SHORT COMMUNICATIONS

ABOUT ONE TWO-DIMENSIONAL SPECIAL LINEAR INTEGRAL EQUATION OF THE THIRD KIND

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Abstract. In the class of Hölder functions a special type of two-dimensional linear integral equations with a coefficient having zero inside an interval of its definition is studied. Using the theory of complex analysis, the necessary and sufficient conditions for solvability of these equations are given.

Key words and phrases: integral equations of the third kind, eigenfunctions, singular integral operator, homogeneous equation.

MSC 2000: 45B05, 45E05.

In present article we consider the linear integral equations of type

$$x\varphi(x,y)+\int_{-1}^{+1}\int_{a}^{b}k(y,y')\varphi(x',y')dx'dy'=f(x,y), \tag{1}$$

$$x\in(-1,+1),\quad y\in[a,b],$$

where $k(y,y')$ is a continuous symmetric function on $[a,b]\times[a,b]$, $f(x,y)$ is a continuous function on $(-1,+1)\times[a,b]$ and satisfies the $\mathbf{H}^*$ condition [1] with respect to x . Such equations are often called equations of the third kind. Methods of the complex analysis are the fundamental methods of our investigation.

We introduce the following singular integral operator

$$(\mathbf{L}u(\star,\cdot))(x,y):=u(x,y)+\int_{-1}^{+1}\int_{a}^{b}\frac{k(y,s)}{t-x}u(x,s)dsdt$$

$$+\int_{-1}^{+1}\int_{a}^{b}\frac{k(y,s)}{t-x}u(t,s)dsdt,\qquad x\in(-1,+1),\quad y\in[a,b].$$

Theorem 1. *The equality*

$$x(\mathbf{L}u)(x,y)+\int_{-1}^{+1}\int_{a}^{b}k(y,y')(\mathbf{L}u)(x',y')dy'dx'=(\mathbf{L}(\star u))(x,y)$$

is correct.

Let $\{m_i(y)\}$ be a set of eigenfunctions corresponding to eigenvalues $\{\lambda_i\}$ of the homogeneous equation

$$m(y) + \lambda \int_a^b k(y, y')m(y')dy' = 0, \quad y \in [a, b]$$

and assume that

$$\phi_i(x, y) = \frac{m_i(y)}{x - \nu_i},$$

where $\ln \frac{\nu_i+1}{\nu_i-1} = \lambda_i$. *Note that* $\nu_i \notin [-1, +1]$.

Theorem 2. *The equalities*

$$x\phi_i(x, y) + \int_{-1}^{+1} \int_a^b k(y, y')\phi_i(x', y')dy'dx' = \nu_i\phi_i(x, y), \quad i = 1, 2, \ldots$$

hold.

Theorem 3. *The equalities*

$$\int_{-1}^{+1} \int_a^b \phi_i(x, y)\phi_j(x, y)dxdy = N_i\delta_{ij}$$

hold.

Let $f_0(x, y)$ be a continuous function on $(-1, +1) \times [a, b]$ and satisfies the $\mathbf{H}^*$ condition with respect to x. By using the methods of complex analysis we can prove

Theorem 4. *The equation*

$$\mathbf{L}(u) = f_0 \tag{2}$$

is solvable if and only if f_0 *satisfies the conditions*

$$\int_{-1}^{+1} \int_a^b f_0\phi_i dydx = 0, \quad i = 1, 2, \ldots.$$

Provided these conditions are satisfied, the equation (2) has one and only one continuous solution $u \in \mathbf{H}^*$ *on* $(-1, +1)$.

Now we introduce the following integral operator

$$(\mathbf{S}u(\star, \cdot))(x, y) := u(x, y) + \int_{-1}^{+1} \int_a^b \frac{k(y, s)}{t - x} u(x, s)dsdt$$

$$+ \int_{-1}^{+1} \int_a^b \frac{k(y, s)}{x - t} u(t, s)dsdt \qquad x \in (-1, +1),\ y \in [a, b].$$

The following property of the introduced operators will be noted. From the preceding considerations it follows that for any two continuous functions $u(x, y)$ and $v(x, y)$, satisfying the $\mathbf{H}^*$ condition with respect to x,

$$\int_{-1}^{+1} \int_a^b u\mathbf{S}vdxdy = \int_{-1}^{+1} \int_a^b v\mathbf{L}udxdy.$$

Consequently, if the equation (2) has a solution, then necessary

$$\int_{-1}^{+1}\int_{a}^{b} v\phi_i dxdy = 0, \quad i = 1,,2,\ldots,$$

where v is any solution of the homogeneous equation

$$\mathbf{S}v = 0.$$

The converse statement is also true. Now it is not difficult to prove the following two theorems

Theorem 5. *The equalities*

$$\mathbf{S}\phi_i = 0, \qquad i = 1,2,\ldots$$

hold.

Theorem 6. *The composition* **SL** *contains no singular part and the equality*

$$(\mathbf{SL}u)(x,y) = u(x,y) + \int_a^b g(x,y,s)u(x,s)ds,$$

where

$$g(x,y,s) = 2\int_{-1}^{+1}\frac{dt}{t-x}k(y,s) + (\pi^2 + (\int_{-1}^{+1}\frac{dt}{t-x})^2)\int_a^b k(y,y')k(y',s)dy'$$

holds.

Taking into account the structure the kernel g, we can prove

Theorem 7. *For every $x \in (-1,+1)$ the homogeneous equation*

$$u_0(x,y) + \int_a^b g(x,y,s)u_0(x,s)ds = 0, \qquad y \in [a,b],$$

admits only trivial solution.

Let $r(x,y,s)$ be the resolvent kernel associated with $g(x,y,s)$ and

$$(\mathbf{T}u)(x,y) := (\mathbf{S}u)(x,y) + \int_a^b r(x,y,s)(\mathbf{S}u)(x,s)ds.$$

After comparison of the results, obtained in the preceding theorems, we conclude

Theorem 8. *Equation (1) is solvable if and only if the function f satisfies the condition*

$$f(0,y) + \int_{-1}^{+1}\int_a^b \frac{f(0,s)-f(x,s)}{x}k(s,y)dsdx = 0, \quad y \in [a,b].$$

Provided these conditions are satisfied, equation (1) has one and only one continuous solution $\varphi(x,y)$ satisfying the condition $\mathbf{H}^*$ with respect to $x \in (-1,+1)$ and this solution may be written as follows

$$\varphi(x,y) = \sum_i \frac{\phi_i(x,y)}{\nu_i N_i} \int_{-1}^{+1} \int_a^b f(x',y')\phi_i(x',y')dy'dx'$$

$$+(\mathbf{L}\frac{1}{\star}(\mathbf{T}f)(\star,\cdot))(x,y).$$

References

[1] Muskhelishvili, N., Singular Integral Equations. Groningen: P.Noordhooff, 1953.

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