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A Partition of an Uncountable Solvable Group into Three Negligible Subsets

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It is proved that every uncountable solvable group $(G, \cdot)$ admits a partition into three G-negligible sets.

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It is well known that if a group $(G, \cdot)$ has a complicated algebraic structure, then it admits various kinds of paradoxical finite decompositions. For instance, if G contains a subgroup isomorphic to F_2 (= the group generated by two independent generators), then paradoxical finite decompositions are realizable for G. Actually, this old result goes back to von Neumann (in this connection, see e.g. [7]). On the other hand, it is also widely known that any solvable (and, in particular, commutative) group $(G, \cdot)$ does not admit such decompositions because it is amenable, i.e., there exists a normalized finitely additive left (right) G-invariant measure whose domain coincides with the power set $\mathcal{P}(G)$ of G (see [7]).

In the present paper we consider one finite decomposition (partition) of an uncountable solvable group $(G, \cdot)$ from the point of view of ordinary (i.e., countably additive) invariant measures. In some sense, the decomposition constructed below may also be regarded as paradoxical.

We will be dealing with nonzero σ-finite measures on G which are invariant (or, more generally, quasi-invariant) under the group of all left translations of G. In the context of such measures, a certain type of "small" subsets of G will be specified and it will be demonstrated that G can be partitioned into three "small" subsets.

The notation and terminology used in this paper is primarily taken from [4] and [5]. All basic facts of modern measure theory can be found in [1]. An extensive survey devoted to measures given on various algebraic-topological structures is presented in [8].

Let E be a base (ground) set and let Γ be some group of transformations of E. In this case, the pair (E, Γ) is usually called a space equipped with a transformation

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group.

We shall say that a set $X \subset E$ is Γ-negligible (in E) if the following two conditions are fulfilled for X:

(a) there exists at least one nonzero σ-finite Γ-invariant (Γ-quasi-invariant) measure μ on E such that $X \in \mathrm{dom}(\mu)$;

(b) for every σ-finite Γ-quasi-invariant measure ν on E such that X belongs to $\mathrm{dom}(\nu)$, the equality $\nu(X) = 0$ holds true.

If $(G, \cdot)$ is a group, then we may consider G as a ground set E and take the group of all left translations of G as a group of transformations of E. Obviously, identifying G with the group of all left translations of G, we may speak of left G-invariant (left G-quasi-invariant) measures on E $(= G)$ and, respectively, we may consider G-negligible subsets of G.

As usual, the symbol ω $(= \omega_0)$ denotes the least infinite cardinal (ordinal) number and ω_1 denotes the least uncountable cardinal (ordinal) number.

For our further purposes, we need several auxiliary propositions.

Lemma 1.1: *Let $(G, +)$ be a commutative group satisfying the following condition: there are three subgroups G_1, G_2 and G_3 of G such that*

$$\mathrm{card}(G_1) = \mathrm{card}(G_2) = \mathrm{card}(G_3) = \omega_1$$

and G is a direct sum of G_1, G_2, G_3.

Then there exist three pairwise disjoint subsets X, Y and Z of G such that $G = X \cup Y \cup Z$ and

(a) for each $g \in G$, the set $X \cap (G_1 + g)$ is finite;

(b) for each $g \in G$, the set $Y \cap (G_2 + g)$ is finite;

(c) for each $g \in G$, the set $Z \cap (G_3 + g)$ is finite.

In fact, Lemma 1.1 is due to Sierpiński (see, for instance, his classical monograph [6]). Notice only that in this monograph Sierpiński formulates the direct analogue of Lemma 1.1 in terms of some equivalent of the Continuum Hypothesis (**CH**). However, we need Sierpiński's result in the form given above.

Lemma 1.2: *Let $(G, +)$ be a commutative group and let H be an uncountable subgroup of G. Suppose also that a set $X \subset G$ satisfies the relation*

$$(\forall g \in G)(\mathrm{card}(X \cap (g + H)) < \omega).$$

Then X is a G-negligible subset of G.

Proof: It is not difficult to check that, for any countable family $\{g_i : i \in I\}$ of elements of G, the inequality

$$\mathrm{card}(G \setminus \cup\{g_i + X : i \in I\}) \geq \omega_1$$

holds true. This implies that $\{X\}$ generates the G-invariant σ-ideal $\mathcal{I}(X)$ in the power set $\mathcal{P}(G)$ of G. Denote by $\mathcal{S}(X)$ the σ-algebra of subsets of G, generated by $\mathcal{I}(X)$. Obviously, $\mathcal{S}(X)$ is G-invariant, too. Let us define a functional μ on $\mathcal{S}(X)$ by putting:

$\mu(Z) = 0$ if $Z \in \mathcal{I}(X)$;

$\mu(Z) = 1$ if $Z \in \mathcal{S}(X) \setminus \mathcal{I}(X)$.

A direct verification shows that μ is a probability G-invariant measure on G for which the relations $X \in \mathrm{dom}(\mu)$ and $\mu(X) = 0$ are valid.

Let now ν be an arbitrary σ-finite G-quasi-invariant measure on G such that $X \in \mathrm{dom}(\nu)$. We have to demonstrate that $\nu(X) = 0$. Suppose otherwise, i.e., $\nu(X) > 0$. By virtue of the σ-finiteness and H-quasi-invariance of ν, there exists a countable subgroup H_0 of H such that

$$(\forall h \in H)(\nu((h + (H_0 + X))\triangle(H_0 + X)) = 0).$$

It follows from the above relation that

$$\nu(\cap\{h_j + (H_0 + X) : j \in J\}) > 0$$

for every countable family $\{h_j : j \in J\}$ of elements of H. On the other hand, keeping in mind the inequality $\mathrm{card}(H/H_0) > \omega$, let us take any partial countably infinite selector $\{h_j : j \in J\}$ of H/H_0 and check that

$$\cap\{h_j + (H_0 + X) : j \in J\} = \emptyset.$$

Indeed, assuming for a moment that

$$z \in \cap\{h_j + (H_0 + X) : j \in J\},$$

we infer that there are two countable families

$$\{h_j^0 : j \in J\} \subset H_0, \quad \{x_j : j \in J\} \subset X$$

such that

$$z - h_j = h_j^0 + x_j \qquad (j \in J).$$

Since there are only finitely many elements of X in the orbit $H + z$, there exist two distinct indices $j \in J$ and $k \in J$ for which we have $x_j = x_k$ and, consequently,

$$z - h_j - h_j^0 = z - h_k - h_k^0, \quad h_k - h_j = h_j^0 - h_k^0 \in H_0$$

contradicting the choice of $\{h_j : j \in J\}$. The contradiction obtained finishes the proof. □

Remark 1: Let $(G, +)$ be a commutative group and let H be an uncountable subgroup of G. Suppose also that a set $X \subset G$ satisfies the relation

$$(\forall g \in G)(\mathrm{card}(X \cap (g + H)) \leq \omega).$$

Then, in general, X is not a G-negligible subset of G. The corresponding examples can be found in [4] and [5].

Lemma 1.3: *Let $(G, \cdot)$ and $(H, \cdot)$ be two groups and let*

$$\phi : (G, \cdot) \to (H, \cdot)$$

be a surjective homomorphism. Suppose also that X is an H-negligible subset of H. Then $\phi^{-1}(X)$ is a G-negligible subset of G.

Proof: Since X is H-negligible in H, there exists a nonzero σ-finite left H-invariant (H-quasi-invariant) measure μ on H such that $X \in \mathrm{dom}(\mu)$ and $\mu(X) = 0$. In the group $(G, \cdot)$ consider the family of sets

$$\mathcal{S}_1 = \{\phi^{-1}(Z) : Z \in \mathrm{dom}(\mu)\}.$$

Clearly, $\mathcal{S}_1$ is a left G-invariant σ-algebra in G and $\phi^{-1}(X) \in \mathcal{S}_1$. We put

$$\mu_1(\phi^{-1}(Z)) = \mu(Z) \qquad (Z \in \mathrm{dom}(\mu)).$$

A straightforward verification shows that μ_1 is well defined, is a nonzero σ-finite left G-invariant (G-quasi-invariant) measure on $\mathcal{S}_1$ and $\mu_1(\phi^{-1}(X)) = 0$.

Let now ν be an arbitrary σ-finite left G-quasi-invariant measure on G such that $\phi^{-1}(X) \in \mathrm{dom}(\nu)$. Without loss of generality, we may assume that ν is a probability measure, and we must demonstrate that $\nu(\phi^{-1}(X)) = 0$. Suppose otherwise, i.e., $\nu(\phi^{-1}(X)) > 0$. In $(H, \cdot)$ consider the family of sets

$$\mathcal{S}_2 = \{Y \subset H : \phi^{-1}(Y) \in \mathrm{dom}(\nu)\}.$$

Clearly, $\mathcal{S}_2$ is a left H-invariant σ-algebra of subsets of H and $X \in \mathcal{S}_2$. We now put

$$\mu_2(Y) = \nu(\phi^{-1}(Y)) \qquad (Y \in \mathcal{S}_2).$$

A direct verification shows that μ_2 is a probability left H-quasi-invariant measure on $\mathcal{S}_2$ and $\mu_2(X) = \nu(\phi^{-1}(X)) > 0$ which gives a contradiction with the fact that X is an H-negligible subset of H. The obtained contradiction completes the proof of Lemma 1.3. □

The next auxiliary propositions is purely algebraic and can be deduced from well-known theorems of the general theory of commutative groups (cf. [2], [3]).

Lemma 1.4: *If $(G, +)$ is an arbitrary uncountable commutative group, then there exist three subgroups G_1, G_2 and G_3 of G such that*

$$\mathrm{card}(G_1) = \mathrm{card}(G_2) = \mathrm{card}(G_3) = \omega_1$$

and $G_1 \cap (G_2 + G_3) = G_2 \cap (G_3 + G_1) = G_3 \cap (G_1 + G_2) = \{0\}$.

Lemma 1.5: *If $(G, +)$ is an uncountable commutative group, then there exist three pairwise disjoint subsets X, Y and Z of G satisfying these two relations:*

(1) $G = X \cup Y \cup Z$;

(2) all the sets X, Y and Z are G-negligible in G.

Proof: By virtue of Lemma 1.4, the group $(G, +)$ contains three subgroups G_1, G_2 and G_3 such that

$$\mathrm{card}(G_1) = \mathrm{card}(G_2) = \mathrm{card}(G_3) = \omega_1,$$

$$G_1 \cap (G_2 + G_3) = G_2 \cap (G_3 + G_1) = G_3 \cap (G_1 + G_2) = \{0\}.$$

Let us introduce the notation

$$G_0 = G_1 + G_2 + G_3.$$

Obviously, $\mathrm{card}(G_0) = \omega_1$ and G_0 is a direct sum of the groups G_1, G_2, G_3. So Lemma 1.1 can be applied to the group G_0. According to Lemma 1.1, there exist three pairwise disjoint subsets X_0, Y_0 and Z_0 of G_0 such that

$$G_0 = X_0 \cup Y_0 \cup Z_0$$

and the following conditions are fulfilled:

(a) for each $g \in G_0$, the set $X_0 \cap (G_1 + g)$ is finite;
(b) for each $g \in G_0$, the set $Y_0 \cap (G_2 + g)$ is finite;
(c) for each $g \in G_0$, the set $Z_0 \cap (G_3 + g)$ is finite.

Fix any selector $\{g_i : i \in I\}$ of the quotient set G/G_0 and put

$$X = \cup\{g_i + X_0 : i \in I\}, \quad Y = \cup\{g_i + Y_0 : i \in I\}, \quad Z = \cup\{g_i + Z_0 : i \in I\}.$$

Then, for X, Y and Z, we readily get the relations

$$G = X \cup Y \cup Z,$$

$$X \cap Y = Y \cap Z = Z \cap X = \emptyset.$$

Further, it follows from the conditions (a), (b) and (c) that:

(a$'$) for each $g \in G$, the set $X \cap (G_1 + g)$ is finite;
(b$'$) for each $g \in G$, the set $Y \cap (G_2 + g)$ is finite;
(c$'$) for each $g \in G$, the set $Z \cap (G_3 + g)$ is finite.

Therefore, by virtue of Lemma 1.2, the sets X, Y and Z are G-negligible in $(G, +)$. Lemma 1.5 has thus been proved. □

Lemma 1.6: *Let $(G, \cdot)$ be a group, let H be a normal subgroup of G such that $\mathrm{card}(G/H) \leq \omega$, and let X be an H-negligible subset of H. Suppose also that $\{g_i : i \in I\}$ is an arbitrary selector of G/H.*

Then the set $X' = \cup\{g_i \cdot X : i \in I\}$ is a G-negligible subset of G.

Proof: According to our assumption, X is H-negligible in H. Consequently, there exists a nonzero σ-finite left H-invariant (H-quasi-invariant) measure μ on H such that $X \in \mathrm{dom}(\mu)$ and $\mu(X) = 0$.

In the group $(G, \cdot)$ consider the family of sets

$$\mathcal{S}' = \{\cup\{g_i \cdot T_i : i \in I\} : (\forall i \in I)(T_i \in \mathrm{dom}(\mu))\}.$$

It can easily be checked that $\mathcal{S}'$ is a left G-invariant σ-algebra of subsets of G.

If $T = \cup\{g_i \cdot T_i : i \in I\}$, where all T_i $(i \in I)$ belong to $\mathrm{dom}(\mu)$, then we put

$$\mu'(T) = \sum\{\mu(T_i) : i \in I\}.$$

The functional μ' defined in this manner turns out to be a nonzero σ-finite left G-invariant (G-quasi-invariant) measure on $\mathcal{S}'$. It also follows from the definition of μ' that $X' \in \mathcal{S}'$ and $\mu'(X') = 0$.

Let now ν be any σ-finite left G-quasi-invariant measure on G such that $X' \in \mathrm{dom}(\nu)$. We must demonstrate that $\nu(X') = 0$. Suppose otherwise, i.e., $\nu(X') > 0$. Then, taking into account the equality

$$X' = \cup\{g_i \cdot X : i \in I\}$$

and G-quasi-invariance of ν, we get $\nu^*(X) > 0$, where the symbol ν^* denotes, as usual, the outer measure associated with ν. In view of the inclusion $X \subset H$, we also have $\nu^*(H) > 0$. Let H^* be a ν-measurable hull of H. It can easily be shown that $\nu(X' \cap H^*) > 0$. Since H is a group, the set H^* is almost left H-invariant, i.e.,

$$(\forall h \in H)(\nu((h \cdot H^*) \triangle H^*) = 0).$$

Further, consider in H the family of sets

$$\mathcal{S}'' = \{H \cap U : U \in \mathrm{dom}(\nu) \ \& \ U \subset H^*\}.$$

Obviously, $\mathcal{S}''$ is a left H-invariant σ-algebra of subsets of H. Moreover, putting

$$\mu''(H \cap U) = \nu(U) \qquad (U \in \mathrm{dom}(\nu), \ U \subset H^*),$$

we are able to define a σ-finite left H-quasi-invariant measure μ'' on $\mathcal{S}''$. Finally, take the index $i_0 \in I$ such that $g_{i_0} \in H$. Then we may write

$$g_{i_0} \cdot X = H \cap (X' \cap H^*) \in \mathcal{S}'', \quad \mu''(g_{i_0} \cdot X) > 0, \quad \mu''(X) > 0,$$

which contradicts the assumption that X is H-negligible in H. This ends the proof of Lemma 1.6. □

Theorem 1.7: *If $(G, \cdot)$ is an uncountable solvable group, then G can be represented in the form $G = X \cup Y \cup Z$, where the sets X, Y and Z are pairwise disjoint and are G-negligible in G.*

Proof: Let e denote the neutral element of G. Since $(G, \cdot)$ is solvable, there exists a finite sequence

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset ... \subset G_{n-1} \subset G_n = G$$

of subgroups of G satisfying these two relations:

(i) for each natural index $k \in [1, n]$, the group G_{k-1} is normal in the group G_k;

(ii) for each natural index $k \in [1, n]$, the quotient group G_k/G_{k-1} is commutative.

To demonstrate the validity of our assertion, we argue by induction on n.

If $n = 1$, then the uncountable group $G = G_n$ is commutative, and we may apply Lemma 1.5 to this G.

Suppose now that the assertion holds true for a natural number $n - 1 \geq 1$ and let us establish its validity for n.

For this purpose, consider the commutative quotient group $H = G_n/G_{n-1}$, where, as above, $G_n = G$. Here only two cases are possible.

(a) the group $H = G_n/G_{n-1}$ is uncountable.

In this case, we take the canonical surjective homomorphism

$$\phi : (G_n, \cdot) \to (H, +).$$

By virtue of Lemma 1.5, there exist three pairwise disjoint H-negligible subsets X_0, Y_0 and Z_0 of H such that

$$H = X_0 \cup Y_0 \cup Z_0.$$

We now put

$$X = \phi^{-1}(X_0), \quad Y = \phi^{-1}(Y_0), \quad Z = \phi^{-1}(Z_0).$$

Then, keeping in mind Lemma 1.3, we see that the sets X, Y and Z are pairwise disjoint, G-negligible in G, and

$$G = \phi^{-1}(H) = \phi^{-1}(X_0 \cup Y_0 \cup Z_0) = X \cup Y \cup Z.$$

(b) the group $H = G_n/G_{n-1}$ is countable.

In this case, in view of the uncountability of $G_n = G$, the group G_{n-1} is necessarily uncountable and we can apply the inductive assumption to this G_{n-1}. So there exist three pairwise disjoint G_{n-1}-negligible subsets X_0, Y_0 and Z_0 of G_{n-1} such that

$$G_{n-1} = X_0 \cup Y_0 \cup Z_0.$$

Let $\{g_i : i \in I\}$ be any selector of G_n/G_{n-1}. We put

$$X = \cup\{g_i \cdot X_0 : i \in I\}, \quad Y = \cup\{g_i \cdot Y_0 : i \in I\}, \quad Z = \cup\{g_i \cdot Z_0 : i \in I\}.$$

In view of Lemma 1.6, the sets X, Y and Z are G-negligible in G. It is also clear that

$$G = X \cup Y \cup Z, \qquad X \cap Y = Y \cap Z = Z \cap X = \emptyset.$$

Theorem 1.7 has thus been proved. □

Remark 2: Let $(G, \cdot)$ be an arbitrary uncountable group. It can easily be seen that there exist no disjoint G-negligible sets X and Y in G such that $G = X \cup Y$.

Theorem 1.8: *If $(G, \cdot)$ is an uncountable solvable group, then there are three pairwise disjoint G-negligible subsets X, Y and Z of G such that, for any nonzero σ-finite left G-quasi-invariant measure μ on G, at least one of the sets X, Y and Z is nonmeasurable with respect to μ.*

Theorem 1.8 is a direct consequence of Theorem 1.7.

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