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Research of Nonlinear Dynamic Systems Describing the Process of Spread of World Religions

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Mathematical modeling of social and political processes, such as globalization, assimilation of peoples, bipolar or multipolar arrangement of the world, is of significant scientific and practical interest. These processes are greatly influenced by the spread and interaction of the main world religions: Christianity (33,9%), Islam (22,7%), Hinduism (13,9%), Buddhism and Chinese folk religions (13,2%). At the same time, these religions and their carriers, in addition to mutual operations, are subjected to great open and secret pressure from atheists (11,5%), who, as a rule, own the main financial and political resources. The remaining 4,8% of the world's population are adherents of small religions that do not have a significant impact on world processes.

As the first mathematical model, a nonlinear three-dimensional dynamic system is considered, describing the interaction of the two main world religions of Christianity and Islam, the dynamics of the number of their carriers, as well as the impact on them of powerful groups consisting of atheists united in various communities, including transnational consortia affecting world processes.

In some particular cases of constant coefficients of a nonlinear dynamic system, the first integrals are found and the three-dimensional dynamic system is reduced to a two-dimensional one. In some cases of a two-dimensional dynamic system, using the principle (theorem) of Bendixson, theorems on the existence of solutions of a closed integral curve in the first quarter of the phase plane are proved, and in another case the problem is reduced to the classical "predator-prey" problem (the Lotka-Volterra system of differential equations). In all cases, the coexistence of three social groups (Christians, Muslims, Atheists) and non-degeneration of any of them are shown.

Keywords and phrases: Mathematical modeling, Nonlinear dynamic system, The spread and interaction of the main world religions, The first integrals, The principle of Bendixson.

AMS subject classification: 93A30, 00A71, 97M10, 97M70.

Introduction

Over the past few decades, mathematical modeling of social processes, such as information warfare, language globalization, ethnic assimilation, political conflicts, the process of state territorial stability, etc. has been of particular interest.

From our point of view, the only scientific approach to an adequate quantitative and qualitative description of these problems is the mathematical modeling of processes, i.e. the creation of mathematical models describing these current problems [1–9].

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The religions existing in the world nowadays have a significant influence on people's lives, either because they are involved in religious practices or because they become affected by their cultural milieu. The relationship between religion and the individual makes us question the reason behind people's concern about the existence and exchange of certain religious beliefs, ideas and opinions, and how these beliefs, ideas, and opinions affect the worldview and behavior of the individuals and decision-making. In fact, one of the most psychological and behavioral functions served by religions is that they provide proper guidance to the individual and society, not only in relation to the general perspectives of life but also with respect to the important choices faced by individuals [10, 11].

Actually, in the last two millennia, many religions disseminated around the globe mainly the four vast religions: Christianity, Islam, Hinduism and Buddhism. These four religions have been introduced repeatedly to groups and societies and have subsequently been accepted by large numbers of people as part of their social identities or have been largely rejected and considered primarily as "foreign" religions [12, 13].

The findings expressed in this survey demonstrate that religious affiliation especially with highly active practice could have a positive impact on such individuals. These advantages of religious practice on people drives us to investigate how religious beliefs spread and to think of the best ways to model the spread of religious beliefs mathematically. Many studies and research in social sciences have focused on this topic and other related topics. But the mathematical studies and research on this topic are still limited and most of them have focused on the statistical aspect of the phenomenon.

The social processes taking place in the modern world, primarily demographic processes, globalization, assimilation of peoples are greatly influenced by the spread and interaction of the main world religions: Christianity (33,9%), Islam (22,7%), Hinduism (13,9%), Buddhism and Chinese folk religions (13,2%). At the same time, these religions and their carriers, in addition to interaction and rivalry, are subjected to great open and secret pressure from atheists (11,5%), who, as a rule, own the main financial and political resources. The remaining 4,8% of the world's population are adherents of small religions that do not significantly affect world processes.

The interaction of three significant representatives of the world population, Christianity, Islam, as well as atheists is considered as a test mathematical model. The mathematical model is described by a three-dimensional nonlinear dynamic system with variable coefficients. The influence of countries where Christianity and Islam are crucial on world politics is significant.

1 Three-dimensional nonlinear dynamical system. Statement of the problem

Let's consider a general nonlinear mathematical model that describes the process of interaction and mutual influence of representatives of two main (major) world religions, Christianity and Islam, as well as a third group that does not profess any religion and are atheists.

The general mathematical model is described by a three-dimensional nonlinear dynamic system with variable coefficients

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1(t)u(t) + (\beta_1(t) - \beta_2(t))u(t)v(t) + (\beta_7(t) - \beta_8(t))u(t)w(t) - \gamma_1(t)u(t) \\ \dfrac{dv(t)}{dt} = \alpha_2(t)v(t) + (\beta_2(t) - \beta_1(t))u(t)v(t) + (\beta_{13}(t) - \beta_{14}(t))v(t)w(t) - \gamma_2(t)v(t) \\ \frac{dw(t)}{dt} = \alpha_5(t)w(t) + (\beta_8(t) - \beta_7(t))u(t)w(t) + (\beta_{14}(t) - \beta_{13}(t))v(t)w(t) \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{aligned}
&u(0) = u_0,\ v(0) = v_0,\ w(0) = w_0, \\
&t \in (0,T),\ u(t), v(t), w(t) \in C^1[0,T], \\
&\beta_1(t), \beta_2(t), \beta_7(t), \beta_8(t), \beta_{13}(t), \beta_{14}(t) > 0, \\
&\gamma_1(t) > 0,\ \gamma_2(t) > 0,\ \gamma_1(t), \gamma_2(t) \in C[0,T], \\
&\alpha_1(t), \alpha_2(t), \alpha_5(t) \in C[0,T],
\end{aligned} \tag{1.2}$$

where

$u(t)$ is the number of Christians in the world at a given time t,

$v(t)$ is the number of Muslims in the world at a given time t,

$w(t)$ is the number of Atheists in the world at a given time t,

$\alpha_1(t), \alpha_2(t), \alpha_5(t)$ are demographic factors, respectively Christians, Muslims and Atheists at a given time t,

$\beta_1(t), \beta_2(t)$ are the coefficients of attraction to one's side in the interaction of Christians and Muslims respectively at a given moment in time t,

$\beta_7(t), \beta_8(t)$ are the coefficients of attraction to one's side in the interaction of Christians and Atheists respectively at a given moment in time t,

$\beta_{13}(t), \beta_{14}(t)$ are the coefficients of attraction to one's side in the interaction of Muslims and Atheists respectively at a given moment in time t,

$\gamma_1(t), \gamma_2(t)$ are coefficients of artificial reduction of Christians and Muslims respectively by atheists at the moment of time t.

Description of the general mathematical model (description of the members of a system of differential equations).

- Coefficients of the first linear terms of the right part of the general system of differential equations (1.1) are demographic factors of the corresponding social groups.

- The last negative linear terms on the right side of the first four equations of the general system are an artificial reduction in the followers of religious groups, for example, due to epidemics, information warfares that develop into interreligious wars, hostilities for natural resources and economic markets, etc.

- The nonlinear terms on the right side of the general system of differential equations are the pairwise mutual influence of four religious groups and a social group of atheists in order to convince them and transfer them to their group.

- The adequacy and non-triviality of the mathematical model assumes the presence in the right parts of the system of equations of at least one both negative and one positive term.

Analytical research of a dynamic system of differential equations with variable coefficients (1.1) is not possible. Consider a three-dimensional nonlinear syste m with constant coefficients, i.e. demographic factors, factors of artificial reduction of representatives of Christianity and Islam, as well as factors of attraction to their religious group are constant functions.

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1 u(t) + (\beta_1 - \beta_2)u(t)v(t) + (\beta_7 - \beta_8)u(t)w(t) - \gamma_1 u(t) \\ \dfrac{dv(t)}{dt} = \alpha_2 v(t) + (\beta_2 - \beta_1)u(t)v(t) + (\beta_{13} - \beta_{14})v(t)w(t) - \gamma_2 v(t) \\ \dfrac{dw(t)}{dt} = \alpha_5 w(t) + (\beta_8 - \beta_7)u(t)w(t) + (\beta_{14} - \beta_{13})v(t)w(t) \end{cases} \tag{1.3}$$

with initial conditions

$$u(0) = u_0,\ v(0) = v_0,\ w(0) = w_0. \tag{1.4}$$

Let us consider the first special case of a three-dimensional nonlinear system with the constant coefficients

$$\alpha_1 = \gamma_1,\ \alpha_2 = \gamma_2,\ \alpha_5 = 0. \tag{1.5}$$

Taking into account (1.5), problem (1.3), (1.4) will be rewritten in the form

$$\begin{cases} \dot{u} = (\beta_1 - \beta_2)uv + (\beta_7 - \beta_8)uw, \\ \dot{v} = (\beta_2 - \beta_1)uv + (\beta_{13} - \beta_{14})vw, \\ \dot{w} = (\beta_8 - \beta_7)uw + (\beta_{14} - \beta_{13})vw, \end{cases} \tag{1.6}$$

with the initial conditions

$$u(0) = u_0,\ v(0) = v_0,\ w(0) = w_0.$$

The first integral (1.6) has the form

$$u(t) + v(t) + w(t) = u_0 + v_0 + w_0 \equiv a. \tag{1.7}$$

Let's express the unknown function $w(t)$ through two other functions, according to (1.7) and substitute it into the system of equations (1.6)

$$\begin{gathered} w = a - u - v, \\ \dot{u} = (\beta_1 - \beta_2)uv + (\beta_7 - \beta_8)u(a - u - v), \\ \dot{v} = (\beta_2 - \beta_1)uv + (\beta_{13} - \beta_{14})v(a - u - v). \end{gathered} \tag{1.8}$$

Let us perform some algebraic transformations and introduce the notation

$$\begin{gathered} \dot{u} = [(\beta_1 - \beta_2) - (\beta_7 - \beta_8)]uv + (\beta_7 - \beta_8)u^2 + a(\beta_7 - \beta_8)u \equiv F_1(u, v), \\ \dot{v} = [(\beta_1 - \beta_2) - (\beta_{13} - \beta_{14})]uv + (\beta_{13} - \beta_{14})v^2 + a(\beta_{13} - \beta_{14})v \equiv F_2(u, v). \end{gathered} \tag{1.9}$$

We rewrite the system of nonlinear differential equations (1.9) in vector form

$$\tfrac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \overrightarrow{\Lambda}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix},\ \overrightarrow{V}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix},\ \overrightarrow{V}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{1.10}$$

Let's calculate the divergence of the vector field $\overrightarrow{\Lambda}$ and introduce the notation

$$\begin{gathered} div\overrightarrow{\Lambda} = \tfrac{\partial F_1}{\partial u} + \tfrac{\partial F_2}{\partial v} = [(\beta_1 - \beta_2) - (\beta_7 - \beta_8) - 2(\beta_{13} - \beta_{14})]v \\ +[(\beta_2 - \beta_1) - (\beta_{13} - \beta_{14}) - 2(\beta_7 - \beta_8)]u + a(\beta_{13} - \beta_{14}) + a(\beta_7 - \beta_8) \equiv G(u, v). \end{gathered} \tag{1.11}$$

Let us define and study curves (straight lines) on which the divergence of the vector field vanishes

$$G(u,v)=0, \tag{1.12}$$

$$\begin{gathered} u[(\beta_2-\beta_1)-(\beta_{13}-\beta_{14})-2(\beta_7-\beta_8)]+v[(\beta_1-\beta_2)-(\beta_7-\beta_8)-2(\beta_{13}-\beta_{14})] \\ +a(\beta_{13}-\beta_{14})+a(\beta_7-\beta_8)=0. \end{gathered} \tag{1.13}$$

We examine all possible cases in detail.

1. $q\equiv(\beta_2-\beta_1)-(\beta_{13}-\beta_{14})-2(\beta_7-\beta_8)\neq 0.$

$$u=\frac{2(\beta_{13}-\beta_{14})+(\beta_7-\beta_8)-(\beta_1-\beta_2)}{(\beta_2-\beta_1)-(\beta_{13}-\beta_{14})-2(\beta_7-\beta_8)}v+\frac{a(\beta_{14}-\beta_{13})+a(\beta_8-\beta_7)}{(\beta_2-\beta_1)-(\beta_{13}-\beta_{14})-2(\beta_7-\beta_8)}. \tag{1.14}$$

Qualitative analysis of the system of equations (1.6), taking into account the adequacy and non-triviality of the mathematical model leads to a system of restrictions on the variable coefficients of the dynamic system

Case A:

$$\begin{cases} \beta_1-\beta_2>0, \\ \beta_7-\beta_8<0, \\ \beta_{13}-\beta_{14}>0, \\ (\beta_{14}-\beta_{13})+(\beta_8-\beta_7)<0. \end{cases} \tag{1.15}$$

Case B:

$$\begin{cases} \beta_1-\beta_2<0, \\ \beta_7-\beta_8>0, \\ \beta_{13}-\beta_{14}<0, \\ (\beta_{14}-\beta_{13})+(\beta_8-\beta_7)>0. \end{cases} \tag{1.16}$$

Let's take a closer look at each of these cases.

Case A:

$$\begin{cases} \beta_1-\beta_2>0, \\ \beta_7-\beta_8<0, \\ \beta_{13}-\beta_{14}>0, \\ (\beta_{14}-\beta_{13})+(\beta_8-\beta_7)<0. \end{cases}$$

A1. $q\equiv(\beta_2-\beta_1)-(\beta_{13}-\beta_{14})-2(\beta_7-\beta_8)>0$ which is equivalent to the inequality $(\beta_1-\beta_2)+(\beta_{13}-\beta_{14})<2(\beta_8-\beta_7)$.

Let us introduce the notation $q_1\equiv 2(\beta_{13}-\beta_{14})+(\beta_7-\beta_8)-(\beta_1-\beta_2)$.

A1.1 $q_1<0$ which is equivalent to the inequality $2(\beta_{13}-\beta_{14})<(\beta_8-\beta_7)+(\beta_1-\beta_2)$ and at the same time, straight line (1.14) does not pass through the physically meaningful first quarter of the phase plane of solutions of system (1.9) (Figure 1).

A1.2 $q_1>0$ which is equivalent to the inequality $2(\beta_{13}-\beta_{14})>(\beta_8-\beta_7)+(\beta_1-\beta_2)$.

In this case straight line (1.14) passes in the physically meaningful first quarter of the phase plane of solutions of system (1.9), (1.10) (Figure 2).

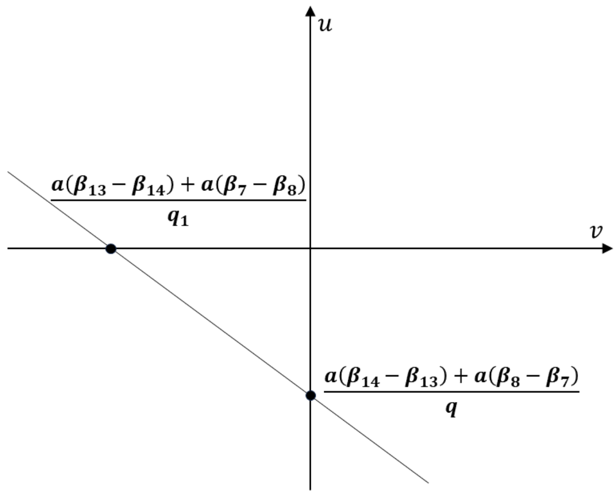

Figure 1

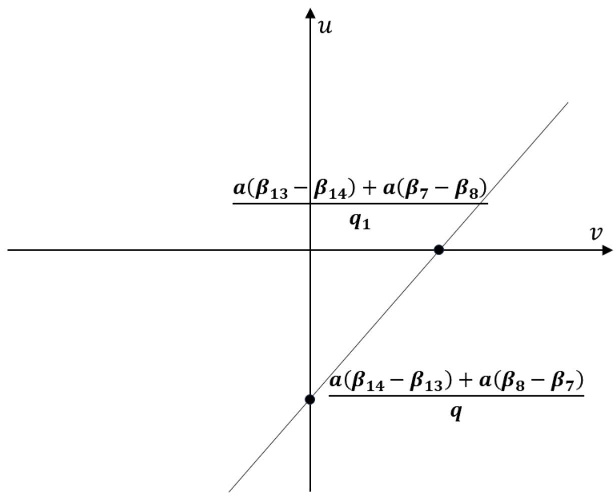

Figure 2

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1 - \beta_2 > 0, \\ \beta_8 - \beta_7 > 0, \\ \beta_{13} - \beta_{14} > 0, \\ (\beta_1 - \beta_2) + (\beta_{13} - \beta_{14}) < 2(\beta_8 - \beta_7), \\ 2(\beta_{13} - \beta_{14}) > (\beta_8 - \beta_7) + (\beta_1 - \beta_2), \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) < 0, \end{cases} \tag{1.17}$$

then the graph (Figure 2) holds.

Now consider the case when the initial conditions (1.10) satisfy the conditions

$$v_0 > \frac{a(\beta_{13}-\beta_{14})+a(\beta_7-\beta_8)}{q_1},\ u_0 = \frac{q_1}{q}v_0 + \frac{a(\beta_{14}-\beta_{13})+a(\beta_8-\beta_7)}{q}. \tag{1.18}$$

The following theorem can now be stated.

Theorem 1.1. *Nonlinear dynamic system (1.9), (1.10) when the system (1.17) is fair and executed (1.18), in some simply connected domain $D \subset (O, v(t), u(t))$ the first quarter of the phase plane $(O, v(t), u(t))$ has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. We show that the divergence of the vector field $\overrightarrow{\Lambda}$ according to (1.11)-(1.13) vanishes on the line

$$u(t) = \frac{q_1}{q}v(t) + \frac{a(\beta_{14}-\beta_{13})+a(\beta_8-\beta_7)}{q} \tag{1.19}$$

phase plane of the solutions $(O, v(t), u(t))$.

Suppose the initial conditions (1.10) satisfy (1.18).

It is clear, that $G(u(t), v(t))$ divergence (1.11)-(1.13) of the vector field $\overrightarrow{\Lambda}\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$, in some simply connected domain $D \subset (O, v(t), w(t))$, containing the $M(v_0, u_0)$ point lying on the line (1.19), changes its sign.

According to the Bendixson principle (criterion), there is a closed integral trajectory of the nonlinear dynamic system (1.9), (1.10), which lies entirely in this domain $D \subset (O, v(t), u(t))$ [14]. □

A.2 $q \equiv (\beta_2-\beta_1)-(\beta_{13}-\beta_{14})-2(\beta_7-\beta_8) < 0$ which is equivalent to the inequality $(\beta_1-\beta_2)+(\beta_{13}-\beta_{14}) > 2(\beta_8-\beta_7)$.

A2.1 $q_1 < 0$ which is equivalent to the inequality $2(\beta_{13}-\beta_{14}) < (\beta_8-\beta_7)+(\beta_1-\beta_2)$.

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1-\beta_2 > 0, \\ \beta_8-\beta_7 > 0, \\ \beta_{13}-\beta_{14} > 0, \\ (\beta_1-\beta_2)+(\beta_{13}-\beta_{14}) > 2(\beta_8-\beta_7), \\ 2(\beta_{13}-\beta_{14}) < (\beta_8-\beta_7)+(\beta_1-\beta_2), \\ (\beta_{14}-\beta_{13})+(\beta_8-\beta_7) < 0. \end{cases} \tag{1.20}$$

In this case straight line (1.14) passes in the physically meaningful first quarter of the phase plane of solutions of system (1.9), (1.10) (Figure 3).

Now consider the case when the initial conditions (1.10) satisfy the conditions

$$v_0 > 0,\ u_0 = \frac{q_1}{q}v_0 + \frac{a(\beta_{14}-\beta_{13})+a(\beta_8-\beta_7)}{q}. \tag{1.21}$$

The following theorem can now be stated.

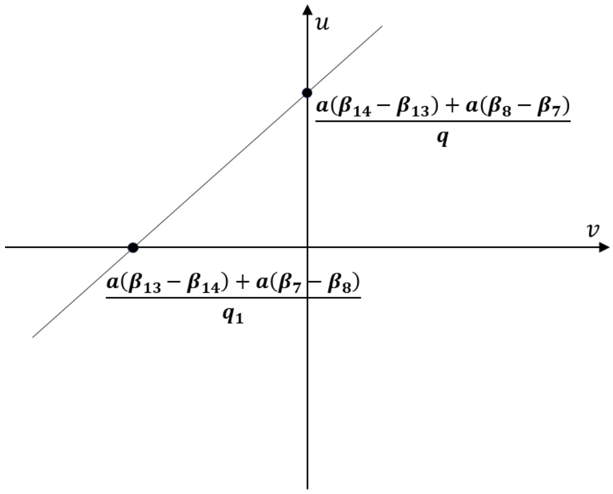

Figure 3

Theorem 1.2. *Nonlinear dynamic system (1.9), (1.10) when the system (1.20) is fair and executed (1.21), in some simply connected domain $D \subset (O, v(t), u(t))$ the first quarter of the phase plane $(O, v(t), u(t))$ has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. We show that the divergence of the vector field $\overrightarrow{\Lambda}$ according to (1.11)-(1.13) vanishes on the line (1.19) phase plane of the solutions $(O, v(t), u(t))$.

Suppose the initial conditions (1.10) satisfy (1.21).

It is clear, that $G(u(t), v(t))$ divergence (1.11)-(1.13) of the vector field $\overrightarrow{\Lambda}\begin{pmatrix}\Lambda_1\\ \\ \Lambda_2\end{pmatrix}$, in some simply connected domain $D \subset (O, v(t), u(t))$, containing the $M(v_0, u_0)$ point lying on the line (1.19), changes its sign.

According to the Bendixson principle (criterion), there is a closed integral trajectory of the nonlinear dynamic system (1.9), (1.10), which lies entirely in this domain $D \subset (O, v(t), u(t))$ [14]. □

A2.2 $q_1 > 0$ which is equivalent to the inequality $2(\beta_{13} - \beta_{14}) > (\beta_8 - \beta_7) + (\beta_1 - \beta_2)$.

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1 - \beta_2 > 0, \\ \beta_8 - \beta_7 > 0, \\ \beta_{13} - \beta_{14} > 0, \\ (\beta_1 - \beta_2) + (\beta_{13} - \beta_{14}) > 2(\beta_8 - \beta_7), \\ 2(\beta_{13} - \beta_{14}) > (\beta_8 - \beta_7) + (\beta_1 - \beta_2), \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) < 0. \end{cases} \tag{1.22}$$

In this case straight line (1.14) passes in the physically meaningful first quarter of the phase plane of solutions of system (1.9), (1.10) (Figure 4).

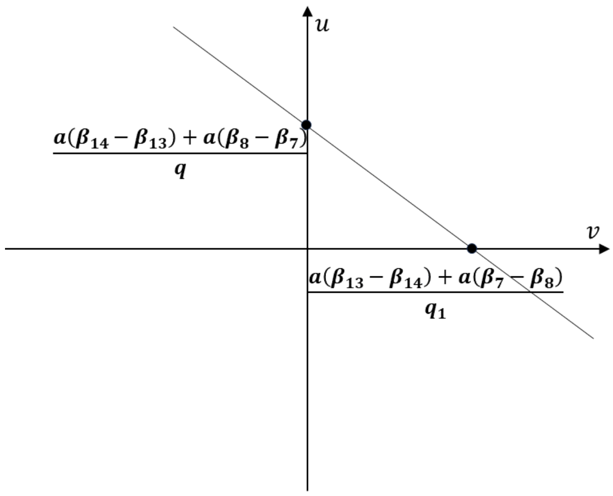

Figure 4

Now consider the case when the initial conditions (1.10) satisfy the conditions

$$0 < v_0 < \frac{a(\beta_{13} - \beta_{14}) + a(\beta_7 - \beta_8)}{q_1},\ u_0 = \frac{q_1}{q} v_0 + \frac{a(\beta_{14} - \beta_{13}) + a(\beta_8 - \beta_7)}{q}. \tag{1.23}$$

The following theorem can now be stated.

Theorem 1.3. *Nonlinear dynamic system (1.9), (1.10) when the system (1.22) is fair and executed (1.23), in some simply connected domain $D \subset (O, v(t), u(t))$ the first quarter of the phase plane $(O, v(t), u(t))$ has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. We show that the divergence of the vector field $\overrightarrow{\Lambda}$ according to (1.11)-(1.13) vanishes on the line (1.19) phase plane of the solutions $(O, v(t), u(t))$.

Suppose the initial conditions (1.10) satisfy (1.23).

It is clear, that $G(u(t), v(t))$ divergence (1.11)-(1.13) of the vector field $\overrightarrow{\Lambda}\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$, in some simply connected domain $D \subset (O, v(t), u(t))$, containing the $M(v_0, u_0)$ point lying on the a segment connecting the points $M_1(0, \frac{a(\beta_{13}-\beta_{14})+a(\beta_7-\beta_8)}{-q})$ and $N(\frac{a(\beta_{13}-\beta_{14})+a(\beta_7-\beta_8)}{q_1}, 0)$, changes its sign.

According to the Bendixson principle (criterion), there is a closed integral trajectory of the nonlinear dynamic system (1.9), (1.10), which lies entirely in this domain $D \subset (O, v(t), u(t))$ [14]. □

Case B:

$$\begin{cases} \beta_1 - \beta_2 < 0, \\ \beta_7 - \beta_8 > 0, \\ \beta_{13} - \beta_{14} < 0, \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) > 0. \end{cases}$$

B1. $q \equiv (\beta_2 - \beta_1) - (\beta_{13} - \beta_{14}) - 2(\beta_7 - \beta_8) > 0$ which is equivalent to the inequality $(\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) > 2(\beta_7 - \beta_8)$.

Let us introduce the notation $q_1 \equiv 2(\beta_{13} - \beta_{14}) + (\beta_7 - \beta_8) - (\beta_2 - \beta_1)$.

B1.1 $q_1 < 0$ which is equivalent to the inequality $2(\beta_{14} - \beta_{13}) > (\beta_7 - \beta_8) + (\beta_2 - \beta_1)$ and at the same time, straight line (1.14) passes through the physically meaningful first quarter of the phase plane of solutions of system (1.9), (1.10) (Figure 5).

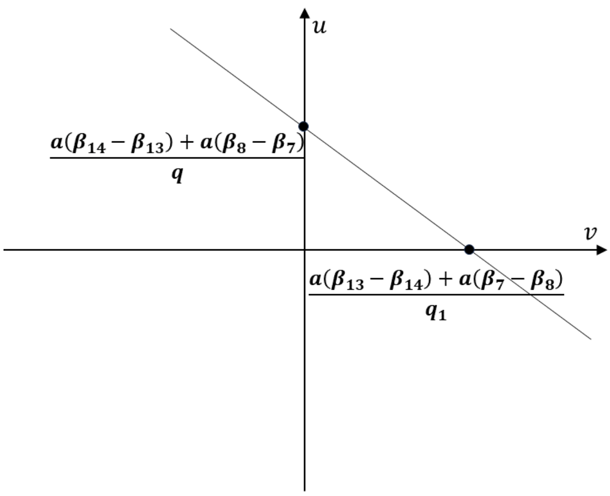

Figure 5

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1 - \beta_2 < 0, \\ \beta_7 - \beta_8 > 0, \\ \beta_{13} - \beta_{14} < 0, \\ (\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) > 2(\beta_7 - \beta_8), \\ 2(\beta_{14} - \beta_{13}) > (\beta_7 - \beta_8) + (\beta_2 - \beta_1), \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) > 0, \end{cases} \tag{1.24}$$

then the graph (Figure 5) holds.

Now consider the case when the initial conditions (1.10) satisfy the conditions

$$0 < v_0 < \frac{a(\beta_{13} - \beta_{14}) + a(\beta_7 - \beta_8)}{q_1}, \; u_0 = \frac{q_1}{q} v_0 + \frac{a(\beta_{14} - \beta_{13}) + a(\beta_8 - \beta_7)}{q}. \tag{1.25}$$

The following theorem can now be stated.

Theorem 1.4. *Nonlinear dynamic system (1.9), (1.10) when the system (1.24) is fair and executed (1.25), in some simply connected domain* $D \subset (O, v(t), u(t))$ *the first quarter of the phase plane* $(O, v(t), u(t))$ *has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. We show that the divergence of the vector field $\overrightarrow{\Lambda}$ according to (1.11)-(1.13) vanishes on the line (1.19) phase plane of the solutions $(O, v(t), u(t))$.

Suppose the initial conditions (1.10) satisfy (1.25).

It is clear, that $G(u(t), v(t))$ divergence (1.11)-(1.13) of the vector field $\overrightarrow{\Lambda}\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$, in some simply connected domain $D \subset (O, v(t), w(t))$, containing the $M(v_0, u_0)$ point lying on the line (1.19), changes its sign.

According to the Bendixson principle (criterion), there is a closed integral trajectory of the nonlinear dynamic system (1.9), (1.10), which lies entirely in this domain $D \subset (O, v(t), u(t))$ [14]. □

B1.2 $q_1 > 0$ which is equivalent to the inequality $2(\beta_{14} - \beta_{13}) < (\beta_7 - \beta_8) + (\beta_2 - \beta_1)$.

In this case straight line (1.14) passes in the physically meaningful first quarter of the phase plane of solutions of system (1.9), (1.10) (Figure 6).

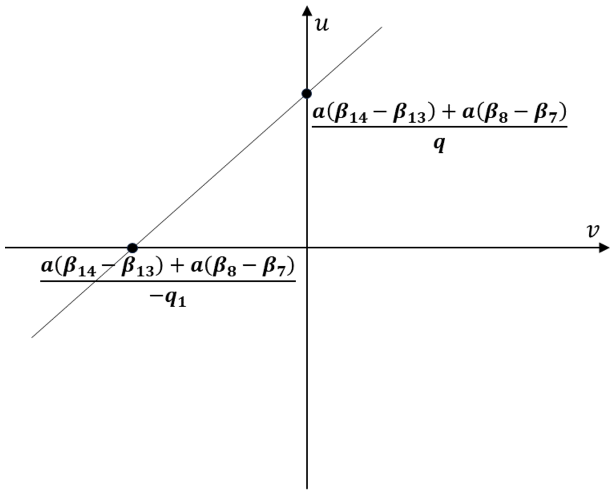

Figure 6

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1 - \beta_2 < 0, \\ \beta_7 - \beta_8 > 0, \\ \beta_{13} - \beta_{14} < 0, \\ (\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) > 2(\beta_7 - \beta_8), \\ 2(\beta_{14} - \beta_{13}) < (\beta_7 - \beta_8) + (\beta_2 - \beta_1), \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) > 0, \end{cases} \tag{1.26}$$

then the graph (Figure 6) holds.

Now consider the case when the initial conditions (1.10) satisfy the conditions

$$0 < v_0,\ u_0 = \frac{q_1}{q} v_0 + \frac{a(\beta_{14} - \beta_{13}) + a(\beta_8 - \beta_7)}{q}. \tag{1.27}$$

The following theorem can now be stated.

Theorem 1.5. *Nonlinear dynamic system (1.9), (1.10) when the system (1.26) is fair and executed (1.27), in some simply connected domain* $D \subset (O, v(t), u(t))$ *the first quarter of the phase plane* $(O, v(t), u(t))$ *has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. We show that the divergence of the vector field $\overrightarrow{\Lambda}$ according to (1.11)-(1.13) vanishes on the line (1.19) phase plane of the solutions $(O, v(t), u(t))$.

Suppose the initial conditions (1.10) satisfy (1.27).

It is clear, that $G(u(t), v(t))$ divergence (1.11)-(1.13) of the vector field $\overrightarrow{\Lambda}\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$, in some simply connected domain $D \subset (O, v(t), w(t))$, containing the $M(v_0, u_0)$ point lying on the line (1.19), changes its sign.

According to the Bendixson principle (criterion), there is a closed integral trajectory of the nonlinear dynamic system (1.9), (1.10), which lies entirely in this domain $D \subset (O, v(t), u(t))$ [14]. □

B.2 $q \equiv (\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) - 2(\beta_7 - \beta_8) < 0$ which is equivalent to the inequality $(\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) < 2(\beta_7 - \beta_8)$.

B2.1 $q_1 < 0$ which is equivalent to the inequality $2(\beta_{13} - \beta_{14}) < (\beta_8 - \beta_7) + (\beta_1 - \beta_2)$.

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1 - \beta_2 < 0, \\ \beta_7 - \beta_8 > 0, \\ \beta_{13} - \beta_{14} < 0, \\ (\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) < 2(\beta_7 - \beta_8), \\ 2(\beta_{14} - \beta_{13}) > (\beta_7 - \beta_8) + (\beta_2 - \beta_1), \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) > 0. \end{cases} \tag{1.28}$$

In this case straight line (1.14) or (1.19) passes in the physically meaningful first quarter of the phase plane of solutions of system (1.9), (1.10) (Figure 7).

Now consider the case when the initial conditions (1.10) satisfy the conditions

$$v_0 > \frac{a(\beta_{13} - \beta_{14}) + a(\beta_7 - \beta_8)}{q_1}, \; u_0 = \frac{q_1}{q} v_0 + \frac{a(\beta_{14} - \beta_{13}) + a(\beta_8 - \beta_7)}{q}. \tag{1.29}$$

The following theorem can now be stated.

Theorem 1.6. *Nonlinear dynamic system (1.9), (1.10) when the system (1.28) is fair and executed (1.29), in some simply connected domain* $D \subset (O, v(t), u(t))$ *the first quarter of the phase plane* $(O, v(t), u(t))$ *has the solution in the form of the closed trajectory which completely lies in this domain.*

Proof. We show that the divergence of the vector field $\overrightarrow{\Lambda}$ according to (1.11)-(1.13) vanishes on the line (1.19) phase plane of the solutions $(O, v(t), u(t))$.

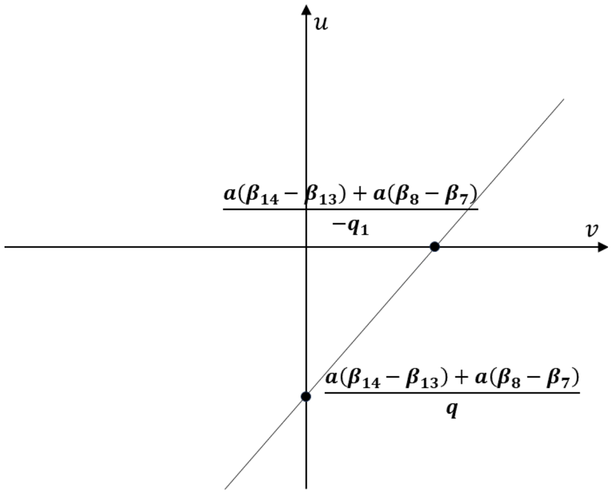

Figure 7

Suppose the initial conditions (1.10) satisfy (1.29).

It is clear, that $G(u(t), v(t))$ divergence (1.11)-(1.13) of the vector field $\overrightarrow{\Lambda}\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$, in some simply connected domain $D \subset (O, v(t), u(t))$, containing the $M(v_0, u_0)$ point lying on the line (1.19), changes its sign.

According to the Bendixson principle (criterion), there is a closed integral trajectory of the nonlinear dynamic system (1.9), (1.10), which lies entirely in this domain $D \subset (O, v(t), u(t))$ [14]. $\square$

B2.2 $q_1 > 0$ which is equivalent to the inequality $2(\beta_{14} - \beta_{13}) < (\beta_7 - \beta_8) + (\beta_2 - \beta_1)$.

Thus, if the system of inequalities is satisfied

$$\begin{cases} \beta_1 - \beta_2 < 0, \\ \beta_7 - \beta_8 > 0, \\ \beta_{13} - \beta_{14} < 0, \\ (\beta_2 - \beta_1) + (\beta_{14} - \beta_{13}) < 2(\beta_7 - \beta_8), \\ 2(\beta_{14} - \beta_{13}) > (\beta_7 - \beta_8) + (\beta_2 - \beta_1), \\ (\beta_{14} - \beta_{13}) + (\beta_8 - \beta_7) > 0. \end{cases} \tag{1.30}$$

In this case straight line (1.14) does not pass through the physically meaningful first quarter of the phase plane of solutions of system (1.9) (Figure 8).

2. Equality of coefficients of attraction to one's side in the interaction of Christians and Muslims. Reduction to the Lotka -Volterra system of differential equations

Let us consider the second special case of the dynamical system (1.3), (1.4), which we rewrite in the following form

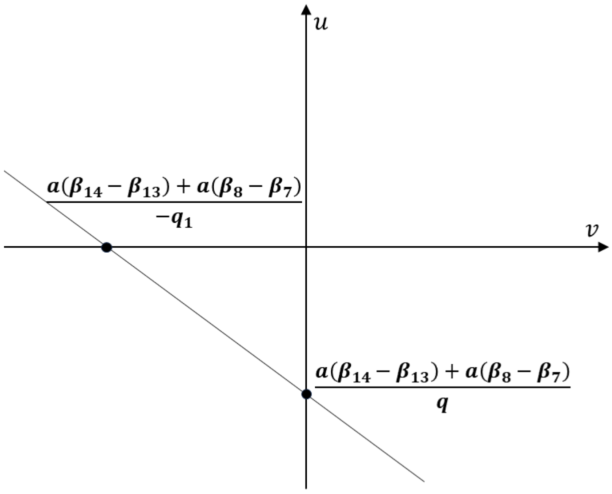

Figure 8

$$\begin{cases} \dfrac{du(t)}{dt} = (\alpha_1-\gamma_1)u(t) + (\beta_1-\beta_2)u(t)v(t) + (\beta_7-\beta_8)u(t)w(t) \\ \dfrac{dv(t)}{dt} = (\alpha_2-\gamma_2)v(t) + (\beta_2-\beta_1)u(t)v(t) + (\beta_{13}-\beta_{14})v(t)w(t) \\ \dfrac{dw(t)}{dt} = \alpha_5 w(t) + (\beta_8-\beta_7)u(t)w(t) + (\beta_{14}-\beta_{13})v(t)w(t) \end{cases} \tag{2.1}$$

with the initial conditions

$$u(0) = u_0, \; v(0) = v_0, \; w(0) = w_0. \tag{2.2}$$

Let us consider the second special case of the dynamic system (2.1), (2.2), which does not contradict the adequacy and non-triviality of the mathematical model and is characterized by the following system of inequalities and equalities

$$\begin{cases} \alpha_5 < 0, \\ \beta_8 - \beta_7 > 0, \\ \beta_{14} - \beta_{13} > 0, \\ \alpha_1 - \gamma_1 > 0, \\ \alpha_2 - \gamma_2 > 0, \\ \beta_1 - \beta_2 = 0. \end{cases} \tag{2.3}$$

When (2.3) is satisfied, system (2.1), (2.2) will take the form

$$\begin{cases} \dfrac{du(t)}{dt} = (\alpha_1-\gamma_1)u(t) + (\beta_7-\beta_8)u(t)w(t), \\ \dfrac{dv(t)}{dt} = (\alpha_2-\gamma_2)v(t) + (\beta_{13}-\beta_{14})v(t)w(t), \\ \dfrac{dw(t)}{dt} = \alpha_5 w(t) + (\beta_8-\beta_7)u(t)w(t) + (\beta_{14}-\beta_{13})v(t)w(t), \end{cases} \tag{2.4}$$

with initial conditions

$$u(0) = u_0,\ v(0) = v_0,\ w(0) = w_0. \tag{2.5}$$

Let us transform (2.4)

$$\begin{cases} \dfrac{du(t)}{udt} = (\alpha_1 - \gamma_1) + (\beta_7 - \beta_8)w(t), \\ \dfrac{dv(t)}{vdt} = (\alpha_2 - \gamma_2) + (\beta_{13} - \beta_{14})w(t), \end{cases} \tag{2.6}$$

and assume that the system of equalities holds

$$\begin{cases} \beta_8 - \beta_7 = \beta_{14} - \beta_{13}, \\ \alpha_1 - \gamma_1 = \alpha_2 - \gamma_2. \end{cases} \tag{2.7}$$

Then from (2.6), we obtain the first integral of the dynamic system (2.4)

$$\begin{gathered} \frac{du(t)}{udt} - \frac{dv(t)}{vdt} = 0, \\ \frac{u(t)}{v(t)} = const = \frac{u_0}{v_0} = p_1, \\ u(t) = p_1 v(t). \end{gathered} \tag{2.8}$$

Substituting (2.8) into the third equation of system (2.4), we can obtain the two-dimensional dynamic system

$$\begin{cases} \dfrac{dv(t)}{dt} = (\alpha_2 - \gamma_2)v(t) + (\beta_{13} - \beta_{14})v(t)w(t), \\ \dfrac{dw(t)}{dt} = \alpha_5 w(t) + (\beta_8 - \beta_7)p_1 v(t)w(t) + (\beta_{14} - \beta_{13})v(t)w(t). \end{cases} \tag{2.9}$$

After some mathematical transformations, we can obtain the first integral of the two-dimensional dynamical system (2.9)

$$\begin{gathered} \frac{dw(t)}{dv(t)} = \frac{w}{v}\frac{\alpha_5 + (\beta_8 - \beta_7)p_1 v(t) + (\beta_{14} - \beta_{13})v(t)}{(\alpha_2 - \gamma_2) + (\beta_{13} - \beta_{14})w(t)}, \\ \int_{w_0}^{w} \frac{(\alpha_2 - \gamma_2) + (\beta_{13} - \beta_{14})w(t)}{w} dw(t) \\ = \int_{v_0}^{v} \frac{(\alpha_5 + (\beta_8 - \beta_7)p_1 v(t) + (\beta_{14} - \beta_{13})v(t)}{v} dv(t), \\ (\alpha_2 - \gamma_2)\ln\frac{w}{w_0} + (\beta_{13} - \beta_{14})(w - w_0) \\ = \alpha_5 \ln\frac{v}{v_0} + [(\beta_8 - \beta_7)p_1 + (\beta_{14} - \beta_{13})](v - v_0). \end{gathered} \tag{2.10}$$

Two-dimensional dynamic system (2.9), under assumption (2.3) is qualitatively similar to the classical 'predator-prey' mathematical model (the Lotka-Volterra system of differential equations) and its first integral (2.10) in the first quarter of the phase plane of solutions is a closed integral curve not touching the coordinate axes.

3. Conclusion

Thus, a new mathematical model in the form of a nonlinear three-dimensional dynamic system is proposed, describing the interaction of the two main world religions of Christianity and Islam, the dynamics of the number of their carriers, as well as the impact on them of powerful groups consisting of atheists united in various communities, including transnational consortia influencing world processes. In some particular cases of constant coefficients of a nonlinear dynamic system, the first integrals are found and the three-dimensional dynamic system is reduced to a two-dimensional one. In some cases of a two-dimensional dynamic system, using the principle (theorem) of Bendixson, theorems on the existence of solutions of a closed integral curve in the first quarter of the phase plane are proved, and in another case the problem is reduced to the classical 'predator-prey' problem (the Lotka-Volterra system of equations). In all cases, the coexistence of three social groups (Christians, Muslims, Atheists) and non-degeneration of any of them are shown.

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