ISSN 1472-2739 (on-line) 1472-2747 (printed) 103

Algebraic & Geometric Topology
Volume 3 (2003) 103–116
Published: 8 February 2003

K-theory of virtually poly-surface groups

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Abstract In this paper we generalize the notion of strongly poly-free group to a larger class of groups, we call them *strongly poly-surface* groups and prove that the Fibered Isomorphism Conjecture of Farrell and Jones corresponding to the stable topological pseudoisotopy functor is true for any virtually strongly poly-surface group. A consequence is that the Whitehead group of a torsion free subgroup of any virtually strongly poly-surface group vanishes.

AMS Classification 19B28, 19A31, 20F99, 19D35; 19J10

Keywords Strongly poly-free groups, poly-closed surface groups, Whitehead group, fibered isomorphism conjecture

1 Introduction

We generalize the class of strongly poly-free groups which was introduced in [1].

Definition 1.1 A discrete group Γ is called *strongly poly-surface* if there exists a finite filtration of Γ by subgroups: $1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$ such that the following conditions are satisfied:

(1) Γ_i is normal in Γ for each i.

(2) Γ_{i+1}/Γ_i is isomorphic to the fundamental group of a surface.

(3) for each $\gamma \in \Gamma$ and i there is a surface F such that $\pi_1(F)$ is isomorphic to Γ_{i+1}/Γ_i and either (a) $\pi_1(F)$ is finitely generated or (b) $\pi_1(F)$ is infinitely generated and F has one end. Also there is a diffeomorphism $f : F \to F$ such that the induced outer automorphism $f_\#$ of $\pi_1(F)$ is equal to c_γ in $Out(\pi_1(F))$, where c_γ is the outer automorphism of $\Gamma_{i+1}/\Gamma_i \simeq \pi_1(F)$ induced by the conjugation action on Γ by γ.

In such a situation we say that the group Γ has *rank* $\leq n$.

Note that in the definition of strongly poly-free group we demanded that the groups Γ_{i+1}/Γ_i be finitely generated free groups. On the other hand in the definition of strongly poly-surface group, Γ_{i+1}/Γ_i can be the fundamental group of any surface other than the surfaces with infinitely generated fundamental groups and with more than one topological ends. We even allow a class of surfaces with infinitely generated fundamental group. Also we remark that if the groups in (2) are fundamental groups of closed surfaces then the condition (3) is always satisfied. This follows from the well-known fact that any automorphism of the fundamental group of a closed surface is induced by a diffeomorphism of the surface. However this fact is very rarely true for surfaces with nonempty boundary ([10]). Thus the class of strongly poly-surface groups contains a class of poly-closed surface groups. Here recall that given a class of groups $\mathcal{G}$, a group Γ is called poly-$\mathcal{G}$ if Γ has a filtration by subgroups $1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$ such that Γ_i is normal in Γ_{i+1} and $\Gamma_{i+1}/\Gamma_i \in \mathcal{G}$ for each i. And a group is called virtually poly-$\mathcal{G}$ if it has a normal subgroup $G \in \mathcal{G}$ of finite index. For a group G, by 'poly-G' we will mean 'poly-$\mathcal{G}$', where $\mathcal{G}$ consists of G only.

In [1] we proved that the Whitehead group of any strongly poly-free group vanishes. Generalizing this result the Fibered Isomorphism Conjecture (FIC) corresponding to the stable topological pseudoisotopy functor ([4]) was proved for any virtually strongly poly-free group in [6]. In this paper we prove FIC for any virtually strongly poly-surface group. The Main Lemma in the next section is the crucial result which makes this generalization possible. The key idea to prove the Main Lemma is that, except for three closed surfaces, the covering space corresponding to the commutator subgroup of the fundamental group of all other closed surfaces have one topological end.

Below we recall the Fibered Isomorphism Conjecture in brief. For details about this conjecture see [4]. Here we follow the formulation given in [5, appendix].

Let $\mathcal{S}$ denote one of the three functors from the category of topological spaces to the category of spectra: (a) the stable topological pseudo-isotopy functor $\mathcal{P}()$; (b) the algebraic K-theory functor $\mathcal{K}()$; (c) and the L-theory functor $\mathcal{L}^{-\infty}()$.

Let $\mathcal{M}$ be the category of continuous surjective maps. The objects of $\mathcal{M}$ are continuous surjective maps $p : E \to B$ between topological spaces E and B. And a morphism between two maps $p : E_1 \to B_1$ and $q : E_2 \to B_2$ is a pair of continuous maps $f : E_1 \to E_2$, $g : B_1 \to B_2$ such that the following diagram commutes.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

There is a functor defined by Quinn [8] from $\mathcal{M}$ to the category of Ω-spectra which associates to the map p the spectrum $\mathbb{H}(B, \mathcal{S}(p))$ with the property that $\mathbb{H}(B, \mathcal{S}(p)) = \mathcal{S}(E)$ when B is a single point. For an explanation of $\mathbb{H}(B, \mathcal{S}(p))$ see [4, section 1.4]. Also the map $\mathbb{H}(B, \mathcal{S}(p)) \to \mathcal{S}(E)$ induced by the morphism: $\mathrm{id}\colon E \to E$; $B \to *$ in the category $\mathcal{M}$ is called the Quinn assembly map.

Let Γ be a discrete group and $\mathcal{E}$ be a Γ space which is universal for the class of all virtually cyclic subgroups of Γ and denote $\mathcal{E}/\Gamma$ by $\mathcal{B}$. For definition of universal space see [4, appendix]. Let X be a space on which Γ acts freely and properly discontinuously and $p : X \times_\Gamma \mathcal{E} \to \mathcal{E}/\Gamma = \mathcal{B}$ be the map induced by the projection onto the second factor of $X \times \mathcal{E}$.

The Fibered Isomorphism Conjecture states that the map

$$\mathbb{H}(\mathcal{B}, \mathcal{S}(p)) \to \mathcal{S}(X \times_\Gamma \mathcal{E}) = \mathcal{S}(X/\Gamma)$$

is an (weak) equivalence of spectra. The equality in the above display is induced by the map $X \times_\Gamma \mathcal{E} \to X/\Gamma$ and using the fact that $\mathcal{S}$ is homotopy invariant.

Let Y be a connected CW-complex and $\Gamma \simeq \pi_1(Y)$. Let X be the universal cover $\tilde{Y}$ of Y and the action of Γ on X is the action by group of covering transformation. If we take an aspherical CW-complex Y' with $\Gamma \simeq \pi_1(Y')$ and X is the universal cover $\tilde{Y}'$ of Y' then by [4, corollary 2.2.1] if the FIC is true for the space $\tilde{Y}'$ then it is true for $\tilde{Y}$ also. Thus whenever we say that FIC is true for a discrete group Γ or for the fundamental group $\pi_1(X)$ of a space X we shall mean it is true for the Eilenberg-MacLane space $K(\Gamma, 1)$ or $K(\pi_1(X), 1)$ and for the functor $\mathcal{S}()$.

Throughout this paper we consider only the stable topological pseudo-isotopy functor; that is the case when $\mathcal{S}() = \mathcal{P}()$. And by FIC we mean FIC for $\mathcal{P}()$.

The main theorem of this article is the following.

Main Theorem *Let Γ be a virtually strongly poly-surface group. Then the Fibered Isomorphism Conjecture is true for Γ.*

Recall that if FIC is true for a torsion free group G then $Wh(G) = \tilde{K}_0(\mathbb{Z}G) = K_{-i}(\mathbb{Z}G) = 0$ for all $i \geq 1$. A proof of this fact is given in several places, e.g., see [6] or [5].

Hence we have the following corollary.

Corollary 1.2 *Let G be a torsion free subgroup of a virtually strongly poly-surface group. Then $Wh(G) = \tilde{K}_0(\mathbb{Z}G) = K_{-i}(\mathbb{Z}G) = 0$ for all $i \geq 1$.*

2 Proof of the Main Theorem

The proof of the Main Theorem appears at the end of this section. Before that we state some known results about the Fibered Isomorphism Conjecture and prove the Main Lemma and some propositions. Apart from being crucial ingredients to the proof of the Main Theorem the Main Lemma and the propositions are also of independent interest.

Recall that the FIC is true for any finite group and for abelian groups ([5, lemma 2.7]).

Lemma A ([4, theorem A.8]) *If the FIC is true for a discrete group* Γ *then it is true for any subgroup of* Γ.

Before we state the next lemma let us recall the following group theoretic definition. Let G and H be two groups. Assume G is finite. Then $H \wr G$ denotes the wreath product with respect to the regular action of G on G. Recall that actually $H \wr G \simeq H^G \rtimes G$ where H^G is product of $|G|$ copies of H indexed by G and G acts on the product via the regular action of G on G. An easily checked fact is that if G_1 is another finite group then $H^{G_1} \wr G$ is a subgroup of $H \wr (G_1 \times G)$. This fact will be used throughout the paper. Another fact we will be using is that for any two groups A and B the group $(A \times B) \wr G$ is a subgroup of $(A \wr G) \times (B \wr G)$.

The Algebraic Lemma from [6] says the following.

Algebraic Lemma *If* G *is an extension of a group* H *by a finite group* K *then* G *is a subgroup of* $H \wr K$.

This lemma is also proved in [2, theorem 2.6A].

Lemma B [9] *Let* Γ *be an extension of the fundamental group* $\pi_1(M)$ *of a closed nonpositively curved Riemannian manifold or a compact surface (may be with nonempty boundary)* M *by a finite group* G *then FIC is true for* Γ. *Moreover FIC is true for the wreath product* $\Gamma \wr G$.

Proof Let us consider the closed case first. By the Algebraic Lemma we have an embedding of Γ in the wreath product $\pi_1(M) \wr G$. Let $U = M \times \cdots \times M$ be the $|G|$-fold product of M. Then U is a closed nonpositively curved Riemannian manifold. By [6, fact 3.1] it follows that FIC is true for $\pi_1(U) \rtimes G \simeq (\pi_1(M))^G \rtimes G \simeq \pi_1(M) \wr G$. Lemma A now proves that FIC is true for Γ.

If M is a compact surface with nonempty boundary then $\pi_1(M) < \pi_1(N)$ where N is a closed nonpositively curved surface. Hence $\Gamma < \pi_1(M) \wr G < \pi_1(N) \wr G$. Using Lemma A and the previous case we complete the proof. □

The above Lemma is also true if M is a compact irreducible 3-manifold with nonempty incompressible boundary and the boundary components are torus or Klein bottle. Indeed in this situation by theorem 3.2 and 3.3 from [7] the interior of M supports a complete nonpositively curved Riemannian metric so that near the boundary the metric is a product metric. Hence the double of M will support a nonpositively curved metric and we argue as in the case of compact surface to deduce the following Corollary.

Corollary B *Let $M_1, \cdots, M_k$ be compact irreducible 3-manifolds with incompressible boundary which has either torus or Klein bottle as components. Then FIC is true for $(\pi_1(M_1) \times \cdots \times \pi_1(M_n)) \wr G$ for any finite group G.*

Lemma C ([4, proposition 2.2]) *Let $f : G \to H$ be a surjective homomorphism. Assume that the FIC is true for H and for $f^{-1}(C)$ for all virtually cyclic subgroup C of H (including $C = 1$). Then FIC is true for G.*

We will use Lemma A, Lemma C and the Algebraic Lemma throughout the paper, sometimes even without referring to them.

We now recall a well-known fact from 2-dimensional real manifold theory.

Lemma D *Let Γ be a finitely generated nonabelian free group. Then Γ is isomorphic to the fundamental group of a compact surface (with nonempty boundary).*

Lemma E *Let Γ be the fundamental group of a surface then FIC is true for $\Gamma \wr G$ for any finite group G.*

Proof If Γ is finitely generated then Γ is the fundamental group of a compact surface and hence the lemma follows from Lemma B. In the infinitely generated case $\Gamma \wr G \simeq \lim_{i\to\infty}(\Gamma_i \wr G)$ where each Γ_i is a finitely generated nonabelian free group. By Lemma B, Lemma D and Theorem F (see below) the proof is complete. $\square$

We quote the following theorem of Farrell and Linnell which will be used throughout the paper.

Theorem F ([5, theorem 7.1]) *Let I be a directed set, and let Γ_n, $n \in I$ be a directed system of groups with $\Gamma = \lim_{n\in I}$; i.e., Γ is the direct limit of the groups Γ_n. If each group Γ_n satisfies FIC, then Γ also satisfies FIC.*

We will also use proposition 2.4 from [4] frequently, sometime without referring to it. This result says that FIC is true for any virtually poly-$\mathbb{Z}$ group.

We need the following crucial proposition to prove the Main Theorem.

Proposition 2.1 *Let S be a surface. If $\pi_1(S)$ is infinitely generated then assume S has one topological end. Let f be a diffeomorphism of S. Then the group $\pi_1(S)\rtimes\mathbb{Z}$ satisfies the FIC. Here, up to conjugation, the action of a generator of $\mathbb{Z}$ on the group $\pi_1(S)$ is induced by the diffeomorphism f.*

Proof There are two cases according as S is compact or not.

If S is compact with nonempty boundary then $\pi_1(S)\rtimes\mathbb{Z}$ is the fundamental group of a compact irreducible 3-manifold M with torus or Klein bottle as boundary component. If $\pi_1(S) = 1$ then there is nothing to prove, otherwise the boundary components of M will be incompressible. Hence Corollary B proves this case.

So assume that either S is closed or a noncompact surface. Note that if the fundamental group is finitely generated free then by Lemma D it falls in the previous case.

Let us consider the closed case first. This case is contained in the following Lemma which was proved in [9] in the case when the fiber is orientable. Here we give a proof for the general situation.

Main Lemma *Let M^3 be a closed 3-dimensional manifold which is the total space of a fiber bundle projection $M^3 \to \mathbb{S}^1$ with fiber F such that $b_1(F) \neq 1$. Then FIC is true for $\pi_1(M)$.*

Proof The following exact sequence is obtained from the long exact homotopy sequence of the fibration $M \to \mathbb{S}^1$.

$$1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\mathbb{S}^1) \to 1$$

Let $[A, A]$ denotes the commutator subgroup of the group A. Then we have

$$1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(F) \to H_1(F, \mathbb{Z}) \to 1$$

Let t be a generator of $\pi_1(\mathbb{S}^1)$. Since $[\pi_1(F), \pi_1(F)]$ is a characteristic subgroup of $\pi_1(F)$ the action (induced by the monodromy) of t on $\pi_1(F)$ leaves $[\pi_1(F), \pi_1(F)]$ invariant. Thus we have another exact sequence

$$1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(F) \rtimes \langle t \rangle \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1$$

Which reduces to the sequence

$$1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1$$

We would like to apply Lemma C to this exact sequence.

Now we have two cases according as the fiber is orientable or nonorientable. Let us first consider the orientable fiber case. If the fiber is $\mathbb{S}^2$ or $\mathbb{T}^2$ then $\pi_1(M)$ is poly-$\mathbb{Z}$ and hence FIC is true for $\pi_1(M)$. So assume that the fiber has genus ≥ 2.

Clearly the group $H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$ is poly-$\mathbb{Z}$. Hence FIC is true for $H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$. Let C be a virtually cyclic subgroup of $H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$. Let $p : \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$ be the above surjective homomorphism. We will show that the FIC is true for $p^{-1}(C)$. Note that C is either trivial or infinite cyclic.

Case $C = 1$ In this case we have that $p^{-1}(C)$ is a nonabelian free group and hence is the fundamental group of a surface. Lemma E proves this case.

Case $C \neq 1$ We have $p^{-1}(C) \simeq [\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle$ where s is a generator of C. Let $\tilde{F}$ be the covering space of F corresponding to the commutator subgroup $[\pi_1(F), \pi_1(F)]$. As F has first Betti number ≥ 2 the group $H_1(F, \mathbb{Z})$ has only one end. Also $H_1(F, \mathbb{Z})$ is the group of covering transformations of the regular covering $\tilde{F} \to F$. Since F is compact, the manifold $\tilde{F}$ has one topological end (see [3]). Figure 1 describes $\tilde{F}$.

We write the manifold $\tilde{F}$ as the union of compact submanifolds. As $\tilde{F}$ has one end there is a connected compact submanifold M_0 of $\tilde{F}$ so that the complement $\tilde{F} - M_0$ has one connected component and for any other connected compact submanifold M containing M_0 the complement $\tilde{F} - M$ also has one component. Consider a sequence M_i of compact submanifolds of $\tilde{F}$ with the following properties.

(1) each M_i has one boundary component

(2) each M_i has the same property as M_0

(3) $\tilde{F} = \cup_i M_i$ and

(4) $M_0 \subset M_1 \subset \cdots$.

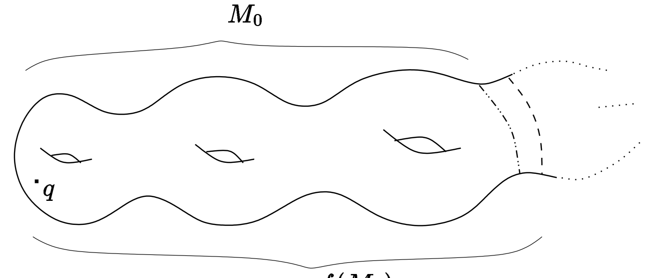

Figure 1

Note that the monodromy diffeomorphism of F lifts to a quadiffeomorphism of $\tilde{F}$ which in turn, up to conjugation, induces the action of t on $[\pi_1(F), \pi_1(F)]$ and also $H_1(F, \mathbb{Z})$ is the group of covering transformation of $\tilde{F} \to F$. Also, the induced action of t on $H_1(F, \mathbb{Z})$ is given by $t(s) = \tilde{f} \circ s \circ \tilde{f}^{-1}$, where $\tilde{f} : \tilde{F} \to \tilde{F}$ is a lift of the monodromy diffeomorphism and $s \in H_1(F, \mathbb{Z})$ acts on $\tilde{F}$ as a covering transformation. From this observation it follows that, up to conjugation, the action of s on $[\pi_1(F), \pi_1(F)]$ is induced by a diffeomorphism (say f) of $\tilde{F}$. Indeed, if $s = (s_1, t^k) \in H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$ then $f = s_1 \circ \tilde{f}^k : \tilde{F} \to \tilde{F}$.

Note that f is also a lift of a diffeomorphism (say f_1) of F. If f_1 is isotopic to a pseudo-Anosov diffeomorphism then by Thurston's theorem $p^{-1}(C)$ is a subgroup of the fundamental group of a closed hyperbolic 3-manifold, namely the mapping torus of f_1. Hence FIC is true for $p^{-1}(C)$. So we can assume that f_1 is isotopic to either a finite order diffeomorphism or to a reducible one. Hence there exists in $\tilde{F}$ simple closed curves so that cutting along them produces a filtration of $\tilde{F}$ with properties (1) to (4).

Using properties (1) to (4) and that f_1 is either of finite order or reducible, it is now easy to see that each $f(M_i)$ is obtained from M_i by attaching an annulus to the boundary component of M_i. So, we can isotope f so that $f(M_i) = M_i$ for each i. Thus we have a filtration $\pi_1(M_0, q) < \pi_1(M_1, q) < \cdots$ of $[\pi_1(F), \pi_1(F)]$ by finitely generated free subgroups so that the action of s on $[\pi_1(F), \pi_1(F)]$ respects this filtration and each $\pi_1(M_i) \rtimes \langle s \rangle$ is the fundamental group of a Haken 3-manifold N_i^s with nonempty incompressible boundary. Indeed, N_i^s is diffeomorphic to the mapping torus of the restriction of f to M_i.

Hence FIC is true for $\pi_1(N_i^s)$ by Corollary B. From above we also get that $[\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle \simeq \lim_{i\to\infty} \pi_1(N_i^s)$. Using Theorem F we conclude that FIC is true for $[\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle$.

This completes the proof of the Main Lemma in the orientable fiber case.

From the above proof we get the following Lemma which is true for nonorientable F also.

Lemma 2.2 *Let F be a closed surface of genus ≥ 2 and $\tilde{F}$ be the covering of F corresponding to the commutator subgroup of $\pi_1(F)$. Let f be a diffeomorphism of F. Then $[\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle \simeq \lim_{i\to\infty} \pi_1(N_i^s)$ where N_i^s are compact Haken 3-manifolds with nonempty incompressible boundary and up to conjugation the action of s on $[\pi_1(F), \pi_1(F)]$ is induced by the lift of f to $\tilde{F}$. Moreover each $\pi_1(N_i^s)$ is a subgroup of the fundamental group of a closed nonpositively curved Riemannian manifold M_i^s.*

Next we deal with the case when the fiber is nonorientable. In this situation $H_1(F, \mathbb{Z})$ has torsion element. Nevertheless $H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$ is a virtually poly-$\mathbb{Z}$ group and hence FIC is true for this group. Thus we can apply Lemma C to the exact sequence.

$$1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1$$

If F is the projective plane then $\pi_1(M)$ is virtually infinite cyclic and FIC is true for this group. Since the fiber is not the Klein bottle we assume that genus of F is ≥ 2.

Again we have two cases.

C **is finite** We have

$$p^{-1}(C) < ([\pi_1(F), \pi_1(F)]) \wr C$$

Hence FIC is true for $p^{-1}(C)$ by Lemma E.

C **is infinite** Let C_1 be an infinite cyclic subgroup of C of finite index. As C_1 is of finite index we can assume that C_1 is normal in C. We have the following exact sequences.

$$1 \to p^{-1}(C_1) \to p^{-1}(C) \to G \to 1$$

and

$$1 \to [\pi_1(F), \pi_1(F)] \to p^{-1}(C_1) \to C_1 \to 1$$

Here G is a finite group. Let C_1 be generated by s. Then we get

$$p^{-1}(C) < ([\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle) \wr G$$

As in the orientable case, up to conjugation, the action of s on $[\pi_1(F), \pi_1(F)]$ is induced by a diffeomorphism of $\tilde{F}$. Also recall that genus of F is ≥ 2. Hence Lemma 2.2 is applicable. Thus we get

$$([\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle) \wr G \backsimeq \lim_{i \to \infty} (\pi_1(N_i^s) \wr G)$$

Lemma B together with Theorem F complete the proof in this case. □

To complete the proof of Proposition 2.1 we need to consider the case when $\pi_1(S)$ is infinitely generated and S has one end. We use Lemma 2.2 to deduce that $\pi_1(S) \rtimes \mathbb{Z} \backsimeq \pi_1(S) \rtimes \langle t \rangle \backsimeq \lim_{i \to \infty} \pi_1(N_i^t)$. Now apply Corollary B and Theorem F to complete the proof of the proposition. □

The proposition below is an application of the method of the proof of the Main Lemma.

Proposition 2.3 *Let M be as in the Main Lemma. Then FIC is true for $\pi_1(M) \wr G$ for any finite group G.*

Proof Recall the following exact sequence.

$$1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1$$

If F is the 2-sphere or the projective plane then $\pi_1(M) \wr G$ is virtually abelian and hence FIC is true by [5, lemma 2.7]. So assume F is not the 2-sphere or the Klein bottle or the projective plane.

Taking wreath product with G the above exact sequence gives the following.

$$1 \to ([\pi_1(F), \pi_1(F)])^G \to \pi_1(M) \wr G \to (H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G \to 1$$

Recall that $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$ is virtually poly-$\mathbb{Z}$ and hence FIC is true for $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$. Applying Lemma C twice and noting that FIC is true for free abelian groups, it is easy to show that if the FIC is true for two torsion free group then it is true for the product of the two groups also. Thus by Theorem F and Lemma E it follows that FIC is true for $([\pi_1(F), \pi_1(F)])^G$.

Let Z be a virtually cyclic subgroup of $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$. If Z is finite then

$$p^{-1}(Z) < ([\pi_1(F), \pi_1(F)])^G \wr Z < ([\pi_1(F), \pi_1(F)]) \wr (G \times Z)$$

Here p is the surjective homomorphism $\pi_1(M) \wr G \to (H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$.

Now Lemma E applies on the right hand side group to show that FIC is true for $p^{-1}(Z)$. If Z is infinite then let Z_1 be the intersection of Z with the torsion free part of $(H_1(F,\mathbb{Z})\rtimes\langle t\rangle)^G$. Hence $Z_1 \backsimeq \langle u\rangle$ is infinite cyclic normal subgroup of Z of finite index. Once again we appeal to the Algebraic Lemma to get

$$p^{-1}(Z) < (p^{-1}(Z_1)) \wr Z/Z_1 \backsimeq (([\pi_1(F),\pi_1(F)])^G \rtimes\langle u\rangle) \wr Z/Z_1$$

$$\backsimeq (([\pi_1(F),\pi_1(F)] \times [\pi_1(F),\pi_1(F)] \cdots \times [\pi_1(F),\pi_1(F)])\rtimes\langle u\rangle) \wr Z/Z_1 = H(\text{say})$$

In the above display there are $|G|$ number of factors of $[\pi_1(F),\pi_1(F)]$. Note that the action of u on $([\pi_1(F),\pi_1(F)])^G$ is factorwise. Let us denote the restriction of the action of u on the j-th factor of $([\pi_1(F),\pi_1(F)])^G$ by u_j. By Lemma 2.2 we get

$$H < (\lim_{i\to\infty}(\pi_1(M_i^{u_1}) \times \pi_1(M_i^{u_2}) \times \cdots \times \pi_1(M_i^{u_{|G|}}))\wr Z/Z_1 \backsimeq \lim_{i\to\infty}(\pi_1(M_i)\wr Z/Z_1)$$

where $M_i^{u_j}$ and hence $M_i = M_i^{u_1} \times M_i^{u_2} \times \cdots \times M_i^{u_{|G|}}$ are closed nonpositively curved Riemannian manifolds. Using Lemma B and Theorem F we complete the proof of the Proposition. □

The following corollary is a consequence of Proposition 2.3.

Corollary 2.4 *Let M_i for $i = 1,\cdots,k$ be 3-manifolds with the same property as M in the Main Lemma. Then FIC is true for $(\pi_1(M_1) \times \cdots \times \pi_1(M_k)) \wr G$ for any finite group G.*

Proof For the proof of the Corollary just note that if A and B be two groups and G is another group acting regularly on itself then $(A\times B)\wr G$ is a subgroup of $(A\wr G) \times (B \wr G)$. Now apply Lemma A and Proposition 2.3. □

Proof of Main Theorem Let Δ be a nontrivial group with Γ a strongly poly-surface normal subgroup of Δ of finite index and $G = \Delta/\Gamma$. We will prove the theorem by induction on the rank of Γ. Note that Δ is a subgroup of $\Gamma \wr G$. Hence it is enough to check that FIC is true for $\Gamma\wr G$.

Induction hypothesis $I(n)$ For any strongly poly-surface group Γ of rank $\leq n$ and for any finite group G, FIC is true for the wreath product $\Gamma \wr G$.

If the rank of Γ is ≤ 0 then $\Gamma\wr G = G$ finite and hence $I(0)$ holds.

Now assume $I(n-1)$. We will show that $I(n)$ holds.

Let Γ be a strongly poly-surface group of rank $\leq n$ and is a normal subgroup of Δ with G as the finite quotient group. So we have a filtration by subgroups

$$1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$$

with all the requirements as in the definition of strongly poly-surface group and there is the exact sequence

$$1 \to \Gamma \to \Delta \to G \to 1$$

We have another exact sequence which is obtained after taking wreath product of the exact sequence $1 \to \Gamma_1 \to \Gamma \to \Gamma/\Gamma_1 \to 1$ with G.

$$1 \to \Gamma_1^G \to \Gamma \wr G \to (\Gamma/\Gamma_1) \wr G \to 1$$

Let p be the surjective homomorphism $\Gamma \wr G \to (\Gamma/\Gamma_1) \wr G$. Note that Γ/Γ_1 is a strongly poly-surface group of rank less or equal to $n-1$.

By induction hypothesis FIC is true for $(\Gamma/\Gamma_1) \wr G$. We would like to apply Lemma C. Let Z be a virtually cyclic subgroup of $(\Gamma/\Gamma_1) \wr G$. Then there are two cases to consider.

Z is finite In this case we have $p^{-1}(Z) < \Gamma_1^G \wr Z < \Gamma_1 \wr (G \times Z)$. Since Γ_1 is a surface group, Lemma E completes the proof in this case.

Z is infinite Let $Z_1 = Z \cap (\Gamma/\Gamma_1)^G$. Then Z_1 is an infinite cyclic normal subgroup of Z of finite index. Let Z_1 be generated by u. We get $p^{-1}(Z) < p^{-1}(Z_1) \wr K$ where K is isomorphic to Z/Z_1.

Also

$$p^{-1}(Z_1) \wr K \simeq (\Gamma_1^G \rtimes \langle u \rangle) \wr K < (\prod_{g \in G} (\Gamma_1 \rtimes_{\alpha_g} \langle u \rangle)) \wr K \qquad \mathbf{2.1}$$

Now we describe the notations in the display (2.1). Let $t \in \Gamma^G$ which goes to u. Then $\alpha_g(\gamma) = t_g \gamma t_g^{-1}$ for all $\gamma \in \Gamma_1$ and t_g is the value of t at g. By definition of strongly poly-surface group each of these actions is induced by a diffeomorphism of a surface S whose fundamental group is isomorphic to Γ_1.

Now there are two cases: (a) Γ_1 is finitely generated and (b) Γ_1 is infinitely generated.

(a) Recall that if the fundamental group of a noncompact surface is finitely generated then the surface is diffeomorphic to the interior of a compact surface with boundary. Thus in this case the right hand side of the display (2.1) is isomorphic to $(\prod_{g \in G} \pi_1(N^g)) \wr K$ where for each g, N^g is a compact 3-manifold fibering over the circle. If S is compact with nonempty boundary or is the interior of a compact surface with nonempty boundary then $\partial N^g \neq \emptyset$ for all g. In this situation use Corollary B to complete the proof of the theorem. If S is closed then so is N^g for each g and hence Corollary 2.4 completes the proof if S is not the Klein Bottle. If S is the Klein bottle then the proof follows from the following lemma and by noting that $\pi_1(S)$ has a finite index rank 2 free abelian subgroup.

Lemma 2.5 *Let $G_1, G_2, \cdots, G_n$ be finitely presented groups so that each G_i contains a finitely generated free abelian subgroup of finite index. For each i let f_i be an automorphism of G_i. Let G be a finite group. Then FIC is true for the group $((G_1 \rtimes_{f_1} \langle t\rangle) \times (G_2 \rtimes_{f_2} \langle t\rangle) \times \cdots \times (G_n \rtimes_{f_n} \langle t\rangle)) \wr G$.*

Proof Recall that for groups A, B and G, $(A \times B) \wr G$ is a subgroup of $(A \wr G) \times (B \wr G)$. Also if FIC is true for two groups then applying Lemma C twice and noting that FIC is true for virtually poly-$\mathbb{Z}$ groups it follows that FIC is also true for the product of the two groups. Thus it is enough to prove the Lemma for $n = 1$.

Note that by taking intersection of all conjugates of the free abelian subgroup of G_i we get a finitely generated free abelian normal subgroup of G_i with a finite quotient group, say K_i. Now since K_i is a finite group and G_i is finitely presented, there are only finitely many homomorphism from G_i onto K_i. Let H_i be the intersection of the kernels of these finitely many homomorphism. Then H_i is a finitely generated free abelian characteristic subgroup of G_i of finite index. Hence we have an exact sequence.

$$1 \to H_i \to G_i \rtimes_{f_i} \langle t\rangle \to L_i \rtimes_{f_i} \langle t\rangle \to 1$$

where $L_i \simeq G_i/H_i$. Taking wreath product with G the above exact sequence reduces to the following.

$$1 \to H_i^G \to (G_i \rtimes_{f_i} \langle t\rangle) \wr G \to (L_i \rtimes_{f_i} \langle t\rangle) \wr G \to 1$$

Note that $(L_i \rtimes_{f_i} \langle t\rangle) \wr G$ is virtually poly-$\mathbb{Z}$ and H_i^G is free abelian and hence FIC is true for these two groups. Let C be a virtually cyclic subgroup of $(L_i \rtimes_{f_i} \langle t\rangle) \wr G$ then $p^{-1}(C)$ is easily shown to be virtually poly-$\mathbb{Z}$ and hence FIC is true for $p^{-1}(C)$. Here p denotes the last surjective homomorphism in the above exact sequence. This completes the proof of the Lemma. □

(b) As Γ_1 is infinitely generated and free, by the definition of strongly poly-surface group, S has one end. Replacing $\tilde{F}$ by S in Lemma 2.2 we get

$$\left(\prod_{g \in G} (\Gamma_1 \rtimes_{\alpha_g} \langle u\rangle)\right) \wr K < \lim_{i \to \infty} \left(\left(\prod_{g \in G} \pi_1(N_i^g)\right) \wr K\right)$$

Now using Corollary B and Theorem F we complete the proof of the theorem.
□

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Received: 25 April 2002 Revised: 15 January 2003