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Rational acyclic resolutions

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Abstract Let X be a compactum such that $\dim_{\mathbb{Q}} X \leq n$, $n \geq 2$. We prove that there is a $\mathbb{Q}$-acyclic resolution $r : Z \longrightarrow X$ from a compactum Z of $\dim \leq n$. This allows us to give a complete description of all the cases when for a compactum X and an abelian group G such that $\dim_G X \leq n$, $n \geq 2$ there is a G-acyclic resolution $r : Z \longrightarrow X$ from a compactum Z of $\dim \leq n$.

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1 Introduction

The cohomological dimension $\dim_G X$ of a separable metric space X with respect to an abelian group G is the least number n such that $\check{H}^{n+1}(X, A; G) = 0$ for every closed subset A of X. It was known long ago that $\dim X = \dim_{\mathbb{Z}} X$ if X is finite dimensional. Solving an outstanding problem in cohomological dimension theory Dranishnikov constructed in 1987 an infinite dimensional compactum (= compact metric space) with $\dim_{\mathbb{Z}} = 3$. A few years earlier a deep relation between $\dim_{\mathbb{Z}}$ and dim was established by the Edwards cell-like resolution theorem [1, 10] saying that a compactum of $\dim_{\mathbb{Z}} \leq n$ can be obtained as the image of a cell-like map defined on a compactum of $\dim \leq n$. A compactum X is cell-like if any map $f : X \longrightarrow K$ from X to a CW-complex K is null homotopic. A map is cell-like if its fibers are cell-like. The reduced Čech cohomology groups of a cell-like compactum are trivial with respect to any group G.

Acyclic resolutions are a natural generalization of the Edwards cell-like resolution. A space is G-acyclic if its reduced Čech cohomology groups modulo G are trivial, a map is G-acyclic if every fiber is G-acyclic. As a consequence of the Vietoris-Begle theorem a surjective G-acyclic map of compacta cannot raise the cohomological dimension $\dim_G$.

Let $\mathcal{P}$ denote the set of prime numbers and let $\mathcal{L} \subset \mathcal{P}$. Recall that

$$\mathbb{Z}_{(\mathcal{L})} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N} \text{ and } n \text{ is not divisible by the elements of } \mathcal{L}\}.$$

Thus $\mathbb{Z}_{(\emptyset)} = \mathbb{Q}$ and $\mathbb{Z}_{(\mathcal{P})} = \mathbb{Z}$. Dranishnikov [4, 5] proved the following:

Theorem 1.1 [4, 5] *Let $\mathcal{L} \subset \mathcal{P}$ and let X be a compactum with $\dim_{\mathbb{Z}_{(\mathcal{L})}} X \leq n$, $n \geq 2$. Then there are a compactum Z with $\dim Z \leq n+1$ and $\dim_{\mathbb{Z}_{(\mathcal{L})}} Z \leq n$ and a $\mathbb{Z}_{(\mathcal{L})}$-acyclic map $r : Z \longrightarrow X$ from Z onto X.*

In [7] Theorem 1.1 was generalized for arbitrary groups.

Theorem 1.2 [7] *Let G be an abelian group and let X be a compactum with $\dim_G X \leq n$, $n \geq 2$. Then there are a compactum Z with $\dim Z \leq n+1$ and $\dim_G Z \leq n$ and a G-acyclic map $r : Z \longrightarrow X$ from Z onto X.*

It is known that in general in Theorem 1.2 the dimension of Z cannot be reduced to n [6]. However, Dranishnikov [5] showed that it can be done for $G = \mathbb{Z}_p$.

Theorem 1.3 ([5], cf. [8]) *Let $\mathcal{L} \subset \mathcal{P}$ and let X be a compactum with $\dim_{\mathbb{Z}_p} X \leq n$ for every $p \in \mathcal{L}$. Then there are a compactum Z with $\dim Z \leq n$ and a map $r : Z \longrightarrow X$ from Z onto X such that r is $\mathbb{Z}_p$-acyclic for every $p \in \mathcal{L}$.*

One of the key problems in the area of acyclic resolutions is whether the dimension of Z in Theorem 1.1 can be reduced to n. The main purpose of this paper is to answer this problem affirmatively by proving:

Theorem 1.4 *Let $\mathcal{L} \subset \mathcal{P}$ and let X be a compactum with $\dim_{\mathbb{Z}_{(\mathcal{L})}} X \leq n$, $n \geq 2$. Then there are a compactum Z with $\dim Z \leq n$ and a $\mathbb{Z}_{(\mathcal{L})}$-acyclic map $r : Z \longrightarrow X$ from Z onto X.*

Theorems 1.3 and 1.4 allow us to give a complete description of all the cases when for an abelian group G and a compactum X such that $\dim_G X \leq n$, $n \geq 2$ there is a G-acyclic resolution $r : Z \longrightarrow X$ from a compactum Z of $\dim \leq n$.

Recall that the Bockstein basis of abelian groups is the following collection of groups $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)} : p \in \mathcal{P}\}$. The Bockstein basis $\sigma(G)$ of an abelian group G is a subcollection of σ defined as follows:

$\mathbb{Z}_{(p)} \in \sigma(G)$ if $G/\mathrm{Tor}G$ is not divisible by p;

$\mathbb{Z}_p \in \sigma(G)$ if $\mathrm{Tor}_p G$ is not divisible by p;

$\mathbb{Z}_{p^\infty} \in \sigma(G)$ if $\mathrm{Tor}_p G \neq 0$ and $\mathrm{Tor}_p G$ is divisible by p;

$\mathbb{Q} \in \sigma(G)$ if $G/\mathrm{Tor}G \neq 0$ and $G/\mathrm{Tor}G$ is divisible by every $p \in \mathcal{P}$.

Let X be a compactum. The Bockstein theorem says that

$$\dim_G X = \sup\{\dim_E X : E \in \sigma(G)\}.$$

It is known [6] that X is G-acyclic if and only if X is E-acyclic for every $E \in \sigma(G)$.

Following [9] define the closure $\overline{\sigma(G)}$ of the Bockstein basis $\sigma(G)$ of G as a collection of abelian groups containing $\sigma(G)$ and possibly some additional groups determined by:

$\mathbb{Z}_p \in \overline{\sigma(G)}$ if $\mathbb{Z}_{p^\infty} \in \overline{\sigma(G)}$;

$\mathbb{Z}_{p^\infty} \in \overline{\sigma(G)}$ if $\mathbb{Z}_p \in \overline{\sigma(G)}$;

$\mathbb{Q} \in \overline{\sigma(G)}$ if $\mathbb{Z}_{(p)} \in \overline{\sigma(G)}$;

$\mathbb{Z}_{(p)} \in \overline{\sigma(G)}$ if $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty} \in \overline{\sigma(G)}$.

Theorem 1.5 *Let G be an abelian group, let X be a compactum and let $n \geq 2$. Then there exist a compactum Z of $\dim \leq n$ and a G-acyclic map $r : Z \longrightarrow X$ from Z onto X if and only if $\dim_E X \leq n$ for every $E \in \overline{\sigma(G)}$.*

Proof Assume that there exist a compactum Z of $\dim \leq n$ and a G-acyclic map $r : Z \longrightarrow X$ from Z onto X. Then r is E-acyclic for every $E \in \sigma(G)$. Consider the exact sequences:

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0;$$

$$0 \longrightarrow \mathbb{Z}_{p^k} \longrightarrow \mathbb{Z}_{p^{k+1}} \longrightarrow \mathbb{Z}_p \longrightarrow 0;$$

$$0 \longrightarrow \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}_p \longrightarrow 0;$$

$$0 \longrightarrow \mathbb{Z}_{(p)} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0.$$

Then it follows from the Bockstein sequences generated by the above exact sequences that in the class of compacta:

the $\mathbb{Z}_{p^\infty}$-acyclicity implies the $\mathbb{Z}_p$-acyclicity;

the $\mathbb{Z}_p$-acyclicity implies the $\mathbb{Z}_{p^k}$-acyclicity for every k and since $\mathbb{Z}_{p^\infty} = \varinjlim \mathbb{Z}_{p^k}$ the $\mathbb{Z}_p$-acyclicity implies the $\mathbb{Z}_{p^\infty}$-acyclicity;

the $\mathbb{Z}_{(p)}$-acyclicity implies the $\mathbb{Z}_p$-acyclicity;

the $\mathbb{Z}_{(p)}$-acyclicity implies the $\mathbb{Q}$-acyclicity;

the $\mathbb{Q}$-acyclicity together with the $\mathbb{Z}_{p^\infty}$-acyclicity imply the $\mathbb{Z}_{(p)}$-acyclicity.

From these properties we obtain that r is E-acyclic for every $E \in \overline{\sigma(G)}$ and therefore by the Vietoris-Begle theorem $\dim_E X \le n$ for every $E \in \overline{\sigma(G)}$.

Now assume that $\dim_E X \le n$ for every $E \in \overline{\sigma(G)}$.

Consider first the case when G is torsion. Let $\mathcal{L} = \{p \in \mathcal{P} : \mathrm{Tor}_p G \neq 0\}$. Then $\dim_{\mathbb{Z}_p} X \le n$ for every $p \in \mathcal{L}$ and the existence of the required resolution follows from Theorem 1.3 and the above properties of acyclicity.

Assume that G is not torsion. Then $\mathbb{Q} \in \overline{\sigma(G)}$. Define $\mathcal{L} = \{p \in \mathcal{P} : \mathbb{Z}_{(p)} \in \overline{\sigma(G)}\}$. By the Bockstein theorem $\dim_{\mathbb{Z}_{(\mathcal{L})}} X \le n$ and therefore by Theorem 1.4 there exists a $\mathbb{Z}_{(\mathcal{L})}$-acyclic resolution $r : Z \longrightarrow X$ from a compactum Z of $\dim \le n$ onto X. Clearly r is $\mathbb{Q}$-acyclic if $\mathcal{L} = \emptyset$. If $\mathcal{L} \neq \emptyset$ then r is $\mathbb{Z}_{(p)}$-acyclic for every $\mathbb{Z}_{(p)} \in \overline{\sigma(G)}$ and by the above properties r is $\mathbb{Q}$-acyclic as well.

Let $\mathbb{Z}_{(p)} \in \sigma(G)$. Then $\mathbb{Z}_{(p)} \in \overline{\sigma(G)}$ and therefore r is $\mathbb{Z}_{(p)}$-acyclic.

Let $\mathbb{Z}_p$ or $\mathbb{Z}_{p^\infty} \in \sigma(G)$. Then $\mathbb{Z}_{p^\infty} \in \overline{\sigma(G)}$ and since $\mathbb{Q} \in \overline{\sigma(G)}$ we have that $\mathbb{Z}_{(p)} \in \overline{\sigma(G)}$. Hence r is $\mathbb{Z}_{(p)}$-acyclic and by the above properties r is also $\mathbb{Z}_p$-acyclic and $\mathbb{Z}_{p^\infty}$-acyclic.

Thus we have shown that r is E-acyclic for every $E \in \sigma(G)$ and hence r is G-acyclic. The theorem is proved. □

Theorem 1.2 was generalized in [9] as follows.

Theorem 1.6 [9] *Let X be a compactum. Then for every integer $n \ge 2$ there are a compactum Z of $\dim \le n+1$ and a surjective map $r : Z \longrightarrow X$ having the property that for every abelian group G and every integer $k \ge 2$ such that $\dim_G X \le k \le n$ we have that $\dim_G Z \le k$ and r is G-acyclic.*

Theorem 1.6 resulted in the following conjecture posed in [9].

Conjecture 1.7 [9] *Let $\mathcal{G}$ be a collection of abelian groups, let X be a compactum X and let $n \ge 2$. Then there exist a compactum Z of $\dim \le n$ and a surjective map $r : Z \longrightarrow X$ such that $\dim_G Z \le \max\{\dim_G X, 2\}$ and r is G-acyclic for every $G \in \mathcal{G}$ if and only if $\dim_E X \le n$ for every $E \in \overline{\sigma(\mathcal{G})} = \cup\{\overline{\sigma(G)} : G \in \mathcal{G}\}$.*

It is easy to derive from the proof of Theorem 1.5 that the condition $\dim_E X \leq n$ for every $E \in \overline{\sigma(\mathcal{G})}$ is necessary for the existence of a resolution required in the conjecture. Theorem 1.5 seems to be an important step toward proving that this condition is also sufficient.

Note that neither the restriction $k \geq 2$ in Theorem 1.6 nor the restriction $\dim_G Z \leq \max\{\dim_G X, 2\}$ in Conjecture 1.7 can be replaced by $k \geq 1$ and $\dim_G Z \leq \dim_G X$ respectively.

Indeed, let a compactum Y be $\mathbb{Q}$-acyclic and $\dim_{\mathbb{Q}} Y \leq 1$. Then for every closed subset A of Y the exact sequence of the pair (Y, A) implies that $\check{H}^1(A;\mathbb{Q}) = 0$ and by Universal Coefficient Theorem $\check{H}^1(A) = 0$. Hence $\check{H}^1(Y) = 0$ and $\dim Y \leq 1$ since every map from A to $\mathbb{S}^1 = K(\mathbb{Z}, 1)$ is null-homotopic and therefore extends over Y. Then Y is $\mathbb{Z}$-acyclic. Thus we obtain that a $\mathbb{Q}$-acyclic map from a compactum of $\dim_{\mathbb{Q}} \leq 1$ is $\mathbb{Z}$-acyclic and hence there is no $\mathbb{Q}$-acyclic resolution $r : Z \longrightarrow X$ for a compactum X of $\dim_{\mathbb{Z}} = \infty$ and $\dim_{\mathbb{Q}} = 1$ from a finite dimensional compactum Z of $\dim_{\mathbb{Q}} \leq 1$.

The last observation also shows that the restriction $n \geq 2$ in Theorem 1.4 cannot be omitted.

2 Preliminaries

The extensional dimension of a compactum X is said not to exceed a CW-complex K, written $\mathrm{e} - \dim X \leq K$, if for every closed subset A of X and every map $f : A \longrightarrow K$ there is an extension of f over X. It is well-known that $\dim X \leq n$ is equivalent to $\mathrm{e} - \dim X \leq \mathbb{S}^n$ and $\dim_G X \leq n$ is equivalent to $\mathrm{e} - \dim X \leq K(G, n)$, where $K(G, n)$ is an Eilenberg-Mac Lane complex of type (G, n). Also note that if $\dim X \leq n + 1$, $n \geq 2$ then $\dim_G X \leq n$ is equivalent to $\mathrm{e} - \dim X \leq M(G, n)$, where $M(G, n)$ is a simply-connected Moore complex of type (G, n) (the condition $\dim X \leq n + 1$ can be relaxed to $\dim X < \infty$ [3]).

A map between CW-complexes is said to be combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain.

Let M be a CW-complex. We say that a triangulation of M is compatible with the CW-structure if every cell is a simplicial subcomplex.

Let $\mathcal{L} \subset \mathcal{P}$. We will call a simplicial complex M an n-dimensional $\mathcal{L}$-sphere if the following conditions are satisfied:

M is a finite n-dimensional simplicial complex;

$\tilde{H}_i(M;\mathbb{Z}_{(\mathcal{L})})=0$ if $i\neq n$ and $\tilde{H}_n(M;\mathbb{Z}_{(\mathcal{L})})=\mathbb{Z}_{(\mathcal{L})}$;

for every n-simplex Δ of M, the subcomplex $B=(M\setminus\Delta)\cup\partial\Delta$ is $\mathbb{Z}_{(\mathcal{L})}$-acyclic. (Note that by Universal Coefficient Theorem the $\mathbb{Z}_{(\mathcal{L})}$-acyclicity of B is equivalent to $\tilde{H}_*(B;\mathbb{Z}_{(\mathcal{L})})=0$.)

It is easy to verify that if M is an n-dimensional $\mathcal{L}$-sphere with respect to a given triangulation then M is also an n-dimensional $\mathcal{L}$-sphere with respect to any barycentric subdivision of the triangulation.

By an n-dimensional $\mathcal{L}$-ball B we mean an n-dimensional $\mathcal{L}$-sphere without the interior of one of its n-simplexes. The boundary of the simplex whose interior is removed is said to be the boundary of the $\mathcal{L}$-ball B and it is denoted by ∂B.

Saying that an n-dimensional simplicial complex N is obtained from an n-dimensional simplicial complex M by replacing an n-simplex Δ of M by an n-dimensional $\mathcal{L}$-ball B we mean that the interior of Δ is removed from M and B is attached to the boundary of Δ by a simplicial homeomorphism of $\partial\Delta$ and ∂B. We regard N as a simplicial complex with respect to the natural triangulation induced by the triangulations of M and B.

It is clear that replacing an n-simplex of an n-dimensional $\mathcal{L}$-sphere by an n-dimensional $\mathcal{L}$-ball we obtain again an n-dimensional $\mathcal{L}$-sphere and replacing an n-simplex of an n-dimensional $\mathcal{L}$-ball by an n-dimensional $\mathcal{L}$-ball we obtain again an n-dimensional $\mathcal{L}$-ball.

Proposition 2.1 *Let $\mathcal{L}\subset\mathcal{P}$, let X be an compactum with $\dim_{\mathbb{Z}_{(\mathcal{L})}}X\leq n$ and $\dim X\leq n+1$, $n\geq 2$ and let $f:X\longrightarrow K$ be a map from X to an $(n+1)$-dimensional finite simplicial complex K. Then for every $\epsilon>0$ there exist an $(n+1)$-dimensional finite CW-complex M and maps $\phi:X\longrightarrow M$, $\psi:M\longrightarrow K$ such that:*

(i) *f and $\psi\circ\phi$ are ϵ-close, that is $\rho(f(x),\psi(\phi(x)))<\epsilon$ for every $x\in X$ where ρ is a metric in K;*

(ii) *ψ is combinatorial; ψ is 1-to-1 over every simplex of K which is not contained in an $(n+1)$-simplex of K; the $(n+1)$-cells of M are exactly the preimages of the $(n+1)$-simplexes of K under ψ such that the interior points of the $(n+1)$-cells of M are sent to the interior of the corresponding $(n+1)$-simplexes of K;*

(iii) *each $(n+1)$-cell C of M has a point $b\in\partial C$ such that b is sent by ψ to the interior of the corresponding $(n+1)$-simplex of K. We will call b a free* *boundary point of C. It is clear from (ii) that C is the only $(n+1)$-cell of M containing b;*

(iv) *M admits a compatible triangulation with respect to which the boundary of every $(n+1)$-cell of M is an n-dimensional $\mathcal{L}$-sphere. Moreover: by (iii) such a triangulation of M can be chosen such that for each cell C of M there is an n-simplex contained in ∂C and consiting only of free boundary points of C.*

Proof Fix an n-simplex Δ of K contained in at least one $(n+1)$-simplex and let $\Delta_1, \Delta_2, ..., \Delta_k$, $k \geq 1$ be the $(n+1)$-simplexes of L containing Δ. Take a small closed n-dimensional ball B contained in Δ such that B does not touch the boundary of Δ and B is centered at the barycenter c of Δ. For every $i = 1, 2, .., k$ take a point p_i sufficiently close to B and contained in the interior of Δ_i. Denote $P = \{p_1, ..., p_k\}$. Consider the join $B * p_i$ as the subset of Δ_i which is the cone over B with the vertex at p_i. Thus we regard $B * P = \cup B * p_i$ as a subset of $\cup \Delta_i$. Let the $(n-1)$-dimensional sphere S be the boundary ∂B of B. Denote $F = f^{-1}(B * P)$ and $A = f^{-1}(S * P)$.

Represent a Moore space of type $(\mathbb{Z}_{(\mathcal{L})}, n-1)$ as an infinite telescope T of a sequence of $(n-1)$-dimensional spheres with bonding maps of non-zero degrees which are not divisible by the elements of $\mathcal{L}$. By a finite subtelescope of T we mean a subspace of T which is the telescope of finitely many consecutive spheres in the sequence. Define the group G as $G = 0$ if $k = 1$ and $G =$ the direct sum $\oplus \mathbb{Z}_{(\mathcal{L})}$ of $k-1$ copies of $\mathbb{Z}_{(\mathcal{L})}$ if $k \geq 2$. Let $D = \{d_1, d_2, ..., d_k\}$ be a discrete space of k singletons. Then $T * D$ is a simply connected Moore space of type (G, n). Let $\tau : S \longrightarrow T$ be an embedding of S as the first sphere of the telescope T and let $\delta : P \longrightarrow D$ be the map sending p_i to d_i.

Consider the map $\alpha = (\tau * \delta) \circ f|_{...} : A \longrightarrow T * D$. By Bockstein Theorem $\dim_G X \leq n$ and since $\dim X \leq n+1$ we have that $\mathrm{e}-\dim X \leq T * D$. Then $\alpha : A \longrightarrow T * D$ extends to $\beta : F \longrightarrow T * D$. Since $\beta(F)$ is compact take a finite subtelescope T_β of T starting at the first sphere and ending at a sphere S_β such that $\beta(F)$ is contained in $T_\beta * D$ and let $\gamma : T_\beta \longrightarrow S_\beta$ be the natural retraction sending T_β to the last sphere of T_β.

Define a CW-complex M_Δ as the quotient space of $L = (K \setminus (B * P)) \cup (S * P)$ obtained by identifying the points of $S * P$ with $S_\beta * D$ according to the map $(\gamma \circ \tau) * \delta : S * P \longrightarrow S_\beta * D$. Denote by $\pi : L \longrightarrow M_\Delta$ the projection and consider $S_\beta * D$ as the subset $\pi(S * P)$ of M_Δ. We will also consider $L \setminus (B * P)$ as a subset of M_Δ. Note that since the identifications on $S_\beta * D$ are induced by the join $(\gamma \circ \tau) * \delta$ we have that $\pi^{-1}(\pi(S * p_i)) = \pi^{-1}(S_\beta * d_i) = S * p_i$, $\pi^{-1}(\pi(p_i)) = \pi^{-1}(d_i) = p_i$ and $\pi^{-1}(\pi(S)) = \pi^{-1}(S_\beta) = S$.

It is easy to see that $(\gamma * id_D)(\alpha(x)) = \pi(f(x)) \in S_\beta * D$ for $x \in A$ where $id_D : D \longrightarrow D$ is the identity map. Then the map $\pi \circ f|_{...} : (X \setminus F) \cup A \longrightarrow M_\Delta$ extends to the map $\phi_\Delta : X \longrightarrow M_\Delta$ defined by $\phi_\Delta(x) = (\gamma * id_D)(\beta(x))$ for $x \in F$.

Define a map $\psi_\Delta : M_\Delta \longrightarrow K$ such that:

> every simplex of K not intersecting $B * P$ and considered as a subset of M_Δ is sent by ψ_Δ to itself by the identity map;
>
> ψ_Δ sends $\pi(\Delta \cap L)$ onto Δ such that $\pi(S) = S_\beta$ is sent to the barycenter c of Δ;
>
> ψ_Δ sends $\pi(\Delta_i \cap L)$ onto Δ_i such that $\pi((\Delta_i \setminus \partial\Delta_i) \cap L)$ is sent into $\Delta_i \setminus \partial\Delta_i$;
>
> $\psi_\Delta \circ \phi$ is sufficiently close to f provided that the ball B and the points $p_i \in \Delta_i$ were chosen to be sufficiently close to the barycenter c of Δ.

Then for every Δ_i, $\psi_\Delta^{-1}(\Delta_i) = \pi(\Delta_i \cap K)$ and since $\Delta_i \cap K = (\Delta_i \setminus (B * p_i)) \cup (S * p_i)$ is homeomorphic to an $(n+1)$-dimensional Euclidean ball in which π makes identifications only on the boundary we may endow M_Δ with a CW-structure so that ψ_Δ becomes a combinatorial map and the $(n+1)$-cells of M_Δ are exactly the preimages of the $(n+1)$-simplexes of K. Moreover, since the identifications on $S * p_i$ are induced by the map $\gamma : S \longrightarrow S_\beta$ of degree not divisible by the elements of $\mathcal{L}$ one can define a compatible triangulation of M_Δ so that the boundary of each cell $C_i = \psi_\Delta^{-1}(\Delta_i)$ becomes an $\mathcal{L}$-sphere. Finally note that $\pi(p_i)$ is a free boundary point of the cell C_i and if a triangulation of M_i is chosen to be sufficiently small then for every i there will be an n-simplex contained in ∂C_i consisting of free boundary points of C_i.

It is easy to see that the procedure of constructing M_Δ and $\psi_\Delta : M_\Delta \longrightarrow K$ described above can be carried out independently for all n-simplexes Δ of L contained in some $(n+1)$-simplexes of K. This way we can construct a CW-complex M and a map $\psi : M \longrightarrow K$ satisfying the conclusions of the proposition. □

Let K' be a simplicial complex and let $\kappa : K \longrightarrow K'$, $\lambda : L \longrightarrow L'$, $\alpha : L \longrightarrow K$ and $\alpha' : L' \longrightarrow K'$ be maps.

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & K \\ \lambda\big\downarrow & & \kappa\big\downarrow \\ L' & \xrightarrow{\alpha'} & K' \end{array}$$

We say that κ, λ, α and α' combinatorially commute if for every simplex Δ of K' we have that $(\alpha' \circ \lambda)((\kappa \circ \alpha)^{-1}(\Delta)) \subset \Delta$. (The direction in which we want the maps κ, λ, α and α' to combinatorially commute is indicated by the first map in the list. Thus saying that α', κ, λ and α combinatorially commute we would mean that $(\kappa \circ \alpha)((\alpha' \circ \lambda)^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of K'.)

Let K and K' be simplicial complexes and let $\kappa : K \longrightarrow K'$ $\kappa' : K \longrightarrow K'$. We say that κ' is a simplicial approximation of κ if κ' is a simplicial map and for every simplex Δ of K', $\kappa'(\kappa^{-1}(\Delta)) \subset \Delta$. Thus if κ' is a simplicial approximation of κ then for every subcomplex N of K' and for every $A \subset K$ such that $\kappa(A) \subset N$ we also have $\kappa'(A) \subset N$.

For a simplicial complex K and $a \in K$, $st(a)$ (the star of a) denotes the union of all the simplexes of K containing a.

The following proposition whose proof is left to the reader is a collection of simple combinatorial properties of maps.

Proposition 2.2

(i) *Let $\kappa : K \longrightarrow K'$ be a combinatorial map of simplicial complexes K and K'. Then κ admits a simplicial approximation.*

(ii) *Let $\kappa : K \longrightarrow K'$ be a map of simplicial complexes K and K'. Then there is a sufficiently small barycentric subdivision of the triangulation of K with respect to which κ admits a simplicial approximation.*

(iii) *Let K and K' be simplicial complexes, let maps $\kappa : K \longrightarrow K'$, $\lambda : L \longrightarrow L'$, $\alpha : L \longrightarrow K$ and $\alpha' : L' \longrightarrow K'$ combinatorially commute and let κ be combinatorial. Then*

$$\lambda(\alpha^{-1}(st(a))) \subset \alpha'^{-1}(st(\kappa(a))) \text{ and } \kappa(st((\alpha(b))) \subset st((\alpha' \circ \lambda)(b))$$

for every $a \in K$ and $b \in L$.

3 Proof of Theorem 1.4

By the Vietoris-Begle theorem the composition of $\mathbb{Z}_{(\mathcal{L})}$-acyclic maps of compacta is again a $\mathbb{Z}_{(\mathcal{L})}$-acyclic map. Then by virtue of Theorem 1.1 we may assume without loss of generality that $\dim X = n+1$.

For every $i = 1, 2, \ldots$ we will construct spaces L_i, W_i, S_i, M_i, K_i, maps represented in the diagram

$$\begin{array}{ccccccc} L_{i+1} & \xrightarrow{\beta_{i+1}} & W_{i+1} \subset S_{i+1} \subset M_{i+1} & \xrightarrow{\psi_{i+1}} & K_{i+1} \\ \lambda_{i+1} \downarrow & & \omega_{i+1} \downarrow \qquad \mu_{i+1} \downarrow & & \kappa_{i+1} \downarrow \\ L_i & \xrightarrow{\beta_i} & W_i \subset S_i \subset M_i & \xrightarrow{\psi_i} & K_i \end{array}$$

and some additional maps including $\phi_i : X \longrightarrow M_i$ and $f_i : X \longrightarrow K_i$ having the following properties.

Remark Since the list of properties (steps of the construction) is rather long we would like to outline a way to get the main idea of the construction omitting some details. The property (9) is the heart of the construction. The fastest way to get to (9) is to skip (8) and to omit all metric estimates in (1), (2), (6). Also note that the main purpose of (8), (11), (12) is to show that the maps $\kappa_{i+1}, \alpha_{i+1}, \lambda_{i+1}$ and α_i combinatorially commute where $\alpha_i = \psi_i \circ \beta_i : L_i \longrightarrow K_i$.

(1) K_i is a finite simplicial complex of dim $\leq n+1$. We fix a metric d_i in K_i such that the distance between any two points of K_i lying in non-intersecting simplexes of K_i is bigger than 1. We also fix a metric in X.

(2) Define ϵ_i to be so small that for every $j \leq i$ and every $x, y \in K_i$ with $d_i(x,y) < \epsilon_i$ we have that $d_j(\kappa_i^j(x), \kappa_i^j(y)) < 1/4^i$ where $\kappa_i^i = id_{K_i}$ and $\kappa_i^j = \kappa_{j+1} \circ \ldots \circ \kappa_i : K_i \longrightarrow K_j$ for $j < i$. By Proposition 2.1 we choose a CW-complex M_i and maps $\psi_i : M_i \longrightarrow K_i$ and $\phi_i : X \longrightarrow M_i$ such that the conclusions of Proposition 2.1 are satisfied with K, M, ϕ, ψ, f, ρ and ϵ replaced by K_i, M_i, ϕ_i, ψ_i, f_i, d_i and $\epsilon_i/2$ respectively. Thus M_i is a finite CW-complex of dim $\leq n+1$ admitting a compatible triangulation. We fix a compatible triangulation of M_i and consider M_i as both a CW-complex and a simplicial complex. Note that the CW-structure of M_i is different from the simplicial structure of M_i and therefore the cells of M_i and simplexes of M_i are not the same. The spaces M_i are the only spaces in which we will consider simultaneously two structures.

(3) Let $W_i =$ the n-skeleton of M_i with respect to the CW-structure of M_i.

Consider W_i as a simplicial complex with a triangulation of W_i to be a sufficiently small barycentric subdivision of the triangulation induced by the triangulation of M_i. We refer to this triangulation of W_i considering simplexes of W_i. Thus the simplexes of W_i and the simplexes of M_i contained in W_i are not the same. However each simplicial subcomplex of M_i contained in W_i will be also a subcomplex of W_i.

(4) Let $S_i =$ the n-skeleton of M_i with respect to the triangulation of M_i.

Clearly $W_i \subset S_i$. Consider S_i as a simplicial complex with the triangulation induced by the triangulation of M_i.

Let $\gamma_i : S_i \longrightarrow W_i$ be a natural retraction built by picking up in each $(n+1)$-cell of M_i an interior point not belonging to S_i and retracting the complement of the point inside the cell to the boundary of the cell. Then by (ii) of Proposition 2.1

$$\gamma(\psi_i^{-1}(\Delta) \cap S_i) \subset \psi_i^{-1}(\Delta) \text{ for every simplex } \Delta \text{ of } K_i.$$

Indeed, it is obvious if Δ is an $(n+1)$-simplex since then $\psi_i^{-1}(\Delta)$ is an $(n+1)$-cell of M_i. If Δ is of dim $\leq n$ then $\psi_i^{-1}(\Delta) \subset W_i$ and we simply have $\gamma(\psi_i^{-1}(\Delta)) = \psi_i^{-1}(\Delta)$.

It is also easy to see that assuming that the triangulation of M_i is sufficiently small γ_i can be constructed such that for every $c \in W_i$ there exists a contractible subcomplex N of M_i contained in W_i such that

$$\gamma_i(st(c, M_i) \cap S_i) \subset N \subset \psi_i^{-1}(st(\psi_i(c)))$$

where $st(c, M_i)$ is the star of c with respect to the triangulation of M_i.

As we already noted in (3), N is also a subcomplex of W_i.

(5) L_i and $\beta_i : L_i \longrightarrow W_i$ are constructed as follows. L_i is a simplicial complex of dim $\leq n$ obtained from W_i by replacing some n-simplexes of W_i by n-dimensional $\mathcal{L}$-balls. Then β_i is a projection of L_i to W_i such that β_i takes the $\mathcal{L}$-balls to the corresponding replaced simplexes such that the non-boundary points of the $\mathcal{L}$-balls are sent to the interior of the replaced simplexes and β_i is 1-to-1 over all the simplexes of W_i which are not replaced;

Let $i = 1$. Set $\epsilon_1 = 1$. Let K_1 be any $(n+1)$-dimensional simplicial complex with any map $f_1 : X \longrightarrow K_1$. Applying Proposition 2.1 define for $i = 1$ all needed spaces and maps satisfying the relevant properties of (2-4). Define L_1 as W_1 with no simplexes replaced by $\mathcal{L}$-balls. We proceed from i to $i+1$ as follows.

(6) Based on $\phi_i : X \longrightarrow M_i$ find a finite simplicial complex K_{i+1} and maps $f_{i+1} : X \longrightarrow K_{i+1}$ and $\nu_{i+1} : K_{i+1} \longrightarrow M_i$ such that:

> $d_i((\psi_i \circ \nu_{i+1} \circ f_{i+1})(x), (\psi_i \circ \phi_i)(x)) < \epsilon_i/2$ for every $x \in X$ with ϵ_i defined in (1);

$f_{i+1}^{-1}(\Delta)$ is of diam $\leq 1/(i+1)$ for every simplex Δ of K_{i+1};

ν_{i+1} is a combinatorial map with respect to the triangulation of M_i.

Define $\kappa_{i+1} = \psi_i \circ \nu_{i+1} : K_{i+1} \longrightarrow K_i$ and suppose that the triangulation of K_{i+1} is so small that

$\kappa_{i+1}^j(\Delta)$ is of diam$\leq 1/(i+1)$ for every simplex Δ of K_{i+1} and $j \leq i$.

(7) Let M_{i+1}, S_{i+1} , W_{i+1} and the corresponding maps be defined as described in (2-4). Since $\nu_{i+1} \circ \psi_{i+1} : M_{i+1} \longrightarrow M_i$ is combinatorial with respect to the simplicial structures of M_{i+1} and M_i, by (i) of Proposition 2.2 there is a simplicial approximation $\mu_{i+1} : M_{i+1} \longrightarrow M_i$ of $\nu_{i+1} \circ \psi_{i+1}$. Then $\mu_{i+1}(S_{i+1}) \subset S_i$. By (ii) of Proposition 2.2 assume that that the triangulation of W_{i+1} is chosen to be so small that the map $\gamma_i \circ \mu_{i+1}|_{\dots} : W_{i+1} \longrightarrow W_i$ admits a simplicial approximation $\omega_{i+1} : W_{i+1} \longrightarrow W_i$.

(8) Now we will verify a technical property which will be used later. Namely we will show that $\kappa_{i+1}, \psi_{i+1}|_{W_{i+1}}, \omega_{i+1}$ and $\psi_i|_{W_i}$ combinatorially commute. This property is equivalent to

$$\omega_{i+1}((\kappa_{i+1} \circ \psi_{i+1})^{-1}(\Delta) \cap W_{i+1}) \subset \psi_i^{-1}(\Delta) \text{ for every simplex } \Delta \text{ of } K_i.$$

Let Δ be a simplex of K_i. Note that $\psi_i^{-1}(\Delta)$ is a subcomplex of M_i with respect to both the CW and the simplicial structures of M_i. Then since μ_{i+1} is a simplicial approximation of $\nu_{i+1} \circ \psi_{i+1}$ and $\psi_i^{-1}(\Delta)$ is a simplicial subcomplex of M_i we have

$$\begin{aligned}\mu_{i+1}((\nu_{i+1} \circ \psi_{i+1})^{-1}(\psi_i^{-1}(\Delta))) &\subset (\nu_{i+1} \circ \psi_{i+1})((\nu_{i+1} \circ \psi_{i+1})^{-1}(\psi_i^{-1}(\Delta))) \\ &= \psi_i^{-1}(\Delta)\end{aligned}$$

and therefore

$$\mu_{i+1}((\nu_{i+1} \circ \psi_{i+1})^{-1}(\psi_i^{-1}(\Delta)) \cap W_{i+1}) \subset \psi_i^{-1}(\Delta) \cap S_i.$$

thus $$(\gamma_i \circ \mu_{i+1})((\nu_{i+1} \circ \psi_{i+1})^{-1}(\psi_i^{-1}(\Delta)) \cap W_{i+1}) \subset \gamma_i(\psi_i^{-1}(\Delta) \cap S_i).$$

By (4) $\gamma_i(\psi_i^{-1}(\Delta) \cap S_i) \subset \psi_i^{-1}(\Delta) \cap W_i$ and since $\psi_i^{-1}(\Delta) \cap W_i$ is a subcomplex of W_i and ω_{i+1} is a simplicial approximation of $\gamma_i \circ \mu_{i+1}|_{\dots} : W_{i+1} \longrightarrow W_i$ we get

$$\omega_{i+1}((\nu_{i+1} \circ \psi_{i+1})^{-1}(\psi_i^{-1}(\Delta)) \cap W_{i+1}) \subset \psi_i^{-1}(\Delta) \cap W_i \subset \psi_i^{-1}(\Delta).$$

Recall that $\kappa_{i+1} = \psi_i \circ \nu_{i+1}$ and finally get that

$$\omega_{i+1}((\kappa_{i+1} \circ \psi_{i+1})^{-1}(\Delta) \cap W_{i+1}) \subset \psi_i^{-1}(\Delta) \text{ for every simplex } \Delta \text{ of } K_i.$$

(9) L_{i+1} is constructed as follows. Let C be an $(n+1)$-cell of M_{i+1}. By (ii) of Proposition 2.1 there is an $(n+1)$-simplex Δ of K_{i+1} such that $C = \psi_{i+1}^{-1}(\Delta)$. By (iv) of Proposition 2.1 there is an n-simplex Δ_C of W_{i+1} contained in ∂C which is sent by ψ_{i+1} to the interior of Δ. Thus for every $(n+1)$-cell C of M_{i+1} we choose an n-simplex Δ_C of W_{i+1} which consisits of free boundary points of C.

We say that an n-simplex Δ_{i+1} of W_{i+1} is marked (for replacement) if Δ_{i+1} is not among the chosen simplexes Δ_C and Δ_{i+1} is mapped by ω_{i+1} onto an n-simplex Δ_i of W_i such that Δ_i was replaced by an n-dimensional $\mathcal{L}$-ball $B_{\Delta_i} = \beta_i^{-1}(\Delta_i)$ while constructing L_i, see (5).

Replace each marked simplex Δ_{i+1} by an n-dimensional $\mathcal{L}$-ball $B_{\Delta_{i+1}}$ which is a copy of B_{Δ_i} attached by the simplicial map of the boundaries of Δ_{i+1} and $B_{\Delta_{i+1}}$ induced by ω_{i+1} and β_i. Such a replacement induces the natural map $\lambda_{\Delta_{i+1}} : B_{\Delta_{i+1}} \longrightarrow B_{\Delta_i} = \beta_i^{-1}(\Delta_i) \subset L_i$.

Let C be an $(n+1)$-cell of M_{i+1} and let T_C be the union of all the simplexes of W_{i+1} which are contained in C and different from Δ_C. Then after the replacement of all marked simplexes of W_{i+1} we will obtain from T_C an n-dimensional $\mathcal{L}$-ball B_C and the map $\lambda_C : B_C \longrightarrow L_i$ induced by ω_{i+1} and β_i for the simplexes of T_C which were not replaced and by the maps $\lambda_{\Delta_{i+1}}$ for the replaced simplexes. Now replace Δ_C by an $\mathcal{L}$-ball B_{Δ_C} which is a copy of B_C attached by the identity map of the boundaries and extend λ_C over B_{Δ_C} first sending the points of B_{Δ_C} to the corresponding points of B_C and from there to L_i using the map λ_C. Thus we construct L_{i+1} and the map $\lambda_{i+1} : L_{i+1} \longrightarrow L_i$ having the property that for every $(n+1)$-cell C of M_{i+1} the map

$$\lambda_{i+1}|_{\dots} : \beta_{i+1}^{-1}(\partial C) \longrightarrow \lambda_{i+1}(\beta_{i+1}^{-1}(\partial C))$$

factors through an n-dimensional $\mathcal{L}$-ball (namely through B_C).

(10) Define $\alpha_i = \psi_i \circ \beta_i : L_i \longrightarrow K_i$.

Then the last property of (9) says that for every $(n+1)$-simplex Δ of K_{i+1} the map $\lambda_{i+1}|_{\dots} : \alpha_{i+1}^{-1}(\Delta) \longrightarrow \lambda_{i+1}(\alpha_{i+1}^{-1}(\Delta))$ factors through a $\mathbb{Z}_{(\mathcal{L})}$- acyclic space.

(11) It follows from (9) that if $a \in \Delta_C$ then $(\beta_i \circ \lambda_{i+1})(\beta_{i+1}^{-1}(a)) \subset \omega_{i+1}(\partial C)$ and if $a \in W_{i+1}$ does not belong to any of the simplexes Δ_C then simply $\beta_i \circ \lambda_{i+1}(\beta_{i+1}^{-1}(a)) = \omega_{i+1}(a)$.

Thus we obtain that for every simplex Δ of K_{i+1}

$$(\beta_i \circ \lambda_{i+1})(\beta_{i+1}^{-1}(\psi_{i+1}^{-1}(\Delta)) = (\beta_i \circ \lambda_{i+1})(\alpha_{i+1}^{-1}(\Delta)) \subset \omega_{i+1}(\psi_{i+1}^{-1}(\Delta) \cap W_{i+1}).$$

(12) Now we will show that the maps $\kappa_{i+1}, \alpha_{i+1}, \lambda_{i+1}$ and α_i combinatorially commute. Let Δ be a simplex of K_i. Then $\kappa_{i+1}^{-1}(\Delta)$ is a subcomplex of K_i and by (11) and (8) we get that

$$(\beta_i \circ \lambda_{i+1})((\kappa_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) = (\beta_i \circ \lambda_{i+1})(\alpha_{i+1}^{-1}(\kappa_{i+1}^{-1}(\Delta)) \\ \subset \omega_{i+1}(\psi_{i+1}^{-1}(\kappa_{i+1}^{-1}(\Delta)) \cap W_{i+1}) \subset \psi_i^{-1}(\Delta).$$

Thus we get that

$$(\alpha_i \circ \lambda_{i+1})((\kappa_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta \text{ for every simplex } \Delta \text{ of } K_i$$

and hence the maps $\kappa_{i+1}, \alpha_{i+1}, \lambda_{i+1}$ and α_i combinatorially commute.

Then by (iii) of Proposition 2.2 for every $a \in K_{i+1}$ and $b \in L_{i+1}$

$$\lambda_{i+1}(\alpha_{i+1}^{-1}(st(a))) \subset \alpha_i^{-1}(st(\kappa_{i+1}(a)))$$

and
$$\kappa_{i+1}(st((\alpha_{i+1}(b))) \subset st((\alpha_i \circ \lambda_{i+1})(b)).$$

(13) Let $a \in K_{i+1}$ be such that $\kappa_{i+1}(a)$ is not contained in the interior of any of the $(n+1)$-simplexes of K_i. Then $\nu_{i+1}(a) \in W_i$. Let us show that the map $\lambda_{i+1}|_{\dots} : \alpha_{i+1}^{-1}(st(a))) \longrightarrow \alpha_i^{-1}(st(\kappa_{i+1}(a)))$ factors through a $\mathbb{Z}_{(\mathcal{L})}$- acyclic space.

By (4) there is a contractible subcomplex N of W_i such that

$$\gamma_i(st(\nu_{i+1}(a)) \cap S_i) \subset N \subset \psi_i^{-1}(st(\kappa_{i+1}(a))$$

where $st(\nu_{i+1}(a))$ is the star of $\nu_{i+1}(a)$ with respect to the triangulation of M_i.

Since ν_{i+1} is combinatorial with respect to the triangulation of M_i we have

$$\nu_{i+1}(st(a)) \subset st(\nu_{i+1}(a))$$

and therefore

$$\nu_{i+1}(\psi_{i+1}(\psi_{i+1}^{-1}(st(a)))) = \nu_{i+1}(st(a)) \subset st(\nu_{i+1}(a)).$$

Since μ_{i+1} is a simplicial approximation of $\nu_{i+1} \circ \psi_{i+1}$ and $st(\nu_{i+1}(a))$ is a simplicial subcomplex of M_i we have

$$\mu_{i+1}(\psi_{i+1}^{-1}(st(a))) \subset st(\nu_{i+1}(a))$$

and therefore

$$\gamma_i(\mu_{i+1}(\psi_{i+1}^{-1}(st(a)) \cap S_i) \subset \gamma_i(st(\nu_{i+1}(a)) \cap S_i) \subset N.$$

Then since N is a subcomplex of W_i and ω_{i+1} is a simplicial approximation of $\gamma_i \circ \mu_{i+1}|_{\dots} : W_{i+1} \longrightarrow W_i$ we have

$$\omega_{i+1}(\psi_{i+1}^{-1}(st(a)) \cap W_i) \subset N$$

and hence
$$\omega_{i+1}(\psi_{i+1}^{-1}(st(a)) \cap W_i) \subset N \subset \psi_i^{-1}(st(\kappa_{i+1}(a)).$$
By (11)
$$\beta_i(\lambda_{i+1}(\alpha_{i+1}^{-1}(st(a)))) \subset \omega_{i+1}(\psi_{i+1}^{-1}(st(a)) \cap W_i)$$
and therefore
$$\lambda_{i+1}(\alpha_{i+1}^{-1}(st(a))) \subset \beta_i^{-1}(N) \subset \beta^{-1}(\psi_i^{-1}(st(\kappa_{i+1}(a))) = \alpha_i^{-1}(st(\kappa_{i+1}(a))).$$
By (5) the preimage under β_i of every simplex of W_i is $\mathbb{Z}_{(\mathcal{L})}$- acyclic and therefore by the combinatorial Vietoris-Begle theorem $\beta_i^{-1}(N)$ is $\mathbb{Z}_{(\mathcal{L})}$- acyclic.

Thus the map $\lambda_{i+1}|_{\dots} : \alpha_{i+1}^{-1}(st(a))) \longrightarrow \alpha_i^{-1}(st(\kappa_{i+1}(a)))$ factors through a $\mathbb{Z}_{(\mathcal{L})}$- acyclic space.

(14) Let $a \in K_{i+2}$. By (12) consider the following maps

$\lambda_{i+2}|_{\dots} : \alpha_{i+2}^{-1}(st(a)) \subset \alpha_{i+1}^{-1}(st(\kappa_{i+2}(a)))$ and

$\lambda_{i+1}|_{\dots} : \alpha_{i+1}^{-1}(st(\kappa_{i+2}(a)) \subset \alpha_i^{-1}(st(\kappa_{i+1}(\kappa_{i+2}(a)))$.

If $\kappa_{i+2}(a)$ is in the interior of an $(n+1)$-simplex Δ of K_{i+1} then $st(\kappa_{i+2}(a)) = \Delta$ and by (10) $\lambda_{i+2}|_{\dots}$ factors through a $\mathbb{Z}_{(\mathcal{L})}$- acyclic space.

If $\kappa_{i+2}(a)$ is not in the interior of any of the $(n+1)$-simplexes of K_{i+1} then by (13) $\lambda_{i+1}|_{\dots}$ factors through a $\mathbb{Z}_{(\mathcal{L})}$- acyclic space.

Thus we get that the map
$$\lambda_{i+1} \circ \lambda_{i+2}|_{\dots} : \alpha_{i+2}^{-1}(st(a)) \longrightarrow \alpha_i^{-1}(st(\kappa_{i+1}(\kappa_{i+2}(a)))$$
always factors through a $\mathbb{Z}_{(\mathcal{L})}$- acyclic space.

(15) Define

$L = \varprojlim(L_i, \lambda_i)$ and $K = \varprojlim(K_i, \kappa_i)$ with the projections

$\pi_i^L : L \longrightarrow L_i$ and $\pi_i^K : K \longrightarrow K_i$.

It is clear that $\dim L \leq n$. For every $b = (b_i) \in L, b_i \in L_i$ define the set $\pi(b) = \cap\{(\pi_i^K)^{-1}(st(\alpha_i(b_i))) : i = 1, 2, 3 \dots\} \subset K$. By (12) $\pi(b) \neq \emptyset$. Recall that by (6) $\operatorname{diam}(\kappa_i^j(\Delta)) < 1/i$ for every simplex Δ of K_i and $j < i$. Then $\pi(b)$ is a singleton and by (12) one can check that the function

$\pi : L \longrightarrow K$ defined by $b \longrightarrow \pi(b)$ is continuous with

$\pi^{-1}(a) = \varprojlim(\alpha_i^{-1}(st(a_i)), \lambda_i|_{\dots})$ for every $a = (a_i) \in K$.

It is easy to see that for every simplex Δ of K_i, $\alpha_i^{-1}(\Delta) \neq \emptyset$. Then we have that for every $a \in K$, $\pi^{-1}(a)$ is not empty and by (14) we get that $\pi^{-1}(a)$ is $\mathbb{Z}_{(\mathcal{L})}$-acyclic.

Thus π is a $\mathbb{Z}_{(\mathcal{L})}$-acyclic surjective map.

(16) Let $x \in X$ and $j \leq i$. Define $f_i^j = \kappa_i^j \circ f_i : X \longrightarrow K_j$.

Then $f_{i+1}^i = \kappa_{i+1} \circ f_{i+1} = \psi_i \circ \nu_{i+1} \circ f_{i+1}$ and by (6) $d_i(f_{i+1}(x), (\psi_i \circ \phi_i)(x)) < \epsilon_i/2$.

By (2) and (i) of Proposition 2.1 we have

$$d_i(f_i(x), (\psi_i \circ \phi_i)(x))) < \epsilon_i/2 \text{ and hence } d_i(f_i(x), f_{i+1}^i(x)) < \epsilon_i.$$

Then, since $f_{i+1}^j = \kappa_i^j \circ f_{i+1}$, (2) implies that $d_j(f_i^j(x), f_{i+1}^j(x)) < 1/4^i$.

Now define $h_j : X \longrightarrow K_j$ as $h_j = \lim_{i\to\infty} f_i^j$.

Then $d_j(h_j(x), f_j(x)) < 1/2$. Let $a = h_j(x)$ and let $y \in X$ be such that $h_j(y) = h_j(x)$. Then $d_j(f_j(y), f_j(x)) < 1$ and hence by (1) $f_j(y) \in st^2(a)$ where $st^2(a)$ is the union of all simplexes of K_j intersecting $st(a)$.

Thus $h_j^{-1}(h_j(x)) \subset f_j^{-1}(st^2(a))$ and hence by (6) $h_j^{-1}(h_j(x))$ is of diam $\leq 4/j$.

It is clear that $h_j = \kappa_{j+1} \circ h_{j+1}$. Then we have that $h : X \longrightarrow K$ defined by $h(x) = (h_j(x)) \in K$ is an embedding of X in K.

Identify X with $h(X)$. Then $r = \pi|_{\dots} : Z = \pi^{-1}(X) \longrightarrow X$ is the required resolution. $\square$

Remark If $n \geq 3$ then in (9) B_C is simply connected and one can use the Hurewicz theorem to replace each simplex Δ_C by an n-cell attached to the boundary by a map of degree not divisible by the elements of $\mathcal{L}$ such that there is a map from the replacing cell to B_C. In our construction we used an exact copy of B_C for replacing Δ_C in order to guarantee that such a map exists for the non-simply connected case $n = 2$.

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