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Conditions for Solvability of a Class of Fourth Order Partial Operator Differential Equations

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In the paper we obtain sufficient conditions for regular solvability of elliptic type fourth order partial operator-differential equations dependent on two variables and whose principal part contains a normal operator. These conditions are expressed by the properties of the coefficients of the operator-differential equation. At the same time the estimates of the norms of intermediate derivatives in abstract Sobolev-type spaces are obtained through the principal part of the operator-differential equation.

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1 Introduction

Let H be a separable Hilbert space, let C be a positive –definite self-adjoint operator. Let $R^2 = R \times R$, and let $L_2\left(R^2; H\right)$ be a Hilbert space of vector-functions $f(x, y)$, determined almost everywhere in R^2, with the values in Í, measurable and quadratically integrable, for which

$$\|f\|_{L_2(R^2;H)} = \left(\int_{R^2} \|f(x, y)\|^2 \, dxdy\right)^{\frac{1}{2}} < \infty.$$

We introduce the linear set $D\left(R^2; H_4\right)$ of infinitely differentiable in H vector-functions $u(x, y)$, with the values $H_4 = D\left(C^4\right)$ $\left((x, y)_{H_4} = \left(C^4 x, C^4 y\right)\right)$, having compact supports in R^2. In the linear set $D\left(R^2; H_4\right)$ we determine the norm

$$\|u\|_{W_2^4(R^2;H)} = \left(\sum_{\substack{k,j=0 \\ 0 \le k+j \le 4}}^{4} \left\| C^{4-(k+j)} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|^2_{L_2(R^2;H)} \right)^{\frac{1}{2}}$$

We denote completion of the linear set $D\left(R^2; H_4\right)$ in the norm $\|u\|_{W_2^4(R^2;H)}$ by $W_2^4(R^2; H)$.

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In the space H we consider the operator-differential equation

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+A^4u+\sum_{\substack{k,j=0\\0\le k+j\le 4}}^{4} A_{k,j}\frac{\partial^{k+j}u}{\partial x^k\partial y^j}=f(x,y),(x,y)\in R^2, \qquad (1)$$

where $f(x,y)$, $u(x,y)$ are vector-functions with the values in H, the operator coefficients satisfy the following conditions:

1) A is a normal invertible operator whose spectrum is contained in the angular sector

$$S_\varepsilon=\left\{\lambda:|arq\lambda|\le\varepsilon,\ 0\le\varepsilon<\frac{\pi}{4}\right\}$$

2) The operators $B_{k,j}=A_{k,j}\ A^{(k+j)-4}$ $(k,j=\overline{0,4},k+j\le 4)$ are bounded in H.

Note that subject to condition 1) the operator A is representable in the form $A=\bigcup C$, where C is a positive –definite self-adjoint operator in H, and $C$is a unitary operator in H.

Definition 1.1. If for $f(x,y)\in L_2\left(R^2;H\right)$ there exists a vector-function $u(x,y)\in W_2^4\left(R^2;H\right)$, satisfying equation (1) almost everywhere in R^2, it will be said a regular solution of equation (1).

Definition 1.2. If for any $f(x,y)\in L_2\left(R^2;H\right)$ there exists the regular solution $u(x,y)$ of equation (1) that has the estimation

$$\|u\|_{W_2^4(R^2;H)}\le const\,\|f\|_{L_2(R^2;H)},$$

equation (1) will be said regularly solvable.

In this paper we find conditions on the coefficients of equation (1), in fulfilling of which equation (1) is regularly solvable. Note that the second order elliptic type partial equation has been studied in [1, 2, 3, 5]. When A is a self-adjoint operator, conditions for regular solvability for (1) have been obtained in [4, 6].

Denote

$$P_0u=\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+A^4u,u(x,y)\in W_2^4\left(R^2;H\right)$$

$$P_1u=\sum_{\substack{k,j=0\\0\le k+j\le 4}}^{4} A_{k,j}\frac{\partial^{k+j}u}{\partial x^k\partial y^j},u(x,y)\in W_2^4\left(R^2;H\right)$$

and

$$Pu=P_0u+P_1u,u(x,y)\in W_2^4\left(R^2;H\right).$$

We have

Theorem 1.3. *Let conditions 1) be fulfilled. Then the operator P_0 isomorphically maps the space $W_2^4\left(R^2;H\right)$ onto $L_2\left(R^2;H\right)$, and we have the estimations*

$$\left\|A^{4-(k+j)}\frac{\partial^{k+j}u}{\partial x^k\partial y^j}\right\|_{L_2(R^2;H)}\le C_{k,j}(\varepsilon)\|P_0u\|_{L_2(R^2;H)} \qquad (2)$$

$(k,j=\overline{0,4}, 0\le k+j\le 4)$, *where*

$$C_{0,0}(\varepsilon)=C_{0,4}(\varepsilon)=C_{4,0}(\varepsilon)=\begin{cases}1, & 0\le\varepsilon\le\frac{\pi}{8}\\ \frac{1}{\sqrt{2}\cos 2\varepsilon}, & \frac{\pi}{8}\le\varepsilon<\frac{\pi}{4}\end{cases} \tag{3}$$

$$C_{k,0}=\left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{4-k}{4}\right)^{\frac{4-k}{4}}, \quad for\ k=1,2,3; j=0 \tag{4}$$

$$C_{0,j}(\varepsilon)=\left(\frac{j}{4}\right)^{\frac{j}{4}}\left(\frac{4-j}{4}\right)^{\frac{4-j}{4}}\left(1+tg^2 2\varepsilon\right)^{\frac{1}{2}}, \quad for\ k=0;\ j=1,2,3 \tag{5}$$

$$C_{k,j}=\left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{j}{4}\right)^{\frac{j}{4}}\left(1+tg^2 2\varepsilon\right)^{\frac{1}{2}}, \ for\ k\ne 0, j\ne 0, k\ne 4, j\ne 4, k+j=4 \tag{6}$$

$$C_{k,j}=\left(\frac{4-(k+j)}{4}\right)^{\frac{4-(k+j)}{4}}\left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{j}{4}\right)^{\frac{j}{4}}\left(1+tg^2 2\varepsilon\right)^{\frac{1}{2}}, \ for\ 2\le k+j\le 3 \tag{7}$$

Proof. Let $\hat{f}(\xi,\eta)$ be the Fourier transform of the vector-function $f(x,y)\in L_2\left(R^2;H\right)$. Then it is easy to see that the vector-function

$$u(x,y)=\frac{1}{2\pi}\int_{R^2}\left(\left(\xi^4+\eta^4\right)E+A^4\right)^{-1}\hat{f}(\xi,\eta)\,e^{i(\xi x+\eta y)}d\xi d\eta \tag{8}$$

satisfies the equation $P_0u=\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+A^4u=f(x,y)$ almost everywhere in R^2. Show that $u(x,y)\in W_2^4\left(R^2;H\right)$. For that is saffices to prove inequality (5)-(7).

Let $k=0$, $j=0$. Then by the Plancherel theorem we find:

$$\left\|A^4u\right\|_{L_2(R^2;H)}=\left\|A^4\hat{u}(\xi,\eta)\right\|^2_{L_2(R^2;H)}$$

$$=\left\|A^4\left(\xi^4E+\eta^4E+A^4\right)^{-1}\hat{f}(\xi,\eta)\right\|^2_{L_2(R^2;H)}$$

$$\le\sup_{(\xi,\eta)\in R^2}\left\|A^4\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\|\cdot\left\|\hat{f}(\xi,\eta)\right\|_{L_2(R^2;H)}$$

$$=\sup_{(\xi,\eta)\in R^2}\left\|A^4\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\|\cdot\left\|f(x,y)\right\|_{L_2(R^2;H)}. \tag{9}$$

For any $(\xi,\eta)\in R$

$$\left\|A^4\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\|=\sup_{\lambda\in\sigma(A)}\left|\lambda^4(\xi^4+\eta^4+\lambda^4)^{-1}\right|$$

$$=\sup_{\substack{\mu>\mu_0>0\\ |\varphi|<\varepsilon}}\left|\mu\left(\xi^4+\eta^4+\mu^4e^{i\varphi}\right)^{-1}\right|$$

$$\le\sup_{\mu>0}\left|\mu^4\left(\left(\xi^4+\eta^4\right)^2+\mu^8+2\left(\xi^4+\eta^4\right)\mu^4\cos 4\varepsilon\right)^{-1}\right| \tag{10}$$

For $0 \le \varepsilon \le \frac{\pi}{8}$, the number $\cos 4\varepsilon > 0$, therefore from (10) we obtain

$$\left\| A^4 \left(\left(\xi^4 + \eta^4 \right) E + A^4 \right)^{-1} \right\| \le \sup_{\mu > 0} \left| \mu^4 \left(\left(\xi^4 + \eta^4 \right)^2 + \mu^8 \right)^{-1} \right| \le 1 \tag{11}$$

For $\frac{\pi}{8} \le \varepsilon \le \frac{\pi}{4}$ using the Cauchy inequality, from (10) we obtain:

$$\left\| A^4 \left(\left(\xi^4 + \eta^4 \right) E + A^4 \right)^{-1} \right\|$$

$$\le \sup_{\mu > 0} \left| \mu^4 \left(\left(\xi^4 + \eta^4 \right)^2 + \mu^8 + (\ \left(\xi^4 + \eta^4 \right)^2 + \mu^4 \ ) \cos 4\varepsilon \right)^{-\frac{1}{2}} \right|$$

$$\le \mu^4 \left(\left(\left(\xi^4 + \eta^4 \right) + \mu^8 \right) \cdot \frac{1}{2\cos^2 2\varepsilon} \right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}\cos 2\varepsilon} \tag{12}$$

Taking into account inequalities (11) and (12) in (10), we obtain

$$\left\| A^4 u \right\|_{L_2(R^2;H)} = \left\| A^4 \hat{u}\left(\xi, \eta \right) \right\|_{L_2(R^2;H)} \le C_0(\varepsilon) \left\| f \right\|_{L_2(R^2;H)} = C_0(\varepsilon) \left\| P_0 u \right\|_{L_2(R^2;H)}$$

Inequality (2) is proved in the same way for $k = 0$, $j = 0$;$k = 4$,$j = 0$.
We now assume that $k = 1, 2, 3$; $j = 0$. Then

$$\left\| A^{4-k} \frac{\partial^k u}{\partial x^k} \right\|_{L_2(R^2;H)} = \left\| A^{4-k} \xi^k \hat{u}\left(\xi, \eta \right) \right\|_{L_2(R^2;H)}$$

$$= \left\| A^{4-k} \xi^k \left(\xi^4 E + \eta^4 E + A^4 \right)^{-1} \hat{f}\left(\xi, \eta \right) \right\|_{L_2(R^2;H)}$$
$$\le \sup_{(\xi,\eta) \in R^2} \left\| A^{4-k} \xi^k \left(\xi^4 E + \eta^4 E + A^4 \right)^{-1} \right\| \cdot \left\| \hat{f}\left(\xi, \eta \right) \right\|_{L_2(R^2;H)}$$

$$= \sup_{(\xi,\eta) \in R^2} \left\| A^{4-k} \xi^k \left(\xi^4 E + \eta^4 E + A^4 \right)^{-1} \right\| \cdot \left\| f\left(x, y \right) \right\|_{L_2(R^2;H)} . \tag{13}$$

Since for $(\xi, \eta) \in R$

$$\left\| A^{4-k} \xi^k \left(\xi^4 E + \eta^4 E + A^4 \right)^{-1} \right\| \le \sup_{\lambda \in \sigma(A)} \left| \lambda^{4-k} \xi^k \left(\xi^4 + \eta^4 + \lambda^4 \right)^{-1} \right|$$

$$\le \sup_{\mu > 0} \left| \mu^{4-k} \xi^k \left(\left(\xi^4 + \eta^4 \right)^2 + \mu^8 + 2\left(\xi^4 + \eta^4 \right) \mu^4 \cos 4\varepsilon \right)^{-\frac{1}{2}} \right|$$

$$\le \sup_{\mu > 0} \frac{\mu^{4-k} \left| \xi \right|^k}{\xi^4 + \eta^4 + \mu^4} \left(\frac{\left(\xi^4 + \eta^4 + \mu^4 \right)^2}{\left(\xi^4 + \eta^4 \right)^2 + \mu^8 + 2\left(\xi^4 + \eta^4 + \mu^4 \cos 4\varepsilon \right)} \right)^{\frac{1}{2}} . \tag{14}$$

Obviously

$$\frac{\left(\xi^4 + \eta^4 + \mu^4 \right)^2}{(\xi^4 + \eta^4)^2 + \mu^8 + 2\left(\xi^4 + \eta^4 \right) \mu^8 \cos 4\varepsilon}$$

$$= \frac{(\xi^4 + \eta^4)^2 + \mu^8 + 2\left(\xi^4 + \eta^4 \right) \mu^8}{(\xi^4 + \eta^4)^2 + \mu^8 + 2\left(\xi^4 + \eta^4 \right) \mu^8 \cos 4\varepsilon}$$

$$= 1 + \frac{2\left(\xi^4+\eta^4\right)\mu^4(1-\cos 4\varepsilon)}{(\xi^4+\eta^4)^2+\mu^8+2\left(\xi^4+\eta^4\right)\mu^8\cos 4\varepsilon}$$

$$\leq 1 + \frac{4(\xi^4+\eta^4)\mu^4\sin^2 2\varepsilon}{2\left(\xi^4+\eta^4\right)\mu^4(1+\cos 4\varepsilon)} = 1 + \frac{\sin^2 2\varepsilon}{\cos^2 2\varepsilon} = 1 + tg^2 2\varepsilon \tag{15}$$

Then allowing for (15) from (14) we obtain

$$\left\| A^{4-k}\xi^k\left(\xi^4 E + \eta^4 E + A^4\right)^{-1}\right\| \leq \sup_{\mu>0}\frac{\mu^{4-k}\left|\xi\right|^k}{\xi^4+\eta^4+\mu^4}\cdot(1+tg^2 2\varepsilon)^{\frac{1}{2}} \tag{16}$$

On the other hand, we have (see [5]):

$$\sup_{\mu>0}\frac{\mu^{4-k}\left|\xi\right|^k}{\xi^4+\eta^4+\mu^4} \leq \sup_{\mu>0}\frac{\mu^{4-k}\left|\xi\right|^k}{\xi^4+\mu^4} = \sup_{\tau>0}\frac{\tau^k}{\tau^4+1} = \left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{4-k}{4}\right)^{\frac{4-k}{4}} \tag{17}$$

Consequently, from inequalities (16) and (17) taking into account inequalities (13), we obtain the validity of inequality (4).

Inequality (5) is proved in a similar way to inequality (3).

We now prove inequality (5). In this case

$$\left\|\frac{\partial^{k+j}u}{\partial x^k\partial y^j}\right\|_{L_{2(R^2;H)}} = \left\|\xi^k\eta^j\hat{u}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}$$

$$= \left\|\xi^k\eta^j\left(\xi^4E+\eta^4E+A^4\right)^{-1}\hat{f}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}$$

$$\leq \sup_{(\xi,\eta)\in R}\left|\xi^k\eta^j\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right|\cdot\left\|f\left(x,y\right)\right\|_{L_2(R^2;H)}. \tag{18}$$

But for $(\xi,\eta)\in R$ and $\delta>0$

$$\left\|\xi^k\eta^j\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\| \leq \sup_{\lambda\in\sigma(A)}\left|\xi^k\eta^j\left(\xi^4+\eta^4+\lambda^4\right)^{-1}\right|$$

$$\leq \sup_{\mu>0}\left|\xi^k\eta^j\left(\left(\xi^4+\eta^4\right)^2+\mu^8+2\left(\xi^4+\eta^4\right)\mu^4\cos 4\varepsilon\right)^{-\frac{1}{2}}\right|$$

$$\leq \sup_{\mu>0}\frac{\mu^{4-k}\left|\xi\right|^k}{\xi^4+\eta^4+\mu^4}\left(\frac{\left(\xi^4+\eta^4+\mu^4\right)^2}{(\xi^4+\eta^4)^2+\mu^8+2\left((\xi^4+\eta^4)\mu^4\cos 4\varepsilon\right)}\right)^{\frac{1}{2}}$$

$$\leq \sup_{\mu>0}\frac{\left|\xi\right|^k\cdot\left|\eta\right|^{4-k}}{\xi^4+\eta^4+\mu^4}\cdot(1+tg^2\varepsilon) \leq \frac{\left(\delta\left|\xi\right|^4\right)^{\frac{k}{4}}\left(\delta^{-\frac{k}{4-k}}\left|\eta^4\right|\right)^{\frac{4-k}{4}}}{\xi^4+\eta^4+\mu_0^4}$$

$$\leq \frac{\frac{k}{4}\delta\left|\xi\right|^4+\frac{4-k}{4}\cdot\delta^{-\frac{k}{4-k}}}{\xi^4+\eta^4+\mu_0^4} \tag{19}$$

Assuming $\delta=\left(\frac{4-k}{k}\right)^{\frac{4-k}{k}}$ we obtain

$$\left\|\frac{\partial^4 u}{\partial x^4\partial y^4}\right\|_{L_{2(R^2;H)}} \leq \left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{j}{4}\right)^{\frac{j}{4}}\left(1+tg^2\varepsilon\right)^{\frac{1}{2}}.$$

Inequality (5) is proved . Prove inequality (6). Obviously,

$$\left\|A^{4-(k+j)}\frac{\partial^{k+j}u}{\partial x^k\partial y^j}\right\|_{L_2(R^2;H)}=\left\|A^{4-(k+j)}\xi^k\eta^j\hat{u}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}$$

$$\leq\sup_{(\xi,\eta)\in R}\left\|A^{4-(k+j)}\xi^k\eta^j\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\|\cdot\left\|\hat{f}\left(\xi,\eta\right)\right\|_{L_2(R^2;H)}$$

$$=\sup_{(\xi,\eta)\in R}\left\|A^{4-(k+j)}\xi^k\eta^j\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\|\cdot\left\|f\left(x,y\right)\right\|_{L_2(R^2;H)}. \tag{20}$$

On the other hand,

$$\left\|A^{4-(k+j)}\xi^k\eta^j\left(\xi^4E+\eta^4E+A^4\right)^{-1}\right\|$$

$$\leq\sup_{\mu>0}\left|\mu^{4-(k+j)}\left(\left(\xi^4+\eta^4\right)^2+\mu^8+2\left(\xi^4+\eta^4\right)\mu^4\cos 4\varepsilon\right)^{-\frac{1}{2}}\right|$$

$$\leq\sup_{\mu>0}\frac{\left|\mu^{4-(k+j)}\xi^k\eta^j\right|}{\xi^4+\eta^4+\mu^4}\left(1+tg^2 2\varepsilon\right)^{\frac{1}{2}} \tag{21}$$

Obviously, for $(\xi_0,\eta_0)\in R^2$, $(\xi_0,\eta_0)\neq(0,0)$and for $\mu>0$

$$\frac{|\xi_0|^k\,|\eta_0|^j\,\mu^{4-(k+j)}}{\xi_0^4+\eta_0^4+\mu^4}\leq\frac{1}{4}\left(\frac{4-(k+j)}{k+j}\right)^{\frac{4-(k+j)}{4}}(k+j)\cdot\frac{|\xi_0|^k\,|\eta_0|^j}{4(\xi_0^4+\eta_0^4)}$$

Then

$$\sup_{(\xi,\eta)\in R}\frac{|\xi|^k\,|\eta|^j\,\mu^{4-(k+j)}}{\xi^4+\eta^4+\mu^4}\leq\left(\frac{4-(k+j)}{k+j}\right)^{\frac{4-(k+j)}{4}}\left(\frac{k}{4}\right)^{\frac{k}{4}}\left(\frac{j}{4}\right)^{\frac{j}{4}}$$

Consequently,

$$\left\|A^{4-(k+j)}\frac{\partial^{k+j}u}{\partial x^k\partial y^j}\right\|_{L_2(R^2;H)}\leq C_{k,j}(\varepsilon)\left\|f\right\|_{L_2(R^2;H)}=C_{k,j}(\varepsilon)\left\|P_0u\right\|_{L_2(R^2;H)}$$

Thus, estimation (2) is proved. It follows from these estimations that $u\left(x,y\right)\in W_2^4\left(R^2;H\right)$. On the other hand,

$$(P_0u)_{W_2^4(R^2;H)}\leq\left\|\frac{\partial^4u}{\partial x^4}\right\|^2_{L_2(R^2;H)}+\left\|\frac{\partial^4u}{\partial y^4}\right\|^2_{L_2(R^2;H)}+\left\|A^4u\right\|^2_{L_2(R^2;H)}\leq\left\|u\right\|^2_{W_2^4(R^2;H)}$$

i.e. the operator $P_0:W_2^4\left(R^2;H\right)\rightarrow L_2\left(R^2;H\right)$ is bounded. Then the theorem statement follows from the Banach theorem on an inverse operator, since $ReP_0=\{0\}$ and $JmP_0=L_2(R^2;H)$.

The Theorem is proved. □

Lemma 1.4. *Let conditions 1), 2) be fulfilled, then $P=P_0+P_1$ is a bounded operator from the space $W_2^4\left(R^2;H\right)$ in $L_2\left(R^2;H\right)$.*

Proof. Obviously, by theorem 1 is suffices to prove the boundedness of the operator $P_1 : W_2^4\left(R^2;H\right) \to L_2\left(R^2;H\right)$.

By the definition, for $u\left(x,y\right) \in W_2^4\left(R^2;H\right)$

$$\|P_1 u\|^2_{L_2(R^2;H)} \leq \sum_{\substack{k,j=0\\0\leq k+j\leq 4}}^{4} \left\| A_{k,j} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|^2_{L_2(R^2;H)}$$

$$\leq \sum_{\substack{k,j=0\\0\leq k+j\leq 4}}^{4} \|B_{k,j}\|^2 \cdot \left\| A^{4-(k+j)} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|^2_{L_2(R^2;H)}$$

$$\sum_{\substack{k,j=0\\0\leq k+j\leq 4}}^{4} \|B_{k,j}\|^2 \cdot \left\| C^{k-j} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|^2_{L_2(R^2;H)}$$

$$\leq \max_{k,j} \|B_{k,j}\|^2 \, \|U\|^2_{W_2^4(R^2;H)}$$

The lemma is proved. □

We now prove the main theorem

Theorem 1.5. *Let conditions 1), 2) and the inequality*

$$q\left(\varepsilon\right) = \sum_{\substack{k+j=0\\0\leq k+j\leq 4}}^{4} C_{k.j}\left(\varepsilon\right) \cdot \|B_{k,j}\| < 1 \left(0 \leq \varepsilon < 1\right)$$

be fulfilled, where the numbers $C_{k,j}(\varepsilon)$ are determined from theorem 1 by inequalities (3)-(6). *Then equation* (1) *is regularly solvable.*

Proof. We write equation (1) in the form: $Pu = P_0 u + P_1 u = f$ where $f \in L_2\left(R^2;H\right)$, $u \in W_2^4\left(R^2;H\right)$. Then after substituting $P_0 u = \omega$ we obtain the equation $\omega + P_1 P_0^{-1}\omega = f$ in $L_2\left(R^2;H\right)$. Since for any $\omega \in L_2\left(R^2;H\right)$ ($P_0 : W_2^4\left(R^2;H\right) \to L_2\left(R^2;H\right)$ is an isomorphic operator)

$$\left\|P_1 P_0^{-1}\omega\right\|_{L_2(R^2;H)} = \|P_1 u\|_{L_2(R^2;H)} = \sum_{\substack{k,j=0\\0\leq k+j\leq 4}}^{4} \left\| A_{k,j} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|_{L_{2,\gamma}(R^2;H)}$$

$$\leq \sum_{\substack{k,j=0\\0\leq k+j\leq 4}}^{4} \|B_{k,j}\| \cdot \left\| A^{4-(k+j)} \frac{\partial^{k+j} u}{\partial x^k \partial y^j} \right\|_{L_2(R^2;H)}$$

$$\leq \sum_{\substack{k,j=0 \\ 0\leq k+j\leq 4}}^{4} C_{k,j}(\varepsilon)\cdot\|B_{k,j}\|\cdot\|P_0 u\|_{L_2(R^2;H)}$$

$$= \sum_{\substack{k,j=0 \\ 0\leq k+j\leq 4}}^{4} C_{k,j}(\varepsilon)\cdot\|B_{k,j}\|\cdot\|\omega\|_{L_2(R^2;H)} = q(\varepsilon)\cdot\|\omega\|_{L_2(R^2;H)}.$$

Since $q(\varepsilon)<1$, then $\omega=\left(E+P_1P_0^{-1}\right)^{-1}f$ while $u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$ it follows that $\|u\|_{W_2^4(R^2;H)}\leq const\,\|f\|_{L_2(R^2;H)}$.

□

In the special case from Theorem 1.3 we have.

Corollary 1.6. *Let A be a self-adjoint operator and the conditions of theorem 2 be fulfilled for $\varepsilon=0$. Then equation* (1) *is regularly solvable.*

This result has been obtained in [4].

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