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Direct Boundary Integral Equations Method for Acoustic Problems in Unbounded Domains

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We investigate some aspects of the so called direct boundary integral equation method in acoustic scattering theory. It is well known that by the direct approach the uniquely solvable exterior boundary value problems for the Helmholtz equation can not be reduced to the boundary integral equations which are uniquely solvable for arbitrary value of the frequency parameter. This implies that for such resonant frequencies the corresponding integral operators are not invertible and consequently solutions to the nonhomogeneous integral equations are not defined uniquely. They are defined modulo a linear combination of the elements of the null spaces of the corresponding integral operators. In the paper, it is shown that among the infinitely many solutions of the corresponding integral equations there is only one solution which has a physical meaning and corresponds either to the boundary trace of the unique solution to the exterior problem or to the boundary trace of its normal derivative. We analyze also modified direct boundary integral equation approaches which reduce the Dirichlet and Neumann boundary value problems to the equivalent uniquely solvable integral or singular integro-differential equations.

Keywords: Helmholtz equation, Direct boundary integral equation method, Sommerfeld radiation conditions.

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1. Introduction

Here we investigate some aspects of the so called direct boundary integral equation method in acoustic scattering theory. The basic interior and exterior boundary value problems for the Helmholtz equation by different methods are studied in scientific literature in various function spaces for smooth and non-smooth domains(see [20], [18], [3], [4], [5], [13]). It is well known that by the direct boundary integral equation approach the uniquely solvable exterior boundary value problems for the Helmholtz equation can not be reduced to the equivalent boundary integral equations which are uniquely solvable for arbitrary value of the frequency parameter. Exceptional values of the frequency parameter are called resonant (exceptional) frequencies. For the resonant frequencies the integral operators associated with

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the integral equations obtained by the direct approach, i.e., obtained by using the general integral representation of solutions, are not invertible. However, as we will show that the corresponding nonhomogeneous equations are always solvable but unfortunately solutions to the nonhomogeneous integral equations are not defined uniquely. They are defined modulo a linear combination of the elements of the null spaces of the corresponding integral operators. The elements of the null spaces are related to the nontrivial solutions of the corresponding interior boundary value problems. In the paper, we show that among the infinitely many solutions of the corresponding integral equations obtained by the direct approach there is only one solution which has a physical meaning and corresponds either to the boundary trace of the unique solution to the exterior problem or to the boundary trace of its normal derivative. For illustration we consider the exterior Dirichlet problem for the Helmholtz equation. However, the arguments applied in the paper can be successfully extended to other boundary value problems including mixed type problems, to the acoustic wave scattering problems in anisotropic media and also to elastic steady state oscillation problems for isotropic and anisotropic solids (cf. [11], [16], [8], [9]).

In the final part of the paper, we also analyze modified direct boundary integral equation approaches which reduce the Dirichlet and Neumann boundary value problems to the equivalent uniquely solvable integral or singular integro-differential (pseudodifferential) equations.

2. Preliminary material

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain with a smooth or Lipschitz boundary S. Further, let $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\partial\Omega^\pm = S$, $\overline{\Omega^\pm} = \Omega^\pm \cup S$. Throughout the paper $n(x)$ stands for the outward unit normal vector at the point $x \in S$. The symbols $\{\cdot\}^\pm$ denote one sided limiting values (traces) on $S = \partial\Omega^\pm$ from $\Omega^\pm$.

By L_2, H_2^s, and W_2^r we denote the well-known Lebesgue, Bessel potential, and Sobolev-Slobodetskii function spaces, respectively (see, e.g., [10], [19]). The corresponding locally integrable and compactly supported function spaces are denoted by the symbols $L_{2,loc}$, $H^s_{2,loc}$, $W^r_{2,loc}$ and $L_{2,comp}$, $H^s_{2,comp}$, $W^r_{2,comp}$ respectively. Recall that $H_2^r = W_2^r$ for any $r \geq 0$. Further, by $C^{k,\alpha}$ with nonnegative integer k and $0 < \alpha \leqslant 1$ is denoted the space of functions whose k-th order partial derivatives are Hölder continuous functions with exponent α.

Consider the Helmholtz equation

$$L(\partial, \omega)\, u(x) := (\Delta + \omega^2)\, u(x) = f(x), \qquad x \in \Omega^\pm, \tag{2.1}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\partial_j = \partial/\partial x_j$, $\partial := (\partial_1, \partial_2, \partial_3)$, $\omega \in \mathbb{R}$ is the so called frequency parameter, and the right hand side function f is compactly supported in the case of an exterior domain, i.e., supp $f \cap \overline{\Omega^-}$ is compact.

We say that u belongs to the Sommerfeld class of radiating functions in the unbounded domain Ω^- and write $u \in \mathcal{S}(\Omega^-)$ if for sufficiently large $|x|$ the relation

$$\frac{\partial u(x)}{\partial r} - i\,\omega\, u(x) = \mathcal{O}(r^{-2}), \quad r = |x|\,, \tag{2.2}$$

holds uniformly in all directions $x/|x|$ (see [20], [18], [3], [4]).

Denote by $\Gamma(x-y,\omega)$ the fundamental solution that corresponds to outgoing waves and satisfies the Sommerfeld radiation condition,

$$\Gamma(x-y,\omega) = -\frac{1}{4\pi}\,\frac{e^{i\,\omega\,|x-y|}}{|x-y|}\,. \tag{2.3}$$

Introduce the volume, single and double layer potentials associated with the fundamental solution (2.3)

$$P_{\Omega^\pm}\big(f\big)\,(x) = \int\limits_{\Omega^\pm} \Gamma(x-y,\omega)\,f(y)\,dy, \qquad x\in\mathbb{R}^3, \tag{2.4}$$

$$V(g)(x) \equiv V_S(g)(x) = \int\limits_S \Gamma(x-y,\omega)\,g(y)\,dS, \qquad x\in\mathbb{R}^3\setminus S, \tag{2.5}$$

$$W(g)(x) \equiv W_S(g)(y) = \int\limits_S \big[\partial_{n(y)}\Gamma(x-y,\omega)\big]\,g(y)\,dS, \qquad x\in\mathbb{R}^3\setminus S, \tag{2.6}$$

where f and g are densities of the potentials and $\partial_n := \frac{\partial}{\partial n}$ denotes the normal derivative.

It is well known that these potentials have the following properties (see, e.g., [14], [20], [3], [3], [7], [5], [13]).

Theorem 2.1 : *Let S be a Lipschitz surface. Then the operators*

$$V : H_2^{-1/2}(S) \longrightarrow H_2^1(\Omega^+), \qquad V : H_2^{-1/2}(S) \longrightarrow H^1_{2,\,loc}(\Omega^-)\cap\mathcal{S}(\Omega^-)\,,$$
$$W : H_2^{1/2}(S) \longrightarrow H_2^1(\Omega^+), \qquad W : H_2^{1/2}(S) \longrightarrow H^1_{2,\,loc}(\Omega^-)\cap\mathcal{S}(\Omega^-)\,,$$
$$P_{\Omega^+} : H_2^0(\Omega^+) \longrightarrow H^2_{2,\,loc}(\mathbb{R}^3)\cap\mathcal{S}(\mathbb{R}^3),\ P_{\Omega^-} : H^0_{2,\,comp}(\Omega^-) \longrightarrow H^2_{2,\,loc}(\mathbb{R}^3)\cap\mathcal{S}(\mathbb{R}^3)$$

are continuous.

If $g\in\big[H_2^{-\frac12}(S)\big]^3$, $h\in\big[H_2^{\frac12}(S)\big]^3$, $f\in L_2(\Omega^+)$ or $f\in L_{2,\,comp}(\Omega^-)$. Then

$$L(\partial,\omega)P_{\Omega^+}\big(f\big)\,(x) = \begin{cases} f(x) & in \quad \Omega^+,\\ 0 & in \quad \Omega^-,\end{cases} \tag{2.7}$$

$$L(\partial,\omega)P_{\Omega^-}\big(f\big)\,(x) = \begin{cases} 0 & in \quad \Omega^+,\\ f(x) & in \quad \Omega^-,\end{cases} \tag{2.8}$$

$$L(\partial,\omega)V\big(g\big)\,(x) = 0, \quad L(\partial,\omega)W\big(g\big)\,(x) = 0, \qquad in \quad \Omega^\pm, \tag{2.9}$$

$$\big\{V(g)(x)\big\}^+ = \big\{V(g)(x)\big\}^- = \mathcal{H}\,g(x) \qquad in \quad S, \tag{2.10}$$

$$\big\{\partial_{n(x)}\,V(g)(x)\big\}^\pm = \big[\,\mp 2^{-1}\,I + \widetilde{\mathcal{K}}\,\big]\,g(x), \qquad in \quad S, \tag{2.11}$$

$$\big\{W(h)\,(x)\big\}^{\pm} = \big[\,\pm\, 2^{-1} I + \mathcal{K}\,\big]\, h(x) \qquad in \quad S, \tag{2.12}$$

$$\big\{\partial_n\, W(h)(x)\big\}^{+} = \big\{\partial_n\, W(h)(x)\big\}^{-} \equiv \mathcal{L}\, h(x) \qquad in \quad S, \tag{2.13}$$

where $\widetilde{\mathcal{K}}$, $\mathcal{K}$, and $\mathcal{H}$ are boundary integral operators

$$\widetilde{\mathcal{K}}\, g(x) := \int\limits_S \big[\partial_{n(x)}\Gamma(x-y,\omega)\big]\, g(y)\, dS, \qquad x \in S, \tag{2.14}$$

$$\mathcal{K}\, g(x) := \int\limits_S \big[\partial_{n(y)}\Gamma(x-y,\omega)\big]\, g(y)\, dS, \qquad x \in S, \tag{2.15}$$

$$\mathcal{H}\, g(x) := \int\limits_S \Gamma(x-y,\omega)\, g(y)\, dS, \qquad x \in S. \tag{2.16}$$

Moreover, the following mappings are bounded

$$\begin{aligned} &\mathcal{H} \;:\; H_2^{-1/2}(S) \longrightarrow H_2^{1/2}(S)\,, &&\widetilde{\mathcal{K}} \;:\; H_2^{-1/2}(S) \longrightarrow H_2^{-1/2}(S)\,,\\ &\mathcal{K} \;:\; H_2^{1/2}(S) \longrightarrow H_p^{1/2}(S)\,, &&\mathcal{L} \;:\; H_2^{1/2}(S) \longrightarrow H_2^{-1/2}(S)\,. \end{aligned}$$

For $S \in C^{1,\alpha}$, $0 < \alpha \leqslant 1$, the operators $\mathcal{H}$, $\widetilde{\mathcal{K}}$, and $\mathcal{K}$ are weakly singular operators, while $\mathcal{L}$ is a singular integro-differential operator.

If $S \in C^{k+1,\alpha}$ with $k \geqslant 1$ and $0 < \beta < \alpha \leqslant 1$. Then the following operators are continuous

$$\begin{aligned} &V \;:\; C^{k,\beta}(S) \longrightarrow C^{k+1,\beta}(\overline{\Omega^{\pm}}), &&W \;:\; C^{k,\beta}(S) \longrightarrow C^{k,\beta}(\overline{\Omega^{\pm}}),\\ &\mathcal{H} \;:\; C^{k,\beta}(S) \longrightarrow C^{k+1,\beta}(S)\,, &&\widetilde{\mathcal{K}}, \mathcal{K} \;:\; C^{k,\beta}(S) \longrightarrow C^{k,\beta}(S)\,,\\ &\mathcal{L} \;:\; C^{k,\beta}(S) \longrightarrow C^{k-1,\beta}(S)\,. \end{aligned}$$

For $\omega = 0$ we equip the corresponding potential and boundary operators by the subindex 0, $P_{_{0,\,\Omega^{\pm}}}$, V_0, W_0, $\widetilde{\mathcal{K}}_0$, $\mathcal{K}_0$, $\mathcal{H}_0$, and they correspond to the fundamental solution $\Gamma(x-y) := \Gamma(x-y,0)$ of the Laplace operator.

For regular functions $u, v \in C^2(\overline{\Omega^+})$ and $S \in C^{1,\alpha}$ the following Green's formulas are valid

$$\int\limits_{\Omega^+}\!\big(\Delta u + \omega^2 u\big)\, v\, dx = \!\int\limits_{\Omega^+}\!\Big[-\sum_{k=1}^{3} \partial_k u\, \partial_k v + \omega^2\, u\;v\Big]\, dx + \!\int\limits_{\partial\Omega^+}\!\{\partial_n u\}^+\,\{v\}^+\, dS, \tag{2.17}$$

$$\int\limits_{\Omega^+}\!\big[(\Delta u + \omega^2 u)\, v - u\,(\Delta v + \omega^2 v)\big]\, dx = \!\int\limits_{\partial\Omega^+}\!\big[\{\partial_n\, u\}^+\,\{v\}^+ - \{u\}^+\,\{\partial_n\, v\}^+\big]\, dS. \tag{2.18}$$

Further, for a regular function $u \in C^2(\overline{\Omega^+})$ we have the following integral representation (Green's third formula)

$$P_{\Omega^+}\big(Lu\big)(x) - V\big(\{\partial_n u\}^+\big)(x) + W\big(\{u\}^+\big)(x) = \begin{cases} u(x), \, x \in \Omega^+, \\ 0, \quad\quad x \in \Omega^-. \end{cases} \tag{2.19}$$

If $u \in C^2(\overline{\Omega^-})$, Lu has a compact support and, in addition, u belongs to the Sommerfeld class $\mathcal{S}(\Omega^-)$ of radiating functions, then the following counterpart of identity (2.19) holds true

$$P_{\Omega^-}\big(Lu\big)\,(x) + V\big(\{\partial_n u\}^-\big)(x) - W\big(\{u\}^-\big)(x) = \begin{cases} 0, \quad\quad x \in \Omega^+, \\ u(x), \, x \in \Omega^-. \end{cases} \tag{2.20}$$

Note that by standard limiting procedure Green's formulas (2.17), (2.18), and the integral representations (2.19), (2.20) can be extended respectively to functions from the Bessel potential spaces $H_2^{1,0}(\Omega^+)$ and $H_{2,loc}^{1,0}(\Omega^-) \cap S(\Omega^-)$, where

$$H_2^{1,0}(\Omega^+) := \{u \in H_2^1(\Omega^+) : \ \Delta u \in H_2^0(\Omega^+)\},$$

$$H_{2,loc}^{1,0}(\Omega^-) := \{u \in H_{2,loc}^1(\Omega^-) : \ \Delta u + \omega^2 u \in H_{2,comp}^0(\Omega^-)\}$$

(for details see, e.g., [6], [13], [7], [5]).
In this case, the surface integrals in (2.17) and (2.18) are understood as dualities between the corresponding function spaces. In particular, for $u, v \in H^{1,0}(\Omega^+)$ we have the following Green's formulas

$$\int\limits_{\Omega^+} \big(\Delta u + \omega^2 u\big)\, v\, dx = \int\limits_{\Omega^+} \Big[-\sum_{k=1}^{3} \partial_k u\, \partial_k v + \omega^2 u\ v \Big]\, dx + \big\langle \{v\}^+ , \{\partial_n u\}^+ \big\rangle_S \tag{2.21}$$

$$\int\limits_{\Omega^+} \big[(\Delta u + \omega^2 u)v - u(\Delta v + \omega^2 v)\big]\, dx = \big\langle \{v\}^+, \{\partial_n u\}^+\big\rangle_S - \big\langle \{u\}^+, \{\partial_n v\}^+\big\rangle_S, \tag{2.22}$$

where $\langle \cdot\, , \cdot \rangle_S$ denotes the duality relation between the spaces $H_2^{1/2}(S)$ and $H_2^{-1/2}(S)$ which extends the usual $L_2(S)$ inner product,

$$\big\langle g\, , h \big\rangle_S = \int\limits_S g(y)\, h(y)\, dS \quad \text{for} \quad g, h \in L_2(S).$$

By relation (2.21) with arbitrary $v \in H_2^1(\Omega^+)$ the generalized trace $\{\partial_n u\}^+ \in H_2^{-1/2}(S)$ is correctly defined for a function $u \in H_2^{1,0}(\Omega^+)$.

Analogously, for a function $u \in H_{2,loc}^{1,0}(\Omega^-)$ the generalized trace $\{\partial_n u\}^- \in$

$H_2^{-1/2}(S)$ is correctly defined by the relation

$$\int\limits_{\Omega^-} (\Delta u + \omega^2 u)\, v\, dx = \int\limits_{\Omega^+} \Big[-\sum_{k=1}^{3} \partial_k u\, \partial_k v + \omega^2\, u\, v \Big]\, dx - \big\langle \{v\}^-, \{\partial_n u\}^- \big\rangle_S \tag{2.23}$$

where $v \in H^1_{2,\,comp}(\Omega^-)$ is an arbitrary function with compact support in $\overline{\Omega^-}$.

3. The Dirichlet and Neumann interior problems

Consider the homogeneous interior Dirichlet problem

$$(\Delta + \omega^2)\, u(x) = 0, \qquad x \in \Omega^+, \tag{3.1}$$

$$\{u(x)\}^+ = 0, \qquad x \in S. \tag{3.2}$$

This problem possesses either only a trivial solution or a finite number of linearly independent solutions $\{u_k\}_{k=1}^N$ with $N \geqslant 1$ being a natural number and due to (2.19) they are representable in the form (see, e.g., [20], [13])

$$u_k(x) = V\big(\{\partial_n u_k\}^+\big)(x), \qquad x \in \Omega^+, \quad k = \overline{1, N}, \tag{3.3}$$

where $\{\partial_n u_k\}^+ \in H_2^{-1/2}(S)$.

Moreover since $0 = \big\{V\big(\{\partial_n u\}^+\big)\big\}^+ = \big\{V\big(\{\partial_n u\}^+\big)\big\}^-$ it follows that

$$V\big(\{\partial_n u_k\}^+\big)(x) = 0, \qquad x \in \Omega^-, \quad k = \overline{1, N}, \tag{3.4}$$

and $\big\{\{\partial_n u_k\}^+\big\}_{k=1}^N$ are linearly independent on S. Equations (3.4) follow from the uniqueness theorem for the exterior Dirichlet problem, while the linear independency of the system $\big\{\{\partial_n u_k\}^+\big\}_{k=1}^N$ is a consequence of linear independency of functions (3.3).

Denote

$$\psi_k = \{\partial_n u_k\}^+, \quad k = \overline{1, N}. \tag{3.5}$$

Evidently

$$\big\{u_k(x),\ x \in \Omega^+\big\}_{k=1}^N \tag{3.6}$$

is a basis in the space of eigenfunctions of the Dirichlet interior problem (3.1)-(3.2).

Quite similarly, for the homogeneous Neumann problem

$$(\Delta + \omega^2)\, u(x) = 0, \qquad x \in \Omega^+, \tag{3.7}$$

$$\{\partial_n u(x)\}^+ = 0, \qquad x \in S \tag{3.8}$$

we have (see, e.g., [20], [13]):

(i) The problem (3.7)-(3.8) possesses either only a trivial solution or a finite dimensional space of eigenfunctions with basis $\{v_k\}_{k=1}^M$ with $M \geqslant 1$ being a natural number and in view of (2.19) they are representable in the form

$$v_k(x) = -W\big(\{v_k\}^+\big)(x), \qquad x \in \Omega^+, \quad k = \overline{1, M}; \tag{3.9}$$

(ii) The following equalities are valid

$$W\big(\{v_k\}^+\big)(x) = 0, \qquad x \in \Omega^-, \quad k = \overline{1, M}, \tag{3.10}$$

due to (2.13) and the uniqueness theorem for the exterior Neumann problem;
(iii) The system of functions $\{\varphi_k\}_{k=1}^M$ with

$$\varphi_k = \big\{v_k\big\}^+, \quad k = \overline{1, M}, \tag{3.11}$$

is linearly independent on S. Indeed, if $\sum\limits_{k=1}^{M} c_k \varphi_k = 0$ on S, then $\sum\limits_{k=1}^{M} c_k v_k(x) = -W\Big(\sum\limits_{k=1}^{M} c_k \varphi_k\Big) = 0$ in Ω^+ which contradicts the linear independence of the system $\{v_k\}_{k=1}^M$ in Ω^+.

Note that if ω is not a resonant frequency for the Dirichlet (Neumann) problem, then $N = 0$ ($M = 0$).

4. Traditional direct approach in exterior problems

Let us consider the nonhomogeneous Dirichlet exterior problem: Find $u \in H^{1,0}_{2,\,loc}(\Omega^-) \cap \mathcal{S}(\Omega^-)$ such that

$$(\Delta + \omega^2)\, u(x) \;=\; f(x), \qquad x \in \Omega^-, \tag{4.1}$$

$$\{u(x)\}^- = \varphi_0(x), \qquad x \in S, \tag{4.2}$$

where

$$f \in L_{2,comp}(\Omega^-), \qquad \varphi_0 \in H_2^{\frac{1}{2}}(S). \tag{4.3}$$

4.1. *The first kind Fredholm integral equation approach*

With the help of the representation formula (2.20) we get the following integral relation

$$u(x) - V\big(\{\partial_n u\}^-\big)(x) = P_{\Omega^-}(f)(x) - W(\varphi_0)(x), \quad x \in \Omega^-, \tag{4.4}$$

whence by taking trace on S from Ω^- we arrive at the first kind Fredholm integral equation for the unknown $\widetilde{\psi} = \{\partial_n u\}^-$,

$$\mathcal{H}\,\widetilde{\psi} = -\big\{P_{_{\Omega^-}}(f)\big\}^- + 2^{-1}\,\varphi_{_0} + \mathcal{K}\,(\varphi_{_0}) \quad \text{on} \quad S. \tag{4.5}$$

Note that for the right hand side expression in (4.5) we have (see (2.12))

$$-\big\{P_{_{\Omega^-}}(f)\big\}^- + 2^{-1}\,\varphi_{_0} + \mathcal{K}\,(\varphi_{_0}) = \big\{-P_{_{\Omega^-}}(f) + W\,(\varphi_{_0})\big\}^+ \quad \text{on} \quad S, \tag{4.6}$$

since for $f \in L_{2,comp}(\Omega^-)$ the volume potential $P_{_{\Omega^-}}(f)$ belongs to the space $H^2_{loc}(\mathbb{R}^3)$ and

$$\big\{P_{_{\Omega^-}}(f)\big\}^+ = \big\{P_{_{\Omega^-}}(f)\big\}^-, \qquad \big\{\partial_n\,P_{_{\Omega^-}}(f)\big\}^+ = \big\{\partial_n\,P_{_{\Omega^-}}(f)\big\}^- \quad \text{on} \quad S. \tag{4.7}$$

Thus the integral equation (4.5) can be rewritten as

$$\mathcal{H}\,\widetilde{\psi} = \Psi \quad \text{on} \quad S \tag{4.8}$$

with

$$\Psi(x) := \big\{-P_{_{\Omega^-}}(f)(x) + W\,(\varphi_{_0})(x)\big\}^+, \quad x \in S. \tag{4.9}$$

Here arise two questions:

- *Question 1.* Is (4.8) solvable in the space $H_2^{-1/2}(S)$ for all $f \in L_{2,comp}(\Omega^-)$ and $\varphi_{_0} \in H_2^{1/2}(S)$?
- *Question 2.* It is well known that the boundary value problem (4.1)-(4.2) is uniquely solvable (see, e.g., [20], [3], [13]) for arbitrary data satisfying the conditions (4.3) and the solution $u \in H^{1,0}_{2,\,loc}(\Omega^-) \cap S(\Omega^-)$. What is the relationship between a solution $\widetilde{\psi}$ of the integral equation (4.8) and the trace of the normal derivative $\{\partial_n u\}^-$? This question becomes essential if ω is a resonant frequency for the interior Dirichlet problem implying that the homogeneous version of the integral equation (4.8) with $\Psi = 0$ possesses nontrivial solutions. Evidently, in this case, if nonhomogeneous equation (4.5) is solvable, then solution is not unique and the problem is how to choose a solution $\widetilde{\psi}$ which has a physical meaning and coincides with the uniquely defined function $\{\partial_n u\}^-$.

Below we analyse both questions.

First, let us note that the operator

$$\mathcal{H} \;:\; H^{-1/2}(S) \to H^{1/2}(S) \tag{4.10}$$

is Fredholm with zero index, since it is a compact perturbation of the invertible operator $\mathcal{H}_{_0}$,

$$\mathcal{H}_0 \;:\; H^{-1/2}(S) \to H^{1/2}(S),$$

that corresponds to $\omega = 0$, i.e., $\mathcal{H}_{_0}$ is the boundary integral operator generated by the harmonic single layer potential (see [6], [7], [13]).

On the one hand, if ω is a resonant frequency for the interior Dirichlet problem for the domain Ω^+, due to (3.4) and (3.5) we then have

$$\mathcal{H}(\psi_k) = 0 \quad \text{on} \quad S, \quad k = 1, \cdots, N, \tag{4.11}$$

and consequently

$$\dim \ker \mathcal{H} \geqslant N. \tag{4.12}$$

On the other hand, if a vector $\widetilde{\psi}$ solves the homogeneous equation $\mathcal{H}\,(\widetilde{\psi}) = 0$ on S, then it follows that, the corresponding single layer potential $V\,(\widetilde{\psi})$ solves the homogeneous interior Dirichlet problem in Ω^+ and therefore

$$V\,(\widetilde{\psi})\,(x) = \sum_{k=1}^{N} c_k\, u_k\,(x) \quad \text{in} \quad \Omega^+, \tag{4.13}$$

since $\{u_k(x),\ x \in \Omega^+\}_{k=1}^{N}$ is a basis in the space of eigenfunctions of the Dirichlet interior problem (3.1)-(3.2) corresponding to the resonant frequency ω.

Further, due to the continuity of the single layer potential $V\,(\widetilde{\psi})$ across the boundary S and the uniqueness theorem for the exterior Dirichlet problem we find

$$V\,(\widetilde{\psi})\,(x) = 0, \quad x \in \Omega^-. \tag{4.14}$$

From (4.13) and (4.14) along with (3.5) we get

$$-\widetilde{\psi} = \{\partial_n\, V\,(\widetilde{\psi})\}^+ - \{\partial_n\, V\,(\widetilde{\psi})\}^- = \sum_{k=1}^{N} c_k\, \{\partial_n u_k\}^+ = \sum_{k=1}^{N} c_k\, \psi_k \quad \text{on} \quad S,$$

whence it follows that $\dim \ker \mathcal{H} \leqslant N$ which together with (4.12) implies that

$$\dim \ker \mathcal{H} = N \quad \text{and} \quad \{\psi_k\}_{k=1}^{N} \quad \text{is a basis of} \quad \ker \mathcal{H}. \tag{4.15}$$

It is easy to see that, if $\widetilde{\psi}$ is a solution to the homogeneous equation

$$\mathcal{H}\,\widetilde{\psi} = 0 \quad \text{on} \quad S$$

then the complex conjugate function $\overline{\widetilde{\psi}}$ solves the adjoint equation

$$\mathcal{H}^*\,\overline{\widetilde{\psi}} = 0 \quad \text{on} \quad S,$$

where $\mathcal{H}^*$ is the adjoint operator to $\mathcal{H}$,

$$\mathcal{H}^*\, g\,(x) = \int_S \overline{\Gamma(x-y,\omega)}\, g\,(y)\, dS = \overline{\mathcal{H}\,\overline{g}}\,(x). \tag{4.16}$$

Therefore the basis $\{\psi_k^*\}_{k=1}^{N}$ of the null space of the operator $\mathcal{H}^* : H_2^{-1/2}(S) \to$

$H_2^{1/2}(S)$ is

$$\{\psi^*_k\}_{k=1}^N \equiv \{\overline{\psi}_k\}_{k=1}^N = \Big\{ \{\overline{\partial_n u_k}\}^+ \Big\}_{k=1}^N. \tag{4.17}$$

The necessary and sufficient conditions for the nonhomogeneous equation (4.5) (i.e. (4.8)) to be solvable read as

$$\langle \Psi \,,\, \overline{\psi^*_k} \rangle_S = \Big\langle \{ -P_{\Omega^-}(f) + W(\varphi_0)\}^+ \,,\, \psi_k \Big\rangle_S = 0, \quad k = \overline{1,N}. \tag{4.18}$$

Using (4.17) rewrite equation (4.18) in the form

$$\Big\langle \{ -P_{\Omega^-}(f) + W(\varphi_0)\}^+ ,\, \{\partial_n u_k\}^+ \Big\rangle = 0, \quad k = \overline{1,N}. \tag{4.19}$$

Denote

$$v\,(x) := -P_{\Omega^-}(f)\,(x) + W\,(\varphi_0)\,(x), \qquad x \in \Omega^+. \tag{4.20}$$

Evidently

$$L(\partial,\omega)v\,(x) = \big(\Delta + \omega^2\big)\,v\,(x) = 0 \quad \text{in} \quad \Omega^+$$

due to (4.20) and Theorem 2.1. Therefore by Green's formula (2.17) we have

$$\int\limits_{\Omega^+} \big[L(\partial,\omega)v\,u_k - v\,L(\partial,\omega)u_k\big]\,dy = \langle \{\partial_n v\}^+, \{u_k\}^+ \rangle_S - \langle \{v\}^+, \{\partial_n u_k\}^+ \rangle_S, \tag{4.21}$$

whence equality (4.19) follows for arbitrary $f \in L_{2,comp}(\Omega^-)$ and $\varphi_0 \in H^{1/2}(S)$.

Thus, the above posed *Question 1* has a positive answer.

Now we go over to the second question, *Question 2*.

From the above analysis in accordance with (4.15) it follows that a general solution of equation (4.5) can be written as

$$\widetilde{\psi} = \psi_0 + \sum_{k=1}^N c_k\, \psi_k, \tag{4.22}$$

where ψ_0 is a particular solution of the nonhomogeneous equation (4.5), c_k are arbitrary complex constants, while $\{\psi_k\}_1^N$ is a basis of the null space $\ker \mathcal{H}$ with ψ_k defined by (3.5), where $\{u_k(x),\ x \in \Omega^+\}_{k=1}^N$ is a basis of the space of eigenfunctions of the interior homogeneous Dirichlet problem (3.1)-(3.2).

Now we show that it is possible to choose efficiently the parameters c_k in (4.22) such that the function $\widetilde{\psi}$ defined by (4.22) coincides with the uniquely defined function $\{\partial_n u\}^-$ on S which in turn is also a solution of (4.5).

Let us construct the function

$$U\,(x) = V\Big(\psi_0 + \sum_{k=1}^N c_k\,\psi_k\Big)\,(x) + P_{\Omega^-}(f)\,(x) - W\,(\varphi_0)\,(x), \quad x \in \Omega^+. \tag{4.23}$$

With the help of (4.5) and (4.22) we find that U solves the homogeneous interior Dirichlet problem

$$\Delta U(x) + \omega^2 U(x) = 0, \quad x \in \Omega^+, \tag{4.24}$$

$$\{U(x)\}^+ = 0, \quad x \in S. \tag{4.25}$$

Consequently U is representable as

$$U(x) = \sum_{k=1}^{N} d_k \, u_k(x) \quad \text{in} \quad \Omega^+, \tag{4.26}$$

where d_k are complex constants and u_k are the elements of the above introduced basis in the space of eigenfunctions of the interior homogeneous Dirichlet problem (3.1)-(3.2).

In accordance with the notations (3.5) we have from (3.3)

$$V(\psi_k) = u_k \quad \text{in} \quad \Omega^+, \tag{4.27}$$

and consequently from (4.23) we conclude

$$U(x) \equiv \sum_{k=1}^{N} c_k \, u_k(x) + V(\psi_0)(x) + P_{\Omega^-}(f)(x) - W(\varphi_0)(x) \quad \text{in} \quad \Omega^+. \tag{4.28}$$

On the other hand, it is evident that if U is orthogonal to all u_k in the $L_2(\Omega^+)$ sense, then it is identically zero in Ω^+ in accordance with (4.26).

In view of (4.28) the orthogonality conditions $(U, u_j)_{L_2(\Omega^+)} = 0$, $j = 1, \cdots, N$, lead to the system of linear algebraic equations for c_k:

$$\sum_{k=1}^{N} b_{jk} \, c_k = b_j, \qquad j = \overline{1, N}, \tag{4.29}$$

where

$$b_{jk} = (u_k, u_j)_{L_2(\Omega^+)}, \qquad b_j = -(F, u_j)_{L_2(\Omega^+)}, \tag{4.30}$$

$$F(x) := V(\psi_0)(x) + P_{\Omega^-}(f)(x) - W(\varphi_0)(x), \quad x \in \Omega^+. \tag{4.31}$$

Note that the Gram determinant $\det [b_{jk}]_{N \times N} \neq 0$, since $\{u_k\}_1^N$ is a set of linearly independent functions. Therefore by the system (4.29) the coefficients c_k are defined uniquely. Denote these constants by $c_k^{(0)}$ and by $U^{(0)}(x)$ the function given by (4.23) with $c_k^{(0)}$ for c_k. Then we evidently have

$$U^{(0)}(x) = V\Big(\psi_0(x) + \sum_{k=1}^{N} c_k^{(0)} \psi_k\Big)(x) + P_{\Omega^-}(f)(x) - W(\varphi_0)(x) = 0, \; x \in \Omega^+. \tag{4.32}$$

We set

$$\psi^{(1)} := \psi_{_0} + \sum_{k=1}^{N} c_{_k}^{(0)} \psi_{_k} \qquad \text{on} \qquad S. \tag{4.33}$$

It is now evident that $\psi^{(1)}$ is a solution of equation (4.5) which has the property

$$V\left(\psi^{(1)}\right)(x) + P_{_{\Omega^-}}(f)(x) - W(\varphi_{_0})(x) = 0, \quad x \in \Omega^+. \tag{4.34}$$

It is easy to show that $\psi^{(1)}$ is uniquely defined by the property (4.34). Indeed, if there are two such solutions $\psi^{(1)}$ and $\psi^{(2)}$ satisfying (4.34), the difference $V(\psi^{(1)}) - V(\psi^{(2)}) = V(\psi^{(1)} - \psi^{(2)})$ vanishes in Ω^+, and by continuity of single layer potential and in view of the uniqueness theorem for the exterior Dirichlet problem, we deduce that $V(\psi^{(1)} - \psi^{(2)}) = 0$ in $\mathbb{R}^3$, implying $\psi^{(1)} - \psi^{(2)} = 0$ on S due to the jump relations (2.11).

Now we show that the function $\psi^{(1)}$ coincides with the function $\{\partial_n u\}^-$ on S. Indeed, if we substitute $\psi^{(1)}$ for $\{\partial_n u\}^-$ in equation (4.4), we get

$$u(x) = \mathcal{P}_{\Omega^-}(f)(x) - W(\varphi_{_0})(x) + V(\psi^{(1)})(x), \quad x \in \Omega^-. \tag{4.35}$$

Taking into account (2.11), (2.13) and keeping in mind that $\mathcal{P}_{\Omega^-} \in H^2_{loc}(\mathbb{R}^3)$ for $f \in L_{2,comp}(\Omega^-))$, we derive

$$\begin{aligned} \{\partial_n u\}^- &= \big\{\partial_n \mathcal{P}_{\Omega^-}(f)\big\}^- - \big\{\partial_n W(\varphi_{_0})\big\}^- + 2^{-1}\,\psi^{(1)} + \widetilde{\mathcal{K}}\psi^{(1)} \\ &= \big\{\partial_n \mathcal{P}_{\Omega^-}(f)\big\}^+ - \big\{\partial_n W(\varphi_{_0})\big\}^+ + \big\{ - 2^{-1}\psi^{(1)} + \widetilde{\mathcal{K}}\psi^{(1)}\big\} + \psi^{(1)} \\ &= \big\{\partial_n \big[\mathcal{P}_{\Omega^-}(f) - W(\varphi_{_0}) + V(\psi^{(1)})\big]\big\}^+ + \psi^{(1)} = \psi^{(1)} \end{aligned} \tag{4.36}$$

due to (4.34).

Thus, we have shown that $\psi^{(1)}$ coincides with $\{\partial_n u\}^-$ and consequently *Question 2* also has a positive answer.

Note that in (4.35) the right hand side will not be changed if instead of $\psi^{(1)}$ we take an arbitrary solution $\widetilde{\psi}$ of equation (4.5), since the difference $\psi^{(1)} - \widetilde{\psi}$ belongs to the linear span of the system $\big\{\psi_{_k}\big\}_1^N \subset \ker \mathcal{H}$ and consequently $V(\psi_{_k})(x) = 0$, $x \in \Omega^-$, due to (3.5) and (3.4). This implies that $V(\psi^{(1)})(x) = V(\widetilde{\psi})(x)$ in Ω^-. But the above obtained results show that among all solutions of (4.5) there is only one function, $\psi^{(1)}$, with the property $\psi^{(1)} = \{\partial_n u\}^-$, where u is the unique solution of the exterior Dirichlet problem (4.1)-(4.2).

4.2. *The second kind Fredholm integral equation approach*

For the same Dirichlet exterior problem (4.1)-(4.2) here we develop an alternative approach which reduces the problem to a second kind boundary integral equation.

From the integral representation formula (2.20) we have

$$u(x) - V\big(\{\partial_n u\}^-\big)(x) = P_{\Omega^-}(f)(x) - W(\varphi_{_0})(x), \quad x \in \Omega^-, \tag{4.37}$$

where f and φ_0 satisfy again the conditions (4.3).

Taking the trace on S from Ω^- of the normal derivative of equation (4.37) we get the relation

$$-2^{-1}\{\partial_n u\}^- + \mathcal{K}^*\{\partial_n u\}^- = \{\partial_n[-P_{\Omega^-}(f) + W(\varphi_0)]\}^- \quad \text{on} \quad S. \tag{4.38}$$

Due to Theorem 2.1 and conditions (4.3) for the right hand side expression in (4.38) the following equality holds

$$\Phi := \{\partial_n[-\mathcal{P}_{\Omega^-}(f) + W(\varphi_0)]\}^- = \{\partial_n[-P_{\Omega^-}(f) + W(\varphi_0)]\}^+ \quad \text{on} \ S\,. \tag{4.39}$$

Denote now the unknown $\{\partial_n u\}^-$ by $\widetilde{\varphi}$,

$$\widetilde{\varphi} := \{\partial_n u\}^- \in H_2^{-1/2}(S)\,. \tag{4.40}$$

Now for the unknown $\widetilde{\varphi}$ we have equation (4.38) which can be rewritten as

$$\big(-2^{-1}I + \widetilde{\mathcal{K}}\big)\,\widetilde{\varphi} = \Phi \quad \text{on} \quad S. \tag{4.41}$$

Note that the operator $-2^{-1}I+\mathcal{K}^*$ is generated by the interior trace of the normal derivative of the single layer potential (see Theorem 2.1) and it is well known that the operator

$$-2^{-1}I + \widetilde{\mathcal{K}} \ : \ H_2^{-1/2}(S) \to H_2^{-1/2}(S) \tag{4.42}$$

is Fredholm with zero index but it is not invertible, in general, if ω is a resonant frequency for the interior Neumann problem for the domain Ω^+ (see, e.g., [20], [13]).

As we have already mentioned in Section 3, the homogeneous interior Neumann problem (3.7)-(3.8) possesses finitely many linearly independent solutions for a resonant frequency ω. Denote the dimension of the null space by M and let $\{v_k\}_{k=1}^M$ be the corresponding basis in $H_2^1(\Omega^+)$. They admit representation by the double layer potential (3.9) in Ω^+. Recall that the system of functions

$$\varphi_k = \{v_k\}^+ \in H_2^{1/2}(S), \quad k = 1, \cdots, M, \tag{4.43}$$

is linearly independent on S and in accordance with (3.10) we have

$$W(\varphi_k)(x) = 0 \quad \text{in} \quad \Omega^-, \quad k = 1, \cdots, M. \tag{4.44}$$

Taking the trace of (4.44) from Ω^- leads to the homogeneous equation

$$\big(-2^{-1}I + \mathcal{K}\big)\,\varphi_k = 0 \quad \text{on} \quad S, \ \ k = 1, \cdots, M. \tag{4.45}$$

For the operator $\mathcal{K}^*$, adjoint to $\mathcal{K}$, we have

$$\mathcal{K}^*\,\psi = \overline{\widetilde{\mathcal{K}}\,\overline{\psi}}\,. \tag{4.46}$$

Therefore keeping in mind that $-2^{-1}\,I+\mathcal{K}$ and $-2^{-1}\,I+\mathcal{K}^*$ are mutually adjoint operators with zero index, in view of (4.46) and (4.45) we conclude that

$$\dim\ker\big(-2^{-1}\,I+\mathcal{K}\big)=\dim\ker\big(-2^{-1}\,I+\widetilde{\mathcal{K}}\big)\geqslant M. \tag{4.47}$$

From (4.47) it follows that equation (4.41) is not unconditionally solvable in general and therefore solution $\widetilde{\varphi}$ of (4.41) is not defined uniquely.

Again, here arise two questions similar to those stated in Subsection 4.1:

- *Question 3.* Is (4.41) solvable in the space $H_2^{-1/2}(S)$ for all $f\in L_{2,comp}(\Omega^-)$ and $\varphi_0\in H_2^{1/2}(S)$?
- *Question 4.* The exterior Dirichlet boundary value problem (4.1)-(4.2) possesses a unique solution $u\in H^{1,0}_{2,\,loc}(\Omega^-)\cap S(\Omega^-)$ for arbitrary data satisfying the conditions (4.3) (see, e.g., [20], [3], [13]). What is the relationship between a solution $\widetilde{\varphi}$ of the integral equation (4.41) and the trace of the normal derivative $\{\partial_n\,u\}^-$? This question becomes again essential if ω is a resonant frequency for the interior Neumann problem implying that the homogeneous version of the integral equation (4.41) with $\Phi=0$ possesses nontrivial solutions. Evidently, in this case, if nonhomogeneous equation (4.41) is solvable, then the solution is not unique and the problem is how to choose a solution $\widetilde{\varphi}$ which has a physical meaning and coincides with the uniquely defined function $\{\partial_n\,u\}^-$.

We start with analysis of *Question 3* and establish that equation (4.41) is always solvable.

To this end, let us show that the null space of the mapping

$$\big(-2^{-1}I+\mathcal{K}\big):H^{1/2}(S)\to H^{1/2}(S), \tag{4.48}$$

is of dimension M and, moreover, $\big\{\varphi_k\big\}_{k=1}^{M}$ is the basis of this null space where M is the dimension of eigenfunctions space of the Neumann problem (3.7)-(3.8).

Indeed, let

$$\big(-2^{-1}I+\mathcal{K}\big)h=0 \qquad \text{on} \qquad S. \tag{4.49}$$

Then it follows that the double layer potential $W(h)$ solves the homogeneous exterior Dirichlet problem, and therefore due to the corresponding uniqueness theorem

$$W(h)(x)=0 \qquad \text{in} \qquad \Omega^-. \tag{4.50}$$

By (2.13) we derive that $W(h)$ solves the homogeneous interior Neumann problem (3.7)-(3.8) and therefore

$$W(h)(x)=\sum_{k=1}^{M}d_{k}v_k(x), \quad x\in\Omega^+. \tag{4.51}$$

In view of (3.9) and (4.44) we have

$$W\Big(h+\sum_{k=1}^{M} d_{k}\varphi_k\Big)=0 \quad \text{in} \quad \Omega^+. \tag{4.52}$$

In view of (4.44) and (4.50) we can write

$$W\Big(h+\sum_{k=1}^{M} d_{k}\varphi_k\Big)=0 \quad \text{in} \quad \Omega^-. \tag{4.53}$$

In turn, from (4.52)-(4.53) it follows

$$h+\sum_{k=1}^{M} d_{k}\varphi_k=0 \quad \text{on} \quad S, \tag{4.54}$$

which implies that

$$\dim\ker\big(-2^{-1}I+\mathcal{K}\big)=\dim\ker\big(-2^{-1}I+\widetilde{\mathcal{K}}\big)=M. \tag{4.55}$$

Evidently, the complete system of linearly independent solutions of the adjoint equation

$$\big(-2^{-1}I+\widetilde{\mathcal{K}}\big)^*g=\overline{\big(-2^{-1}I+\mathcal{K}\big)\overline{g}}=0 \tag{4.56}$$

is

$$\Big\{\overline{\varphi_{k}}\Big\}_{k=1}^{M}\equiv\Big\{\overline{\{v_{k}\}^+}\Big\}_{k=1}^{M}. \tag{4.57}$$

Then the necessary and sufficient conditions of solvability of equation (4.41) read as

$$\langle\Phi,\varphi_{k}\rangle_{S}=0, \qquad k=\overline{1,M}, \tag{4.58}$$

where Φ is defined in (4.39).

The conditions (4.58) can be checked again with the help of Green's identity (4.21) with v_{k} for u_{k} and with v as in (4.20), $v=-P_{\Omega^-}(f)+W(\varphi_{0})$ in Ω^+, and keeping in mind that $\{\partial_n v\}^+=\Phi$ on S.

Thus the equation (4.38) (i.e. (4.41)) is solvable.

Denote the complete system of linearly independent solutions of the homogeneous equation

$$\big(-2^{-1}I+\widetilde{\mathcal{K}}\big)\widetilde{\varphi}=0 \quad \text{on} \quad S \tag{4.59}$$

by $\{\widetilde{\varphi}_{k}\}_{k=1}^{M}$. Evidently, the corresponding single layer potentials

$$v_{k}(x)=V(\widetilde{\varphi}_{k})(x), \quad x\in\Omega^+, \quad k=\overline{1,M}, \tag{4.60}$$

satisfy the conditions (3.7)-(3.8).

One can easily check that the system

$$\{v_k\}_{k=1}^{M} \quad \text{is linearly independent in} \quad \Omega^+. \tag{4.61}$$

Therefore, the system $\big\{\{v_k\}^+\big\}_{k=1}^{M} \equiv \big\{\mathcal{H}\widetilde{\varphi}_k\big\}_{k=1}^{M}$ is also linearly independent on S. Indeed, if

$$\sum_{k=1}^{M} a_k \{v_k\}^+ = 0 \quad \text{on } S \quad \text{with} \quad \sum_{k=1}^{M} |a_k| \neq 0,$$

with the help of the equality

$$\sum_{k=1}^{M} a_k \big\{\partial_n v_k\big\}^+ = 0 \quad \text{on } S$$

and Green's third formula (2.19) we conclude that

$$\sum_{k=1}^{M} a_k v_k(x) = 0 \quad \text{in } \Omega^+$$

which contradicts (4.61).

So, a general solution to the equation (4.41) can be written as

$$\widetilde{\varphi} = \widetilde{\varphi}_0 + \sum_{k=1}^{M} c_k \widetilde{\varphi}_k, \tag{4.62}$$

where $\widetilde{\varphi}_0$ is some particular solution of (4.41).

Now let us analyse Question 4: How to choose the constants c_k in (4.62) to obtain the desired "physical" relation $\widetilde{\varphi} = \big\{\partial_n u\big\}^-$. In other words, it means that we have to choose the constants c_k such that in the formula (see (4.37))

$$u(x) = P_{\Omega^-}(f)(x) - W(\varphi_0)(x) + V\Big(\widetilde{\varphi}_0 + \sum_{k=1}^{M} c_k \widetilde{\varphi}_k\Big)(x)\,, \tag{4.63}$$

the density of the single layer potential coincides with the normal derivative of the right hand side expression in (4.63), i.e. the following equality must be satisfied

$$\big\{\partial_n u\big\}^- = \widetilde{\varphi} \equiv \widetilde{\varphi}_0 + \sum_{k=1}^{M} c_k \widetilde{\varphi}_k \quad \text{on} \quad S. \tag{4.64}$$

To this end, we need that the right hand side in (4.63) vanishes in Ω^+, i.e., the following relation holds

$$U := P_{\Omega^-}(f) - W(\varphi_0) + V(\widetilde{\varphi}) = 0 \quad \text{in} \quad \Omega^+. \tag{4.65}$$

Evidently $(\Delta+\omega^2)\,U=0$ in Ω^+ and due to (4.41) and (4.39)

$$\big\{\partial_n U\big\}^+=0 \qquad \text{on} \qquad S. \tag{4.66}$$

Therefore U solves the interior homogeneous Neumann problem and consequently U belongs to the space of eingenfunctions of this problem whose basis is $\{v_k\}_{k=1}^M$. Employing (4.60), rewrite U in the following form

$$U=P_{\Omega^-}(f)-W(\varphi_0)+V(\widetilde{\varphi}_0)+\sum_{k=1}^{M}c_k\,v_k \qquad \text{in} \qquad \Omega^+. \tag{4.67}$$

Now let us choose the constants c_k such that U is orthogonal to the functions v_k, $k=\overline{1,M}$.

The relations $(U,\,v_k)_{L_2(\Omega^+)}=0$ lead to the system of linear algebraic equations,

$$\sum_{k=1}^{M}(v_k,\,v_j)_{L_2(\Omega^+)}\,c_k=\big(-P_{\Omega^-}(f)+W(\varphi_0)-V(\psi_0)\,,\,v_j\big)_{L_2(\Omega^+)},\ \ j=\overline{1,M}. \tag{4.68}$$

Since Gram's matrix $[((v_k,\,v_j)_{L_2(\Omega^+)})]_{M\times M}$ is nonsingular, the parameters c_k are defined uniquely from (4.68).

Thus there are constants $c_k^{(0)}$ such that (4.65) holds. By these $c_k^{(0)}$ we construct a solution $\widetilde{\varphi}^{(1)}$ to (4.41)

$$\widetilde{\varphi}^{(1)}=\widetilde{\varphi}_0+\sum_{k=1}^{M}c_k^{(0)}\,\widetilde{\varphi}_k\,. \tag{4.69}$$

Evidently, relation (4.65) holds true.

As in the previous case we can show that $\widetilde{\varphi}^{(1)}$ is defined uniquely.

Now we show that just this function $\widetilde{\varphi}^{(1)}$ corresponds to $\big\{\partial_n u\big\}^-$. In other words, if we construct a solution function of the Dirichlet boundary value problem in the form (cf. (4.37)-(4.38))

$$u(x)=P_{\Omega^-}(f)(x)-W(\varphi_0)(x)+V(\widetilde{\varphi}^{(1)})(x), \quad x\in\Omega^-, \tag{4.70}$$

then

$$\big\{\partial_n u\big\}^-=\widetilde{\varphi}^{(1)}) \qquad \text{on} \qquad S. \tag{4.71}$$

Indeed, with the help of (4.39) and (4.65) we get

$$\begin{aligned}
\big\{\partial_n u\big\}^- &= \big\{\partial_n\big[P_{\Omega^-}(f)-W(\varphi_0)+V(\widetilde{\varphi}^{(1)})\big]\big\}^-\\
&= \big\{\partial_n\big[P_{\Omega^-}(f)-W(\varphi_0)\big]\big\}^+ + 2^{-1}\widetilde{\varphi}^{(1)}+\widetilde{\mathcal{K}}\,\widetilde{\varphi}^{(1)}\\
&= \widetilde{\varphi}^{(1)}+\Big\{\partial_n\big[P_{\Omega^-}(f)-W(\varphi_0)\big]\Big\}^+ - 2^{-1}\widetilde{\varphi}^{(1)}+\widetilde{\mathcal{K}}\,\widetilde{\varphi}^{(1)}\\
&= \widetilde{\varphi}^{(1)}+\big\{\partial_n\big[P_{\Omega^-}(f)-W(\varphi_0)+V(\widetilde{\varphi}^{(1)}\big]\big\}^+=\widetilde{\varphi}^{(1)}.
\end{aligned}$$

Thus we have analysed the traditional direct boundary equation approaches for the exterior Dirichlet boundary value problem for the Helmholtz equation which reduce the boundary value problem either to the first kind Fredholm integral equation or to the second kind Fredholm integral equation. In both cases the corresponding integral equations are not equivalent to the uniquely solvable boundary value problem since the corresponding operators are not invertible for all values of the frequency parameter, in particular for resonant frequencies. However we have shown that the obtained integral equations are always solvable for arbitrary data, solutions of the integral equations are not defined uniquely and among the solutions of the integral equations there is only one solution which has a physical sense and coincides with the trace of the normal derivative of the radiating solution on S.

Remark 1 : Quite similarly can be analysed the traditional direct approach for the exterior Neumann problem.

Remark 2 : If ω is not a resonant frequency, then the corresponding boundary integral equations are uniquely solvable and the unique solution of the boundary integral equation automatically satisfies the desired property: in the case of the Dirichlet BVP it coincides with the normal derivative of the radiating solution, while in the case of the Neumann BVP it coincides with the trace of the solution on S.

5. Modified direct boundary integral equation method

Applying the approach introduced in the references [1], [12], and [17] (see also [3] and [2]), we can use again the direct boundary integral equation method and reduce the exterior Dirichlet and Neumann boundary value problems to the equivalent, uniquely solvable integral equations.

5.1. *Modified direct approach for the Dirichlet problem*

First we deal with the Dirichlet exterior problem (4.1)-(4.2).

From the integral representation formula (4.4) we get:

$$u(x) - V(\{\partial_n u\}^-)(x) = P_{\Omega^-}(f)(x) - W(\varphi_0)(x), \quad x \in \Omega^-, \tag{5.1}$$

$$\mathcal{H}(\{\partial_n u\}^-) = \big\{ - P_{\Omega^-}(f) + W(\varphi_0)\big\}^- + \varphi_0 \quad \text{on} \quad S, \tag{5.2}$$

$$- 2^{-1}\{\partial_n u\}^- + \widetilde{\mathcal{K}}\left(\{\partial_n u\}^-\right) = \big\{\partial_n\big[- P_{\Omega^-}(f) + W(\varphi_0)\big]\big\}^- \quad \text{on} \quad S\,. \tag{5.3}$$

Our goal is to define the unknown function $\{\partial_n u\}^-$.

Let us introduce the notation:

$$\psi := \{\partial_n u\}^- \quad \text{on} \quad S, \tag{5.4}$$

$$\Psi := \big\{ \big[-P_{\Omega^-}(f) + W(\varphi_0) \big] \big\}^- + \varphi_0 \quad \text{on} \quad S, \tag{5.5}$$

$$\Phi := \big\{ \partial_n \big[-P_{\Omega^-}(f) + W(\varphi_0) \big] \big\}^- \quad \text{on} \quad S\,. \tag{5.6}$$

Let us multiply (5.2) by $i\,\alpha$ and add to (5.3) to obtain

$$\big[-2^{-1} I + \widetilde{\mathcal{K}} + i\,\alpha\,\mathcal{H} \big]\, \psi = \Phi + i\,\alpha\,\Psi \quad \text{on} \quad S\,. \tag{5.7}$$

Here α is a real constant different from zero.

It can be shown that the operator

$$-2^{-1} I + \widetilde{\mathcal{K}} + i\,\alpha\,\mathcal{H} \,:\, H^{-1/2}(S) \to H^{-1/2}(S) \tag{5.8}$$

is Fredholm with zero index (see [1], [3], [17], [12], [8], [9]). Now we show that its kernel is trivial. Indeed, if ψ_0 solves the homogeneous equation

$$\big[-2^{-1} I + \widetilde{\mathcal{K}} + i\,\alpha\,\mathcal{H} \big]\, \psi_0 = 0 \quad \text{on} \quad S, \tag{5.9}$$

then the single layer potential $u_0(y) := V(\psi_0)(x)$ is a solution to the interior Robin problem (due to (5.9) and Theorem 2.1)

$$(\Delta + \omega^2)\, u_0 \;=\; 0 \qquad \text{on} \quad \Omega^+, \tag{5.10}$$

$$\{\partial_n u_0 + i\,\alpha\, u_0\}^+ = 0 \qquad \text{on} \quad S. \tag{5.11}$$

With the help of Green's formula (2.17) with $u = u_0$ and $v = \overline{u_0}$ we have

$$\int\limits_{\Omega^+} \Big[\sum_{k=1}^{3} |\partial_k u_0|^2 - \omega^2\, |u_0|^2 \Big]\, dx = -i\,\alpha \int\limits_{\partial\Omega^+} |\{u_0\}^+|^2\, dS$$

Since $\alpha \neq 0$ and $\alpha \in \mathbb{R}$ we conclude $\{u_0\}^+ = 0$ on S. Then by (5.11) we get $\{\partial_n\, u_0\}^+ = 0$ on S and from Green's third formula (2.19) it follows that $u_0(x) = V(\psi_0)(x) = 0$ for $x \in \Omega^+$, whence $\psi_0 = 0$ on S follows due to continuity of the single layer potential, uniqueness theorem for the exterior Dirichlet problem and the jump relations (2.11).

Thus, from (5.7) the unknown ψ can be defined uniquely and the solution to the BVP (4.2) can be constructed by (5.1)

$$u(x) = P_{\Omega^-}(f)(x) - W(\varphi_0)(x) + V(\psi)(x), \quad x \in \Omega^-. \tag{5.12}$$

Now we have to show that relation (5.4) holds true for the function u defined by

(5.12). By Theorem 2.1 and using (5.12) and (5.7) we derive

$$\begin{aligned}\{\partial_n u\}^- &= \{\partial_n[P_{\Omega^-}(f) - W(\varphi_0)]\}^- + 2^{-1}\psi + \widetilde{\mathcal{K}}\psi \\ &= -\Phi + 2^{-1}\psi + \widetilde{\mathcal{K}}\psi = \psi - i\,\alpha\,\mathcal{H}\psi + i\,\alpha\,\Psi \\ &= \psi - i\,\alpha\,[\mathcal{H}\psi + \{P_{\Omega^-}(f) - W(\varphi_0)\}^- + \varphi_0] \\ &= \psi - i\,\alpha\,[\mathcal{H}\psi + \{P_{\Omega^-}(f)\}^+ - 2^{-1}\varphi_0 - \mathcal{K}\varphi_0] \\ &= \psi - i\,\alpha\,\{V(\psi) + P_{\Omega^-}(f) - W(\varphi_0)\}^+ . \end{aligned} \tag{5.13}$$

Now we show that the function

$$U(x) := V(\psi)(x) + P_{\Omega^-}(f)(x) - W(\varphi_0)(x), \quad x \in \Omega^+, \tag{5.14}$$

vanishes identically in Ω^+. We proceed as follows. It is evident that U solves the Helmholtz equation

$$(\Delta + \omega^2)\,U(x) = 0, \qquad x \in \Omega^+, \tag{5.15}$$

and moreover in view of (5.7)

$$\begin{aligned}\{\partial_n U + i\,\alpha\,U\}^+ =&\big(-2^{-1}I + \widetilde{\mathcal{K}} + i\,\alpha\,\mathcal{H}\big)\psi + \{\partial_n[P_{\Omega^-}(f) - W(\varphi_0)]\}^+ \\ &+ i\,\alpha\,\{P_{\Omega^-}(f) - W(\varphi_0)\}^+ \\ =&\big(-2^{-1}I + \widetilde{\mathcal{K}} + i\,\alpha\,\mathcal{H}\big)\psi - \Phi - i\,\alpha\,\Psi = 0. \end{aligned} \tag{5.16}$$

Here we employed Theorem 2.1 and the following evident relations

$$\{\partial_n[P_{\Omega^-}(f) - W(\varphi_0)]\}^+ = \{\partial_n[P_{\Omega^-}(f) - W(\varphi_0)]\}^- = -\Phi,$$

$$\begin{aligned}\{P_{\Omega^-}(f) - W(\varphi_0)\}^+ &= \{P_{\Omega^-}(f)\}^+ - \mathcal{K}\varphi_0 - 2^{-1}\varphi_0 = \\ &= \{P_{\Omega^-}(f) - W(\varphi_0)\}^- - \varphi_0 = -\Psi.\end{aligned}$$

Therefore the function U solves the homogeneous interior Robin BVP (5.15)-(5.16). By Green's formula (2.17) it is easy to show that the homogeneous Robin BVP (5.15)-(5.16) possesses only the trivial solution. Therefore the function U defined by (5.14) vanishes identically in Ω^+, and consequently

$$\Big\{V(\psi) + P_{\Omega^-}(f) - W(\varphi_0)\Big\}^+ = 0 \quad \text{on} \quad S. \tag{5.17}$$

Then from (5.13) we get the desired equality $\{\partial_n u\}^- = \psi$.

Thus, the above described direct approach reduces the exterior Dirichlet boundary value problem (4.1)-(4.2) to the equivalent (i.e. uniquely solvable) boundary integral equation (5.7) for the unknown function $\psi = \{\partial_n u\}^-$.

5.2. *Modified direct approach for the Neumann problem*

Now we consider the exterior Neumann problem

$$\Delta u + \omega^2 u = f \quad \text{in} \quad \Omega^-, \tag{5.18}$$

$$\{\partial_n u\}^- = \psi_0 \quad \text{on} \quad S. \tag{5.19}$$

Here we assume that

$$f \in L_{2,comp}(\Omega^-), \quad \psi_0 \in H^{-1/2}(S). \tag{5.20}$$

We apply again Green's third formula (2.20) and rewrite it in the following form

$$u(x) + W\big(\{u\}^-\big)(x) = P_{\Omega^-}(f)(x) + V(\psi_0)(x)\,, \quad x \in \Omega^-. \tag{5.21}$$

From (5.21) we derive the following two boundary relations

$$2^{-1}\{u\}^- + \mathcal{K}\{u\}^- = \Phi \quad \text{on} \quad S, \tag{5.22}$$

$$\mathcal{L}\{u\}^- = \Psi \quad \text{on} \quad S, \tag{5.23}$$

where

$$\Phi := \big\{P_{\Omega^-}(f) + V(\psi_0)\big\}^- \quad \text{on} \quad S, \tag{5.24}$$

$$\Psi := \big\{\partial_n\big[P_{\Omega^-}(f) + V(\varphi_0)\big]\big\}^- - \varphi_0 \quad \text{on} \quad S. \tag{5.25}$$

Note that, in view of Theorem 2.1, the functions Φ and Ψ can be represented also as

$$\Phi = \big\{P_{\Omega^-}(f) + V(\psi_0)\big\}^+ \quad \text{on} \quad S, \tag{5.26}$$

$$\Psi = \big\{\partial_n\big[P_{\Omega^-}(f) + V(\varphi_0)\big]\big\}^+ \quad \text{on} \quad S. \tag{5.27}$$

Further, let us introduce the notation

$$\varphi := \{u\}^-. \tag{5.28}$$

Multiply (5.22) by $i\alpha$ and add to (5.23) to obtain

$$\big[\mathcal{L} + i\,\alpha\,\big(2^{-1}\,I + \mathcal{K}\big)\big]\,\varphi = \Psi + i\,\alpha\,\Phi \quad \text{on} \quad S, \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0. \tag{5.29}$$

The operator

$$\mathbf{K} := \mathcal{L} + i\,\alpha\,\big(2^{-1}\,I + \mathcal{K}\big) \;\; : \;\; H^{1/2}(S) \to H^{-1/2}(S) \tag{5.30}$$

is Fredholm with zero index. In the case of a smooth boundary S, the operator $\mathbf{K}$ is a strongly elliptic pseudodifferential operator of order 1; actually $\mathbf{K}$ is a singular integro-differential operator with a positive symbol (see, e.g., [7], [8], [9]).

Further we show that the null-space of the operator (5.30) is trivial. Indeed, let

$$\mathbf{K}\varphi_0 = 0 \quad \text{on} \quad S \tag{5.31}$$

and consider the function

$$u_0(x) = W(\varphi_0)(x), \quad x \in \Omega^{\pm}. \tag{5.32}$$

Evidently u_0 solves the homogeneous Helmholtz equation in $\Omega^{\pm}$ and due to (5.31) satisfies the homogeneous Robin condition

$$\{\partial_n u_0 + i\,\alpha\, u_0\}^+ = 0 \quad \text{on} \quad S.$$

Therefore $u_0(x) = W(\varphi_0)(x) = 0$ in Ω^+. By the relation (2.13) it then follows that the radiating function $u_0 = W(\varphi_0)$ solves the homogeneous exterior Neumann problem and consequently $u_0(x) = W(\varphi_0)(x) = 0$ in Ω^-, implying $\varphi_0 = 0$ on S.

Therefore the operator (5.30) is invertible and consequently (5.29) is uniquely solvable. From (5.21) via (5.28) we get

$$u(x) = P_{\Omega^-}(f)(x) + V(\psi_0)(x) - W(\varphi)(x) \quad \text{in} \quad \Omega^-, \tag{5.33}$$

where φ solves (5.29).

Let us show that relation (5.28) holds true for the function u defined by (5.33).

Using the same arguments as above and keeping in mind relations (5.26)-(5.27), from (5.33) we derive

$$\begin{aligned}\{u\}^- =& \big\{P_{\Omega^-}(f) + V(\psi_0)\big\}^- + 2^{-1}\varphi - \mathcal{K}\varphi = \varphi + \Phi - \big(2^{-1}\varphi + \mathcal{K}\varphi\big)\\ =& \varphi + \big\{P_{\Omega^-}(f) + V(\psi_0) - W(\varphi)\big\}^+.\end{aligned} \tag{5.34}$$

As in the previous subsection, we can show that the function

$$U(x) = P_{\Omega^-}(f)(x) + V(\psi_0)(x) - W(\varphi)(x), \quad x \in \Omega^+, \tag{5.35}$$

vanishes identically in Ω^+, since it solves the homogeneous Helmholtz equation and satisfies the homogeneous Robin condition on S in accordance with (5.29),

$$\big\{\partial_n U + i\alpha U\big\}^+ = \Psi + i\,\alpha\,\Phi - \mathcal{L}\varphi - i\alpha\big(2^{-1}\varphi + \mathcal{K}\varphi\big) = 0 \quad \text{on} \quad S. \tag{5.36}$$

Therefore from (5.34) we get

$$\{u\}^- = \varphi \quad \text{on} \quad S.$$

Thus, for arbitrary value of the frequency parameter ω, the above described direct approach reduces the exterior Neumann problem (5.18)-(5.19) to the equivalent (i.e. uniquely solvable) boundary integral equation (5.29) for the unknown function $\varphi = \{u\}^-$.

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