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Limit points of lines of minima in Thurston's boundary of Teichmüller space

Raquel Díaz
Caroline Series

Abstract Given two measured laminations μ and ν in a hyperbolic surface which fill up the surface, Kerckhoff [8] defines an associated *line of minima* along which convex combinations of the length functions of μ and ν are minimised. This is a line in Teichmüller space which can be thought as analogous to the geodesic in hyperbolic space determined by two points at infinity. We show that when μ is uniquely ergodic, this line converges to the projective lamination $[\mu]$, but that when μ is rational, the line converges not to $[\mu]$, but rather to the barycentre of the support of μ. Similar results on the behaviour of Teichmüller geodesics have been proved by Masur [9].

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1 Introduction

Let S be a surface of hyperbolic type, and denote its Teichmüller space by $\mathrm{Teich}(S)$. Given a measured geodesic lamination μ on S (see Section 2 for definitions), there is a function $l_\mu\colon \mathrm{Teich}(S) \to \mathbb{R}^+$ which associates to each $\rho \in \mathrm{Teich}(S)$ the hyperbolic length $l_\mu(\rho)$ of μ in the hyperbolic structure ρ. If μ, ν are two measured laminations which fill up the surface, Kerckhoff [8] proved that for any number $s \in (0,1)$, the function $F_s = (1-s)l_\mu + sl_\nu$ has a global minimum at a unique point $m_s \in \mathrm{Teich}(S)$. The set of all these minima, when s varies in the interval $(0,1)$, is called a *line of minima* $\mathcal{L}_{\mu,\nu}$.

The Teichmüller space of a surface is topologically a ball which, as shown by Thurston, can be compactified by the space $\mathrm{P}\mathcal{ML}$ of projective measured laminations on S. Various analogies between Teichmüller space and hyperbolic space have been studied, for example *earthquake paths* in Teichmüller space are analogous to horocycles in hyperbolic space. In [8], Kerckhoff studied some properties of the lines of minima analogous to properties of geodesics in hyperbolic space. For example, two projective measured laminations determine a unique line of minima, in analogy to the fact that two different points in the boundary of hyperbolic space determine a unique geodesic. He warns, however, that lines of minima do not always converge to the point corresponding to μ in Thurston's compactification of $\mathrm{Teich}(S)$, mentioning that examples can be constructed by taking μ rational (that is, such that its support consists entirely of closed leaves). In this paper we make this explicit by showing that any line of minima $\mathcal{L}_{\mu,\nu}$ where $\mu = \sum a_i\alpha_i$, with $a_i > 0$, converges to the projective lamination $[\sum \alpha_i]$, rather than to $[\sum a_i\alpha_i]$.

Theorem 1.1 *Let $\mu = \sum_{i=1}^N a_i\alpha_i$ be a rational measured lamination (that is, α_i is a collection of disjoint simple closed curves on S and $a_i > 0$ for all i) and ν any measured lamination so that μ, ν fill up the surface. For any $0 < s < 1$, consider the function $F_s\colon \mathrm{Teich}(S) \to \mathbb{R}$ defined by $F_s(\rho) = (1-s)l_\mu(\rho) + sl_\nu(\rho)$, and denote its unique minimum by m_s. Then*

$$\lim_{s\to 0} m_s = [\alpha_1 + \cdots + \alpha_N] \in \mathrm{P}\mathcal{ML}.$$

By contrast, if μ is uniquely ergodic and maximal (see Section 2 for the definition), we prove:

Theorem 1.2 *Let μ and ν be two measured laminations which fill up the surface and such that μ is uniquely ergodic and maximal. With m_s as above,*

$$\lim_{s\to 0} m_s = [\mu] \in \mathrm{P}\mathcal{ML}.$$

Exactly similar results have been proved by Masur [9] for Teichmüller geodesics. In this case, a geodesic ray is determined by a base surface ρ and a quadratic differential ϕ on ρ. Roughly speaking, the end of this ray depends on the horizontal foliation F of ϕ. Masur shows that if F is a Jenkins–Strebel differential, that is, if its horizontal foliation has closed leaves, then the associated ray converges in the Thurston boundary to the barycentre of the leaves (the foliation with the same closed leaves all of whose cylinders have unit height), while if F is uniquely ergodic and every leaf (apart from saddle connections) is dense in S, it converges to the boundary point defined by F.

Our interest in lines of minima arose from the study of the space $\mathcal{QF}(S)$ of quasifuchsian groups associated to a surface S. The *pleating plane* determined by a pair of projective measured laminations is the set of quasifuchsian groups whose convex hull boundary is bent along the given laminations with bending measure in the given classes. It is shown in [14], see also [15], that if μ, ν are measured laminations, then the closure of their pleating plane meets fuchsian space exactly in the line of minima $\mathcal{L}_{\mu,\nu}$.

From this point of view, it is often more natural to look at the collection of all groups whose convex hulls are bent along a specified set of closed curves. That is, we forget the proportions between the bending angles given by the measured lamination μ and look only at its support. This led us in [3] to study the *simplex of minima* determined by two systems of disjoint simple curves on the twice punctured torus, where direct calculation of some special examples led to our results here.

The *simplex of minima* $\mathcal{S}_{\mathcal{A},\mathcal{B}}$ associated to systems of disjoint simple curves $\mathcal{A} = \{\alpha_1, \dots, \alpha_N\}$ and $\mathcal{B} = \{\beta_1, \dots, \beta_M\}$ which fill up the surface, is the union of lines of minima $\mathcal{L}_{\mu,\nu}$, where $\mu, \nu \in \mathcal{ML}(S)$ are strictly positive linear combinations of $\{\alpha_i\}$ and $\{\beta_i\}$, respectively. We can regard $\mathcal{S}_{\mathcal{A},\mathcal{B}}$ as the image of the affine simplex $S_{\mathcal{A},\mathcal{B}}$ in $\mathbb{R}^{N+M-1}$ spanned by independent points $A_1, \dots, A_N, B_1, \dots, B_M$, under the map Φ which sends the point $(1-s)(\sum_i a_i A_i) + s(\sum_j b_j B_j)$ (with $0 < s, a_i, b_j < 1, \sum a_i = 1, \sum b_j = 1$) to the unique minimum of the function $(1-s)(\sum_i a_i l_{\alpha_i}) + s(\sum_j b_j l_{\beta_j})$.

As observed in [3], the methods of [8] show that the map Φ is continuous and proper. It may or may not be a homeomorphism onto its image; in [3] we give a necessary and sufficient condition and show by example that both cases occur. The map extends continuously to the faces of $S_{\mathcal{A},\mathcal{B}}$ which correspond to curves $\{\alpha_{i_1}, \dots, \alpha_{i_k}\}$, $\{\beta_{j_1}, \dots, \beta_{j_l}\}$ that still fill up the surface. Nevertheless, as a consequence of Theorem 1.1, Φ does not necessarily extend to a function from the closure of $S_{\mathcal{A},\mathcal{B}}$ into the Thurston boundary.

Corollary 1.3 *Let $\mathcal{A}, \mathcal{B}$ be as above and suppose that $\{\alpha_1, \dots, \alpha_{N-1}\}$ and $\mathcal{B} = \{\beta_1, \dots, \beta_M\}$ also fill up S. Then, the map $\Phi\colon S_{\mathcal{A},\mathcal{B}} \to \mathrm{Teich}(S)$ does not extend continuously to a function $\overline{S_{\mathcal{A},\mathcal{B}}} \to \mathrm{Teich}(S) \cup \mathrm{P}\mathcal{ML}(S)$.*

Proof Let $\{x_n\}$ be a sequence of points in $S_{\mathcal{A},\mathcal{B}}$, and $\{y_n\}$ another sequence in the face spanned by $A_1, \dots, A_{N-1}, B_1, \dots, B_M$, both converging to $(A_1 + \dots + A_{N-1})/(N-1)$. Then, by Theorem 1.1, $\Phi(x_n)$ converges to $[\alpha_1 + \dots + \alpha_N]$ while $\Phi(y_n)$ converges to $[\alpha_1 + \dots + \alpha_{N-1}]$. □

We remark that examples of curve systems as in the corollary are easy to construct.

The paper is organised as follows. The main work is in proving Theorem 1.1. In Section 2 we recall background and give the (easy) proof of Theorem 1.2. In Section 3 we study an example which illustrates Theorem 1.1 and its proof. The general proof is easier when $\alpha_1, \dots, \alpha_N$ is a pants decomposition. We work this case in Sections 4 and 5, leaving the non-pants decomposition case for Section 6.

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2 Background

We take the Teichmüller space Teich(S) of a surface S of hyperbolic type to be the set of faithful and discrete representations $\rho\colon \pi_1(S) \to PSL(2,\mathbb{R})$ which take loops around punctures to parabolic elements, up to conjugation by elements of $PSL(2,\mathbb{R})$. An element ρ of Teich(S) can be regarded as a *marked* hyperbolic structure on S. The space Teich(S) is topologically a ball of dimension $2(3g-3+b)$, where g is the genus and b the number of punctures of S. A *pants decomposition* of S is a set of disjoint simple closed curves, $\{\alpha_1, \dots, \alpha_N\}$ which decompose the surface into pairs of pants ($N = 3g-3+b$). Given a pants decomposition $\{\alpha_1, \dots, \alpha_N\}$ on S, the *Fenchel-Nielsen* coordinates give a global parameterization of Teich(S). Given a marked hyperbolic structure on S, these coordinates consist of the lengths l_{α_i} of the geodesics representing the curves α_i, and the *twist parameters* t_{α_i}. The lengths l_{α_i} determine uniquely the geometry on each pair of pants, while the twist parameters are real numbers determining the way these pairs of pants are glued together to build up the hyperbolic surface. We need to specify a set of base points in Teich(S), namely a subset of Teich(S) where the twist parameters are all equal to zero. This can be done by choosing a set of curves $\{\delta_i\}$ *dual* to the $\{\alpha_i\}$, in the sense that each δ_i intersects α_i either once or twice and is disjoint from α_j for all $j \neq i$. For each fixed set of values of l_{α_i}, the base point is then the marked hyperbolic structure in which each δ_i is orthogonal to α_i, when they intersect once, or in which the two intersection angles (measured from δ_i to α_i) sum to π, when they intersect twice.

A *geodesic lamination* in a hyperbolic surface ρ is a closed subset of ρ which is disjoint union of simple geodesics, called its *leaves*. A geodesic lamination is *measured* if it carries a transverse invariant measure (for details, see for example [4, 10] and the appendix to [11]). The space $\mathcal{ML}$ of measured laminations is given the weak topology. If $\mu \in \mathcal{ML}$, then $|\mu|$ will denote its underlying support. To exclude trivial cases, we assume that each leaf l of $|\mu|$ is a density point of μ, meaning that any open interval transverse to l has positive μ-measure. In this paper, we shall mainly use *rational* measured laminations, denoted by $\sum_i a_i\alpha_i$, where $a_i \in \mathbb{R}^+$ and α_i are disjoint simple closed geodesics. This measured lamination assigns mass a_i to each intersection of a transverse arc with α_i. The length of a rational lamination $\sum_i a_i\alpha_i$ on a hyperbolic surface ρ is defined to be $\sum_i a_i l_{\alpha_i}(\rho)$, where $l_{\alpha_i}(\rho)$ is the length of α_i at ρ. Rational measured laminations are dense in $\mathcal{ML}$ and the length of a measured lamination can be defined as the limit of the lengths of approximating rational measured laminations, see [6]. This construction appears to depend on ρ, however a homeomorphism between hyperbolic surfaces transfers geodesic laminations canonically from the first surface to the second. Thus, given a measured lamination μ, there is a map $l_\mu\colon \operatorname{Teich}(S) \to \mathbb{R}^+$ which assigns to each point $\rho \in \operatorname{Teich}(S)$ the length $l_\mu(\rho)$ of μ on the hyperbolic structure ρ. The map l_μ is real analytic with respect to the real analytic structure of $\operatorname{Teich}(S)$, see [6] Corollary 2.2.

Two measured laminations are *equivalent* if they have the same underlying support and proportional transverse measures. The equivalence class of a measured lamination μ is called a *projective measured lamination* and is denoted by $[\mu]$. A measured lamination is *maximal* if its support is not contained in the support of any other measured lamination. A lamination is *uniquely ergodic* if every lamination with the same support is in the same projective equivalence class. (Thus the lamination $\sum_i a_i\alpha_i$ is uniquely ergodic if and only if the sum contains exactly one term.) The geometric intersection number $i(\gamma,\gamma')$ of two simple closed geodesics is the number of points in their intersection. This number extends by bilinearity and continuity to the intersection number of measured laminations, see [12, 6, 2]. The following characterisation of uniquely ergodic laminations is needed in the proof of Theorem 1.2.

Lemma 2.1 *A lamination $\mu \in \mathcal{ML}$ is uniquely ergodic and maximal if and only if, for all $\nu \in \mathcal{ML}$, $i(\mu,\nu) = 0$ implies $\nu \in [\mu]$.*

Proof If $i(\mu,\nu) = 0$ implies $\nu \in [\mu]$, it is easy to see that μ must be uniquely ergodic and maximal. The converse follows using the definition of intersection number as the integral over S of the product measure $\mu\times\nu$; see for example [6]. Since we are assuming μ is uniquely ergodic, it is enough to show that the supports of μ and ν are the same.

Let ω be the lamination consisting of leaves (if any) which are common to $|\mu|$ and $|\nu|$. Let μ_ω and ν_ω denote the restrictions of μ and ν to ω. Clearly ω

is closed, and hence (using the decomposition of laminations into finitely many minimal components, see for example [1],[11]), one can write $\mu = \mu_\omega + \mu'$, $\nu = \nu_\omega + \nu'$ where μ', ν' are (measured) laminations disjoint from ω such that every leaf of μ' is transverse to every leaf of ν'. Since μ is uniquely ergodic, one or other of μ' or μ_ω is zero. In the former case, maximality of μ forces $\nu' = 0$, and we are done.

Thus we may assume that every leaf of $|\mu|$ intersects every leaf of $|\nu|$ transversally; let X be the set of intersection points of these leaves. Cover X by small disjoint open 'rectangles' R_i, each with two 'horizontal' and two 'vertical' sides, in such a way that $|\mu| \cap R$ consists entirely of arcs with endpoints on the horizontal sides and similarly for $|\nu| \cap R$ replacing horizontal by vertical. Put a product measure on R by using the transverse measure μ on 'horizontal' arcs and ν on 'vertical' ones. Then $i(\mu, \nu) = \sum_i \int_{R_i} d\mu \times d\nu$. Our assumption that each leaf of $|\mu|$ and $|\nu|$ is a density point means that the contribution to $i(\mu, \nu)$ is non-zero whenever $R \cap X$ is non-empty. Thus $i(\mu, \nu) = 0$ implies that $X = \emptyset$. Since μ is maximal, every leaf of $|\nu|$ either coincides with or intersects some leaf of $|\mu|$, and we conclude that the leaves of $|\nu|$ and $|\mu|$ coincide as before. $\square$

There is a similar characterisation of uniquely ergodic foliations due to Rees [12], see also [9] Lemma 2, in which the assumption that μ is maximal is replaced by the assumption that every leaf, other than saddle connections, is dense. (Notice that the above proof shows that a uniquely ergodic lamination is also minimal, in the sense that every leaf is dense in the whole lamination.)

2.1 The Thurston Boundary

We denote the set of all non-zero projective measured laminations on S by $\mathrm{P}\mathcal{ML}(S)$. Thurston has shown that $\mathrm{P}\mathcal{ML}(S)$ compactifies $\mathrm{Teich}(S)$ so that $\mathrm{Teich}(S) \cup \mathrm{P}\mathcal{ML}(S)$ is homeomorphic to a closed ball. We explain this briefly; for details see [4]. A sequence $\{\rho_n\} \subset \mathrm{Teich}(S)$ converges to $[\mu] \in \mathrm{P}\mathcal{ML}$ if the lengths of simple closed geodesics on ρ_n converge projectively to their intersection numbers with μ; more precisely, if there exists a sequence $\{c_n\}$ converging to infinity, so that $l_\gamma(\rho_n)/c_n \to i(\gamma, \mu)$, for any simple closed geodesic γ. The following lemma summarises the consequences of this definition we shall need.

Lemma 2.2 *Let $\alpha_1, \ldots, \alpha_N$ be a pants decomposition on S and let $\{\rho_n\} \subset \mathrm{Teich}(S)$ so that $\rho_n \to [\mu] \in \mathrm{P}\mathcal{ML}(S)$. Then:*

(a) *if $\nu \in \mathcal{ML}$ with $i(\mu, \nu) \neq 0$ then $l_\nu(\rho_n) \to \infty$,*

(b) *if $l_{\alpha_i}(\rho_n)$ is bounded for all $i = 1, \dots, N$, then there exist $a_1, \dots, a_N \geq 0$ so that $[\mu] = [a_1\alpha_1 + \dots + a_N\alpha_N]$.*

The proofs are immediate from the definitions. Part (b) gives a sufficient condition for convergence to a rational lamination $[\Sigma_i a_i \alpha_i]$, when the $\{\alpha_i\}$ is a pants decomposition. To compute the coefficients a_i we take another system of curves $\{\delta_i\}$ dual to the $\{a_i\}$. From the definition,

$$\frac{l_{\delta_j}(\rho_n)}{l_{\delta_k}(\rho_n)} \to \frac{i(\Sigma a_i \alpha_i, \delta_j)}{i(\Sigma a_i \alpha_i, \delta_k)} = \frac{a_j i(\alpha_j, \delta_j)}{a_k i(\alpha_k, \delta_k)},$$

and we know that $i(\alpha_i, \delta_i)$ is either 1 or 2, so this gives the proportion a_j/a_k.

2.2 Lines of minima

Two measured laminations μ, ν are said to *fill up* a surface S if for any other lamination η we have $i(\mu, \eta) + i(\nu, \eta) \neq 0$. It is proved in [8] that for any two such laminations, the function $l_\mu(\rho) + l_\nu(\rho)$ has a unique minimum on $\mathrm{Teich}(S)$. Thus μ and ν determine the *line of minima* $\mathcal{L}_{\mu,\nu}$, namely the set of points $m_s \in \mathrm{Teich}(S)$ at which the function $F_s(\rho) = (1-s)l_\mu(\rho) + sl_\nu(\rho)$ reaches its minimum as s varies in $(0,1)$.

Given this definition, we can immediately prove Theorem 1.2.

Proof of Theorem 1.2 Observe that $l_\mu(m_s)$ is bounded as $s \to 0$, because $l_\mu \leq 2((1-s)l_\mu + sl_\nu) = 2F_s$ for $s < 1/2$ and $F_s(m_s) \leq F_s(\rho_0)$ where ρ_0 is some arbitrary point in $\mathrm{Teich}(S)$. By compactness of $\mathrm{Teich}(S) \cup \mathrm{P}\mathcal{ML}$, we can choose some sequence $s_n \to 0$ for which m_{s_n} is convergent. Moreover, it is proved in [8] that the map $s \to m_s$ is proper, and so $m_{s_n} \to [\eta] \in \mathrm{P}\mathcal{ML}$. By Lemma 2.2 (a) we have that $i(\mu, \eta) = 0$ and from Lemma 2.1 we deduce that $[\eta] = [\mu]$. The result follows. $\square$

We now turn to the more interesting rational case. In general, the minimum m_s is in fact the unique critical point of F_s, so a point $p \in \mathcal{L}_{\mu,\nu}$ if and only if the 1-form $\mathrm{d}F_s = (1-s)\mathrm{d}l_\mu + s\mathrm{d}l_\nu$ vanishes at p for some s. If $\mu = \sum_i^N a_i \alpha_i$ is rational and $\{\alpha_1, \dots, \alpha_N\}$ is a pants decomposition, this enables us to find equations for $\mathcal{L}_{\mu,\nu}$. In fact, applying $\mathrm{d}F_s = (1-s)\sum_i a_i \mathrm{d}l_{\alpha_i} + s\mathrm{d}l_\nu$ to the tangent vectors $\frac{\partial}{\partial t_{\alpha_i}}$, we get

$$\frac{\partial l_\nu}{\partial t_{\alpha_i}} = 0, \quad \text{for all} \quad i = 1, \dots, n. \tag{1}$$

Similarly, applying $\mathrm{d}F_s$ to the tangent vectors $\frac{\partial}{\partial l_{\alpha_i}}$, we get $\partial l_\nu/\partial l_{\alpha_i} = -a_i(1-s)/s$, so that the line of minima satisfies the equations

$$\frac{1}{a_i}\frac{\partial l_\nu}{\partial l_{\alpha_i}} = \frac{1}{a_j}\frac{\partial l_\nu}{\partial l_{\alpha_j}} \quad \text{for all} \quad i, j. \tag{2}$$

Since $\frac{\partial}{\partial l_{\alpha_i}}, \frac{\partial}{\partial t_{\alpha_i}}$ form a basis of tangent vectors ([6] Proposition 2.6), the equations (1) and (2) completely determine $\mathcal{L}_{\mu,\nu}$.

3 Example

Let $S = S_{1,2}$ be the twice punctured torus. Consider two disjoint, non-disconnecting simple closed curves α_1 and α_2, and let β be a simple closed curve intersecting each of α_1 and α_2 once. For positive numbers a_1, a_2, denote by μ the measured lamination $a_1\alpha_1 + a_2\alpha_2$. (When S has a hyperbolic structure and $\gamma \in \pi_1(S)$, we abuse notation by using γ to mean also the unique geodesic in the homotopy class of γ.) We shall compute the equation of the line of minima $\mathcal{L}_{\mu,\beta}$, in terms of Fenchel-Nielsen coordinates relative to the pants decomposition $\{\alpha_1, \alpha_2\}$ and dual curves $\{\delta_1, \delta_2\}$ (see Figure 1), and we shall show that this line converges in Thurston's compactification to $[\alpha_1 + \alpha_2]$.

As explained above, the line of minima $\mathcal{L}_{\mu,\beta}$ is determined by the equations

$$\frac{\partial l_\beta}{\partial t_{\alpha_1}} = 0, \quad \frac{\partial l_\beta}{\partial t_{\alpha_2}} = 0 \quad \text{and} \quad \frac{1}{a_1}\frac{\partial l_\beta}{\partial l_{\alpha_1}} = \frac{1}{a_2}\frac{\partial l_\beta}{\partial l_{\alpha_2}}.$$

For two simple closed geodesics α, β, Kerckhoff's derivative formula [7] states

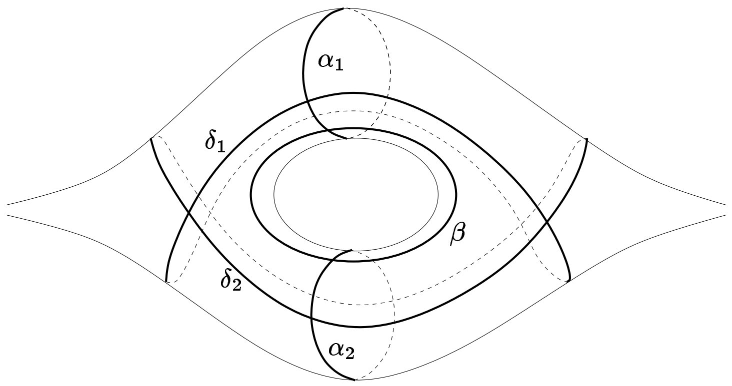

Figure 1: The curves α_1, α_2 and β on a twice punctured torus

that $(\partial l_\beta/\partial t_\alpha)|_\rho = \sum \cos\psi_i(\rho)$, where ψ_i are the intersection angles from β to α at each intersection point. Thus the first two equations mean that at a point in the line of minima the geodesic β is orthogonal to α_1 and α_2. Let P, P' be the two pairs of pants into which $\{\alpha_1, \alpha_2\}$ split S; denote by H_{12}, H_{11} the perpendicular segments in P from the geodesic α_1 to α_2 and from α_1 to itself, respectively; and denote by H'_{12}, H'_{11} the analogous perpendiculars in P'. Since β is orthogonal to α_1 and α_2, P and P' are glued so that the segments H_{12} and H'_{12} match up. Since P, P' are isometric (each is determined by the lengths $(l_{\alpha_1}, l_{\alpha_2}, 0)$), the segments H_{11} and H'_{11} also match, so that the union of both segments is the geodesic δ_1. Therefore, δ_1 intersects α_1 orthogonally, and so the twist parameter t_{α_1} is zero. In the same way, δ_2 intersects α_2 orthogonally and $t_{\alpha_2} = 0$.

It is not difficult to find the expression for the length of β in the Fenchel-Nielsen coordinates $(l_{\alpha_1}, l_{\alpha_2}, t_{\alpha_1}, t_{\alpha_2})$, either using trigonometry or by looking at the trace of the element representing β in the corresponding fuchsian group. This is done in detail in [3]. We have

$$\cosh\frac{l_\beta}{2} = \frac{1+\cosh\frac{l_{\alpha_1}}{2}\cosh\frac{l_{\alpha_2}}{2}}{\sinh\frac{l_{\alpha_1}}{2}\sinh\frac{l_{\alpha_2}}{2}}\cosh\frac{t_{\alpha_1}}{2}\cosh\frac{t_{\alpha_2}}{2} + \sinh\frac{t_{\alpha_1}}{2}\sinh\frac{t_{\alpha_2}}{2}.$$

Computing the derivatives $\partial l_\beta/\partial l_{\alpha_i}$ directly from this formula we get

$$-\sinh\frac{l_\beta}{2}\frac{\partial l_\beta}{\partial l_{\alpha_1}} = \frac{\cosh\frac{l_{\alpha_1}}{2}+\cosh\frac{l_{\alpha_2}}{2}}{\sinh^2\frac{l_{\alpha_1}}{2}\sinh\frac{l_{\alpha_2}}{2}}\cosh\frac{t_{\alpha_1}}{2}\cosh\frac{t_{\alpha_2}}{2},$$

$$-\sinh\frac{l_\beta}{2}\frac{\partial l_\beta}{\partial l_{\alpha_2}} = \frac{\cosh\frac{l_{\alpha_1}}{2}+\cosh\frac{l_{\alpha_2}}{2}}{\sinh\frac{l_{\alpha_1}}{2}\sinh^2\frac{l_{\alpha_2}}{2}}\cosh\frac{t_{\alpha_1}}{2}\cosh\frac{t_{\alpha_2}}{2}.$$

Therefore, the equations determining the line of minima are:

$$t_{\alpha_1} = t_{\alpha_2} = 0, \quad \frac{a_1}{a_2} = \frac{\sinh(l_{\alpha_2}/2)}{\sinh(l_{\alpha_1}/2)}.$$

By allowing a_1, a_2 to vary among all positive numbers, we observe that the corresponding lines of minima are pairwise disjoint and in fact foliate the whole plane $\{(l_{\alpha_1}, l_{\alpha_2}, t_{\alpha_1}, t_{\alpha_2}) \mid t_{\alpha_1} = 0, t_{\alpha_2} = 0\}$ in $\mathrm{Teich}(S_{1,2})$.

Clearly, at one end of $\mathcal{L}_{\mu,\beta}$ the lengths $l_{\alpha_1}, l_{\alpha_2}$ tend to zero. This cannot happen when $s \to 1$ because if $l_{\alpha_1}, l_{\alpha_2} \to 0$, then l_β and hence $(1-s)l_\mu(m_s) + sl_\beta(m_s)$ tend to ∞, and this contradicts the fact that m_s is the minimum. Thus $l_{\alpha_1}, l_{\alpha_2} \to 0$ as $s \to 0$ and therefore by Lemma 2.2 (b), the line $\mathcal{L}_{\mu,\beta}$ converges to a point of the form $[a'_1\alpha_1 + a'_2\alpha_2]$, for some $a'_1, a'_2 \geq 0$. To compute these numbers, we compute the lengths of the dual curves δ_1, δ_2. By hyperbolic trigonometry we get

$$\cosh \frac{l_{\delta_1}}{4} = \sinh \frac{l_{\alpha_2}}{2} \sinh \frac{l_\beta}{2}, \quad \cosh \frac{l_{\delta_2}}{4} = \sinh \frac{l_{\alpha_1}}{2} \sinh \frac{l_\beta}{2}.$$

Thus comparing the lengths of δ_1, δ_2 along the line of minima $\mathcal{L}_{\mu,\beta}$ we find

$$\lim_{s\to 0} \frac{\cosh (l_{\delta_1}/4)}{\cosh (l_{\delta_2}/4)} = \lim_{s\to 0} \frac{\sinh (l_{\alpha_2}/2)}{\sinh (l_{\alpha_1}/2)} = \frac{a_1}{a_2}.$$

Since $l_{\alpha_i} \to 0$, we have that $l_{\delta_i} \to \infty$, so that

$$\frac{a_1}{a_2} = \lim_{s\to 0} e^{(l_{\delta_1} - l_{\delta_2})/2}.$$

Taking logarithms, we get that $\lim_{s\to 0}(l_{\delta_1} - l_{\delta_2})$ is a constant, and this implies that $\lim_{s\to 0}(l_{\delta_1}/l_{\delta_2}) = 1$. Hence, $\mathcal{L}_{\mu,\beta} \to [\alpha_1 + \alpha_2]$ as $s \to 0$.

4 Statements of main results: pants decomposition case

In order to prove Theorem 1.1, we need to show that the lengths of all the simple closed geodesics converge projectively to their intersection numbers with $\sum \alpha_i$. So we want to estimate the length of any simple closed geodesic along the line of minima. We first prove that along the line of minima the lengths of α_i tend to zero and the twist parameters about α_i are bounded (Proposition 4.1 (a) and (b)). When α_i is a pants decomposition, these two properties allow one to give a nice estimate of the length of a closed geodesic (Proposition 4.2): the main contribution is given by the arcs going through the collars around the curves α_i. Finally, to compare the length of two closed geodesics, we need to compare the orders of the lengths $l_{\alpha_i}, l_{\alpha_j}$. This is done in Proposition 4.1 (c).

In this section we state these two propositions in the case in which $|\mu|$ is a pants decomposition and ν is rational. The propositions will be proved in the next section. Both propositions remain true when ν (in Proposition 4.1) and γ (in Proposition 4.2) are arbitrary measured laminations. We will comment on the proof of these stronger versions in Section 5.4.

Recall that two real functions $f(s), g(s)$ have the *same order* as $s \to s_0$, denoted by $f \sim g$, when there exist positive constants $k < K$ so that $k < f(s)/g(s) < K$ for all s near enough to s_0. Write $f(s) = O(1)$ if $f(s)$ is bounded.

Proposition 4.1 *Suppose that $\mu = \sum a_i\alpha_i$ and $\nu = \sum b_i\beta_i$ are two measured laminations where $\{\alpha_i\}$ is a pants decomposition, and $a_i > 0$ for all i. Let m_s be the minimum point of the function F_s. Then*

(a) *for any i, $\lim_{s\to 0} l_{\alpha_i}(m_s) = 0$;*

(b) *for any i, $|t_{\alpha_i}(m_s)|$ is bounded when $s \to 0$;*

(c) *for all i,j, $l_{\alpha_i}(m_s) \sim l_{\alpha_j}(m_s) \sim s$ as $s \to 0$.*

The proof of Proposition 4.1 (a) is direct and could be read now.

Proposition 4.2 *Let $\{\alpha_1,\dots,\alpha_N\}$ be a pants decomposition of S and γ a closed geodesic. Let $\rho_n \in \mathrm{Teich}(S)$ be a sequence so that, when $n \to \infty$, all the lengths $l_{\alpha_i}(\rho_n)$ are bounded above and the twists $t_{\alpha_i}(\rho_n)$ are bounded for all i. Then, as $n \to \infty$, we have*

$$l_\gamma(\rho_n) = 2\sum_{j=1}^{N} i(\alpha_j,\gamma)\log\frac{1}{l_{\alpha_j}(\rho_n)} + O(1).$$

In view of this proposition, it is enough to work with collars around α_j of width $2\log(1/l_{\alpha_j})$, even if they are not the maximal embedded collars. The more relaxed hypothesis about the lengths l_{α_i} being bounded above is not needed if $\{\alpha_1,\dots,\alpha_N\}$ is a pants decomposition, but will be useful in the general case in Section 6.

Proof of Theorem 1.1 for the pants decomposition case
Suppose $\alpha_1,\dots,\alpha_N$ is a pants decomposition system. By Proposition 4.1 (a), the lengths l_{α_i} tend to zero as $s \to 0$. Therefore, by Lemma 2.2 (b), $m_s \to [a'_1\alpha_1 + \dots + a'_N\alpha_N]$, as $s \to 0$, for some $a'_i \geq 0$. By Proposition 4.1 (b), along the line of minima the twists t_{α_i} are bounded. Then we can use Proposition 4.2 to estimate the length of two simple closed curves γ,γ': the proportion between their lengths is

$$\frac{l_\gamma}{l_{\gamma'}} = \frac{2\sum i(\alpha_j,\gamma)\log\frac{1}{l_{\alpha_j}} + O(1)}{2\sum i(\alpha_j,\gamma')\log\frac{1}{l_{\alpha_j}} + O(1)}. \tag{3}$$

Now, by Proposition 4.1(c), $l_{\alpha_i} \sim l_{\alpha_j}$ as $s \to 0$; this implies that $\log\frac{1}{l_{\alpha_i}}/\log\frac{1}{l_{\alpha_j}} \to 1$ as $s \to 0$ (see Lemma 5.3 below). Dividing numerator and denominator of (3) by $\log(1/l_{\alpha_1})$, we get that $\lim_{s\to 0}(l_\gamma/l_{\gamma'}) = i(\sum\alpha_j,\gamma)/i(\sum\alpha_j,\gamma')$. Hence $a'_i = 1$ for all i. $\square$

5 Proof: pants decomposition case

We shall estimate the length of a geodesic γ comparing it with the length of a "broken arc" relative to a pants decomposition. Broken arcs are a main tool in [13], and we refer there for details. The idea is that there is a unique curve freely homotopic to γ with no backtracking, made up of arcs which wrap around a pants curve, alternating with arcs which cross a pair of pants from one boundary to another following the common perpendiculars between the boundaries. This collection of mutually perpendicular arcs constitute the broken arc, whose length, as shown in Lemma 5.1, approximates the length of γ.

To determine the length of the broken arc, we study the geometry of a pair of pants. By using the trigonometric formulae for right angle hexagons and pentagons, we can compute the length of the segments perpendicular to two boundary components, and estimate this length when the lengths of the boundary components tend to zero. This is done in Lemma 5.4.

5.1 Broken arcs

A *broken arc* in $\mathbb{H}^2$ is a sequence of oriented segments such that the final point of one segment is equal to the initial point of the next, and such that consecutive arcs meet orthogonally. Labelling the segments in order $V_1, H_1, \dots, V_r, H_r, V_{r+1}$, we also require that for $1 \leq i \leq r-1$ the segments H_i, H_{i+1} are contained in opposite halfplanes with respect to V_{i+1}. We call the V_i the 'vertical arcs' and the H_i the 'horizontal' ones.

Lemma 5.1 *Consider a broken arc in hyperbolic plane with endpoints R, R' and with side lengths $s_1, d_1, \dots, s_r, d_r, s_{r+1}$. For any $D > 0$, there exists a constant $K = K(D, r)$, depending only on D and the number r of horizontal arcs, so that, if $d_j > D$ for all j, we have*

$$d(R, R') > \sum d_j + \sum s_j - K.$$

If $D' > D$, then $K(D', r) < K(D, r)$.

In the proof we use the following facts about universal constants for hyperbolic triangles, which can be deduced from the property that hyperbolic triangles are thin, see for example [5]. (I) There exists a positive constant $K(\theta_0)$ so that for any hyperbolic triangle with side lengths a, b, c and angle θ opposite to c satisfying $\theta \geq \theta_0 > 0$, we have $c > a + b - K(\theta_0)$. Moreover, if $\theta_0' > \theta_0$, then

$K(\theta_0') < K(\theta_0)$. (II) Given $D > 0$, there exists a constant $\theta_0 = \theta_0(D)$ so that for any hyperbolic triangle with one side of length $d \geq D$ and angles $\pi/2, \theta$ on this side, we have $\theta \leq \theta_0$. If $D' > D$, then $\theta_0(D') < \theta_0(D)$.

Proof of Lemma 5.1 Consider $D > 0$. The proof will be by induction on r. For $r = 1$ we have a broken arc with three arcs V_1, H_1, V_2 with lengths s_1, d_1, s_2; denote by Q, Q' the vertices of the arc H_1. Since $d_1 > D$, by (II), there exists θ_0 so that the angle $QQ'R$ is less than θ_0; therefore the angle $RQ'R'$ is greater than $\theta_1 = \pi/2 - \theta_0$. Applying (I) to the triangles RQQ' and $RQ'R'$, we have

$$d(R, R') > d(R, Q') + d(Q', R') - K(\theta_1) > s_1 + d_1 + s_2 - K(\pi/2) - K(\theta_1)$$

so that we can take $K(D, 1) = K(\pi/2) + K(\theta_1)$.

Now consider a broken arc with arc lengths $s_1, d_1, \ldots, s_r, d_r, s_{r+1}$ and $d_i > D$. Denote by Q, Q' the vertices of the arc d_r. Since $d_r \geq D$, the angle $R'QQ'$ is smaller than θ_0. Since R, R' are on different sides of the line containing the vertical segment V_r, the angle RQR' is greater than $\theta_1 = \pi/2 - \theta_0$. Applying (I) to the triangles RQR' and $QQ'R'$ and using the induction hypothesis we get

$$d(R, R') > \sum_{j=1}^{r} d_j + \sum_{j=1}^{r+1} s_j - K(D, r-1) - K(\pi/2) - K(\theta_1).$$

So we can take $K(D, r) = r\left(K(\frac{\pi}{2}) + K(\theta_1)\right)$. If $D' > D$, by (I) and (II), we have that $K(D', r) < K(D, r)$. □

Now let γ be a closed geodesic on a hyperbolic surface ρ, and let the geodesics $\{\alpha_i\}$ be a pants decomposition. We shall use the $\{\alpha_i\}$ to construct a broken arc $BA_\gamma(\rho)$ associated to γ, as illustrated in Figure 2. Fix an orientation on γ and let Q be an intersection point of γ with a pants curve. Let $\tilde{\gamma}$ be the lift of γ through a lift $\tilde{Q}$ of Q. Let $\tilde{C}_1, \ldots, \tilde{C}_{r+1}$ be the lifts of the geodesics $\{\alpha_i\}$ which are intersected, in order, by $\tilde{\gamma}$, so that $\tilde{C}_1 \cap \tilde{\gamma} = \tilde{Q}$ and $\tilde{C}_{r+1}$ is the image of $\tilde{C}_1$ under the covering translation τ corresponding to γ. Thus, if we denote by $\tilde{Q}'$ the intersection of $\tilde{\gamma}$ with $\tilde{C}_{r+1}$, the geodesic segment $\tilde{Q}\tilde{Q}'$ projects onto γ. For $i = 1, \ldots, r$, consider the common perpendicular segment to $\tilde{C}_i, \tilde{C}_{i+1}$, with endpoints denoted by Q_i^-, Q_i^+; and finally, let $Q_0^+ = \tau^{-1}(Q_r^+)$. Then we define $BA_\gamma(\rho)$ to be the broken arc with vertical arcs the segments $Q_0^+Q_1^-, Q_1^+Q_2^-, \ldots, Q_{r-1}^+Q_r^-$, and horizontal arcs the segments $Q_1^-Q_1^+, \ldots, Q_r^-Q_r^+$. Denote by s_i the lengths of the vertical arcs and by d_i the lengths of the horizontal arcs. The horizontal segments project onto

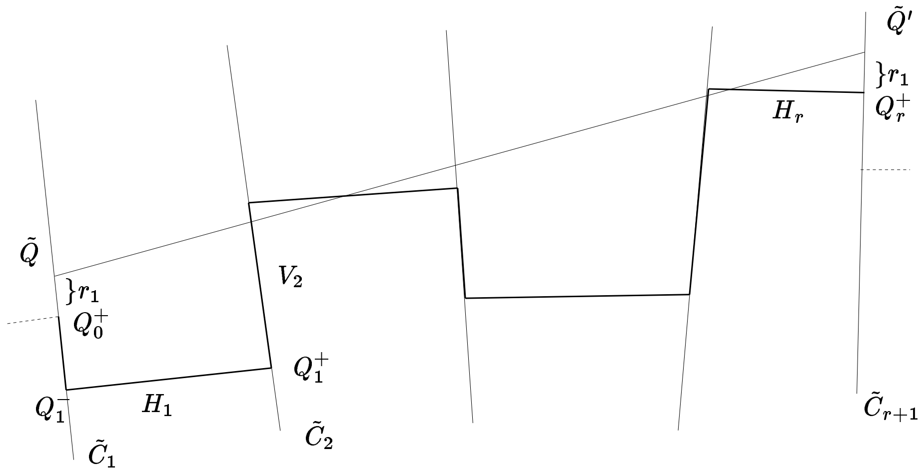

Figure 2: Broken arc with $r = 4$

geodesic segments which are the perpendiculars either between two boundary components, or from one boundary component to itself, of one of the pairs of pants. Their length will be studied in the next subsection. The vertical segments project onto arcs contained in the pants geodesics α_j. If the segment $Q_i^+Q_{i+1}^-$ projects, say, onto α_1, then its length is of the form

$$s_i = |n_i l_{\alpha_1} + t_{\alpha_1} + e_i| \tag{4}$$

where $n_i \in \mathbb{Z}$ depends on the combinatorics of γ relative to the pants decomposition (related to how many times γ wraps around α_1), and e_i is a number smaller in absolute value than l_{α_1} which depends on the combinatorics of γ and on the geometry of the two pairs of pants meeting along α_1. For our purposes we will not need more details about e_i, see [13] for more explanation.

We remark that the endpoints of $BA_\gamma(\rho)$ do not necessarily coincide with those of $\tilde{\gamma}$, but we can consider another broken arc $\overline{BA}_\gamma$ with the same endpoints as $\tilde{\gamma}$ by just changing the first vertical segment $Q_0^+Q_1^-$ to $\tilde{Q}Q_1^-$ and adding at the end the vertical segment $Q_r^+\tilde{Q}'$. To control the lengths of these two new segments, we use the following lemma.

Lemma 5.2 *With the above notation, suppose that $\tilde{C}_i$ projects onto α_k and denote $\tilde{Q}_i = \tilde{\gamma} \cap \tilde{C}_i$. Then, either $\tilde{Q}_i$ is between Q_{i-1}^+ and Q_i^-, or the minimum of the distances $d(Q_{i-1}^+, \tilde{Q}_i), d(Q_i^-, \tilde{Q}_i)$ is less than l_{α_k}.*

Proof Suppose that $\tilde{Q}_i$ is not between Q_{i-1}^+ and Q_i^- and that both distances $d(Q_{i-1}^+, \tilde{Q}_i), d(Q_i^-, \tilde{Q}_i)$ are greater than l_{α_k}. Then, applying the covering transformation corresponding to α_k (or to α_k^{-1}) to the segments H_{i-1}, H_i, we obtain two new segments H'_{i-1}, H'_i which are closer to $\tilde{\gamma}$ than H_{i-1}, H_i. The lines containing these segments are disjoint from $\tilde{C}_{i-1}, \tilde{C}_{i+1}$ respectively and hence necessarily both intersect $\tilde{\gamma}$. Therefore they determine, together with the lines $\tilde{\gamma}, \tilde{C}_i$, two right-angled triangles. One of them has angle sum greater than π, so we have a contradiction. (There is a similar argument when one of the distances $d(Q^+_{i-1}, \tilde{Q}_i), d(Q^-_i, \tilde{Q}_i)$ is equal to l_{α_k}.) □

As a consequence, if $\tilde{C}_1$ projects over α_{i_1} and if $\bar{s}_1, \bar{s}_{r+1}$ are the lengths of the first and last vertical segments of $\overline{BA}_\gamma$, then $\bar{s}_1 + \bar{s}_{r+1}$ is either equal to s_1 or to $s_1 + 2r_1$, where $r_1 < l_{\alpha_{i_1}}$. Thus, by Lemmas 5.1 and 5.2, we can approximate the length of γ by the length of the broken arc BA_γ within an error of $K(D,r) + 2l_{\alpha_{i_1}}$.

Remark A straightforward generalisation of the above construction allows one to associate a broken arc with any (not necessarily closed) geodesic, and also with a geodesic arc with endpoints on the pants curves. (For a geodesic arc, to determine the first vertical arc, prolong the geodesic in the negative direction until it crosses the next pants curve.) Then, we can use Lemmas 5.1 and 5.2 to estimate this length from the length of this broken arc. This is useful when ν is irrational, see Section 5.4.

5.2 Geometry of a pair of pants

We now estimate the lengths of the common perpendicular segments between two curves of the pants decomposition. In the situation to be considered, these segments will be sufficiently long to apply Lemma 5.1.

It is useful to refine slightly the notation $f(s) \sim g(s)$ as $s \to s_0$ defined on p. 216. For f, g real valued functions we write $f \approx g$ to mean that $\lim_{s\to s_0} f/g$ exists and is strictly positive. Clearly, $f \approx g$ is slightly stronger than $f \sim g$. However even if the limit does not exist, if $f \sim g$, and if both functions tend to either 0 or ∞, then $\lim_{s\to s_0} \log f/\log g$ does exist and equals 1. This fact is crucial for our results. We collect this and other elementary properties in the next lemma. We also recall the notation $f = O(g)$ as $s \to s_0$ meaning that f/g is bounded when $s \to s_0$, and $f = o(g)$ as $s \to s_0$ meaning that $f/g \to 0$ when $s \to s_0$.

Lemma 5.3 (a) *$f \sim g$ is equivalent to* $\log f = \log g + O(1)$.

(b) *If* $f, g \to \infty$ *or* 0 *and* $f \sim g$, *then* $\lim(\log f / \log g) = 1$.
(c) $f \approx g$ *is equivalent to* $f = ag + o(g)$, *with* $a > 0$.

Proof (a) If there exists $0 < k < K$ with $k < f/g < K$, then taking logarithms we get that $\log k < \log f - \log g < \log K$. The converse is also clear by exponentiating $\log f = \log g + O(1)$.

(b) Since $g \to 0$ or ∞, then $\log g \to -\infty$ or $+\infty$ respectively, and in both cases $O(1)/\log g \to 0$. Then, dividing $\log f = \log g + O(1)$ by $\log g$, we get the result. Part (c) is immediate from the definitions. □

Lemma 5.4 *Consider a pair of pants* P *with boundary components* B_1, B_2, B_3 *of lengths* l_1, l_2, l_3. *For any* $i, j \in \{1, 2, 3\}$, *let* H_{ij} *be the common perpendicular arc to the boundary components* B_i, B_j, *with length* d_{ij}. *Suppose that each of* l_1, l_2, l_3 *either tends to zero or is bounded above. Then, for any* i, j, *we have*

$$d_{ij} = \log \frac{1}{l_i} + \log \frac{1}{l_j} + O(1).$$

Proof The pair of pants P is made up by gluing two isometric right angle hexagons with alternate sides of lengths $l_1/2, l_2/2, l_3/2$. For $i \neq j$, the segments H_{ij} are the remaining sides. The segment H_{ii} is the union of the common perpendicular segments in the two hexagons between the side contained in B_i and its opposite side. We therefore obtain the trigonometric formulae:

$$\cosh d_{ij} = \frac{\cosh \frac{l_k}{2} + \cosh \frac{l_i}{2} \cosh \frac{l_j}{2}}{\sinh \frac{l_i}{2} \sinh \frac{l_j}{2}}, \quad \cosh \frac{d_{ii}}{2} = \sinh d_{ij} \sinh \frac{l_j}{2}.$$

For $i \neq j$ we deduce that $\cosh d_{ij} \approx \frac{1}{l_i l_j}$ as $(l_1, l_2, l_3) \to (0, 0, 0)$. Thus there exits $a > 0$ so that

$$\cosh d_{ij} = \frac{a}{l_i l_j} + o\left(\frac{1}{l_i l_j}\right).$$

Since $d_{ij} \to \infty$, $e^{-d_{ij}}$ is bounded, and so

$$e^{d_{ij}} = \frac{2a}{l_i l_j} + o\left(\frac{1}{l_i l_j}\right).$$

The result follows from Lemma 5.3 (a).

For the case $i = j$, we have that $\sinh d_{ij} \approx \frac{1}{l_i l_j}$ as $(l_1, l_2, l_3) \to (0, 0, 0)$ (because $d_{ij} \to \infty$ and in that case $\sinh d_{ij} \approx \cosh d_{ij}$). Then, from the above formula for $\cosh(d_{ii}/2)$, we have that $\cosh(d_{ii}/2) \approx 1/l_i$. As before, $d_{ii}/2 = \log(1/l_i) + O(1)$ and so we get the result.

We can check that the same works when some or all the l_i do not tend to zero but are still bounded above. □

5.3 Proof of Propositions 4.1 and 4.2

Proposition 4.2 is used in the proof of Proposition 4.1 (c), so we follow this order below. We remark that in our proofs the hypothesis of being in the line of minima is always used in the same way: simply compare the value of F_s at its minimum m_s and at some other point.

Proof of Proposition 4.1(a) Consider i with $a_i \neq 0$. Given $\epsilon > 0$, consider a hyperbolic surface ρ_ϵ so that the length of any α_i is equal to $(a_i/4)\epsilon$. Take $s_0 = \min\{\frac{1}{2}, \frac{a_i \epsilon}{4 l_\nu(\rho_\epsilon)}\}$. Then, for $s < s_0$, we have

$$\begin{aligned} F_s(\rho_\epsilon) &= (1-s)\textstyle\sum_{j=1}^N a_j l_{\alpha_j}(\rho_\epsilon) + s l_\nu(\rho_\epsilon) \\ &= (1-s)(\textstyle\sum_{j=1}^N a_j)\frac{a_i}{4}\epsilon + s l_\nu(\rho_\epsilon) < \frac{a_i}{4}\epsilon + \frac{a_i}{4}\epsilon = \frac{a_i}{2}\epsilon. \end{aligned}$$

Hence, for the minimum point m_s of F_s, we have

$$(1-s) a_i l_{\alpha_i}(m_s) \leq F_s(m_s) < (a_i/2)\epsilon.$$

Since $1 - s > 1/2$, we have that $l_{\alpha_i}(m_s) < \epsilon$. Since, by hypothesis, $a_i \neq 0$ for all i, we have the result. □

Proof of Proposition 4.1(b) We shall prove that all the twists about α_r are bounded when $s \to 0$. Suppose not; renumbering, we may assume that $|t_{\alpha_1}(m_s)|$ is not bounded. Suppose moreover that, up to subsequence, $t_{\alpha_1} \to +\infty$. (The proof is the same if $t_{\alpha_1} \to -\infty$.)

For each s, consider the point ρ_s obtained from m_s by twisting (earthquaking) by $-t_{\alpha_1}(m_s)$ about α_1. This new surface has the same Fenchel-Nielsen coordinates with respect to $\{\alpha_i\}$ as m_s (for a fixed choice of dual curves) except that $t_{\alpha_1}(\rho_s) = 0$. We shall prove that, if s is small enough, then $l_\nu(m_s) > l_\nu(\rho_s)$. Since the lengths of α_i are the same at both points, we will have that $F_s(m_s) > F_s(\rho_s)$, which is a contradiction.

We make the following argument for all the curves β in the support of ν. Since the lengths of all the pants curves tend to zero, by Lemma 5.4 there exists $s_0 > 0$ so that for all $s < s_0$, all the horizontal arc lengths d_{ij} of the broken arc BA_β are bounded below by some constant D, and therefore by Lemma 5.1 (and Lemma 5.2), there exists a constant K so that $l_{BA_\beta}(m_s) - l_\beta(m_s) < K$ for all $s < s_0$. We will prove that

$$l_{BA_\beta}(m_s) - l_{BA_\beta}(\rho_s) \to \infty \ \text{ as } \ s \to 0. \tag{5}$$

Assuming (5), there exists $0 < s_1 < s_0$ so that for all $s < s_1$, $l_{BA_\beta}(m_s) - l_{BA_\beta}(\rho_s) > K$, and then

$$l_\beta(m_s) > l_{BA_\beta}(m_s) - K > l_{BA_\beta}(\rho_s) > l_\beta(\rho_s).$$

Summing over all curves β_i, we obtain $l_\nu(m_s) > l_\nu(\rho_s)$.

It is left to prove (5). We compare the broken arcs BA_β at the points m_s and ρ_s. The horizontal arcs have the same length in both broken arcs. In fact only the vertical arcs projecting over the curve α_1 change length. There are $i(\beta,\alpha_1)$ of such segments, with lengths $s_1,\dots,s_{i(\beta,\alpha_1)}$ at m_s, where $s_j = |n_j l_{\alpha_1} + t_{\alpha_1} + e_j|$. On the other hand, the lengths of these segments at ρ_s are $s'_j = |n_j l_{\alpha_1} + e_j|$. (We remark that $n_1 + \cdots + n_{i(\beta,\alpha_1)}$ is called the *wrapping number* of β around α_1.) Then

$$l_{BA_\beta}(m_s) - l_{BA_\beta}(\rho_s) = \sum (s_j - s'_j) \geq i(\beta,\alpha_1)t_{\alpha_1} - 2\sum_{j=1}^{i(\beta,\alpha_1)} |n_j l_{\alpha_1} + e_j|.$$

Since $|n_j l_{\alpha_1} + e_j|$ is bounded for all j, this expression tends to infinity as $s \to 0$ as required. □

Proof of Proposition 4.2 Consider the broken arc $BA_\gamma(\rho_n)$ associated to γ. Since all the lengths l_{α_i} tend to 0 or are bounded above when $n \to \infty$, then, for n big enough, all horizontal arcs of this broken arc are greater than some given D. Then we can use Lemmas 5.1 and 5.2 to estimate the length of γ and we have $\sum d_i + \sum s_i - K < l_\gamma(\rho_n) < \sum d_i + \sum s_i$. Since all the twists t_{α_i} are bounded, the lengths of the vertical arcs are bounded. On the other hand, using Lemma 5.4 to estimate the lengths d_i of the horizontal arcs, and collecting terms together, we have

$$l_\gamma(\rho_n) = \sum d_i + O(1) = \sum_j r_j \log \frac{1}{l_{\alpha_j}(\rho_n)} + O(1),$$

where r_j is the number of times that the projections of the horizontal arcs end in the geodesic α_j. Then this number is equal to $2i(\alpha_j,\gamma)$, and we get the desired result. □

Proof of Proposition 4.1(c) We need to prove that, on the line of minima $\mathcal{L}_{\mu,\nu}$, all the lengths l_{α_i} have the same order as s, when $s \to 0$. That is, there exists $\epsilon > 0$ and positive constants $k_i < K_i$ for any i, so that

$$k_i < \frac{l_{\alpha_i}(m_s)}{s} < K_i$$

for all $s < \epsilon$. Suppose not; then, up to subsequence, there exists some curve α_j so that, when $s \to 0$, then $l_{\alpha_j}(m_s)/s$ tends either to 0 or to ∞.

We construct a new sequence of surfaces ρ_s defined by the Fenchel-Nielsen coordinates $l_{\alpha_i} = s$ and $t_{\alpha_i} = 0$, for all i. We compare $F_s(m_s)$ with $F_s(\rho_s)$, and show that $F_s(m_s) - F_s(\rho_s) > 0$, which will be a contradiction because m_s is the minimum of the function F_s.

By Proposition 4.1 (a) and (b), along m_s the lengths l_{α_i} tend to 0 and the twists t_{α_i} are bounded. Applying Proposition 4.2 to estimate the length of the curves β_i we find

$$F_s(m_s) = (1-s)\sum_i a_i l_{\alpha_i}(m_s) + s\sum_i b_i\left(2\sum_j i(\alpha_j,\beta_i)\log\tfrac{1}{l_{\alpha_j}(m_s)} + O(1)\right).$$

On the other hand, the sequence ρ_s also satisfies the hypothesis of Proposition 4.2, and so we can also use this proposition to estimate the length of β_i, giving

$$F_s(\rho_s) = (1-s)\sum_i a_i s + s\sum_i b_i\left(2\sum_j i(\alpha_j,\beta_i)\log\tfrac{1}{s} + O(1)\right).$$

Then

$$\begin{aligned}
&\frac{F_s(m_s) - F_s(\rho_s)}{s}\\
&\quad = (1-s)\textstyle\sum_i a_i \frac{l_{\alpha_i}(m_s) - s}{s}\\
&\qquad\qquad + \textstyle\sum_i b_i\left(2\sum_j i(\alpha_j,\beta_i)\left(\log\frac{1}{l_{\alpha_j}(m_s)} - \log\frac{1}{s}\right) + O(1)\right)\\
&\quad = (1-s)\textstyle\sum_i a_i\frac{l_{\alpha_i}(m_s)}{s} + \sum_i b_i\left(2\sum_j i(\alpha_j,\beta_i)\left(\log\frac{s}{l_{\alpha_j}(m_s)}\right)\right) + O(1)\\
&\quad = \textstyle\sum_i\left((1-s)a_i\frac{l_{\alpha_i}(m_s)}{s} + C_i\log\frac{s}{l_{\alpha_i}(m_s)}\right) + O(1),
\end{aligned}$$

where in the last equality we have rearranged the second group of summands, and C_i are some positive coefficients.

Now, if there is some i so that $l_{\alpha_i}(m_s)/s \to 0$, then $\log(s/l_{\alpha_i}(m_s)) \to \infty$. On the other hand, if there is some i so that $l_{\alpha_i}(m_s)/s \to \infty$ then $\log(s/l_{\alpha_i}(m_s)) \to -\infty$, but any positive linear combination of $l_{\alpha_i}(m_s)/s$ and $\log(s/l_{\alpha_i}(m_s))$ tends to infinity.

Hence, $(F_s(m_s) - F_s(\rho_s))/s$ tends to $+\infty$ and therefore $F_s(m_s) - F_s(\rho_s) > 0$ for sufficiently small s, obtaining the desired contradiction. □

5.4 The proof of Theorem 1.1 when ν is irrational

Finally we discuss the proof of Theorem 1.1 when ν is irrational. All that is needed is to extend Propositions 4.1 and 4.2 to the case in which ν and γ respectively are general measured laminations.

First, consider the effect of replacing γ in Proposition 4.2 by an irrational lamination η. There exists a sequence of rational laminations $c_k\gamma_k$ converging to η, with γ_k simple closed curves and $c_k \to 0$. Then $l_\eta(\rho_n) = \lim_{k\to\infty} c_k l_{\gamma_k}(\rho_n)$. We can compute this limit by using the expression obtained in Proposition 4.2 for closed curves. Since $\lim_{k\to\infty} i(c_k\gamma_k, \alpha) = i(\eta, \alpha)$, then we only need to check that the error (which is a function $f_k(n)$, bounded for fixed k as $n \to \infty$) stays bounded when $k \to \infty$. By careful inspection of this error (in the proofs of Lemmas 5.1, 5.4 and Proposition 4.2), we see that it depends linearly on the intersection number of γ_k with $\sum \alpha_i$ and on the wrapping numbers of γ_k around α_i. In both cases, these numbers, after scaling with c_k, converge when $k \to \infty$ (to the intersection number of η with $\sum \alpha_i$ and to the *twisting numbers* of η around α_i respectively). This proves Proposition 4.2.

The proof of Proposition 4.1 (b) for ν irrational, uses the same kind of arguments. The proof of part (c) is unchanged, once we have the stronger version of Proposition 4.2.

Alternatively, we can compute the length of an irrational lamination η from its definition (see [6]), as the integral over the surface of the product measure $d\eta \times dl$, where dl is the length measure along the leaves of η. We can cover the surface with thin rectangles, so that the length of arcs of η intersecting one rectangle are almost equal, and we approximate the length of one of these geodesic arcs by the length of a broken arc (notice that we need the remark after Lemma 5.2 to do this). In this way we can prove both Propositions 4.2 and 4.1 (b). □

6 Non-pants decomposition case

We now investigate the modifications needed to the above work if $\alpha_1, \dots, \alpha_N$ is not a pants decomposition. The problem is that we no longer have full control over the geometry of the complement in S of the curve system $\alpha_1, \dots, \alpha_N$. The hyperbolic structures on at least some components of the complement might themselves diverge, in other words, the estimate of Proposition 4.2 for the lengths of arbitrary closed geodesics may no longer hold. Without ruling out this possibility, we show that the divergences in question must be of a lower order than those caused by the shrinking of the curves α_i. A precise statement is made in Corollary 6.6.

In more detail, we proceed as follows. With minimal changes we still can prove that $l_{\alpha_i} \to 0$, with the same order as s, and that the twists t_{α_i} are bounded

(Proposition 6.1). This implies that, if there exists a limiting lamination η, then its support is either disjoint from or contains the curves α_i. Next we prove that there is in fact no other lamination contained in the limiting lamination, and therefore $[\eta] = [a'_1\alpha_1 + \cdots + a'_N\alpha_N]$, for some coefficients $a'_i \geq 0$ (Proposition 6.2). To compute these coefficients we need to compare the lengths of two closed geodesics. Even though we no longer have Proposition 4.2, we can still extend the curves α_i to a pants decomposition and estimate the length of the dual curves, (Proposition 6.5). This is enough to compute the coefficients a'_i and prove Theorem 1.1. A posteriori we obtain an estimate for the length of closed geodesics along the line of minima in Corollary 6.6.

Proposition 6.1 *Suppose that $\mu = \sum a_i\alpha_i$, $\nu = \sum b_i\beta_i$ are two measured laminations (where $\{\alpha_1, \ldots, \alpha_N\}$ is not necessarily a pants decomposition), and that $a_i > 0$ for all i. Let m_s be the minimum point of the function F_s. Then:*

(a) *for any i, $\lim_{s\to 0} l_{\alpha_i}(m_s) = 0$;*

(b) *for any i, $|t_{\alpha_i}(m_s)|$ is bounded when $s \to 0$;*

(c) *for all i, j, $l_{\alpha_i}(m_s) \sim l_{\alpha_j}(m_s) \sim s$ as $s \to 0$.*

Remark The twist parameter t_{α_i} is a real parameter which determines how the surface is glued along α_i. It is determined up to the choice of an initial surface on which the twist is zero. Changing the initial surface results in an additive change to the twist parameter, so (b) above is independent of this choice.

Proof The proof of (a) is exactly the same as that of Proposition 4.1(a).

For (b), consider a broken arc associated to the curve β_i relative to the curves $\alpha_1, \ldots, \alpha_N$. Even if these curves are not a pants decomposition, the definition of broken arc given in Section 5.1 makes sense. The horizontal segments now project onto arcs perpendicular to two of the α_i. If the length of each α_i is sufficiently small, the horizontal segments are greater that some given positive constant D and we can apply Lemmas 5.1 and 5.2 to approximate the length of β_i with the length of the corresponding broken arc. The length of the vertical arc projecting over α_i is $|t_{\alpha_i} + r|$, where r depends only on the curve β_i and the system $\alpha_1, \ldots, \alpha_N$. (To see this, think about obtaining the given surface from an initial surface with $t_{\alpha_i} = 0$ by twisting about α_i.) We can therefore argue exactly as in the proof of Proposition 4.1 (b): consider points ρ_s obtained from the points m_s on the line of minima by twisting by $-t_{\alpha_i}(m_s)$ about each α_i. As in that proposition, we obtain that $F_s(m_s) > F_s(\rho_s)$ for small enough s.

For (c) we follow the same argument as in the proof of Proposition 4.1 (c). Let $\gamma_1, \dots, \gamma_K$ be simple closed curves so that α_i, γ_j are a pants decomposition. Consider the surfaces ρ_s defined by $l_{\alpha_i}(\rho_s) = s$, $l_{\gamma_i}(\rho_s) = 1$, $t_{\alpha_i}(\rho_s) = t_{\gamma_i}(\rho_s) = 0$. The family ρ_s satisfies the hypothesis of Proposition 4.2, and so we can estimate the length of β_i as

$$l_{\beta_i}(\rho_s) = 2\sum_j i(\alpha_j, \beta_i) \log \frac{1}{s} + O(1).$$

Now each curve α_i is contained in an embedded annular collar of width at least $2\log(1/l_{\alpha_i})$. Using the contribution of the these collars gives the following rough estimation for the length of β_i at m_s:

$$l_{\beta_i}(m_s) = 2\sum_j i(\alpha_j, \beta_i) \log \frac{1}{l_{\alpha_j}(m_s)} + f(s),$$

where $f(s)$ is a positive function that might tend to infinity. Now, as in the proof of Proposition 4.1 (c), we have

$$\frac{F_s(m_s) - F_s(\rho_s)}{s} = \sum_i \left((1-s)a_i \frac{l_{\alpha_i}(m_s)}{s} + C_i \log \frac{s}{l_{\alpha_i}(m_s)} \right) + f(s) - O(1)$$

where $f(s) - O(1)$ may be negative but is nevertheless bounded below. The conclusion follows as in that proposition since, if l_{α_i} does not have the same order as s, the above group of summands always tends to $+\infty$. □

Proposition 6.2 *Let μ, ν and m_s be as in* Proposition 6.1 *and suppose that m_s converges to a projective measured lamination $[\eta]$. Then the support of $[\eta]$ is contained in the union of $\alpha_1, \dots, \alpha_N$.*

Proof If the conclusion is false, then using Proposition 6.1 (a) we must have $|\eta| \subset \alpha_1 \cup \dots \cup \alpha_N \cup |\delta|$, where δ is a measured lamination whose support is disjoint from the α_i. Since μ and ν fill up the surface and $i(\mu, \delta) = 0$, it follows that $i(\nu, \delta) \neq 0$, and therefore some curve $\beta \subset |\nu|$ intersects $|\delta|$. Let $\bar{\kappa}$ be a geodesic arc contained in β, intersecting δ, running from some α_i to some α_j (where possibly $\alpha_i = \alpha_j$) and not intersecting any other α_l. We take open collar neighbourhoods A_i of the curves α_i, of width $2\log(1/l_{\alpha_i})$, and let $\kappa = \bar{\kappa} - \bar{\kappa} \cap (A_i \cup A_j)$, so that κ is a geodesic segment with endpoints on the relevant components $\tilde{\alpha}_i, \tilde{\alpha}_j$ of the boundaries ∂A_i and ∂A_j. Note that the boundary curves $\tilde{\alpha}_i, \tilde{\alpha}_j$ of the collars A_i, A_j have length $O(1)$.

We are going to prove that l_κ, the length of the geodesic segment κ, tends to infinity by comparing to the length of a simple closed curve or curves we call the *double* of κ (or $\bar{\kappa}$). If $i \neq j$ (or if $i = j$ but κ meets both boundary components of the collar A_i), then the double is the simple closed curve $\tilde{\beta}$ created by going around $\tilde{\alpha}_i$, then parallel to κ, around $\tilde{\alpha}_j$, and back parallel to κ. In the case that $\alpha_i = \alpha_j$ and κ intersects only one boundary component of the collar A_i, then κ splits this boundary component, $\tilde{\alpha}_i$, into two arcs $\tilde{\alpha}_i'$ and $\tilde{\alpha}_i''$; we create two simple closed curves $\tilde{\beta}' = \kappa \cup \tilde{\alpha}_i'$ and $\tilde{\beta}'' = \kappa \cup \tilde{\alpha}_i''$, and designate $\tilde{\beta} = \tilde{\beta}' \cup \tilde{\beta}''$ the double. In Lemma 6.3 below, we show that $i(\tilde{\beta}, \delta) \neq 0$, where in the second case we define $i(\tilde{\beta}, \delta) = i(\tilde{\beta}', \delta) + i(\tilde{\beta}'', \delta)$.

Since $m_s \to [\sum a_i' \alpha_i + \delta]$, the length on the surface m_s of the geodesic(s) isotopic to $\tilde{\beta}$ must tend to infinity as $s \to 0$. We claim that the length of the arc κ at m_s tends to infinity with $\tilde{\beta}$. If $\alpha_i \neq \alpha_j$ then $2l_\kappa + l_{\tilde{\alpha}_i} + l_{\tilde{\alpha}_j} > l_{\tilde{\beta}}$. The lengths of the boundary curves $\tilde{\alpha}_i, \tilde{\alpha}_j$ are bounded above (and below); since $l_{\tilde{\beta}} \to \infty$, this forces $l_\kappa \to \infty$. A similar proof works if $\alpha_i = \alpha_j$.

Finally, we use the hypothesis that m_s is the minimum of F_s to arrive to a contradiction. The argument is similar to others used above. Let $\gamma_1, \dots, \gamma_K$ be simple closed curves extending $\alpha_1, \dots, \alpha_N$ to a pants decomposition, and fix a set of dual curves. For each s let ρ_s be the surface whose Fenchel-Nielsen coordinates with respect to these choices are

$$l_{\alpha_i}(\rho_s) = l_{\alpha_i}(m_s),\ l_{\gamma_i}(\rho_s) = 1,\ t_{\alpha_i}(\rho_s) = t_{\gamma_i}(\rho_s) = 0.$$

The surfaces ρ_s satisfy the hypothesis of Proposition 4.2, and therefore $l_{\beta_i}(\rho_s) = \sum_j 2i(\beta_i, \alpha_j) \log(1/l_{\alpha_j}) + O(1)$. On the other hand, we have

$$l_{\beta_i}(m_s) = \sum_j 2i(\beta_i, \alpha_j) \log(1/l_{\alpha_j}) + f(s),$$

where $f(s)$ is a positive function which tends to infinity for those curves β_i intersecting $|\delta|$, since, by the above argument, some arcs of some β_i outside the collars A_j tend to infinity. Thus

$$F_s(m_s) - F_s(\rho_s) = s \textstyle\sum_i b_i \left(l_{\beta_i}(m_s) - l_{\beta_i}(\rho_s)\right) = s \sum_i b_i (f(s) - O(1))$$

which is positive for small enough s. □

The following lemma was used in the above proof. We provide a proof, although the result describes a well-known construction.

Lemma 6.3 *Let ρ be a hyperbolic surface, let α_1, α_2 be two disjoint simple closed geodesics, and $\bar{\kappa}$ be a geodesic arc from α_1 to α_2. Let δ be a geodesic intersecting $\bar{\kappa}$ and $\tilde{\beta}$ be the 'double' of $\bar{\kappa}$, as constructed in the proof of* Proposition 6.2. *Then $i(\tilde{\beta}, \delta) \neq 0$.*

Proof Suppose that $\alpha_1 \neq \alpha_2$; by the construction of the curve $\tilde{\beta}$, the curves $\alpha_1, \alpha_2, \tilde{\beta}$ bound a pair of pants, made up of a thin strip around the arc κ (the part of $\bar{\kappa}$ outside the annuli A_i), and the sub-annuli of A_1 and A_2 with boundaries α_i and $\tilde{\alpha}_i$ for $i = 1, 2$. Correspondingly, there is a pair of pants P in our hyperbolic surface bounded by α_1, α_2 and the geodesic representing $\tilde{\beta}$. Now, $\bar{\kappa}$ is an arc contained in P joining α_1 to α_2. The lamination δ intersects P in arcs which do not meet α_1, α_2, so running from the geodesic representative of $\tilde{\beta}$ to itself; therefore each of these arcs intersects $\bar{\kappa}$ once and $\tilde{\beta}$ twice so that $i(\tilde{\beta}, \delta) = 2i(\bar{\kappa}, \delta) > 0$.

In the case that $\alpha_1 = \alpha_2$, remember that the 'double' of $\bar{\kappa}$ is the union of the two simple closed curves $\tilde{\beta}', \tilde{\beta}''$ described above. We have to show that $i(\tilde{\beta}', \delta) + i(\tilde{\beta}'', \delta) \neq 0$. Arguing much as above, we have that α_1 and the geodesic representatives of $\tilde{\beta}'$ and $\tilde{\beta}''$ bound a pair of pants containing $\bar{\kappa}$, and $\bar{\kappa}$ joins α_1 to itself. Since δ intersects this pair of pants in geodesic arcs not intersecting α_1, each such arc intersects $\tilde{\beta}' \cup \tilde{\beta}''$ twice and the result follows. $\square$

Finally, we estimate the length along the line of minima of curves dual to the α_i. Suppose α_i, γ_j is a pants decomposition of the surface S, and let δ_i be the dual curves. If δ_i is dual to α_i, then these two curves intersect either once or twice. If $i(\alpha_i, \delta_i) = 1$, then α_i is on the boundary of just one pair of pants P (two boundary components of P are glued together along α_i). We denote the other boundary component of P by ω. If $i(\alpha_i, \delta_i) = 2$, then α_i is on the boundary of two different pants P, P'; let $\omega_1, \omega_2, \omega_1', \omega_2'$ the other boundary components of P, P', respectively. To simplify notation, in the following proposition we drop the indices in α_i, δ_i.

Lemma 6.4 *With the above notation, let δ be the dual curve to α.*

(a) *Suppose that $i(\alpha, \delta) = 1$ and that ρ_n is a sequence of surfaces so that $l_\alpha(\rho_n) \to 0$ and $|t_\alpha(\rho_n)|$ is bounded. Then*

$$l_\delta(\rho_n) = 2i(\alpha, \delta) \log(1/l_\alpha(\rho_n)) + (1/2) l_\omega(\rho_n) + O(1).$$

(b) *Suppose that $i(\alpha, \delta) = 2$ and suppose that ρ_n is a sequence of surfaces so that $l_\alpha(\rho_n) \to 0$ and $|t_\alpha(\rho_n)|$ is bounded. Then*

$$l_\delta(\rho_n) = 2i(\alpha, \delta) \log(1/l_\alpha(\rho_n)) + l_{|\omega|}(\rho_n) + l_{|\omega'|}(\rho_n) + O(1),$$

where we denote by $l_{|\omega|}(\rho_n)$ the maximum of $l_{\omega_1}(\rho_n)$ and $l_{\omega_2}(\rho_n)$, and similarly with $l_{|\omega'|}(\rho_n)$.

Proof In both cases, the length of δ depends on the lengths of α and of the neigbouring pants curves and on the twist about α. One can calculate δ explicitly, however it is easier to use the broken arc Lemma 5.1 to simplify the estimates.

If $i(\alpha,\delta)=1$, let d be the distance in P between the two boundary components α',α'' projecting over α. The dual curve δ can be approximated by a broken arc which wraps part-way round α' and then follows the common perpendicular from α' to α'', and finally wraps part-way round α''. Since we are assuming that $l_\alpha \to 0$ and $|t_\alpha(\rho_n)|$ is bounded, Lemma 5.1 gives the approximation $l_\delta = d + O(1)$.

If $i(\alpha,\delta)=2$, let b,b' be the lengths of the common perpendicular arcs from α to itself in P and P'. In this case the approximating broken arc has five segments; three vertical segments which each wrap part-way round arcs which project to α, and two horizontal segments which are just the common perpendiculars from α to itself in P and P'. Thus in this case Lemma 5.1 gives the approximation $l_\delta = b + b' + O(1)$.

The proof is completed by using the trigonometric formulae in the proof of Lemma 5.4 to estimate d, b and b'. For (a), if l_ω is bounded above, then $d = 2\log(1/l_\alpha(\rho_n)) + O(1)$; while if $l_\omega \to \infty$, then $\cosh d \approx e^{l_\omega/2}/l_\alpha^2$, so the result still holds.

For case (b), note that P is made up of two right-angled hexagons with alternate sides of lengths $l_\alpha/2, l_{\omega_1}/2, l_{\omega_2}/2$ and that $b/2$ is the distance between a and its opposite side. Let d_1 be the length of the side between the sides of lengths $l_\alpha/2$ and $l_{\omega_1}/2$. We claim that $b = 2\log(1/l_\alpha) + l_{|\omega|} + O(1)$, from which, combined with a similar estimate for b', part (b) follows.

Since $l_\alpha \to 0$, we have $d_1 \to \infty$, and so $\cosh d_1 \approx \sinh d_1$. Thus

$$\cosh\frac{b}{2} \approx \frac{\cosh\frac{l_{\omega_2}}{2} + \cosh\frac{l_\alpha}{2}\cosh\frac{l_{\omega_1}}{2}}{\sinh\frac{l_\alpha}{2}\sinh\frac{l_{\omega_1}}{2}}\sinh\frac{l_{\omega_1}}{2} \approx \frac{\cosh\frac{l_{\omega_2}}{2} + \cosh\frac{l_{\omega_1}}{2}}{l_\alpha}.$$

Since $b \to \infty$, we have $\cosh b/2 \approx e^{b/2}$, so by Lemma 5.3,

$$b/2 = \log\left(\cosh(l_{\omega_2}/2) + \cosh(l_{\omega_1}/2)\right) + \log(1/l_\alpha) + O(1). \qquad (6)$$

Now, expressing $\cosh(l_{\omega_2}/2)+\cosh(l_{\omega_1}/2)$ as $\max\{\cosh(l_{\omega_2}/2), \cosh(l_{\omega_1}/2)\}+\min\{\cosh(l_{\omega_2}/2), \cosh(l_{\omega_1}/2)\}$, we easily obtain that

$$\log\left(\cosh(l_{\omega_2}/2) + \cosh(l_{\omega_1}/2)\right) = l_{|\omega|}/2 + O(1).$$

Applying this to (6) gives the claim. $\square$

Proposition 6.5 *Let $\alpha_1, \dots, \alpha_N, \gamma_1, \dots, \gamma_K$ be a pants decomposition of S and let δ_i be the dual curve to α_i. Let ρ_n be a sequence so that $l_{\alpha_i} \to 0$, $|t_{\alpha_i}|$ is bounded and $\rho_n \to [a'_1 a_1 + \cdots + a'_N \alpha_N]$. Then, for any j with $a'_j \neq 0$ we have*

$$l_{\delta_j}(\rho_n) = 2i(\alpha_j, \delta_j) \log(1/l_{\alpha_j}) + o(\log(1/l_{\alpha_j})).$$

Proof Suppose that α_j, δ_j intersect twice (the proof is similar if they intersect once). From Lemma 6.4 (b) we have

$$l_{\delta_j} = 2i(\alpha_j, \delta_j) \log(1/l_{\alpha_j}) + l_{|\omega|} + l_{|\omega'|} + O(1).$$

Since $\rho_n \to [a'_1 a_1 + \cdots + a'_N \alpha_N]$, we have $l_{\delta_j}/c_n \to a'_j i(\alpha_j, \delta_j)$, for some sequence $c_n \to \infty$. If ω_1 is one of the curves α_i, then $l_{\omega_1}(\rho_n) \to 0$. Otherwise, ω_1 is one of the curves γ_i, so it is disjoint from the curves α_i, in which case $l_{\omega_1}(\rho_n)/c_n \to i(\omega_1, a'_1\alpha_1 + \cdots + a'_N\alpha_N) = 0$. The same holds for $\omega_2, \omega'_1, \omega'_2$. Thus

$$\lim_n \frac{l_{\delta_j}(\rho_n)}{c_n} = \lim_n \frac{2i(\alpha_j, \delta_j)\log(1/l_{\alpha_j}(\rho_n))}{c_n} = a'_j i(\alpha_j, \delta_j),$$

which implies that $c_n \approx \log(1/l_{\alpha_j}(\rho_n))$, and therefore $\frac{l_{|\omega|}(\rho_n) + l_{|\omega'|}(\rho_n) + O(1)}{\log(1/l_{\alpha_j})} \to 0$. Thus $l_{|\omega|}(\rho_n) + l_{|\omega'|}(\rho_n) + O(1) = o(\log(1/l_{\alpha_j}))$, which completes the proof. □

We can now complete the proof of Theorem 1.1. First, continuing with the assumption that ν is rational, we follow the method used in Section 4 for the pants decomposition case. Proposition 6.2 shows that the limit of any convergent subsequence of minima m_s is a projective lamination $[a'_1\alpha_1 + \cdots + a'_N\alpha_N]$ for some $a'_i \geq 0$. Proposition 6.1 (c) implies that $\lim \frac{\log(1/l_{\alpha_i})}{\log(1/l_{\alpha_j})} = 1$, for all i, j and therefore, using Proposition 6.5, we argue as in Section 4 to get that $\lim_{s\to 0} \frac{l_{\delta_i}(m_s)}{l_{\delta_j}(m_s)} = i(\sum_k \alpha_k, \delta_i)/i(\sum_k \alpha_k, \delta_j)$, so that $a'_k = 1$ for all k. Thus the limit is independent of the subsequence, and the result follows by compactness of $\mathrm{Teich}(S) \cup \mathrm{P}\mathcal{ML}$. Finally, to complete the proof when ν is irrational, we follow the outline sketched in Section 5.4.

As a corollary, we obtain an estimate of the length of *any* closed geodesic along the line of minima $\mathcal{L}_{\mu,\nu}$. This should be compared with the almost identical estimate on p.190 in [9].

Corollary 6.6 *Let $\mu = a_1\alpha_1 + \cdots + a_N\alpha_N$ and ν be two measured laminations which fill up S (where $\alpha_1, \dots, \alpha_N$ is not necessarily a pants decomposition).*

Let m_s be the minimum of the function F_s and let γ be any simple closed curve. Then

$$l_\gamma(m_s) = 2\sum_{j=1}^{N} i(\alpha_j,\gamma)\log\frac{1}{l_{\alpha_j}(m_s)} + o(\log\frac{1}{s}).$$

Proof By Theorem 1.1, $m_s \to [\sum \alpha_j]$. This means that

$$\lim_{s\to 0}\frac{l_\gamma(m_s)}{c_s} = i(\gamma, \sum \alpha_j).$$

If γ is any closed geodesic, then $l_\gamma(m_s) = 2\sum i(\gamma,\alpha_j)\log\left(1/l_{\alpha_j}(m_s)\right) + f(s)$, where $f(s) > 0$. On the other hand, it is shown in the proof of Proposition 6.5 that

$$\lim \frac{2\log\left(1/l_{\alpha_j}(m_s)\right)}{c_s} = a'_j,$$

but we know that $a'_j = 1$. Therefore $\lim(f(s)/c_s) = 0$ and the result follows. □

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Deparmento Geometría y Topología, Fac. CC. Matemáticas
Universidad Complutense, 28040 Madrid, Spain
and
Mathematics Institute, University of Warwick
Coventry CV4 7AL, UK

Email: `radiaz@eucmos.sim.ucm.es` and `cms@maths.warwick.ac.uk`

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