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Iteration Method for the Neumann Boundary Value Problem for the Laplace Equation

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We construct a convergent recurrence scheme for a solution of the Neumann boundary value problem for the Laplace equation. By a special approach, using the potential method we construct a uniquely solvable boundary integral equation containing a selfadjoint compact operator. The single layer potential constructed by the solution of the integral equation gives a particular solution of the Neumann problem if the corresponding necessary condition is satisfied. First, we construct a sequence of successive approximations which converges to the solution of the boundary integral equation in appropriate Bessel-potential spaces of functions defined on the boundary. Afterwards, using these approximations as densities of the single layer potential, we formulate another iteration which converges to a particular solution of the Neumann boundary value problem in the appropriate Sobolev-Slobodetskii spaces of functions defined in the three-dimensional domain under consideration. A general solution of the Neumann boundary value problem is obtained then by adding an arbitrary constant.

Keywords and phrases: Laplace equation, first kind Fredholm equation, layer potentials, Neumann problem, iteration method, method of successive approximations.

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1 Introduction

As it is well known, by the single layer potential the Neumann boundary value problem for a bounded three-dimensional domain $\Omega \subset \mathbb{R}^3$ can be reduced to the Fredholm type boundary integral equation on $S = \partial\Omega$, which is solvable if the corresponding necessary condition for the Neumann datum is satisfied. The null space of the integral operator is not trivial and therefore the integral equation is not uniquely solvable. Solutions of the Neumann problem is defined modulo a constant summand (see, e.g., [7], [8], [14] and the references therein).

We modify the boundary integral equation in such a way that the null space of the modified operator is trivial and, consequently, it is invertible in appropriate function spaces. Moreover,

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if the corresponding necessary condition is satisfied, then the solution of the modified equation is also a solution to the original integral equation generated by the single layer potential.

The modified operator is a compact selfadjoint injective operator. First, we construct a convergent iteration scheme for a solution of the modified integral equation assuming that the above mentioned necessary condition is satisfied. Afterwards, we use the obtained approximations as densities of the single layer potential and construct another explicit iteration which converges to a particular solution of the Neumann boundary value problem in the appropriate Sobolev-Slobodetskii spaces of functions defined in Ω. By adding an arbitrary constant to the particular solution one can obtain a general solution to the Neumann problem under consideration.

By evident modifications, the approach treated here can be extended to the interior Neumann type boundary value problems for formally self-adjoint differential equations of mathematical physics and mechanics, in particular, for the boundary value problem of the elasticity theory when the stress vector is prescribed on the boundary $S = \partial\Omega$.

2 Formulation of the problem and reduction to boundary integral equation

Let $\Omega = \Omega^+ \subset \mathbb{R}^3$ be a bounded three-dimensional domain with connected boundary $S = \partial\Omega$ and let $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$ be the unbounded complement of the domain Ω. Throughout the paper we assume that the boundary S is a Lipschitz surface if not otherwise stated and n stands for the outward directed unite normal vector to S.

By L_2, W_2^r, and H_2^s with $r \geq 0$ and $s \in \mathbb{R}$, we denote the standard Lebesgue space, Sobolev–Slobodetskii space, and Bessel potential space of real-valued functions, respectively. Recall that $H_2^r = W_2^r$ for $r \geq 0$. In what follows, we will drop the subscript 2 and use the notation $W^r = W_2^r$ and $H^s = H_2^s$. By $\langle f\,,\, g\rangle_S$ we denote the duality pairing on $H^{-r}(S) \times H^r(S)$ that extends the standard $L_2(S)$ inner product,

$$\langle f\,,\, g\rangle_S = \int\limits_S f(x)\, g(x)\, dS = (f\,,\, g)_{L_2(S)} \quad \text{for} \quad g, h \in L_2(S). \tag{2.1}$$

We omit the subscript S in the dualiy pairing when no ambiguity can arise.

The symbols $\{\,\cdot\,\}^+$ and $\{\,\cdot\,\}^-$ denote the standard one-sided traces of functions on the surface $S = \partial\Omega^\pm$ from Ω^+ and Ω^- respectively.

Let us consider the Neumann boundary value problem: Find a function $u \in H^1(\Omega)$ satisfying the Laplace equation in Ω and the Neumann boundary condition on S,

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\Big\{ \frac{\partial u}{\partial n} \Big\}^+ = f \quad \text{on} \quad S, \tag{2.3}$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative and $f \in H^{-\frac{1}{2}}(S)$.

Let us look for a solution of the Neumann problem (2.2)-(2.3) in the form of single layer potential (cf. [5], [7, Ch. 11], [14, Ch. 8])

$$u(x) = V(\psi)(x) = \int\limits_S \Gamma(x-y)\, \psi(y)\, dS_y, \quad x \in \Omega, \tag{2.4}$$

with the unknown density $\psi \in H^{-\frac{1}{2}}(S)$ and the fundamental solution

$$\Gamma(x-y) \;=\; -\frac{1}{4\pi}\,\frac{1}{|x-y|}\,.$$

Using the properties of the single layer potential, the boundary condition (2.3) leads to the boundary integral equation (see Appendix A)

$$\Big(-\frac{1}{2}I+\mathcal{K}^*\Big)\psi = f \quad \text{on} \;\; S, \tag{2.5}$$

where

$$\mathcal{K}^*\psi = \int\limits_S \Big[\frac{\partial}{\partial n(x)}\,\Gamma(x-y)\Big]\,\psi(y)\,dS_y, \qquad x\in S. \tag{2.6}$$

Introduce the boundary integral operators generated by the direct values on S of the single and double layer potentials:

$$\mathcal{H}\,\psi(x) = \int\limits_S \Gamma(x-y)\,\psi(y)\,dS_y, \qquad x\in S, \tag{2.7}$$

$$\mathcal{K}\,\psi(x) = \int\limits_S \Big[\frac{\partial}{\partial n(y)}\,\Gamma(x-y)\Big]\,\psi(y)\,dS_y, \qquad x\in S. \tag{2.8}$$

Mapping properties of these operators are collected in Appendix 5. Evidently, $\mathcal{K}^*$ and $\mathcal{K}$ are mutually adjoint singular integral operators, while $\mathcal{H}$ is a self-adjoint operator with weakly singular kernel function. Note that, there holds the relation (see, e.g., [12])

$$\mathcal{H}\,\mathcal{K}^* = \mathcal{K}\,\mathcal{H}. \tag{2.9}$$

Throughout the paper, the adjoint of an operator $\mathcal{A}$ we denote by $\mathcal{A}^*$.
From relation (2.9) we have

$$\big(\mathcal{H}\mathcal{K}^*\big)^* = \mathcal{K}\,\mathcal{H} = \mathcal{H}\,\mathcal{K}^*, \tag{2.10}$$

and consequently $\mathcal{H}\,\mathcal{K}^*$ is a self-adjoint operator.
Since the operator

$$\mathcal{H}: H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S)$$

is an isomorphism (see Appendix A), equation (2.5) is equivalent to the following equation

$$\mathcal{H}\Big(-\frac{1}{2}\,I+\mathcal{K}^*\Big)\psi = \mathcal{H}\,f \quad \text{on} \;\; S. \tag{2.11}$$

The homogeneous equation

$$\Big(-\frac{1}{2}\,I+\mathcal{K}^*\Big)\psi = 0$$

has only one linearly independent solution $\psi_0 \in H^{-\frac{1}{2}}(S)$ and we have (see Appendix A)

$$V(\psi_0)(x) = const = c_0 \neq 0, \qquad x\in\Omega.$$

We can choose ψ_0 such that

$$V(\psi_0)(x) = 1 \quad \text{for} \;\; x\in\Omega, \quad \text{implying} \;\; \mathcal{H}(\psi_0)(x) = 1 \quad \text{for} \;\; x\in S. \tag{2.12}$$

Equation (2.5) is solvable iff (see, e.g., [27, Theorem 3.3])

$$\big\langle f,\, 1\big\rangle := \int\limits_S f(y)\, dS = 0. \tag{2.13}$$

Remark 2.1. Note that if S is a Lipschitz surface, then

$$\psi_0 \in H^1(S).$$

If S is a sufficiently smooth surface, then the function ψ_0 is smoother, e.g., if $S \in C^\infty$, then $\psi_0 \in C^\infty(S)$.

If (2.13) is satisfied, then

$$\big\langle f,\, 1\big\rangle = \big\langle \mathcal{H}^{-1}\,\mathcal{H}\, f,\, 1\big\rangle = \big\langle \mathcal{H}\, f,\, \mathcal{H}^{-1}\, 1\big\rangle = 0$$

and (2.11) is solvable if and only if

$$\big\langle \mathcal{H}\, f,\, \mathcal{H}^{-1}\, 1\big\rangle = 0, \tag{2.14}$$

which is equivalent to (2.13). Further, let us introduce the operator $\mathcal{N}$,

$$\mathcal{N}\,\psi(x) := \mathcal{H}\Big(-\frac{1}{2}\, I + \mathcal{K}^*\Big)\psi(x), \qquad x \in S.$$

Using (2.10) and Theorem 5.1 one can show that the operator $\mathcal{N}$ is self-adjoint and has the following mapping properties:

$$\mathcal{N} : H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \qquad \mathcal{N} : H^0(S) \to H^1(S).$$

If we substitute the single layer potential $u = V(\psi)$ with $\psi \in H^{-\frac{1}{2}}(S)$ in the Green formula

$$\int\limits_\Omega \Delta u\ u\, dx = -\int\limits_\Omega |\nabla u|^2\, dx + \int\limits_S \{\,\partial_n u\,\}^+\,\{\,u\,\}^+ dS \tag{2.15}$$

and use that $\mathcal{H}$ is a self-adjoint operator, we find

$$\int\limits_\Omega |\nabla V(\psi)|^2\, dx = \int\limits_S \Big(-\frac{1}{2}\, I + \mathcal{K}^*\Big)\,\psi\ \mathcal{H}\,\psi\, dS = \Big\langle \mathcal{N}\,\psi,\, \psi\Big\rangle \geq 0,\ \ \forall \psi \in H^{-\frac{1}{2}}(S).$$

Consequently, $\mathcal{N}$ is a nonnegative operator. Moreover, if

$$\Big\langle \mathcal{N}\,\psi,\, \psi\Big\rangle = 0, \qquad \text{then} \qquad \psi \in \ker\Big(-\frac{1}{2}\, I + \mathcal{K}^*\Big), \tag{2.16}$$

that is,

$$\psi(x) = c\ \psi_0(x), \quad c = const. \tag{2.17}$$

Rewrite equation (2.11) as

$$\mathcal{N}\psi = \mathcal{H}f \quad \text{on} \ \ S, \tag{2.18}$$

and instead of (2.18) let us consider the modified equation

$$\mathcal{N}\psi + \Big\langle \mathcal{H}\, 1,\, \psi\Big\rangle \mathcal{H}\, 1 = \mathcal{H}\, f \quad \text{on} \ \ S \tag{2.19}$$

with

$$\mathcal{H}\,1(x) = \int_S \Gamma(x-y)\,dS_y, \quad x \in S.$$

Now, let us introduce the operator $\mathcal{M}$ generated by the left-hand side expression in (2.19),

$$\mathcal{M}\,\psi := \mathcal{N}\psi + \Big\langle \mathcal{H}\,1,\,\psi \Big\rangle \mathcal{H}\,1 = \mathcal{N}\psi + \Big\langle 1,\,\mathcal{H}\,\psi \Big\rangle \mathcal{H}\,1. \tag{2.20}$$

It is evident that $\mathcal{M}$ has the following mapping properties:

$$\mathcal{M}\,:\,H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S), \qquad \mathcal{M}\,:\,H^0(S) \to H^1(S). \tag{2.21}$$

Moreover, $\mathcal{M}$ is a symmetric operator,

$$\begin{aligned}\langle\,\mathcal{M}\,\psi,\,\chi\,\rangle &= \langle\,\mathcal{N}\,\psi,\,\chi\,\rangle + \langle\,\mathcal{H}\,1,\,\psi\,\rangle\langle\,\mathcal{H}\,1,\,\chi\,\rangle \\ &= \langle\,\psi,\,\mathcal{N}\,\chi\,\rangle + \langle\,\psi,\,\langle\,\mathcal{H}\,1,\,\chi\,\rangle\,\mathcal{H}\,1\rangle \\ &= \langle\,\psi,\,\mathcal{N}\,\chi\,+\langle\mathcal{H}\,1,\,\chi\,\rangle\,\mathcal{H}\,1\,\rangle = \langle\,\psi,\,\mathcal{M}\,\chi\,\rangle.\end{aligned} \tag{2.22}$$

From the relation

$$\langle\mathcal{M}\,\psi,\,\psi\rangle = \langle\mathcal{N}\psi,\,\psi\rangle + \langle\mathcal{H}\,1,\,\psi\rangle\langle\mathcal{H}\,1,\,\psi\rangle = \langle\mathcal{N}\psi,\,\psi\rangle + \langle\mathcal{H}\,1,\,\psi\rangle^2, \quad \psi \in H^{-\frac{1}{2}}(S),$$

it follows that $\mathcal{M}$ is a positive operator: $\langle\mathcal{M}\,\psi,\,\psi\rangle > 0$ for $\psi \neq 0$. Indeed, assume that $\langle\mathcal{M}\psi,\,\psi\rangle = 0$ for $\psi \neq 0$. Then $\langle\mathcal{H}\,1,\,\psi\rangle^2 = 0$ and $\langle\mathcal{N}\psi,\,\psi\rangle = 0$. From the later equality, in view of (2.16) and (2.17) we have $\psi = c_0\,\psi_0$ with $c_0 \neq 0$. In view of (2.12)

$$\big\langle\mathcal{H}\,1,\,\psi\big\rangle^2 = \big\langle 1,\,\mathcal{H}\,\psi\big\rangle^2 = \big\langle 1,\,c_0\,\mathcal{H}\,\psi_0\big\rangle^2 = c_0^2\,\big\langle 1,\,\mathcal{H}\,\psi_0\big\rangle^2 = c_0^2\,\big\langle 1, 1\big\rangle^2 = c_0^2\,|S|^2 > 0,$$

where $|S|$ is the area of the surface S. This contradiction shows that $\mathcal{M}$ is a positive operator.

Remark 2.2. Note that if condition (2.14) is fulfilled and $\psi \in H^{-\frac{1}{2}}(S)$ solves the equation

$$\mathcal{M}\,\psi \equiv \mathcal{N}\,\psi + \langle\mathcal{H}1,\,\psi\rangle\,\mathcal{H}1 = \mathcal{H}f, \tag{2.23}$$

then the ψ will be also a solution of equations (2.18) and (2.5). Indeed, let ψ be a solution to equation (2.23) under the condition $\langle\mathcal{H}\,f,\,\mathcal{H}^{-1}1\rangle = \langle f,\,1\rangle = 0$. Let us show that then ψ meets the condition

$$\langle\mathcal{H}\,1,\,\psi\rangle = 0.$$

Applying the operator $\mathcal{H}^{-1}$ to equation (2.23) we get

$$\mathcal{H}^{-1}\,\mathcal{M}\,\psi = \Big(-\frac{1}{2}I + \mathcal{K}^*\Big)\psi + \langle\mathcal{H}\,1,\,\psi\rangle\cdot 1 = f. \tag{2.24}$$

Keeping in mind that $\big\langle\big\{\frac{\partial}{\partial n}V(\psi)\big\}^+,\,1\big\rangle = \big\langle\big(-\frac{1}{2}I + \mathcal{K}^*\big)\psi\,,\,1\big\rangle = 0$ and $\langle\mathcal{H}\,1,\,\psi\rangle$ is constant, from (2.24) we find

$$\langle\langle\mathcal{H}\,1,\,\psi\rangle\,,\,1\rangle = \langle\mathcal{H}\,1,\,\psi\rangle\;\langle 1\,,\,1\rangle = \langle\mathcal{H}\,1,\,\psi\,\rangle|S| = \langle f,\,1\rangle = 0.$$

This proves that ψ solves both equations (2.18) and (2.5).

Remark 2.3. Let us show that if S is a Lipschitz surface then the operator

$$\mathcal{P} \equiv -\frac{1}{2}I + \mathcal{K}^* + \langle \mathcal{H}\,1,\,\cdot\,\rangle \;:\; H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S) \tag{2.25}$$

is an isomorphism, implying that the equation

$$\mathcal{P}\,\psi \equiv \Big(-\frac{1}{2}I + \mathcal{K}^*\Big)\psi + \langle \mathcal{H}\,1,\,\psi\,\rangle = f$$

is uniquely solvable in the space $H^{-\frac{1}{2}}(S)$ for arbitrary $f \in H^{-\frac{1}{2}}(S)$.

Note that the operator defined by the relation

$$\langle \mathcal{H}\,1,\,\psi\,\rangle = \langle\,1,\,\mathcal{H}\,\psi\,\rangle = \int\limits_S \Big(\int\limits_S \Gamma(x-y)\,\psi(y)\,dS_y\Big)dS_x$$

is a compact perturbation of the operator

$$\Big(-\frac{1}{2}I + \mathcal{K}^*\Big) : H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S),$$

which is a Fredholm operator with zero index (see Theorem 5.1(iii) in Appendix A). Therefore, it remains to show that the null-space of the operator $\mathcal{P}$ is trivial, $\ker\mathcal{P} = \{0\}$. Let $\psi \in H^{-\frac{1}{2}}(S)$ be a solution to the homogeneous equation $\mathcal{P}\psi = 0$. Then the single layer potential $u_0 = V(\psi) \in H^1(\Omega)$ solves the following nonlocal boundary value problem

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \ \Omega,\\ &\Big\{\frac{\partial u}{\partial n}\Big\}^+ + \Big\langle\,1,\,\{u\}^+\,\Big\rangle = 0 \qquad \text{on } \ S. \end{aligned}$$

Using Green's formula we get

$$\begin{aligned} \int\limits_\Omega |\nabla u_0(x)|^2\,dx &= \int\limits_S \Big\{\frac{\partial u_0}{\partial n}\Big\}^+\{u_0\}^+\,dS\\ &= -\Big\langle\,1,\,\{u_0\}^+\,\Big\rangle \int\limits_S \{u_0\}^+\,dS = -\Big\langle\,1,\,\{u_0\}^+\,\Big\rangle^2. \end{aligned}$$

Therefore, $\nabla u_0(x) = 0$ in Ω and $\Big\langle\,1,\,\{u_0\}^+\,\Big\rangle = 0$, that is,

$$u_0(x) = const = C_0 \quad \text{ in } \ \Omega$$

and

$$\Big\langle\,1,\,\{u_0\}^+\,\Big\rangle = \Big\langle\,1,\,C_0\,\Big\rangle = \int\limits_S C_0\,dS = C_0\,|S| = 0.$$

Thus, $C_0 = 0$ and $u_0(x) = V(\psi_0)(x) = 0$ in Ω, implying $\mathcal{H}\psi_0 = 0$ on S. Consequently, $\psi_0 = 0$ on S, that is, the homogeneous equation $\mathcal{P}\psi = 0$ has only the trivial solution and the operator (2.25) is invertible.

Remark 2.4. Using Theorem 5.1(iii) we can easily show that if S is a Lipschitz surface, then the operator

$$\mathcal{P} \;:\; L_2(S) \to L_2(S) \tag{2.26}$$

is an isomorphism.

Now, we prove the following assertion.

Theorem 2.5. *The operator $\mathcal{M}$ is coercive, i.e., there is a positive constant δ such that*

$$\big\langle \mathcal{M}\,\psi,\,\psi \big\rangle \geq \delta\,\|\psi\|^2_{H^{-\frac{1}{2}}(S)}, \qquad \forall\,\psi \in H^{-\frac{1}{2}}(S). \tag{2.27}$$

Proof. By contradiction, assume that (2.27) is not true. Then for every natural number $k \in \mathbb{N}$ there exists a function $\psi_k \in H^{-1/2}(S)$ satisfying the inequality

$$\big\langle \mathcal{M}\,\psi_k,\,\psi_k \big\rangle \leq \frac{1}{k}\,\|\psi_k\|^2_{H^{-\frac{1}{2}}(S)}, \quad k = 1, 2, 3, ...,$$

that is,

$$\big\langle \mathcal{N}\,\psi_k,\,\psi_k \big\rangle + \big\langle 1,\,\mathcal{H}\,\psi_k \big\rangle^2 \leq \frac{1}{k}\,\|\psi_k\|^2_{H^{-\frac{1}{2}}(S)}. \tag{2.28}$$

Without loss of generality we can assume that

$$\|\psi_k\|_{H^{-\frac{1}{2}}(S)} = 1.$$

Then inequality(2.28) takes the form

$$\big\langle \mathcal{N}\,\psi_k,\,\psi_k \big\rangle + \big\langle 1,\,\mathcal{H}\,\psi_k \big\rangle^2 \leq \frac{1}{k}, \quad k = 1, 2, 3, ... \tag{2.29}$$

Let us construct the single layer potentials with densities ψ_k,

$$u_k = V(\psi_k) \in H^1(\Omega), \quad k = 1, 2, 3, ... \tag{2.30}$$

and rewrite (2.29) as follows

$$\Big\langle \mathcal{H}\Big(-\frac{1}{2}I + \mathcal{K}^*\Big)\psi_k, \psi_k \Big\rangle + \Big\langle 1,\,\mathcal{H}\psi_k \Big\rangle^2 = \\ = \Big\langle \Big\{ \frac{\partial V(\psi_k)}{\partial n} \Big\}^+,\, \big\{ V(\psi_k) \big\}^+ \Big\rangle + \Big\langle 1,\, \big\{ V(\psi_k) \big\}^+ \Big\rangle^2 \leq \frac{1}{k}\,,$$

that is,

$$\Big\langle \Big\{ \frac{\partial u_k}{\partial n} \Big\}^+,\, \big\{ u_k \big\}^+ \Big\rangle + \Big\langle 1,\, \big\{ u_k \big\}^+ \Big\rangle^2 \leq \frac{1}{k}\,. \tag{2.31}$$

From Green's formula we have (see (2.15))

$$\Big\langle \Big\{ \frac{\partial u_k}{\partial n} \Big\}^+,\, \big\{ u_k \big\}^+ \Big\rangle = \int\limits_{\Omega} |\nabla u_k|^2\,dx. \tag{2.32}$$

Therefore, due to (2.31) and (2.32)

$$\int\limits_{\Omega} |\nabla u_k|^2\,dx \to 0, \qquad \text{that is ,} \qquad \nabla u_k \to 0, \quad \text{as} \;\; k \to \infty \;\; \text{in the sense of } L_2(\Omega) \tag{2.33}$$

and

$$\big\langle 1,\, \{u_k\}^+ \big\rangle^2 = \big\langle 1,\, \mathcal{H}\,\psi_k \big\rangle^2 \to 0, \quad \text{as} \;\; k \to \infty. \tag{2.34}$$

Let us introduce the notation

$$v_k(x) = V(\psi_k)(x) - c_k = u_k(x) - c_k,$$

where

$$c_k = \frac{1}{|\Omega|} \int\limits_{\Omega} V(\psi_k)(x)\, dx = \frac{1}{|\Omega|} \int\limits_{\Omega} u_k(x)\, dx. \tag{2.35}$$

It is evident that $v_k \in H^1(\Omega)$ and

$$\nabla v_k(x) = \nabla u_k(x) = \nabla V(\psi_k)(x). \tag{2.36}$$

By the Poincar inequality we have

$$\begin{aligned}\|v_k\|_{L_2(\Omega)} = \|u_k - c_k\|_{L_2(\Omega)} = \|V(\psi_k) - c_k\|_{L_2(\Omega)} \le \\ \le c^* \, \|\nabla u_k\|_{L_2(\Omega)} = c^* \, \|\nabla v_k\|_{L_2(\Omega)} = c^* \, \|\nabla V(\psi_k)\|_{L_2(\Omega)},\end{aligned}$$

where $c^* = \mathit{const} > 0$ does not depend on k. Further, using (2.33) and (2.36) we conclude

$$\|v_k\|_{L_2(\Omega)} = \|u_k - c_k\|_{L_2(\Omega)} = \|V(\psi_k) - c_k\|_{L_2(\Omega)} \to 0, \quad \text{as} \;\; k \to \infty.$$

Thus

$$\|v_k\|_{H^1(\Omega)} = \|u_k - c_k\|_{H^1(\Omega)} = \|V(\psi_k) - c_k\|_{H^1(\Omega)} \to 0, \quad \text{as} \;\; k \to \infty.$$

Let us show that

$$\|\{v_k\}^+\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|\{u_k\}^+ - c_k\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|\mathcal{H}\psi_k - c_k\|_{H^{\frac{1}{2}}(\partial\Omega)} \to 0, \tag{2.37}$$

$$\Big\|\Big\{\frac{\partial v_k}{\partial n}\Big\}^+\Big\|_{H^{-\frac{1}{2}}(\partial\Omega)} = \Big\|\Big\{\frac{\partial u_k}{\partial n}\Big\}^+\Big\|_{H^{-\frac{1}{2}}(\partial\Omega)} = \Big\|\Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k\Big\|_{H^{-\frac{1}{2}}(\partial\Omega)} \to 0. \tag{2.38}$$

Indeed, relation (2.37) follows from the trace theorem, $\|\{u\}^+\|_{H^{\frac{1}{2}}(S)} \le c\, \|u\|_{H^1(\Omega)}$ with c independent of u, whereas (2.38) is a consequence of definition of the generalized trace $\{\frac{\partial V(\psi_k)}{\partial n}\}^+ \in H^{-\frac{1}{2}}(S)$ on S of the normal derivative of the harmonic single layer potential $V(\psi_k)$, which is defined by the relation (see, e.g., [14])

$$\Big\langle\Big\{\frac{\partial V(\psi_k)}{\partial n}\Big\}^+,\, \varphi\Big\rangle_S = \int\limits_{\Omega} \nabla V(\psi_k) \cdot \nabla E(\varphi)\, dx, \quad \forall \varphi \in H^{\frac{1}{2}}(S), \tag{2.39}$$

where $E(\varphi)$ is a bounded "extension" of φ into Ω, that is, $E : H^{-\frac{1}{2}}(S) \to H^1(\Omega)$ is a retraction operator with the trace operator as the corretraction operator belonging to E ([26, Section 1.2.4]): $E(\varphi) \in H^1(\Omega)$ and $\|E\varphi\|_{H^1(\Omega)} \le C\|\varphi\|_{H^{\frac{1}{2}}(S)}$. Here the central dot denotes the dot-product in $\mathbb{R}^3$. Further, since

$$\Big\{\frac{\partial V(\psi_k)}{\partial n}\Big\}^+ = \Big\{\frac{\partial u_k}{\partial n}\Big\}^+ = \Big\{\frac{\partial v_k}{\partial n}\Big\}^+,$$

from (2.39) we have

$$\Big\|\Big\{\frac{\partial v_k}{\partial n}\Big\}^+\Big\|_{H^{-\frac{1}{2}}(S)} = \Big\|\Big\{\frac{\partial u_k}{\partial n}\Big\}^+\Big\|_{H^{-\frac{1}{2}}(S)} = \Big\|\Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k\Big\|_{H^{-\frac{1}{2}}(S)}$$
$$\leq C^*\,\|\nabla V(\psi_k)\|_{L_2(\Omega)} = C^*\,\|\nabla v_k\|_{L_2(\Omega)} \leq C_1^*\,\|v_k\|_{H^1(\Omega)} \to 0, \quad \text{as} \;\; k \to \infty.$$

Thus (2.38) holds true.

For the harmonic function $v_k(x) = u_k(x) - c_k = V(\psi_k)(x) - c_k$, where u_k is the single layer potential (2.30) and c_k is given by (2.35), let us introduce the notation

$$\begin{aligned} f_k :=& \Big\{\frac{\partial v_k}{\partial n}\Big\}^+ + \Big\langle 1,\, \{v_k\}^+\Big\rangle = \Big\{\frac{\partial u_k}{\partial n}\Big\}^+ + \Big\langle 1,\, \{u_k\}^+ - c_k\Big\rangle \\ =& \Big\{\frac{\partial V(\psi_k)}{\partial n}\Big\}^+ + \Big\langle 1,\, \mathcal{H}\,\psi_k - c_k\Big\rangle \\ =& \Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k + \Big\langle 1,\, \mathcal{H}\,\psi_k\Big\rangle - \Big\langle 1,\, c_k\Big\rangle. \end{aligned} \tag{2.40}$$

Since

$$\langle 1,\, c_k\rangle = \int\limits_S c_k\, dS = |S|\, c_k,$$

we can rewrite (2.40) as

$$\Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k + \Big\langle 1,\, \mathcal{H}\,\psi_k\Big\rangle - |S|\,c_k = f_k. \tag{2.41}$$

Due to relations (2.34), (2.37), and (2.38), we have

$$b_k := \big\langle 1,\, \mathcal{H}\,\psi_k\big\rangle \to 0, \quad \text{as} \;\; k \to \infty, \tag{2.42}$$
$$\|f_k\|_{H^{-\frac{1}{2}}(S)} \to 0, \quad \text{as} \;\; k \to \infty, \tag{2.43}$$
$$\Big\|\Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k\Big\|_{H^{-\frac{1}{2}}(S)} \to 0, \quad \text{as} \;\; k \to \infty. \tag{2.44}$$

Remark that the norm of the constant b_k in space $H^{-\frac{1}{2}}(S)$ is equivalent to the following expression ([25], [19]):

$$\begin{aligned} \|b_k\|^2_{H^{-\frac{1}{2}}(S)} &\sim \langle \mathcal{H}\,b_k,\, b_k\rangle = \int\limits_S (\mathcal{H}\,b_k)\,b_k\; dS = \int\limits_S\int\limits_S \Gamma(x-y)\,b_k\,b_k\; dS_x\, dS_y \\ &= b_k^2 \int\limits_S\int\limits_S \Gamma(x-y)\; dS_x\, dS_y \to 0, \quad \text{as} \;\; k \to \infty. \end{aligned} \tag{2.45}$$

Therefore from (2.41) in view of (2.42) - (2.45) we have

$$c_k\,|S| = \Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k + \big\langle 1,\, \mathcal{H}\,\psi_k\big\rangle - f_k,$$

that is,

$$\|\, c_k\,|S|\,\|_{H^{-\frac{1}{2}}(S)} \leq \Big\|\Big(-\frac{1}{2}\,I + \mathcal{K}^*\Big)\psi_k\,\Big\|_{H^{-\frac{1}{2}}(S)} + \|b_k\|_{H^{-\frac{1}{2}}(S)} + \|f_k\|_{H^{-\frac{1}{2}}(S)} \to 0, \quad \text{as} \;\; k \to \infty.$$

Using the equivalence

$$\| c_k \, |S| \, \|^2_{H^{-\frac{1}{2}}(S)} \sim \big\langle \mathcal{H} \, c_k \, |S|, \, c_k \, |S| \, \big\rangle = c_k^2 \, |S|^2 \int\limits_S \int\limits_S \Gamma(x-y) \, dS_x \, dS_y$$

and the strict inequality $\int\limits_S \int\limits_S \Gamma(x-y) \, dS_x \, dS_y \geq \delta > 0$, we eventually conclude

$$c_k \to 0, \quad \text{as} \ \ k \to \infty.$$

Now, rewrite (2.41) in the following form

$$\Big(-\frac{1}{2} \, I + \mathcal{K}^* \Big) \psi_k + \big\langle 1, \, \mathcal{H} \, \psi_k \big\rangle = f_k + |S| \, c_k,$$

that is,

$$\mathcal{P} \, \psi_k = f_k + |S| \, c_k.$$

Using invertibility of the operator (2.25) we deduce

$$\psi_k = \mathcal{P}^{-1} \, (f_k + |S| \, c_k), \tag{2.46}$$

where

$$\mathcal{P}^{-1} \ : \ H^{-\frac{1}{2}}(S) \to H^{-\frac{1}{2}}(S)$$

is a bounded operator. Therefore, from (2.46) we finally conclude

$$\|\psi_k\|_{H^{-\frac{1}{2}}(S)} \ \leq \ C^* \, \| \, f_k + |S| \, c_k \, \|_{H^{-\frac{1}{2}}(S)} \ \to \ 0 \quad \text{as} \ \ k \to \infty.$$

This contradicts the condition $\|\psi\|_{H^{-\frac{1}{2}}(S)} = 1$. Thus the operator $\mathcal{M}$ is coercive and inequality (2.27) holds true. □

The coercivity property (2.27) and relation (2.22) along with the embedding theorems for the Bessel potential spaces lead to the following assertion.

Corollary 2.6. *The operator*

$$\mathcal{M} : H^{-\frac{1}{2}}(S) \to H^{\frac{1}{2}}(S)$$

is invertible.
Consequently, the equation

$$\mathcal{M} \, \psi = \mathcal{H} \, f \quad \textit{on} \ \ S \tag{2.47}$$

possesses a unique solution $\psi \in H^{-\frac{1}{2}}(S)$ *for arbitrary* $f \in H^{-\frac{1}{2}}(S)$.

Remark 2.7. From the results obtained above it follows that if $\psi \in H^{-\frac{1}{2}}(S)$ is a solution to the uniquely solvable integral equation (2.47) and the orthogonality condition $\big\langle f, \, 1 \big\rangle = 0$ is fulfilled, then $u = V(\psi) \in H^1(\Omega)$ is a particular solution of the Neumann problem (2.2)-(2.3). The general solution to the Neumann problem, which is defined modulo a constant summand, is given then by the formula

$$v(x) = V(\psi)(x) + C,$$

where C is an arbitrary constant.

Remark 2.8. From Remark 2.4 and invertibility of operators (2.26) and (5.1) it follows that that the operator

$$\mathcal{M} : L_2(S) \to H^1(S) \tag{2.48}$$

is invertible (cf., [17, Proposition 2.1]).

3 Iteration scheme for the boundary integral equation

Here we construct a convergent iteration for a unique solution ψ of equation (2.47) assuming that $\langle f, 1\rangle = \langle \mathcal{H} f, \mathcal{H}^{-1} 1\rangle = 0$. Based on this approximation we will construct another iteration, which converges to a particular solution of the Neumann problem $u = V(\psi) \in H^1(\Omega)$. The appropriate function spaces will be specified below.

3.1 Auxiliary material

It is well known that the Bessel potential spaces $H^{-\frac{1}{2}}(S)$ is a separable Hilbert space with an inner product of a special type defined with the help of the orthogonal projection operator (see, e.g., [14, Ch. 3]). For our analysis, introduction of another inner product, which generates an equivalent norm, is more appropriate.

Let us define a bilinear functional in the space $H^{-\frac{1}{2}}(S) \times H^{-\frac{1}{2}}(S)$ (cf. [25])

$$((f, g)) = \big\langle \mathcal{M} f,\, g \big\rangle_S, \quad f, g \in H^{-\frac{1}{2}}(S). \tag{3.1}$$

Lemma 3.1. *The bilinear functional* (3.1) *defines an inner product in the space $H^{-\frac{1}{2}}(S)$, that is, the following conditions hold for arbitrary $h, g, f \in H^{-\frac{1}{2}}(S)$:*

$$\begin{aligned}
&((h, ag + bf)) = a\,((h, g)) + b\,((h, f)) \quad \text{for arbitrary } \ a, b \in \mathbb{R},\\
&((h, g)) = \big\langle \mathcal{M} h,\, g \big\rangle_S = \big\langle h,\, \mathcal{M} g \big\rangle_S = \big\langle \mathcal{M} g,\, h \big\rangle_S = ((g, h)),\\
&((h, h)) \geq 0 \quad \text{and} \quad ((h, h)) = 0 \quad \text{implies} \ \ h = 0.
\end{aligned}$$

Moreover, there are positive constants δ_1 and δ_2, such that for arbitrary $h \in H^{-\frac{1}{2}}(S)$ the following inequality holds

$$\delta_2 \, \|h\|^2_{H^{-\frac{1}{2}}(S)} \geqslant ((h, h)) = \big\langle \mathcal{M} h,\, h \big\rangle_S \geqslant \delta_1 \, \|h\|^2_{H^{-\frac{1}{2}}(S)}. \tag{3.2}$$

Proof. It directly follows from relations (2.20), (2.22), Theorem 2.5 and Theorem 5.1. □

Evidently, the inner product (3.1) generates an equivalent norm in the space $H^{-\frac{1}{2}}(S)$, due to inequality (3.2). Therefore, in what follows, we can consider the totality of functions from the space $H^{-\frac{1}{2}}(S)$ as a separable Hilbert space with the inner product defined by (3.1), which defines the norm (cf. [2, Proposition 5.1], [14, Chapter 3], [25])

$$||h||^2_* = ((h, h)) = \big\langle \mathcal{M} h,\, h \big\rangle_S \quad \text{for arbitrary} \quad h \in H^{-\frac{1}{2}}(S).$$

Denote this separable Hilbert space by $\mathbf{H}^{-\frac{1}{2}}(S)$. There holds the following assertion.

Lemma 3.2. (i) *The operator*

$$\mathcal{M} : \mathbf{H}^{-\frac{1}{2}}(S) \to \mathbf{H}^{-\frac{1}{2}}(S) \tag{3.3}$$

is a compact symmetric positive definite operator.
(ii) *There exists a countable decreasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive eigenvalues of the operator* (3.3) *(by taking into account multiplicities of the eigenvalues)*

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_k \geqslant \cdots \tag{3.4}$$

converging to zero and an orthonormal basis in the Hilbert space $\mathbf{H}^{-\frac{1}{2}}(S)$ *of the corresponding eigenfunctions* $\{\varphi_k\}_{k=1}^{\infty}$ *with* $\mathcal{M}\varphi_k(x) = \lambda_k\varphi_k(x)$ *on* S.

(iii) *For an arbitrary function* $h \in \mathbf{H}^{-\frac{1}{2}}(S)$ *the following series are convergent in the sense of the norm of the Hilbert space* $\mathbf{H}^{-\frac{1}{2}}(S)$*:*

$$\mathcal{M}h(x) = \sum_{k=1}^{\infty} \lambda_k\, b_k\, \varphi_k(x), \quad h(x) = \sum_{k=1}^{\infty} b_k\, \varphi_k(x) \quad \text{with} \quad b_k = ((h, \varphi_k)).$$

Moreover,

$$||\mathcal{M}h||_*^2 = \sum_{k=1}^{\infty} \lambda_k^2 b_k^2, \quad ||h||_*^2 = \sum_{k=1}^{\infty} b_k^2\,.$$

Proof. It follows from the relations

$$((\mathcal{M}h,\, g)) = \big\langle \mathcal{M}^2 h,\, g\big\rangle_S = \big\langle \mathcal{M}h,\, \mathcal{M}g\big\rangle_S = ((h,\, \mathcal{M}g)),$$
$$((\mathcal{M}h,\, h)) = \big\langle \mathcal{M}h,\, \mathcal{M}h\big\rangle_S = ||\mathcal{M}h||^2_{L_2(S)} > 0 \quad \text{for} \quad h \neq 0,$$

along with definition (3.1), mapping property (2.21), compact embedding theorems for the Bessel potential spaces, the well-known Hilbert-Schmidt theorem for separable Hilbert spaces and Parseval's identity (see, e.g., [7, Chapter VIII], [9, Chapter 6], [24, Part V, Chapter 38], [26]). □

Remark 3.3. The elements of the orthonormal basis of eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ of the operator (3.3) satisfy the equation $\mathcal{M}\varphi_k = \lambda_k\varphi_k$ and due to Theorem 5.1(iii) the following inclusions are valid

$$\varphi_k \in H^{\frac{1}{2}}(S) \subset L_2(S), \quad k = 1, 2, 3, ...$$

Observe that Theorem 5.1(iii) yields the following possible maximal smoothness

$$\varphi_k \in H^1(S) \subset L_2(S), \quad k = 1, 2, 3, ...$$

Consequently, the orthonormality property of the system $\{\varphi_k(x)\}_{k=1}^{\infty}$ with respect to the inner product (3.1) can be written in the classical integral form

$$((\varphi_k, \varphi_j)) = \big\langle \mathcal{M}\varphi_k,\, \varphi_j\big\rangle = \lambda_k \left\langle \varphi_k,\, \varphi_j\right\rangle = \lambda_k \left(\varphi_k,\, \varphi_j\right)_{L_2(S)} = 0, \quad k \neq j, \;\; k, j = 1, 2, 3, ...$$
$$||\varphi_k||_*^2 = ((\varphi_k, \varphi_k)) = \langle \mathcal{M}\varphi_k, \varphi_k\rangle = \lambda_k\, ||\varphi_k||^2_{L_2(S)} = 1, \quad k = 1, 2, 3, ...$$

Hence, the system $\{\varphi_k\}_{k=1}^{\infty}$ is orthogonal with respect to the $L_2(S)$ inner product sense as well and the sequence of eigenfunctions $\{\omega_k(x)\}_{k=1}^{\infty}$ with

$$\omega_k(x) = \frac{\varphi_k(x)}{||\varphi_k||_{L_2(S)}} = \sqrt{\lambda_k}\, \varphi_k(x), \;\; k = 1, 2, 3, ... \tag{3.5}$$

is an orthonormal basis in the Hilbert space $L_2(S)$.
Therefore, any function $g \in L_2(S)$ is representable in the form

$$g(x) = \sum_{k=1}^{\infty} c_k\, \omega_k(x) \quad \text{with} \quad c_k = (g, \omega_k)_{L_2(S)} = \sqrt{\lambda_k}(g, \varphi_k)_{L_2(S)}, \tag{3.6}$$

and in view of Parseval's identity we have

$$||g||^2_{L_2(S)} = \sum_{k=1}^{\infty} c_k^2 = \sum_{k=1}^{\infty} \lambda_k (g, \varphi_k)^2_{L_2(S)}.$$

Taking into account relation (3.5), from (3.6) we get

$$g(x) = \sum_{k=1}^{\infty} \lambda_k (g,\, \varphi_k)_{L_2(S)}\, \varphi_k(x) = \sum_{k=1}^{\infty} ((g,\, \varphi_k))\, \varphi_k(x).$$

3.2 Iteration scheme in the space $H^{-\frac{1}{2}}(S)$

Consider the boundary integral equation

$$\mathcal{M}\psi = \mathcal{H}f \quad \text{on} \quad S, \tag{3.7}$$

where S is an arbitrary Lipschitz surface and $f \in H^{-\frac{1}{2}}(S)$. In view of Corollary 2.6 this equation is uniquely solvable and the solution ψ belongs to the space $H^{-\frac{1}{2}}(S)$.
Let us consider the following iteration

$$\psi^{(n)}(x) = \psi^{(n-1)}(x) + \tau \left(\mathcal{H}f(x) - \mathcal{M}\psi^{(n-1)}(x)\right), \quad n = 1, 2, 3, \ldots, \tag{3.8}$$

where $\psi^{(0)} \in H^{\frac{1}{2}}(S)$ is an arbitrary function and the parameter τ satisfies the inequality

$$0 < \tau\, \lambda_1 < 2 \tag{3.9}$$

with λ_1 being the greatest eigenvalue of the operator (3.3) (see (3.4)). A rough estimate of the first eigenvalue λ_1 for an arbitrary Lipschitz surface is given in Appendix B (see (6.10)).

Theorem 3.4. *Let S be a Lipschitz surface, let $\psi \in H^{-\frac{1}{2}}(S)$ be a unique solution of equation (3.7) with $f \in H^{-\frac{1}{2}}(S)$, and let $\psi^{(0)} \in H^{\frac{1}{2}}(S)$ be an arbitrary function. Then the sequence $\{\psi^{(n)}\}_{n=0}^{\infty}$ defined by (3.8) converges to ψ in the sense of the space $H^{-\frac{1}{2}}(S)$.*

Proof. Observe that $\psi^{(n)} \in H^{\frac{1}{2}}(S) \subset L_2(S) \subset H^{-\frac{1}{2}}(S)$ for all $n = 0, 1, 2, 3, \ldots$ Define the following sequence of functions

$$\zeta^{(n)} = \psi^{(n)} - \psi \in H^{-\frac{1}{2}}(S), \quad n = 0, 1, 2, 3, \ldots, \tag{3.10}$$

where ψ is a unique solution of equation (3.7).
In what follows, our goal is to show that the sequence $\zeta^{(n)}$ tends to zero in the space $\mathbf{H}^{-\frac{1}{2}}(S)$,

$$\lim_{n\to\infty} ||\zeta^{(n)}||^2_* = \lim_{n\to\infty} ((\zeta^{(n)}, \zeta^{(n)})) = 0. \tag{3.11}$$

In turn, (3.11) implies that the sequence $\zeta^{(n)}$ tends to zero in the space $H^{-\frac{1}{2}}(S)$ by virtue of the norms equivalence relation (3.2) and, consequently, $\psi^{(n)}$ tends to ψ in the sense of the space $H^{-\frac{1}{2}}(S)$ due to (3.10).

We proceed as follows. Substitute $\psi^{(n)} = \zeta^{(n)} + \psi$ and $\psi^{(n-1)} = \zeta^{(n-1)} + \psi$ into (3.8) to obtain

$$\zeta^{(n)} = \zeta^{(n-1)} - \tau\, \mathcal{M}\zeta^{(n-1)}, \quad n = 1, 2, 3, \ldots \tag{3.12}$$

In accordance with Lemma 3.2, we have the following expansions for the functions $\zeta^{(n)}(x)$ and $\mathcal{M}\zeta^{(n)}$ with respect to the orthonormal basis of eigenfunctions $\{\varphi_k\}_1^\infty$ in the space $\mathbf{H}^{-\frac{1}{2}}(S)$:

$$\zeta^{(n)}(x) = \sum_{k=1}^{\infty} a_k^{(n)} \varphi_k(x), \quad n = 0, 1, 2, 3, ... \tag{3.13}$$

$$\mathcal{M}\zeta^{(n)}(x) = \sum_{k=1}^{\infty} \lambda_k \, a_k^{(n)} \, \varphi_k(x), \quad n = 0, 1, 2, 3, ...$$

where

$$a_k^{(n)} = ((\zeta^{(n)}, \varphi_k)).$$

From (3.12) we have

$$((\zeta^{(n)}, \varphi_k)) = ((\zeta^{(n-1)}, \varphi_k)) - \tau((\mathcal{M}\zeta^{(n-1)}, \varphi_k)),$$

which implies the following recurrence relation between the corresponding Fourier coefficients

$$a_k^{(n)} = (1 - \tau\,\lambda_k)\, a_k^{(n-1)} \quad \text{for} \quad n, k = 1, 2, 3, \dots \tag{3.14}$$

From (3.14) we deduce

$$a_k^{(n)} = (1 - \tau\,\lambda_k)^n \, a_k^{(0)} \quad \text{for} \quad n, k = 1, 2, 3, \dots, \tag{3.15}$$

where the numbers $a_k^{(0)}$ are the Fourier coefficients in the expansion of the fixed initial function $\zeta^{(0)} = \psi^{(0)} - \psi \in H^{-\frac{1}{2}}(S)$,

$$\zeta^{(0)} = \sum_{k=1}^{\infty} a_k^{(0)} \varphi_k(x), \qquad ||\zeta^{(0)}||_*^2 = \sum_{k=1}^{\infty} \big[a_k^{(0)}\big]^2, \;\; a_k^{(0)} = ((\zeta^{(0)}, \varphi_k)), \;\; k = 1, 2, 3, \dots$$

Therefore, (3.13) can be rewritten as

$$\zeta^{(n)}(x) = \sum_{k=1}^{\infty} (1 - \tau\,\lambda_k)^n a_k^{(0)} \varphi_k(x).$$

By Lemma 3.2(iii) along with (3.15) we have

$$||\zeta^{(n)}||_*^2 = \sum_{k=1}^{\infty} \big[a_k^{(n)}\big]^2 = \sum_{k=1}^{\infty} (1 - \tau\,\lambda_k)^{2n} \big[a_k^{(0)}\big]^2. \tag{3.16}$$

Using the decreasing property of the sequence of eigenvalues λ_k, see (3.4), from (3.9) it follows that

$$0 < \tau\,\lambda_k \leqslant \tau\,\lambda_1 < 2 \quad \text{for} \quad k = 1, 2, 3, ...$$

Consequently,

$$-1 < 1 - \tau\,\lambda_k < 1, \quad (1 - \tau\,\lambda_k)^{2n} < 1 \quad \text{for} \quad k = 1, 2, 3, ... \tag{3.17}$$

$$\lim_{k\to\infty} (1 - \tau\lambda_k)^{2n} = 1.$$

Therefore, the series (3.16) is majorized by the convergent series $\sum_{k=1}^{\infty}\big[a_k^{(0)}\big]^2$ and for an arbitrarily small number $\varepsilon > 0$ there is a positive integer $N_1(\varepsilon)$, such that

$$\sum_{k=N_1(\varepsilon)}^{\infty} \big[a_k^{(n)}\big]^2 \leqslant \sum_{k=N_1(\varepsilon)}^{\infty} \big[a_k^{(0)}\big]^2 < \frac{\varepsilon}{2} \quad \text{for} \quad n = 1, 2, 3, ... \tag{3.18}$$

Evidently, $N_1(\varepsilon)$ does not depend on n, it depends only on ε.
Further, let us estimate the finite sum

$$\sum_{k=1}^{N_1(\varepsilon)-1} \big[a_k^{(n)}\big]^2 = \sum_{k=1}^{N_1(\varepsilon)-1} (1 - \tau\,\lambda_k)^{2n}\big[a_k^{(0)}\big]^2.$$

In view of relations (3.17) and positivity of the eigenvalues λ_k, we have the strict inequality

$$q := \max_{1\leqslant k\leqslant N_1(\varepsilon)-1} |1 - \tau\,\lambda_k| < 1.$$

Therefore, for an arbitrarily small number $\varepsilon > 0$ we can choose a positive integer $N_2(\varepsilon)$, such that

$$\begin{aligned}\sum_{k=1}^{N_1(\varepsilon)-1} \big[a_k^{(n)}\big]^2 &= \sum_{k=1}^{N_1(\varepsilon)-1} (1 - \tau\,\lambda_k)^{2n}\big[a_k^{(0)}\big]^2 \leqslant q^{2n} \sum_{k=1}^{N_1(\varepsilon)-1} \big[a_k^{(0)}\big]^2 \\ &\leqslant q^{2n} \sum_{k=1}^{\infty}\big[a_k^{(0)}\big]^2 = q^{2n}\|\zeta^{(0)}\|_*^2 < \frac{\varepsilon}{2} \quad \text{for } n > N_2(\varepsilon),\end{aligned} \tag{3.19}$$

where $N_2(\varepsilon)$ depends only on ε. Now, combining inequalities (3.18) and (3.19), from (3.16) we finally conclude

$$||\zeta^{(n)}(x)||_*^2 < \varepsilon \quad \text{for } n > N_2(\varepsilon),$$

which shows that (3.11) holds. This completes the proof. □

3.3 *Iteration scheme in the space* $L_2(S)$

Now, let S be again a Lipschitz surface and $f \in H^0(S) = L_2(S)$. Invertibility of the operator (2.48) implies that the integral equation

$$\mathcal{M}\psi = \mathcal{H}f \quad \text{on} \quad S \tag{3.20}$$

possesses a unique solution $\psi \in L_2(S)$. In view of the mapping property (2.48), it is evident that the operator

$$\mathcal{M} : L_2(S) \to L_2(S) \tag{3.21}$$

is a compact, positive, and symmetric operator with respect to the standard inner product in the space $L_2(S)$ defined by (2.1). Therefore, we have the following counterparts of Lemma 3.2 and Theorem 3.4.

Lemma 3.5. (i) *There exist a countable decreasing sequence of positive eigenvalues of the operator* (3.21) *(by taking into account multiplicities of the eigenvalues)*

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_k \geqslant \cdots \tag{3.22}$$

converging to zero and an orthonormal basis in $L_2(S)$ of the corresponding eigenfunctions $\{\omega_k\}_{k=1}^{\infty}$ *with* $\mathcal{M}\omega_k(x) = \lambda_k \omega_k(x)$ *on* S.
(ii) *For an arbitrary function* $g \in L_2(S)$ *the following series are convergent in the space* $L_2(S)$*:*

$$g(x) = \sum_{k=1}^{\infty} d_k\, \omega_k(x) \quad \textit{with} \quad d_k = (g, \omega_k)_{L_2(S)},$$
$$\mathcal{M}g(x) = \sum_{k=1}^{\infty} \lambda_k\, d_k\, \omega_k(x).$$

Moreover,

$$||g||^2_{L_2(S)} = \sum_{k=1}^{\infty} d_k^2, \quad ||\mathcal{H}g||^2_{L_2(S)} = \sum_{k=1}^{\infty} \lambda_k^2 d_k^2.$$

Proof. The claims of the lemma follow from the relations

$$(\mathcal{M}h,\, g) = (h,\, \mathcal{M}g) \quad \text{for} \quad g, h \in L_2(S),$$
$$(\mathcal{M}h,\, h) > 0 \quad \text{for} \quad h \neq 0,$$

along with the mapping property (2.48), compact embedding theorems for the Bessel potential spaces, the well-known Hilbert-Schmidt theorem for separable Hilbert spaces and Parseval's identity (see, e.g., [7, Chapter VIII], [9, Chapter 6], [24, Part V, Chapter 38], [26]). □

Theorem 3.6. *Let* S *be a Lipschitz surface,* $f \in L_2(S)$*, and let* $\psi \in L_2(S)$ *be a unique solution of the integral equation* (3.20)*. Then the recurrence sequence* $\{\psi^{(n)}\}_{n=0}^{\infty}$ *defined by the relation*

$$\psi^{(n)}(x) = \psi^{(n-1)}(x) + \tau\left(\mathcal{H}f(x) - \mathcal{M}\psi^{(n-1)}(x)\right), \quad n = 1, 2, 3, \dots,$$

where $\psi^{(0)} \in H^1(S)$ *is an arbitrary function and the parameter* τ *satisfies the inequality*

$$0 < \tau\, \lambda_1 < 2$$

with λ_1 *being the greatest eigenvalue of the operator* (3.21) *(see* (3.22)*), converges to the unique solution* ψ *of equation* (3.20) *in the sense of the space* $L_2(S)$*.*

Proof. The proof can be performed with the help of exactly the same arguments applied in the proof of Theorem 3.4. One needs only to replace the basis of the eigenfunctions $\{\varphi_k(x)\}_{k=1}^{\infty}$ by the orthonormal basis of eigenfunctions $\{\omega_k(x)\}_{k=1}^{\infty}$. □

Remark 3.7. If $S \in C^{m,\alpha}$and $f \in C^{k+1,\beta}(S)$, where m is a positive integer, k is a nonnegative integer, $k \leqslant m-1$, and $0 < \beta < \alpha \leqslant 1$, then the unique solution of equation (3.20) belongs to the space $C^{k,\beta}(S)$ (see, e.g., [13, Ch. 5] and [22, Ch. 1, §6] for more general cases). Evidently, in this case, all the eigenfunctions of the operators (3.3) and (3.21) belong to the space $C^{m,\beta}(S)$.

4 Iteration scheme for solutions to the Neumann BVP

Using the results obtained in the previous section here we construct the successive approximation scheme for the Neumann boundary value problem (2.2)-(2.3).

Theorem 4.1. *Let S be a Lipschitz surface, let $f \in H^{-\frac{1}{2}}(S)$ satisfy the orthogonality condition* (2.13)*, and let $\psi \in H^{-\frac{1}{2}}(S)$ be a unique solution to the integral equation $\mathcal{M}\psi = \mathcal{H}f$ on S. Then $u = V(\psi) \in H^1(\Omega)$ is a particular solution to the interior Neumann boundary value problem* (2.2)-(2.3) *and the sequence of functions $\{u^{(n)}\}_{n=1}^{\infty}$ given by the single layer potentials*

$$u^{(n)}(x) = V(\psi^{(n)})(x), \quad x \in \Omega, \quad n = 1, 2, 3, \dots, \tag{4.1}$$

converges to the particular solution $u = V(\psi)$ in the sense of the space $H^1(\Omega)$. Here $\psi^{(n)}$ is defined by the recurrence relation

$$\psi^{(n)}(x) = \psi^{(n-1)}(x) + \tau\,(\mathcal{H}f - \mathcal{M}\psi^{(n-1)}(x)), \quad n = 1, 2, 3, \dots, \tag{4.2}$$

where $\psi^{(0)} \in H^{\frac{1}{2}}(S)$ is an arbitrary function and the real parameter τ satisfies the inequality $0 < \tau\,\lambda_1 < 2$ with λ_1 being the greatest eigenvalue of the operator (3.3).

Proof. By Theorem 3.4 the sequence $\{\psi^{(n)}\}_{n=1}^{\infty}$ tends to the solution ψ of the equation $\mathcal{M}\psi = \mathcal{H}f$ on S in the sense of the space $H^{-\frac{1}{2}}(S)$. Due to Remark 2.2 then ψ is a particular solution of equation (2.5) and $V(\psi)$ is a particular solution of the Neumann boundary value problem (2.2)-(2.3). In view of Theorem 5.1(i) there is a positive constant C such that

$$||u - u^{(n)}||_{H^1(\Omega)} = ||V(\psi - \psi^{(n)})||_{H^1(\Omega)} \leqslant C\,||\psi - \psi^{(n)}||_{H^{-\frac{1}{2}}(S)} \to 0, \quad \text{as } n \to \infty,$$

which completes the proof. □

Remark 4.2. Note that the function $u^{(n)}$ given by (4.1) can be rewritten as follows

$$u^{(n)}(x) = \tau\,V(\mathcal{H}f)(x) + V(\psi^{(n-1)} - \tau\mathcal{M}\psi^{(n-1)})(x),\ x \in \Omega,\ n = 1, 2, 3, \dots$$

which makes a good basis for creating an efficient numerical algorithms.

Remark 4.3. Let S be a Lipschitz surface and $f \in H^0(S) = L_2(S)$ satisfies the orthogonality condition (2.13). Due to Theorem 5.1(iii) and Remark 2.8 it follows that the equation $\mathcal{M}\psi = \mathcal{H}f$ possesses a unique solution $\psi \in H^0(S) = L_2(S)$. It is known that the single layer potential $V(\psi)$ with $\psi \in L_2(S)$ has the nontangential limiting boundary value $\{V(\psi)\}_{nt}^{+}$ on S, which belongs to $H^1(S)$, and, moreover, $V(\psi)$ and $\frac{\partial}{\partial x_j}V(\psi)$, $j = 1, 2, 3$, have square integrable maximal functions. Consequently, the single layer potential operator

$$V : L_2(S) \to H^{\frac{3}{2}}(\Omega)$$

is bounded (for details see [2, Theorem 5.3, Theorem 5.4], [6, p.796 and Theorem 3.7], [11, Theorem 4.2, Theorem 3.1], [17, Proposition 3.1], [27, Corollary 3.5]).

Now, let $u^{(n)}$ and let $\psi^{(n)}$ be defined by relations (4.1) and (4.2). Then by Theorem 3.6 the sequence $\{\psi^{(n)}\}_{n=1}^{\infty}$ tends to the function ψ in the sense of the space $L_2(S)$. Therefore, there is a positive constant C_1, such that

$$||u - u^{(n)}||_{H^{\frac{3}{2}}(\Omega)} = ||V(\psi - \psi^{(n)})||_{H^{\frac{3}{2}}(\Omega)} \leqslant C_1\,||\psi - \psi^{(n)}||_{L_2(S)} \to 0, \quad \text{as } n \to \infty.$$

5 Appendix A: properties of the single layer potential and boundary integral operators

Here we collect the known properties of the single layer potential operator V and the boundary integral operators $\mathcal{H}$, $\pm\frac{1}{2}I+\mathcal{K}^*$, and $\pm\frac{1}{2}I+\mathcal{K}$, defined by relations (2.4) and (2.6)-(2.8) (for details see, e.g., [1], [2], [3], [4], [5], [8], [11], [14], [15], [16], [17], [25], [27]).

Theorem 5.1. *Let $S=\partial\Omega$ be a Lipschitz boundary.*
(i) *The following single layer potential operator is continuous*

$$V: H^{-\frac{1}{2}}(S)\to H^1(\Omega).$$

(ii) *If $\psi\in H^{-\frac{1}{2}}(S)$, then the following jump relations hold across the surface S:*

$$\{V(\psi)\}^+=\{V(\psi)\}^-=\mathcal{H}\psi\in H^{\frac{1}{2}}(S) \quad \text{on } \ S,$$
$$\Big\{\frac{\partial V(\psi)}{\partial n}\Big\}^{\pm}=\Big(\mp\frac{1}{2}I+\mathcal{K}^*\Big)\psi\in H^{-\frac{1}{2}}(S) \quad \text{on } \ S,$$

where $\mathcal{K}^*$ and $\mathcal{H}$ are defined by (2.6) *and* (2.7) *respectively.*
(iii) *The following boundary integral operators are continuous:*

$$\mathcal{H}: H^{-\frac{1}{2}}(S)\to H^{\frac{1}{2}}(S), \qquad \mathcal{H}: L_2(S)\to H^1(S), \tag{5.1}$$
$$-\frac{1}{2}I+\mathcal{K}^*: H^{-\frac{1}{2}}(S)\to H^{-\frac{1}{2}}(S), \qquad -\frac{1}{2}I+\mathcal{K}^*: L_2(S)\to L_2(S), \tag{5.2}$$
$$-\frac{1}{2}I+\mathcal{K}: H^{\frac{1}{2}}(S)\to H^{\frac{1}{2}}(S), \qquad -\frac{1}{2}I+\mathcal{K}: H^1(S)\to H^1(S), \tag{5.3}$$
$$\frac{1}{2}I+\mathcal{K}^*: H^{-\frac{1}{2}}(S)\to H^{-\frac{1}{2}}(S), \qquad \frac{1}{2}I+\mathcal{K}^*: L_2(S)\to L_2(S), \tag{5.4}$$
$$\frac{1}{2}I+\mathcal{K}: H^{\frac{1}{2}}(S)\to H^{\frac{1}{2}}(S), \qquad \frac{1}{2}I+\mathcal{K}: H^1(S)\to H^1(S). \tag{5.5}$$

Moreover, the operators (5.1), (5.4), *and* (5.5) *are invertible, while the operators* (5.2) *and* (5.3) *are Fredholm operators with zero index having one-dimensional null-spaces.*
(iv) *There is a positive constant δ_1 such that*

$$\big\langle -\mathcal{H}\psi,\,\psi\big\rangle_S\geqslant\delta_1\,\|\psi\|^2_{H^{-\frac{1}{2}}(S)} \quad \textit{for arbitrary} \quad \psi\in H^{-\frac{1}{2}}(S).$$

Remark 5.2. If $S\in C^{m,\alpha}$, where m is a positive integer, k is a nonnegative integer, $k\leqslant m-1$, and $0<\beta<\alpha\leqslant 1$, then the operator

$$\mathcal{H}: C^{k,\beta}(S)\to C^{k+1,\beta}(S)$$

is invertible (see, e.g., [4], [13], [18]).

Remark 5.3. In the case of C^∞-smooth surface S, the operator $-\mathcal{H}$ is a pseudodifferential operator of order -1 with positive definite principal homogeneous symbol matrix and has the mapping property (see, e.g., [3], [4], [13], [18], [21], [23])

$$\mathcal{H}: H^t(S)\to H^{t+1}(S) \qquad \text{for arbitrary} \quad t\in\mathbb{R}. \tag{5.6}$$

Moreover, operator (5.6) is invertible for all $t\in\mathbb{R}$.

6 Appendix B: A rough estimate of λ_1

It is evident that the operators (3.3) and (3.21) have the same sequence of eigenvalues, in particular, they have the same greatest eigenvalue λ_1 (see Remark 3.3). Since λ_1 equals to the norm of the corresponding operator (see, e.g., [9, Ch. 6]), it follows that the norms of the operators (3.3) and (3.21) equal to each other. To choose an explicit bound for the parameter τ in the iteration relation (3.8), one needs at least a rough estimate of the eigenvalue λ_1. To obtain such a rough estimate in the case of a Lipschitz surface S, we consider operator (3.21) in $L_2(S)$, where

$$\begin{aligned}\mathcal{M}\psi(x) &= \mathcal{H}\Big(-\frac{1}{2}\,I+\mathcal{K}^*\Big)\psi(x)+\big(\mathcal{H}\,1,\,\psi\big)_{L_2(S)}\,\mathcal{H}\,1\\ &= \int\limits_S\int\limits_S \Gamma(z-y)\psi(z)dS_z dS_y\int\limits_S \Gamma(x-y)\,dS_y\\ &\quad -\frac{1}{2}\,\mathcal{H}\psi(x)+\mathcal{H}\mathcal{K}^*\psi(x),\quad x\in S,\quad \psi\in L_2(S). \qquad (6.1)\end{aligned}$$

Denote by $\mathbf{M}=||\mathcal{M}||_{L_2(S)\to L_2(S)}$ the norm of this operator.
To find an upper boud of $\mathbf{M}$, let us introduce the functions:

$$\begin{aligned}F_1(x) &= \int\limits_S\int\limits_S \Gamma(z-y)\psi(z)dS_z dS_y\int\limits_S \Gamma(x-y)\,dS_y,\quad x\in S,\\ F_2(x) &= -\frac{1}{2}\,\mathcal{H}\psi(x),\quad x\in S, \qquad (6.2)\\ F_3(x) &= \mathcal{H}\mathcal{K}^*\psi(x),\quad x\in S. \qquad (6.3)\end{aligned}$$

It is evident that if

$$B_1=\sup_{x\in S}\Big|\int\limits_S\frac{dS_y}{|x-y|}\Big|, \qquad (6.4)$$

then

$$\|F_1\|_{L_2(S)}\leqslant\frac{B_1^2\,|S|}{16\pi^2}\,\|\psi\|_{L_2(S)}, \qquad (6.5)$$

where $|S|$ is the area of the surface S.

From (6.2) using the Cauchy-Schwartz inequality we get

$$\begin{aligned}64\pi^2\,[F_2(x)]^2 &= \Big[\int_S\frac{1}{|x-y|}\,|\psi(y)|dS_y\Big]^2=\Big[\int_S\frac{1}{|x-y|^{\frac12}}\,\frac{|\psi(y)|}{|x-y|^{\frac12}}dS_y\Big]^2\\ &\leqslant\int_S\frac{dS_y}{|x-y|}\int_S\frac{|\psi(y)|^2}{|x-y|}dS_y\leqslant B_1\int_S\frac{|\psi(y)|^2}{|x-y|}dS_y. \qquad (6.6)\end{aligned}$$

Now, by Fubini's theorem we derive the following estimate

$$\begin{aligned}\|F_2\|^2_{L_2(S)} &\leqslant\frac{B_1}{64\pi^2}\int_S\int_S\frac{|\psi(y)|^2}{|x-y|}\,dS_y dS_x\\ &\leqslant\frac{B_1^2}{64\pi^2}\int_S|\psi(y)|^2\,dS_y=\frac{B_1^2}{64\pi^2}\,\|\psi\|^2_{L_2(S)},\end{aligned}$$

i.e.,

$$\|F_2\|_{L_2(S)} \leqslant \frac{B_1}{8\pi} \|\psi\|_{L_2(S)}. \tag{6.7}$$

Further, from (6.3) we have

$$\begin{aligned}
\big[F_3(x)\big]^2 &= \frac{1}{16\pi^2}\Big[\int\limits_S \frac{1}{|x-y|}\Big(\int\limits_S \frac{\partial}{\partial n(y)}\frac{1}{|y-z|}\,\psi(z)\,dS_z\Big)dS_y\Big]^2 \\
&= \frac{1}{16\pi^2}\Big[\int\limits_S \Big(\int\limits_S \frac{1}{|x-y|}\Big(\frac{\partial}{\partial n(y)}\frac{1}{|y-z|}\Big)\,dS_y\Big)\psi(z)\,dS_z\Big]^2 \\
&= \frac{1}{16\pi^2}\Big[\int\limits_S K(x,z)\,\frac{\psi(z)}{|x-z|}\,dS_z\Big]^2,
\end{aligned} \tag{6.8}$$

where

$$K(x,z) = |x-z| \int\limits_S \frac{1}{|x-y|}\Big(\frac{\partial}{\partial n(y)}\frac{1}{|y-z|}\Big)\,dS_y, \quad x,z \in S.$$

In view of the estimate (see, e.g., [10], [20])

$$\int\limits_S \frac{1}{|x-y|}\Big(\frac{\partial}{\partial n(y)}\frac{1}{|y-z|}\Big)\,dS_y = \mathcal{O}\big(|x-z|^{-1}\big), \quad x,z \in S,$$

the function $K(x,z)$ is bounded and let

$$B_2 = \sup_{x,z\in S} |K(x,z)|.$$

Then, from (6.8) we get

$$\big[F_3(x)\big]^2 \leqslant \frac{B_2^2}{16\pi^2}\Big[\int\limits_S \frac{|\psi(z)|}{|x-z|}\,dS_z\Big]^2,$$

and using (6.4) we arrive at the estimate

$$\|F_3\|_{L_2(S)} \leqslant \frac{B_1 B_2}{4\pi} \|\psi\|_{L_2(S)}. \tag{6.9}$$

Keeping in mind (6.1) and combining the inequalities (6.5), (6.7), and (6.9), we finally obtain the following estimate of the norm $\mathbf{M}$ of the operator $\mathcal{M}$,

$$\lambda_1 = \mathbf{M} = ||\mathcal{M}||_{L_2(S)\to L_2(S)} \leqslant \frac{B_1(2\pi + B_1|S| + 4\pi B_2)}{16\pi^2}. \tag{6.10}$$

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