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SHORT COMMUNICATIONS

Insert Roots to Extract the Roots of a Quartic Polynomial

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In this paper, we propose a new method of determining the roots of a quartic polynomial. In this method two unknown roots are inserted to the quartic polynomial, which results in a sextic polynomial; this polynomial is then transformed into an even powered sextic polynomial in a novel fashion. The six roots of the even powered sextic polynomial are determined by employing the methods for solving cubic equations. Consequently, the two inserted roots along with the four roots of the given quartic polynomial are determined. A numerical example is solved using the proposed method.

Keywords and phrases: Quartic polynomial, resolvent cubic equation, roots, sextic polynomial.

AMS subject classification: 12E12.

1 Introduction

The subject of finding the roots of polynomials has been a matter of great interest. The quartic equation was first solved by the Italian mathematician, Ferrari in 1545. Thereafter, several methods of solving quartic equations have appeared in the literature like Descartes' method, Tschirnhaus' method, Euler's method, and so on. A brief account of these methods is given below; for more details one may see the literature [1, 2, 3, 4].

In the method proposed by Ferrari, the quartic equation is rearranged on either side of the equal sign along with an expression containing some unknown number such that the both sides become perfect squares, provided that unknown number satisfies certain cubic equation, termed as the resolvent cubic equation. Descartes (1637) employed a method, in which the depressed quartic polynomial is equated to the product of two quadratic polynomial factors with unknown coefficients, resulting in four equations in four unknowns. Elimination of three unknowns from these equations yields a resolvent cubic equation, which is in fact is an even powered sextic equation in the fourth unknown.

Tschirnhaus (1683) introduced a transformation (bearing his name) for solving the polynomial equations. To solve a quartic equation with this method, one uses a quadratic Tschirnhaus transformation, $x^2 + Ax + B = y$, with two unknown numbers, A and B, which transforms the quartic equation in x to a quartic equation in y; the coefficients of y^3 and y in the new quartic

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equation are equated to zero, yielding a resolvent cubic equation in one of the unknowns. Euler (1733) solved the quartic equation in an innovative way by proposing that the solution is equal to the sum of the square roots of three numbers, and these three numbers happen to be the roots of the resolvent cubic equation.

Lagrange (1771) was looking for an universal method for solving polynomial equations of any degree. He noticed that by a proper choice of intermediate parameters, which are simple functions (polynomials) of the solutions, one can obtain the resolvent equations of lower degree [3]. For the quartic equation, he proposed three intermediate parameters as: $z_1 = (x_1x_2 + x_3x_4)/2$, $z_2 = (x_1x_3 + x_2x_4)/2$, and $z_3 = (x_1x_4 + x_2x_3)/2$, where x_1, x_2, x_3, and x_4 are the solutions of the depressed quartic equation, $x^4 + ax^2 + bx + c = 0$. While Lagrange's method successfully solved the cubic and the quartic equations, it resulted in a sextic resolvent equation for the quintic equation; and as a result quintic equation could not be solved. It was then left to Abel (1826) and later Galois (1832) to show that quintic and the higher degree equations can not be solved using the radicals alone, which were sufficient for solving the lower degree equations [3].

This paper proposes a new method of determining the roots of a quartic polynomial, wherein two unknown roots are inserted to the quartic polynomial, resulting in a sextic polynomial, which is first transformed to a sextic polynomial without fifth degree term, and then to an even powered sextic polynomial, by determining the two inserted (unknown) roots. In this process (of determining the unknowns), we encounter an even powered sextic equation (effectively a cubic equation) in one of the unknowns, which is known as the resolvent cubic equation. Solving the resolvent cubic equation paves the way for determining the six roots of the sextic polynomial, from which the four roots of the given quartic polynomial are extracted; the details are described in the next section.

2 The proposed method

Since we know that power-three term in a general quartic polynomial, $u^4+a_3u^3+a_2u^2+a_1u+a_0$, can be removed using a simple linear transformation, $u = x-(a_3/4)$, let us consider the following depressed quartic polynomial without loss of any generality,

$$A(x) = x^4 + ax^2 + bx + c, \tag{1}$$

where a, b, and c are real numbers and $bc \neq 0$. It is required to find the roots of $A(x) = 0$ using the method proposed here; for this purpose, we insert two (unknown) roots, $-d$ and $-f$, to the quartic polynomial, $A(x)$, which is equivalent to multiplying $A(x)$ with the quadratic polynomial $(x + d)(x + f)$. This yields a sextic polynomial, $B(x)$, as shown below.

$$\begin{aligned} B(x) = x^6 + (d + f)x^5 + (a + df)x^4 + [b + a(d + f)]x^3 \\ +[c + b(d + f) + adf]x^2 + [c(d + f) + bdf]x + cdf \end{aligned} \tag{2}$$

Notice that (2) contains only the sum and product terms of d and f, i.e., $d + f$ and df; so in order to simplify the algebraic manipulations we let,

$$g = d + f \;\text{ and }\; h = df, \tag{3}$$

and use them in (2), which yields,

$$B(x) = x^6 + gx^5 + (a + h)x^4 + [b + ag]x^3 + [c + bg + ah]x^2 + (cg + bh)x + ch. \tag{4}$$

The sextic polynomial $B(x)$ contains two new unknown numbers, g and h. In order to solve $B(x) = 0$ algebraically, it is essential that $B(x)$ be an even powered sextic polynomial, which implies the coefficients of x, x^3, and x^5 in (4) must be zero, and it means three conditions are to be met, while there are only two unknowns. This results in $g = 0$, and then $b = 0$, which is not a valid result, since $b \neq 0$.

So our next plan is to transform $B(x)$ to a depressed sextic polynomial (in which the fifth degree term is absent), so that only the third degree and the first degree terms need to be eliminated by determining the proper values of g and h. For this purpose, consider the term, $x^6 + gx^5$, appearing in (4), which can be written as,

$$\begin{aligned} x^6 + gx^5 = [x + (g/6)]^6 - (5g^2/12)x^4 - (5g^3/54)x^3 \\ -(5g^4/432)x^2 - (g^5/1296)x - (g^6/46656), \end{aligned} \tag{5}$$

and we make use of it in (4) resulting in a new expression for $B(x)$ as,

$$\begin{aligned} B(x) = [x + (g/6)]^6 + [a + h - (5g^2/12)]x^4 + [b + ag - (5g^3/54)]x^3 \\ +(c + bg + ah - (5g^4/432)]x^2 + [cg + bh - (g^5/1296)]x + ch - (g^6/46656). \end{aligned} \tag{6}$$

Let us define a new variable y as,

$$y = x + (g/6), \tag{7}$$

and use it in (6), which yields a sextic polynomial in y, say $C(y)$,

$$\begin{aligned} C(y) = y^6 + [a + h - (5g^2/12)][y - (g/6)]^4 \\ +[b + ag - (5g^3/54)][y - (g/6)]^3 + [c + bg + ah - (5g^4/432)][y - (g/6)]^2 \\ +[cg + bh - (g^5/1296)][y - (g/6)] + ch - (g^6/46656). \end{aligned} \tag{8}$$

Expanding (8) and rearranging it in descending powers of y yields,

$$C(y) = y^6 + jy^4 + ky^3 + my^2 + ny + p, \tag{9}$$

where the coefficients j, k, m, n, and p are given by:

$$j = a + h - (5g^2/12), \tag{10}$$

$$k = b + ag - (5g^3/54) - (2g/3)[a + h - (5g^2/12)], \tag{11}$$

$$\begin{aligned} m = c + bg + ah - (5g^4/432) - (g/2)[b + ag - (5g^3/54)] \\ +(g^2/6)[a + h - (5g^2/12)], \end{aligned} \tag{12}$$

$$\begin{aligned} n = cg + bh - (g^5/1296) - (g/3)[c + bg + ah - (5g^4/432)] \\ +(g^2/12)[b + ag - (5g^3/54)] - (g^3/54)[a + h - (5g^2/12)], \end{aligned} \tag{13}$$

$$\begin{aligned} p = ch - (g^6/46656) - (g/6)[cg + bh - (g^5/1296)] \\ +(g^2/36)[c + bg + ah - (5g^4/432)] - (g^3/216)[b + ag - (5g^3/54)] \\ +(g^4/1296)[a + h - (5g^2/12)]. \end{aligned} \tag{14}$$

Observe that $C(y)$ is a depressed sextic polynomial in y [see (9)], and determination of the roots of $C(y) = 0$ by algebraic methods is possible only if $C(y)$ is transformed to an even powered sextic polynomial. This can be achieved by equating the coefficients of y^3 and y (i.e., k and n) in (9) to zero, which results in the following two equations in the two unknowns, g and h [see (11) and (13)]:

$$5g^3 - 18gh + 9ag + 27b = 0, \tag{15}$$

$$g^5 - 6g^3h + 21ag^3 - 81bg^2 - 108agh + 216cg + 324bh = 0. \tag{16}$$

As a result, $C(y)$ becomes a cubic polynomial in y^2 as shown below,

$$C(y) = y^6 + jy^4 + my^2 + p, \tag{17}$$

and its roots can be determined only if j, m, and p, which are functions of g and h, become known quantities. To accomplish this, let us consider (15) and (16), which contain g and h. Elimination of h from (15) and (16) and simplifying results in the following even powered sextic equation in g,

$$g^6 + 18ag^4 + 81(a^2 - 4c)g^2 - 729b^2 = 0. \tag{18}$$

Notice that (18) is a cubic equation in g^2, which we will call the resolvent cubic equation. Solving the cubic equation (18) by any of the well known methods [1, 3, 5], determines g^2 and subsequently g. Using the value of g in (15), we determine h. So, all the coefficients, j, m, and p, mentioned in (17) are now determined using (10), (12), and (14). Further, by solving $C(y) = 0$ [see (17)], the three roots of y^2 are determined, and thereafter we determine the six roots of y.

Next, using the relation (7), which connects x and y, the six roots of the sextic polynomial $B(x)$ [see (4)] are determined, in which two are the roots ($-d$ and $-f$) inserted in the polynomial $A(x)$, and the remaining four are the roots of $A(x)$. However, observe that d and f are not yet determined; so we do not know which ones are the inserted roots. We proceed to determine them as follows.

Since g and h are the sum and the product terms respectively of d and f [see (3)], we determine d and f as the solutions of a quadratic equation, say: $z^2 - gz + h = 0$, as shown below.

$$d,\ f\ = \left(g \pm \sqrt{g^2 - 4h}\right) \Big/ 2 \tag{19}$$

So, now it is possible to identify the inserted roots and the roots of the quartic polynomial (1). We solve one numerical example based on the proposed method.

3 Numerical example

Let us solve one numerical example; consider the following quartic polynomial,

$$A(x) = x^4 - 6x^2 + 12x - 8,$$

for determining its roots by the proposed method. The resolvent cubic equation (18) in g^2 is,

$$g^6 - 108g^4 + 5508g^2 - 104976 = 0.$$

We solve the above resolvent cubic equation using any of the well-known methods, and the three values of g^2 are:

$$36,\ \ 36 + 18\sqrt{5}i,\ \ \text{and}\ \ 36 - 18\sqrt{5}i.$$

We can choose any one of the above three values of g^2 to obtain the roots of the quartic polynomial; different values of g^2 yield different sequential order for the roots, but yield the same four roots. It means what we call as root x_1 for a certain value of g^2, the same value of the root may appear as x_2 for some other value of g^2.

Let us select one value of g^2 as: 36, and the corresponding two values of g are determined as: ± 6. Let $g = 6$; using this value in (15) we determine h as: 10. Using these values of g and h in (10), (12), and (14), we obtain j, m, and p as: -11, 55, and -125. The cubic equation (17) in y^2 is solved and the three values of y^2 are obtained as: 5, $3 + 4i$, and $3 - 4i$. Taking square roots of y^2 values, we determine six values of y [which are the roots of (17)] as:

$$\sqrt{5},\ \ -\sqrt{5},\ \ 2+i,\ \ -2-i,\ \ 2-i,\ \ \text{and}\ \ -2+i.$$

Using (7), we determine six values of x, which are the roots of sextic polynomial (4), as:

$$-1+\sqrt{5},\ \ -1-\sqrt{5},\ \ 1+i,\ \ -3-i,\ \ 1-i,\ \ \text{and}\ \ -3+i.$$

From (19), we determine d and f as: $3 + i$ and $3 - i$, so the inserted roots are identified as: $-3 - i$ and $-3 + i$. Hence the roots of the given quartic polynomial, $A(x)$ are: $-1 + \sqrt{5}$, $-1 - \sqrt{5}$, $1 + i$, and $1 - i$.

Conclusions

This paper has presented a new method of determining the roots of a quartic polynomial, wherein two unknown roots are inserted to the quartic polynomial, making it a sextic polynomial, which is first transformed to a depressed one (i.e., without fifth degree term). Then, the third degree and the first degree terms are eliminated from the depressed sextic polynomial yielding two equations in two unknowns. One of the unknowns is eliminated from these equations, yielding an even powered sextic equation in the other unknown (essentially a cubic equation in the square of the unknown), which is termed as the resolvent cubic equation. This equation is solved leading to the determination of the unknowns, and subsequently the six roots of the sextic polynomial are determined, identifying the inserted roots and the four roots of the given polynomial.

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