space contains a non-split link. The classification of these graphs, in terms of excluded minors, was settled by Robertson, Seymour and Thomas in [10]. Study of intrinsically linked graphs has turned to finding conditions that guarantee more complex structures within every embedding of the graph. For example, one could require that every embedding of a graph contains two disjoint links [2], or a link with three or more components [5] [1], or a link where one component is a nontrivial knot [7], or a two component link whose linking number is larger than some constant [4] [12].

It is this last case that interests us here. We will prove that every embedding of certain complete graphs always contains links (of various numbers of components) whose linking numbers are multiples of some constant (generally a power of two). These results are Ramsey theoretic in nature, in that they imply certain structures arise in sufficiently large complete graphs. In essence, we are striving to prove that given stringent requirements on a link, we can find a graph so that every embedding of that graph contains a link meeting those requirements.

In the papers of Flapan [4] and Shirai and Taniyama [12], the authors study graphs that always contain two component links with linking number greater than or equal to k, where here, in our investigation of two component links, we demand that the linking number be a multiple of a power of two. This is a much more restrictive condition and hence it should not be surprising that our bounds are much larger.

For two component links, we have the following theorems:

Theorem 2.5 *Every embedding of K_{β_r} contains a two component link with $lk(A,L)=2^r k$, for some $k\neq 0$.*

Theorem 4.1 *Every embedding of K_{35} contains a link of two components with $lk(A,L)=3k$, for some $k\neq 0$.*

We can obtain a similar result for three component links, but we would like to see it strengthened so that all pairwise linking numbers are zero mod 2^r.

Theorem 3.1 *Every embedding of K_{δ_r} contains a non-split three component link L,W,A with $lk(L,W)=2^r k$, $lk(L,A)=2^r k'$ and $lk(W,A)\equiv 0 \mod 2$, for some $k,k'\neq 0$.*

There are also a number of results for links of arbitrarily many components.

Corollary 2.6 *Every embedding of $K_{\beta'_{n,r}}$ contains an $n+1$ component link $L,Z_1,\dots,Z_n$ with $lk(L,Z_i)=2^r k$, for some $k\neq 0$.*

Corollary 4.2 *Every embedding of $K_{7\alpha'_{3n}}$ contains an $n+1$ component link $L,Z_1,\dots,Z_n$ with $lk(L,Z_i)=3k$, for some $k\neq 0$.*

Theorem 3.2 *Every embedding of K_{ϵ_n} contains a non-split link of n+1 components where all the pairwise linking numbers are even.*

The sequence $\gamma_n=\Pi_{i=0}^{n-1}(2^i+1)$ plays a significant role in determining the size of a complete graph required for some of these properties. Unfortunately, γ_n

grows faster than n^n. Obviously, we would like to see better bounds on the number of vertices required. We also note that in this paper we provide upper bounds for the number of vertices required for these behaviors. Obvious lower bounds are seven for all cases [3], and ten for cases involving links of three or more components [6]. Finding better lower bounds or even the minimum number of vertices required for these properties remain open questions.

2 The basic linking construction

Our primary technique in this paper is to take two cycles that link with a third and use them to produce new cycles with various properties. To that end, the following technical lemmas will be used extensively in later results.

Lemma 2.1 *Let a graph G be embedded such that there exists a three component link with components L, Z, W which have $lk(L,Z) = q_1 > 0$, $lk(L,W) = q_2 > 0$ with $q_i \equiv 0 \bmod 2^r$. Assume that there exist $2^{r+1}+1$ paths P_i from Z to W with distinct end points such that the interior of the P_i are disjoint from L, W, Z and from the other P_i. Further assume that when traversing W in the direction of its orientation, we meet the P_i in ascending order, and when traversing Z in a similar fashion, we meet them in descending order. Then this embedding of G contains a two component link $A \cup L$ with $lk(A,L) = k2^{r+1}$, for some $k \neq 0$.*

Proof Clearly, if q_1 or q_2 is zero mod 2^{r+1}, we are done, so we may assume that $q_1 \equiv q_2 \equiv 2^r \bmod 2^{r+1}$.Orient the P_i such that going from Z to W is the positive direction.Let A_1 be the oriented simple closed curve formed by traversing P_1 in the positive direction, along W in the positive direction from P_1 to P_2, in the negative direction along P_2, and finally along Z in the positive direction from P_2 to P_1. See Figure 1. Define the other A_i similarly.

Note that now, we have that in $H_1(\mathbf{R}^3 - L; \mathbf{Z})$;

$$[Z] + [W] = \sum [A_i]$$

Notice $[Z] + [W] = q_1 + q_2$, which is 0 mod 2^{r+1} and non-zero integrally.

For $1 \leq i \leq 2^{r+1}$, let $S_i = \sum_{j=1}^{i} [A_i]$.

Suppose $S_i \equiv 0 \bmod 2^{r+1}$ for some i. If $S_i \neq 0$ integrally, let $A = A_1 \cup A_2 \cup \ldots \cup A_i$. If $S_i = 0$ integrally, then $\sum_{i+1}^{2^{r+1}+1} [A_j] = q_1 + q_2$. Thus by setting $A = A_{i+1} \cup \ldots \cup A_{2^{r+1}+1}$, we have the desired result.

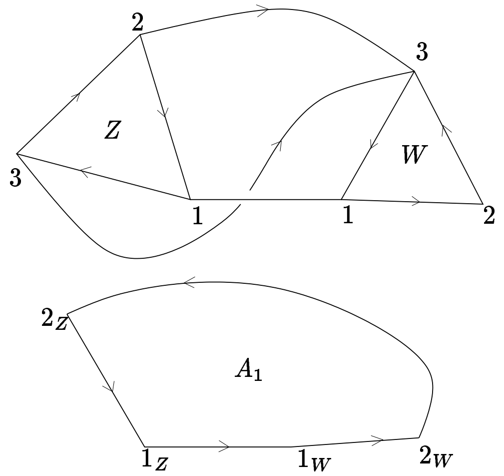

Figure 1: Constructing the A_i for r=1

Thus we may now assume that $S_i \neq 0 \bmod 2^{r+1}$ for all i. However, we have 2^{r+1} values, so for some $i < j$, we have that $S_i \equiv S_j \bmod 2^{r+1}$. Thus $0 \equiv S_j - S_i \equiv \sum_{k=i+1}^{j}[A_k] \bmod 2^{r+1}$. Now, as before, if $S_j - S_i \neq 0$ integrally, let $A = A_{i+1} \cup \dots A_j$. If $S_j - S_i = 0$ integrally, then let $A = A_1 \cup \dots \cup A_i \cup A_{j+1} \cup \dots \cup A_{2^{r+1}+1}$. □

Define $\alpha_1 = 6$, $\alpha_2 = 10$, and for $m \geq 1, \alpha_{2m+1} = 2\alpha_{2m} + 6$ and $\alpha_{2m+2} = 2\alpha_{2m+1}$. Then for $m \geq 1$ we have $\alpha_{2m+1} = 6(\sum_{j=0}^{m} 4^j) - 4^m$. Note that $\alpha_1 = 6\alpha'_1$ and for $n \geq 2$, $\alpha_n < 6\alpha'_n$, where α'_n is the sequence defined before Corollary 2.4.

Lemma 2.2 *Every embedding of K_{α_n} contains a nonsplit link of n+1 components labeled $L, Z_1, \dots, Z_n$, where $lk(L, Z_i) \neq 0$ for all i.*

Proof The case $n = 1$ was shown in [3].

We will proceed by induction. The base case is $n = 2$, and the desired result for K_{10} was shown in [6]. The inductive step for the even and odd cases must be handled separately.

Now suppose that $K_{\alpha_{n-1}}$ contains such of link of n components, where n is even. The graph K_{α_n} is $K_{2\alpha_{n-1}}$ and we may partition it into two disjoint copies of $K_{\alpha_{n-1}}$. Thus we have found two cycles, L and L' which each have non-zero linking number with $n - 1$ other cycles. Clearly if $lk(Z_i, L') \neq 0$ or

$lk(Z_i', L) \neq 0$ for some i, we are done. So we may assume that all such linking numbers are zero.

Now, choose two vertices l_1 and l_2 in L and two vertices l_1' and l_2' in L'. Let P be a path in L connecting l_1 to l_2, and P' be a path in L' connecting l_1' to l_2'. Since this is a complete graph, there are edges connecting l_1 to l_1' and l_2 to l_2'. Note that the interiors of these edges are disjoint from L, L', Z_i, and Z_i'. Let V denote the cycle formed by these edges and P and P'. See Figure 2.

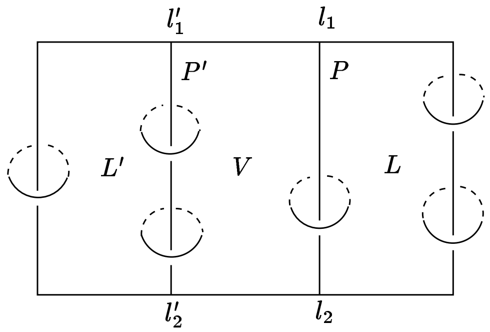

Figure 2: The cycles L, L' and V in the case $n=4$. The Z_i are shown as ovals.

Now, examine $lk(V, Z_i)$ and $lk(V, Z_i')$. If n of these are non-zero, then we have constructed the desired link. If fewer than $n-1$ of these are non-zero, then we take the cycle $L \cup V \cup L'$, and since linking number is additive, we have the desired link once again.

If exactly $n-1$ of the Z_i, Z_i' have non-zero linking number with V, then as $n-1$ is odd, more than half of them came from one set, say the Z_i'. Again using the additivity of linking number, the cycle $V \cup L$, has $lk(V \cup L, Z_i) \neq 0$ for at least $\frac{n}{2}$ of the Z_i, and similarly for the Z_i'. This gives a link with $n+1$ components with the desired properties when n is even.

When n is odd, the inductive step requires more care. In the graph K_{α_n} we may find a two copies of $K_{\alpha_{n-1}}$ as well as a copy of K_6. The latter will tip the balance in our favor.

Again choose vertices l_i and l_i'. Choose also vertices x and y on one of the two linked triangles in K_6, call this triangle L'', the other T. Let P and P' be the paths in L and L' as before. Now let V be the cycle formed by P, the edge $l_2 - l_2'$, P', $l_1' - x$, $x - y$, $y - l_1$. See Figure 3.

Now, clearly $lk(Z_i, L')$, $lk(Z_i', L)$, $lk(L, T)$, and $lk(L', T)$ must be zero, or we have the desired link. Once again, if n or more of the Z_i, Z_i' have non-zero

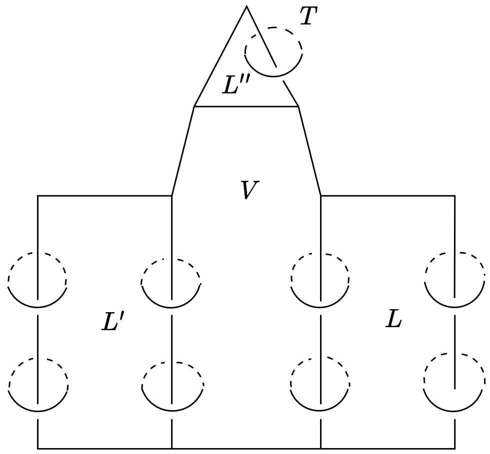

Figure 3: The cycles L, L', L'' and V in the case $n = 5$. The Z_i and T are shown as ovals.

linking number with V we are done, and if fewer than $n-1$, then we take $L \cup V \cup L'$ as before. This leaves only the case where $n-1$ of the Z_i, Z_i' have non-zero linking number with V.

If $lk(T, V) \neq 0$, then take the curve V. This has non-zero linking number with $n-1$ of the Z_i, Z_i', and non-zero linking with T, so we are done in this case as well.

Thus, we may assume $lk(T, V)$ is zero. Let C be the subset of $\{Z_i, Z_i'\}$ which have zero linking number with V, and D be the subset which have non-zero linking number with V. Suppose r elements of C and s elements of D have non-zero linking with L''. Of these s elements, suppose t satisfy the equation $lk(Z_i, L) + lk(Z_i, V) + lk(Z_i, L'') = 0$ (or the similar version for Z_i', L'). Then at least $n + r - s + t$ elements have non-zero linking with $V \cup L''$ and at least $n - r + s - t$ have non-zero linking with $V \cup L \cup L' \cup L''$. Clearly one of these is greater than or equal to n, so we are done. $\square$

Lemma 2.3 *Every embedding of K_{α_n} contains a nonsplit link of n+1 components labeled $L, Z_1, \ldots, Z_n$, where $lk(L, Z_i) \equiv 1 \bmod 2$ for all i.*

The proof of Lemma 2.3 proceeds exactly as for Lemma 2.2, replacing the phrase (non-)zero with (non-)zero mod two.

Let $\alpha_1' = 1$, $\alpha_{2m}' = 2\alpha_{2m-1}'$, and $\alpha_{2m+1}' = 2\alpha_{2m}' + 1$. Again, a calculation gives $\alpha_{2m-1}' = \frac{4^m - 1}{3}$.

Corollary 2.4 *If every embedding of G contains a two component link with non-zero linking number, then every embedding of $*^{\alpha'_n}G$ contains an $n+1$ component link L, Z_i with $lk(L, Z_i) \neq 0$.*

Proof Note that the sequence α'_n satisfies the following recursive relations: $\alpha'_n = 2\alpha'_{n-1}$ if n is even, $\alpha'_n = 2\alpha'_{n-1} + 1$ if n is odd, and $\alpha'_1 = 1$.

The proof is now the same as the proof of Lemma 2.2, but replace K_{10} with $G * G$ and K_6 with G. □

Naturally, for Lemma 2.3 we have a corollary similar to Corollary 2.4.

We will now obtain a sequence of links that can be thought of as the "mod two Whitehead links."

Let $\beta_0 = 6$, $\beta_1 = 10$ and $\beta_r = \alpha'_{2^r}(\gamma_r + 3)$ for higher r. Here $\gamma_r = \Pi_{i=0}^{r-1}(2^i + 1)$.

Theorem 2.5 *Every embedding of K_{β_r} contains a two component link with $lk(A, L) = 2^r k$, for some $k \neq 0$.*

Proof Notice that the case $r = 0$ was proved for K_6 in [3].

The case $r = 1$ is direct. By [6] every embedding of K_{10} contains a three component link, with $lk(L, Z) = q_1, lk(L, W) = q_2$ with q_i odd. Choose an orientation of $L \cup W \cup Z$ so that the q_i are positive. Clearly Z and W contain at least three vertices. Since K_{10} is a complete graph, we have edges from each vertex of Z to all vertices of W, and the interiors of these edges are disjoint from L, W, Z. We may choose P_i from among these edges that satisfy the conditions of Lemma 2.1, so we can find the desired link.

We will now give the construction for general r.

Every embedding of K_{m+3} contains a link composed of a triangle and an m-cycle with nonzero linking number by [9]. By Corollary 2.4, we know that $K_{m+3} * K_{m+3}$ contains a three component link where at least two of the three pairwise linking numbers are nonzero. Choosing the triangles to act as L, we know that the Z_i have m vertices.

Thus, in K_{2m+6} we may find a three component link L, Z_1, Z_2 with $lk(L, Z_i) \neq 0$, and with each Z_i containing m vertices. Now, by Corollary 2.4 and the preceding discussion, if we want a $n+1$ component link with these properties, we need only take $K_{\alpha'_n(m+3)}$.

Suppose n is a large power of two, and m is large. Pair up the Z_i. If $lk(Z_i, L)$ is even for some i, ignore that pair. For the remaining pairs (where $lk(Z_i, L) \equiv 1 \mod 2$) choose three evenly spaced paths between the members of that pair. Applying Lemma 2.1 to these pairs, we have new cycles A_i with $lk(L, A_i) = 2k$ $(k \neq 0)$. Now including the Z_i with even linking number, we have $\frac{n}{2}$ cycles which have nonzero even linking number with L and each cycle has at least $2(\frac{m}{3}+1)$ vertices. As long as we have chosen m and n large enough, we may iterate this process to obtain the desired link.That is, suppose $lk(Z_1, L) \equiv lk(Z_2, L) \equiv 0 \mod 2^{s-1}$ but are non-zero integrally, and that the Z_i have at least m' vertices. If $lk(Z_i, L) \equiv 0 \mod 2^s$, we are done. So we may assume that $lk(Z_1, L) \equiv lk(Z_2, L) \equiv 2^{s-1} \mod 2^s$. Choosing $2^s + 1$ paths from Z_1 to Z_2 allows us to apply Lemma 2.1 to obtain a cycle Z with $lk(Z, L) = k2^s, k \neq 0$, such that Z has at least $2(\frac{m'}{2^s+1}+1)$ vertices.

Thus, if we want a two component link with $lk(A, L) \equiv 0 \mod 2^r$, we choose $n = 2^r$, and $m = \gamma_r$. Clearly we need $n = 2^r$ components linked with L, as after each pairing, we have increased the linking number of Z_i with L by a power of two, but have halved the number of components. However, on iteration s in this procedure (moving from 2^{s-1} to 2^s) we need to have at least $2^s + 1$ vertices in each Z_i to successfully apply Lemma 2.1. Since the number of vertices is strictly decreasing in this process, it suffices to guarantee that we have c_r vertices at the rth iteration, where $c_r \geq 2^r + 1$. If we have c_s vertices at the beginning of iteration s, we will be left with $c_{s+1} = 2(\frac{c_s}{2^s+1}+1)$ vertices at the beginning of iteration $s+1$. Thus, when c_{s+1} is even, we have $c_s = (2^s+1)(\frac{c_{s+1}}{2}-1)$, and we want to choose $m \geq c_1$. Choosing $c_r = 2^r + 2$ gives $c_{r-1} = (2^{r-1}+1)(2^{r-1})$. Since we are looking only for a bound, let us take $c_{r-1} = (2^{r-1}+1)(2^{r-1}) + 2$. This gives $c_{r-2} = (2^{r-2}+1)(2^{r-1}+1)(2^{r-2})$. Continuing this process gives $c_1 = \gamma_r$. Thus, by taking $m = \gamma_r$ we ensure that at each stage, we have a sufficient number of vertices to apply Lemma 2.1. □

We have calculated the actual number of vertices required for some small cases, and γ_r provides a very reasonable estimate. Denoting the required number of vertices v_r, we note that $\gamma_1 = v_1 = 3$; $\gamma_2 = v_2 = 6$;$\gamma_3 = 30, v_3 = 27$;$\gamma_4 = 270, v_4 = 261$.

Let $\beta'_{n,r} = \alpha'_{n2^r}(\gamma_r + 3)$.

Corollary 2.6 *Every embedding of $K_{\beta'_{n,r}}$ contains an $n+1$ component link $L, Z_1, \ldots, Z_n$ with $lk(L, Z_i) = 2^r k$, for some $k \neq 0$.*

Proof By Lemma 2.2, we can find an $n2^r+1$ component link where $lk(Z'_i, L) \neq 0$ and each of the Z_i has at least γ_r vertices. Applying the construction of

Theorem 2.5 to each set of 2^r components, we obtain the desired link. Note that the construction of one such Z_i does not affect L or the other Z_i so we may treat them independently. □

Note that Theorem 2.5 implies that every embedding of K_{10} contains a non-split two component link with even linking number. In fact, we can improve upon that result.

Proposition 2.7 *Every embedding of K_{10} contains a two component link A, L with $lk(A,L) \equiv 2 \mod 4$.*

Proof The proof of Thoerem 2.4 in [8] shows that every embedding of K_{10} contains a two component link with odd unoriented Sato-Levine invariant. A pair of intersection points of opposite sign contribute an even quantity to β^*; a pair of intersection points with matching sign contribute an odd amount. Thus, it is easy to see that $\beta^*(A,L)$ is even if and only if $lk(A,L) \equiv 0 \mod 4$. Since the constructed link has odd unoriented Sato-Levine invariant, it must have linking number 2 mod 4. □

3 Multiple component constructions

We would like to see the following theorem extended to have all three pairwise linking numbers be zero mod 2^r.

Let $\delta_r = \alpha'_{3(2^r)}((2^{2r-1} + 2^r)\gamma_r + 3)$.

Theorem 3.1 *Every embedding of K_{δ_r} contains a non-split three component link L, W, A with $lk(L,W) = 2^r k$, $lk(L,A) = 2^r k'$ and $lk(W,A) \equiv 0 \mod 2$, for some $k, k' \neq 0$.*

Proof By Corollary 2.6, we may find a four component link L, W, Z_1, Z_2, with $lk(L,W) \equiv lk(L,Z_i) \equiv 0 \mod 2^r$, but nonzero integrally. In building up the Z_i as we did in the proof of Theorem 2.5, at each stage the linking number is multiplied by two, but the number of vertices is reduced from c_j to c_{j+1}. If we had begun with γ_r vertices in each cycle, the construction would end with the Z_i containing at least four vertices. However, we will require $(2^r+1)^2$ vertices, so we multiply γ_r by the smallest constant that ensures this result. Thus, each of W, Z_i has at least $2(2^{2r-1} + 2^r + 1) > (2^r+1)^2$ vertices. This number of vertices is necessary to form a cycle A from the Z_i so that the three component link L, W, A will have the desired property. This number of vertices is not necessary in W, but unfortunately Lemma 2.2 does not allow us to control the size of the components independently.

If $lk(Z_i, W) \equiv 0 \bmod 2$ we are done, so suppose that $lk(Z_i, W) \equiv 1 \bmod 2$. The Z_i have $(2^r+1)^2$ vertices by construction. Find $(2^r+1)^2$ paths P_i and construct cycles A_i' as in the proof of Lemma 2.1. Now, write these in a list. Group the first 2^r cycles, then leave one, then group the next 2^r cycles, etc. Look at the first 2^r cycles in $H_1(\mathbf{R}^3 - L; \mathbf{Z})$; we know that there is some subsequence whose sum is 0 mod 2^r. Glue these cycles together. Look at the next block of 2^r cycles. Find its subsequence. Now identify all cycles between these two. The result will be a sequence of cycles A_i' where $[A_i'] \equiv 0 \bmod 2^r$ for i odd, and takes arbitrary values for i even. Any cycles coming before A_1' may be pushed around to the end of the sequence since it is cyclic.

The result is a list of $2(2^r+1)$ cycles, half of which are zero mod 2^r. Now, ignoring the zeros, there is a list of 2^r+1 numbers, and as before there must be a proper consecutive subsequence that is zero mod 2^r. Let A_2'' be the cycle which corresponds to this sequence, A_1'' be the cycle which is 0 mod 2^r that immediately precedes it, and A_3'' be the complement of these two cycles. Since $\sum [A_i'']_L = [Z_1]_L + [Z_2]_L = k2^r, k \neq 0$, we know $[A_3'']_L \equiv 0 \bmod 2^r$.

Now examine $[A_i''] \in H_1(\mathbf{R}^3 - W; \mathbf{Z})$. Again, there is a subsequence of these that sums to zero mod 2. Call the corresponding cycle A. Since $\sum [A_i'']_W = [Z_1]_W + [Z_2]_W \equiv 0 \bmod 2$, the complement of A is also zero mod 2. Call it A'.

By construction, $lk(A, L) \equiv 0 \bmod 2^r$ and $lk(A, W) \equiv lk(A', W) \equiv 0 \bmod 2$. If $lk(A, L) = 0$, then as $[A]_L + [A']_L = \sum [A_i'']_L$, we have that $lk(A', L) = k2^r, k \neq 0$ and we are done. □

Let $\epsilon_1 = 10$ and $\epsilon_n = \alpha'_{2^{n+1}-2}(\gamma'_n + 3)$. Here $\gamma'_n = \Pi_{i=1}^n 3(2^{i-1}) = 3^n 2^{\frac{n(n-1)}{2}}$.

Then extending our construction yet again, we can show the following.

Theorem 3.2 *Every embedding of K_{ϵ_n} contains a non-split link of n+1 components where all the pairwise linking numbers are even.*

Proof The case $n = 1$ is proved in Lemma 2.1 and $n = 2$ is proved in Theorem 3.1. For larger n, we may think of K_{ϵ_n} as the star of $\alpha'_{2^{n+1}-2}$ copies of $K_{\gamma'_n+3}$, and so it contains a link of $2^{n+1}-1$ components, labeled L, Z_i, where $lk(L, Z_i) \neq 0$ and the Z_i have γ'_n vertices. Since $2^{n+1} - 2 = \sum_{i=1}^n 2^i$, we may break the Z_i into sets of size 2^i for each $1 \leq i \leq n$.

Take Z_1 and Z_2. If $lk(Z_i, L) \equiv 0 \mod 2$, label that cycle V_1 and move on. If $lk(Z_i, L) \equiv 1 \mod 2$ for $i = 1, 2$, choosing three paths between them and forming A_i as usual, we may produce V_1, which has $lk(V_1, L) \equiv 2k_1$, with $k_1 \neq 0$.

Take Z_3, Z_4, Z_5, Z_6. Pair these and construct Z_3', Z_4' which have even (and positive) linking number with L as we just did for Z_1 and Z_2. If $lk(Z_i', V_1) \equiv 0 \mod 2$, label it V_2 and continue to the next set of Z_i. If not, choose six paths from Z_3' to Z_4' and construct $A_1, \dots, A_6$. Examine these cycles in $H_1(\mathbf{R}^3 - L; \mathbf{Z/2})$. Since $lk(Z_i', L) \equiv 0 \mod 2$ this is a binary sequence which sums to zero. Cut this sequence into the maximum number of blocks, where the sum in each block is zero. Since we can cyclicly permute the entries, there must be at least three such blocks. Produce the cycles A_1', A_2', A_3' by fusing the cycles in each block. Look at the A_s' in $H_1(\mathbf{R}^3 - V_1; \mathbf{Z/2})$. There is a proper subsequence whose sum is zero mod two. Call the union of those cycles A, and their complement A'. Now, $lk(A, V_1) \equiv lk(A', V_1) \equiv 0 \mod 2$. If $lk(A, L) = 0$, then as $[A]_L + [A']_L = [Z_3'] + [Z_4']$, we have that $lk(A', L) = 2k_2, k_2 \neq 0$. Call this cycle V_2.

Take $Z_{(\sum_{s=1}^{j-1} 2^s)+1} \dots Z_{\sum_{s=1}^{j} 2^s}$. Pair them to produce $Z_i^{(1)}$ that have even, nonzero linking number with L as we did for Z_1 and Z_2. Now, pair the $Z_i^{(1)}$ and produce $Z_i^{(2)}$ that have $lk(Z_i^{(2)}, V_1) \equiv 0 \mod 2$, and $lk(Z_i^{(2)}, L) = k2, k \neq 0$ as we did for Z_3, Z_4, Z_5, Z_6. Now, pair the $Z_i^{(2)}$. If either member of a pair has $lk(Z_i^{(2)}, V_2) \equiv 0 \mod 2$, it advances to the next round. Otherwise, choose twelve paths connecting the members of each pair and produce $A_1 \dots A_{12}$. Examine these in $H_1(\mathbf{R}^3 - L; \mathbf{Z/2})$, then $H_1(\mathbf{R}^3 - V_1; \mathbf{Z/2})$, breaking it into the maximum number of blocks each time. We will be left with at least three cycles that we examine in $H_1(\mathbf{R}^3 - V_2; \mathbf{Z/2})$, allowing us to construct $Z_i^{(3)}$.

Continue the process, connecting the paired cycles by $3(2^{k-1})$ paths in the kth step and producing the A_s. We have chosen our cycles Z_i to have γ_n' vertices, which ensures that we may find these paths in up to n iterative steps. Examine these A_s in $H_1(\mathbf{R}^3 - L; \mathbf{Z/2})$, then $H_1(\mathbf{R}^3 - V_i; \mathbf{Z/2})$ for $i < k-1$ fusing cycles as necessary each time. We began with $3(2^{k-1})$ cycles labeled A_s, so there must be at least three distinct cycles left after fusing $k-1$ times. Examine these three cycles in $H_1(\mathbf{R}^3 - V_{k-1}; \mathbf{Z/2})$. There is a subsequence that sums to zero, we take the corresponding cycle W_1 and its complement W_2. Now, by construction, $lk(W_l, V_i) \equiv 0 \mod 2$ for $i < k$, and at least one of $lk(W_l, L)$ is even and nonzero. Choose that W_l to be $Z_i^{(k)}$.

Since we began with 2^j cycles, we may repeat this process j times until we have V_j, a cycle with $lk(V_j, V_i) \equiv 0 \mod 2$ for $i < j$ and $lk(V_j, L) = 2k_j$ with $k_j \neq 0$.

Iterating this process produces the desired link. □

4 Mod 3 constructions

It is unclear if similar constructions will work modulo integers other than 2. We provide a first step in that direction.

Theorem 4.1 *Every embedding of K_{35} contains a link of two components with $lk(A, L) = 3k$, $k \neq 0$.*

Proof The graph K_{35} has $*^5 K_7$ as a subgraph. By Corollary 2.4 and the discussion in the proof of Theorem 2.5, every embedding of K_{35} contains a four component link L, Z_1, Z_2, Z_3 with $lk(L, Z_i) \neq 0$, and the Z_i each have four vertices. Orient the Z_i and L so that $lk(L, Z_i) = q_i > 0$.

If $lk(L, Z_i) \equiv 0 \mod 3$ for some i, then we are done.

If $lk(L, Z_i) \equiv 1 \mod 3$ and $lk(L, Z_j) \equiv 2 \mod 3$, then we may construct four cycles A_i as in Lemma 2.1. Here $q_1 + q_2 \equiv 0 \mod 3$, and taking S_1, S_2, S_3 as before, the construction of A proceeds in exactly the same manner.

Thus, we may assume that $lk(L, Z_1) \equiv lk(L, Z_2) \equiv lk(L, Z_3) \mod 3$.

We now construct cycles A_1, A_2, A_3, A_4, A_5 as in Figure 4. Note that in $H_1(\mathbf{R}^3 - L; \mathbf{Z})$,

$$\sum [Z_i] = \sum [A_i] = q_1 + q_2 + q_3 \equiv 0 \mod 3$$

If $[A_i] \equiv 0$ for some $i \geq 4$, then we take $A = A_i$ if $[A_i] \neq 0$. If $[A_i] = 0$ integrally, then let $A = \cup_{j \neq i} A_j$.

Suppose that $[A_i] \equiv 0$ for some $i \leq 3$. Again, take $A = A_i$ if $[A_i] \neq 0$ integrally. If $[A_i] = 0$, look at the remaining A_j. We have four contiguous cycles whose sum is zero mod 3, see Figure 4. Forming the partial sums S_1, S_2, and S_3, either two of these are equal, or one is zero, and we may proceed as in Lemma 2.1.

Now, if $[A_i] \neq 0 \mod 3$ for all i, then two of $[A_1], [A_2], [A_3]$ are equal. Say $[A_1] = [A_2]$. Take the following sums. Let $S_1 = [A_4]$, $S_2 = [A_4] + [A_1]$, $S_3 = [A_4] + [A_1] + [A_5]$, all taken mod 3. We can ignore the cases $S_1 \equiv 0, S_1 \equiv S_2$ and

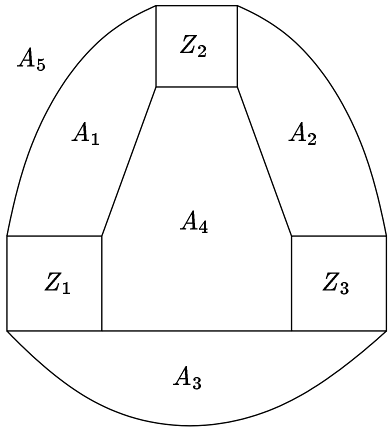

Figure 4: The cycles A_i when using all the Z_i. The A_i (except A_5) are oriented counter-clockwise, A_5 and the Z_i clockwise.

$S_2 \equiv S_3$ as each of these would imply that some $[A_i]$ was zero. Now, if $S_1 = S_3$, we take the cycle representing their difference, and its complement. Since both cycles are connected, one of the two must satisfy the desired conditions. If S_2 is zero mod 3, the same argument applies. The hard case is when $S_3 = 0$ integrally.

The cycle corresponding to S_3 is connected, but its complement, $A_2 \cup A_3$ is not. Since $S_3 = 0$, we know that $[A_1] = -([A_4] + [A_5])$. Now, since $[A_i]$ is non-zero mod 3 for all i, $[A_4] \equiv [A_5]$ mod 3 (otherwise $[A_1] \equiv 0$). But now $[A_1] \equiv -2[A_4] \equiv [A_4]$ mod 3.

Thus $[A_1] \equiv [A_2] \equiv [A_4] \equiv [A_5] \equiv \sum_{i \neq 3}[A_i]$ mod 3. So, as $\sum[A_i] \equiv 0$ mod 3, $[A_3] + [A_5] \equiv 0$ mod 3. Now both $A_3 \cup A_5$ and its complement $A_1 \cup A_2 \cup A_4$ are connected, so one of the two is the desired cycle A. □

Corollary 4.2 *Every embedding of $K_{7\alpha'_{3n}}$ contains an $n+1$ component link $L, Z_1, \ldots, Z_n$ with $lk(L, Z_i) = 3k$, $k \neq 0$.*

Proof The proof is essentially the same as the proof of Corollary 2.6.

We can think of $K_{7\alpha'_{3n}}$ as α'_{3n} copies of K_7. By Corollary 2.4 we have a $3n+1$ component link L, Z_i with $lk(L, Z_i) \neq 0$. The Z_i have at least four vertices each, so breaking them into groups of three, we may now apply construction from the proof of Theorem 4.1 to each group individually to obtain the desired link. □

References

[1] **G Bowlin**, **J Foisy**, *Some new intrinsically 3-linked graphs*, J. Knot Theory Ramifications 13 (2004) 1021–1027

[2] **S Chan**, **A Dochtermann**, **J Foisy**, **J Hespen**, **E Kunz**, **T Lalonde**, **Q Loney**, **K Sharrow**, **N Thomas**, *Graphs with disjoint links in every spatial embedding*, J. Knot Theory Ramifications 13 (2004) 737–748

[3] **J H Conway**, **C M Gordon**, *Knots and links in spatial graphs*, J. Graph Theory 7 (1983) 445–453

[4] **E Flapan**, *Intrinsic knotting and linking of complete graphs*, Algebr. Geom. Topol. 2 (2002) 371–380

[5] **E Flapan**, **J Pommersheim**, **J Foisy**, **R Naimi**, *Intrinsically n-linked graphs*, J. Knot Theory Ramifications 10 (2001) 1143–1154

[6] **E Flapan**, **R Naimi**, **J Pommersheim**, *Intrinsically triple linked complete graphs*, Topology Appl. 115 (2001) 239–246

[7] **T Fleming**, *Intrinsically linked graphs with knotted components*, preprint

[8] **T Fleming**, **J Pommersheim** *Intrinsically linked graphs and the Sato-Levine invariant*, preprint

[9] **B Johnson**, **W Johnson**, *On the size of links in $K_{n,n}$, $K_{n,n,1}$, and K_n*, J. Knot Theory Ramifications 11 (2002) 145–151

[10] **N Robertson**, **P Seymour**, **R Thomas**, *Sachs' linkless embedding conjecture*, J. Combin. Theory Ser. B 64 (1995) 185–227

[11] **H Sachs**, *On spatial representations of finite graphs*, from: "Finite and infinite sets, Vol. I, II (Eger, 1981)", Colloq. Math. Soc. János Bolyai 37, North-Holland, Amsterdam (1984) 649–662

[12] **M Shirai**, **K Taniyama**, *A large complete graph in a space contains a link with large link invariant*, J. Knot Theory Ramifications 12 (2003) 915–919

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