Bulletin of TICMI
Vol. 29, No. 1, 2025, 59–62

A Note on the Law of Large Numbers

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(Received April 17, 2025; Revised June 13, 2025; Accepted June 25, 2025)

Let us assume that a sequence of random variables satisfies the requirements of the well-known Chebyshev's theorem of the law of large numbers. Then will such a sequence always satisfy the strong law of large numbers? We show that the answer to this question is negative.

Keywords and phrases: Law of large numbers, Chebyshev theorem. strong law of large numbers, Prokhorov theorem.

AMS subject classification: 60F05, 60F15.

1 Introduction

Let $\xi_1, \xi_2, ..., \xi_n, ...$ be a sequence of real random variables defined on the probability space (Ω, A, P), with finite mathematical expectations $E\xi_n < \infty, n = 1, 2, ...$; denote by $S_n = \sum_{k=1}^{n} \xi_k, n = 1, 2, ...$ We say that the given sequence (ξ_k) of random variables satisfies the **Law of Large Numbers** (LLN), if the sequence $\frac{S_n - ES_n}{n}$ converges in probability to zero as $n \to \infty$, i.e. for every $\varepsilon > 0$

$$\lim_{n\to\infty} P[|\frac{S_n - ES_n}{n}| > \varepsilon] = 0.$$

A special form of the LLN was first proved by Jacob Bernoulli [1], which historically was the first and simplest formulation of the LLN. It is also worth noting the work of S.D. Poisson, who in 1837 proved a more general form of the LLN than that of J. Bernoulli. After Bernoulli and Poisson many famous mathematicians also contributed to refinement of the LLN including Chebyshev, Borel, Kolmogorov, Khinchin etc. Let us formulate some results, necessary for further discussion. We begin with the well-known Chebyshev theorem and formulate it in a more general form [2], p.62 (see also [3], p. 35).

Theorem 1.1. *Let $\xi_1, \xi_2, ..., \xi_n, ...$ be a sequence uncorrelated random variables and assume that*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} D\xi_k = 0, \tag{1.1}$$

where $D\xi_k$ is the variance of ξ_k. Then the sequence $\xi_1, \xi_2, ..., \xi_n, ...$ satisfies the **LLN.**

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ISSN: 1512-0082 print

Let $\xi_1, \xi_2, ..., \xi_n, ...$ be again a sequence of real random variables with finite mathematical expectations $E|\xi_n| < \infty$ and $S_n = \sum_{k=1}^{n} \xi_k$. As we can see, the law of large numbers means that the sequence $\frac{S_n - ES_n}{n}$ converges to zero in probability. If this sequence converges to zero almost surely (a.s.), then we say that the sequence (ξ_k) satisfies the Strong Law of Large Numbers (SLLN).

Let us pose the following question. Let us assume that a sequence of random variables satisfies the requirements of Chebyshev's theorem. Then will such a sequence always satisfy the SLLN? The answer to this question is negative, but it requires some effort to justify it. The main purpose of this paper is to prove this fact.

The following theorem gives a criterion for the fulfillment of the SLLN for independent centered Gaussian random variables [4].

Theorem 1.2 (Yu.V. Prokhorov, 1950). *Let $\xi_1, \xi_2, ..., \xi_n, ...$ be a sequence of independent Gaussian centered random variables and*

$$b_n = \frac{1}{2^{2n}} \sum_{k=2^n+1}^{2^{n+1}} E\xi_k^2, n = 1, 2, ...$$

Then the sequence satisfies the SLLN if and only if for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} e^{-\frac{\varepsilon}{b_n}} < \infty. \tag{1.2}$$

Remark 1.3. If (b_n) is a monotonically decreasing sequence, then condition (1.2) is equivalent to the following condition

$$\lim_{n\to\infty} (b_n \ln n) = 0. \tag{1.3}$$

Proof. Indeed, since the sequence (b_n) of positive numbers is monotonically decreasing, it is clear that $(e^{-\frac{\varepsilon}{b_n}})$ will also be monotonically decreasing. Then from the convergence of the series $\sum_{n=1}^{\infty} e^{-\frac{\varepsilon}{b_n}}$ it follows that

$$\lim_{n\to\infty} (ne^{-\frac{\varepsilon}{b_n}}) = 0.$$

This means that there exists a positive integer n_0, such that for any $n > n_0$ the following relations hold:

$$\ln (ne^{-\frac{\varepsilon}{b_n}}) = \ln n - \frac{\varepsilon}{b_n} < 0 \Rightarrow b_n \ln n < \varepsilon \Leftrightarrow \lim_{n\to\infty} (b_n \ln n) = 0.$$

Now we will show the validity of the converse statement. Let us assume that $n > 1$ and denote $b_n \cdot \ln n = a_n$, i.e. $b_n = \frac{a_n}{\ln n}$. It is clear that

$$e^{-\frac{\varepsilon}{b_n}} = e^{-\frac{\varepsilon}{a_n} \ln n} = (e^{\ln n})^{-\frac{\varepsilon}{a_n}} = \frac{1}{n^{\frac{\varepsilon}{a_n}}}.$$

Hence,

$$\sum_{n=1}^{\infty} e^{-\frac{\varepsilon}{b_n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{\varepsilon}{a_n}}} < \infty,$$

which completes the proof of Remark 1.3. □

Now we will formulate and prove our main result.

2 Main result

Theorem 2.1. *There is a sequence of independent Gaussian centered random variables that satisfies the conditions of Theorem 1.1 and does not satisfy the SLLN.*

Proof. Let $\xi_1, \xi_2, ..., \xi_n, ...$ be a sequence of independent Gaussian centered random variables and let the variance of ξ_k be x_k, where x_k is defined as follows

$$x_k = \begin{cases} 1, & 1 \le k < 27, \\ \frac{k}{\ln\ln k}, & k \ge 27. \end{cases}$$

In addition, let us denote

$$b_n = \frac{1}{2^{2n}} \sum_{k=2^n}^{2^{n+1}-1} x_k.$$

It is easy to see that the sequence $(\frac{k}{\ln\ln k})$ is increasing for $k \ge 27$. Bearing in mind that M and N are arbitrary natural numbers with $27 < M < N$, we get the following estimates:

$$\int_M^N \frac{M-1}{\ln\ln(M-1)} dx < \sum_{k=M}^{N} \frac{k}{\ln\ln k} \equiv S_M^N < \int_M^N \frac{N+1}{\ln\ln(N+1)} dx. \tag{2.1}$$

From where

$$\frac{(M-1)(N-M)}{\ln\ln(M-1)} < S_M^N < \frac{(N+1)(N-M)}{\ln\ln(N+1)}. \tag{2.2}$$

If we substitute $M = 28$ and $N = n$, then the right-hand side of (2.2) gives us

$$\frac{1}{n^2} S_{28}^n < \frac{(n+1)(n-28)}{\ln\ln(n+1)n^2},$$

which obviously converges to zero. This means that condition (1.1) is satisfied.

If we use the left-hand side of (2.2) for the case $M = 2^n + 1$ and $N = 2^{n+1}$ we get

$$b_n = \frac{1}{2^{2n}} \cdot S_{2^n+1}^{2^{n+1}} > \frac{1}{2^{2n}} \frac{(2^n+1-1)(2^{n+1}-2^n-1)}{\ln\ln(2^n)} =$$
$$= \frac{2^n-1}{2^n \cdot \ln\ln(2^n)} > \frac{1}{2 \cdot \ln(n \ln 2)} = \frac{1}{2(\ln n + \ln\ln 2)}.$$

Thus,

$$b_n \cdot \ln n \ge \frac{\ln n}{2(\ln n + \ln\ln 2)} \to \frac{1}{2} > 0$$

and therefore condition (1.3) is not satisfied.

Now if we show that the sequence (b_n) is decreasing, it turns out that condition (1.2) is not satisfied, because as we have already shown, in this case condition (1.2) implies that condition (1.3) is satisfied, but we have just shown that condition (1.3) is not satisfied.

To satisfy the inequality

$$b_n > b_{n+1} \Leftrightarrow \frac{1}{2^{2n}} \cdot S_{2^n+1}^{2^{n+1}} > \frac{1}{2^{2n+2}} \cdot S_{2^{n+1}+1}^{2^{n+2}},$$

the following inequality must be satisfied:

$$\frac{S_{2^{n+1}+1}^{2^{n+2}}}{S_{2^n+1}^{2^{n+1}}} < 4.$$

Let us find the asymptotic estimate $S_{2^n+1}^{2^{n+1}}$. To do this, we will estimate from above and from below expression $(\ln\ln k)$ for $k = 2^n+1, 2^n+2, ..., 2^{n+1}$. Let $k = 2^n + m$, where $1 \leq m \leq 2^n$. Then,

$$\ln\ln k = \ln\ln(2^n \cdot \frac{2^n+m}{2^n}) = \ln(n\ln 2 + \ln p),$$

where $1 < p = \frac{2^n+m}{2^n} \leq 2$. As $p > 1$, then $\ln p > 0$ and we have

$$\ln(n\ln 2) < \ln\ln k < \ln((n+1)\ln 2)$$

for any $k = 2^n+1, 2^n+2, ..., 2^{n+1}$. Therefore,

$$\frac{1}{\ln((n+1)\ln 2)} \sum_{k=2^n+1}^{2^{n+1}} k < \sum_{k=2^n+1}^{2^{n+1}} \frac{k}{\ln\ln k} < \frac{1}{\ln(n\ln 2)} \sum_{k=2^n+1}^{2^{n+1}} k.$$

That means

$$\frac{3\cdot 4^n + 2^n}{\ln((n+1)\ln 2)\cdot 2} < S_{2^n+1}^{2^{n+1}} < \frac{3\cdot 4^n + 2^n}{\ln(n\ln 2)\cdot 2}.$$

By this inequality

$$\frac{S_{2^{n+1}+1}^{2^{n+2}}}{S_{2^n+1}^{2^{n+1}}} < \frac{3\cdot 4^{n+1} + 2^{n+1}}{\ln((n+1)\ln 2)\cdot 2} \cdot \frac{\ln((n+1)\ln 2)\cdot 2}{3\cdot 4^n + 2^n} = \frac{3\cdot 4^{n+1} + 2^{n+1}}{3\cdot 4^n + 2^n} < 4.$$

Thus, we have shown that the sequence (b_n) decreases starting from some n, which completes the proof of the theorem. □

Acknowledgments

The author is grateful to the members of the seminar of the Department of Probability and statistical methods of Muskhelishvili Institute of Computational Mathematics of GTU, for useful discussions and suggestions.

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