

# OPERATORS ON CONFIGURATIONS WITH SINGULAR GEOMETRY

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**ABSTRACT.** The analysis on manifolds with singular geometry (e.g., with conical points, edges, corners, or higher polyhedral singularities) gives rise to a new machinery of operators with symbolic structures which is able to express the regularity of solutions to elliptic boundary value problems via parametrices. Examples are mixed problems of Zaremba type, crack problems, or models when potentials in an operator have singularities at special points or along interfaces within the domain (for instance, Coulomb or other potentials in Schroedinger operators, coming from the position of particles).

We outline a calculus on corresponding stratified spaces, where the strata (boundaries, interfaces, etc.) contribute their own principal (and complete) symbols to the problem which determine ellipticity and induce adequate new scales of weighted spaces and subspaces with asymptotics.

The problems including the symbolic structures will be interpreted in the framework of algebras of operators with a specific dependence on parameters and with a typical degenerate behaviour in the distance variables to the singularities. We illustrate an iterative process of building up the calculus for higher singularities.

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## INTRODUCTION

Elliptic operators on spaces with singular geometry appear in connection with numerous applications of mechanics, elasticity theory, crack theory, and mathematical physics, but also in pure mathematics such as geometry and topology.

The approach that we are discussing here, is to a large extent inspired by singular integral equations and pseudo-differential analysis. Both the concrete models themselves and the theoretical insight have been promoted by the work of Vekua and other Georgian scientists, especially, Muskhelishvili, Kupradze, and many others, and there is now a long tradition in this field in Georgia, with strong activities up to the present. Models with singularities in the geometry of a manifold or in the coefficients of operators have been studied by many authors worldwide. Despite of the achievements of the ‘classical’ period, connected with the names of Kondratiev [16], Agranovich and Vishik [1], Grisvard [11], and many others, there is now a broad international development and a new enthusiasm in studying singular problems by new means and with realistic chances to understand them in terms of pseudo-differential algebras, index theory, and geometric and topological ideas. An overview on the latter aspects and a comprehensive bibliography is given in [20].

There is an enormous variety of different problems with specific challenges and individual properties that affect the nature of solvability, up to numerical processes. It is therefore desirable to point out a number of common features (‘axioms’) that apply to sufficiently general situations of applications. This is the primary goal of our approach.

## 1. DEGENERATE OPERATORS AND SYMBOLIC HIERARCHIES

**1.1. Fuchs type and edge-degenerate operators.** The basic (local) model of a (regular) conical singularity is the cone

$$X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \quad (1.1)$$

with  $X$  being a  $C^\infty$  manifold (here compact, with or without boundary), where  $\{0\} \times X$  represents the conical singularity  $v$  in the quotient space. We then have the open stretched cone

$$X^\wedge := \mathbb{R}_+ \times X \cong X^\Delta \setminus \{0\}$$

with a splitting of variables  $(r, x)$ . An example is  $X^\Delta = \mathbb{R}^{n+1}$  with  $v = \{0\}$  as the conical singularity and  $X = S^n$ , the unit sphere. Expressing a differential operator  $\tilde{A}$  in  $\mathbb{R}^{n+1}$  of order  $\mu \in \mathbb{N}$  with  $C^\infty$  coefficients in  $\mathbb{R}^{n+1}$  in polar coordinates we obtain an operator of the form

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left(-r \frac{\partial}{\partial r}\right)^j \quad (1.2)$$

with coefficients  $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \text{Diff}^{\mu-j}(X))$ . Here  $\text{Diff}^\nu(X)$  is the space of differential operators on  $X$  of order  $\nu$ , with smooth coefficients in local coordinates.

Another example of an operator (1.2) with  $\mu = 2$  is the Laplace-Beltrami operator, associated with a Riemannian metric on  $X^\wedge$  of the form

$$dr^2 + r^2 g_X(r),$$

with  $g_X(r)$  being a family of Riemannian metrics on  $X$ , smooth in  $r$  up to  $r = 0$ .

Operators of the form (1.2) will be called of Fuchs type. Given a manifold  $M$  with conical singularity  $v$ , i.e.,  $M \setminus \{v\}$  is  $C^\infty$ , and  $M$  is locally near  $v$  modelled on (1.1), we have a space  $\text{Diff}_{\text{deg}}^\mu(M)$ , defined to be the set of all  $A \in \text{Diff}^\mu(M \setminus \{v\})$  that are locally near  $v$  of the form (1.2). (In the singular set-up the notation ‘manifold’ is used for convenience; clearly, it happens that  $M$  is not a topological manifold, namely, when the base of the cone is not a sphere.)

In this exposition we freely employ some basic tools on pseudo-differential operators on a  $C^\infty$  manifold  $X$ . By  $L_{\text{cl}}^\mu(X; \mathbb{R}^l)$  for an open  $C^\infty$  manifold  $X$  we denote the space of classical pseudo-differential operators of order  $\mu \in \mathbb{R}$  with parameters  $\lambda \in \mathbb{R}^l$  that are (mod  $L^{-\infty}(X; \mathbb{R}^l)$ ) locally given in the form

$$\text{Op}(a)(\lambda)u(x) = \iint e^{i(x-x')\xi} a(x, \xi, \lambda) u(x') dx' d\xi,$$

$d\xi = (2\pi)^{-n} d\xi$ , with an amplitude function  $a(x, \xi, \lambda)$  in Hörmanders (1,0)-space of classical symbols in the covariable  $(\xi, \lambda) \in \mathbb{R}^{n+l}$ . We set  $L^{-\infty}(X; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$ , with  $L^{-\infty}(X)$  being the space of all operators on  $X$  with  $C^\infty$  kernel (referring to a measure  $dx$  belonging to a Riemannian metric on  $X$ ).

By definition every  $A \in L_{\text{cl}}^\mu(X; \mathbb{R}^l)$  has a parameter-dependent homogeneous principal symbol, locally given by the homogeneous principal part  $a_{(\mu)}(x, \xi, \lambda)$  of  $a(x, \xi, \lambda)$  in  $(\xi, \lambda) \neq 0$  of order  $\mu$ . For  $l = 0$  the homogeneous principal symbol of an  $A \in L_{\text{cl}}^\mu(X)$  will also be denoted by  $\sigma_\psi(A)(x, \xi)$ . In particular, any  $A \in \text{Diff}_{\text{deg}}^\mu(M)$ , regarded as an element of  $\text{Diff}^\mu(M \setminus \{v\}) \subset L_{\text{cl}}^\mu(M \setminus \{v\})$ , has a corresponding principal symbol  $\sigma_\psi(A)$ . Locally near  $v$  in the variables  $(r, x)$  with covariables  $(\varrho, \xi)$  we have a ‘reduced’ symbol

$$\tilde{\sigma}_\psi(A)(r, x, \varrho, \xi) := r^\mu \sigma_\psi(A)(r, x, r^{-1} \varrho, \xi) \quad (1.3)$$

which is smooth up to  $r = 0$ .

A manifold  $M$  with conical singularity  $v$  is a simple example of a stratified space, with  $M \setminus \{v\}$  being the ‘main’ stratum and  $v$  another stratum (called ‘mimal’ later on), such that  $M$  is the disjoint union

$$M = (M \setminus \{v\}) \cup \{v\}.$$

The symbol  $\sigma_\psi(A)$  is associated with  $M \setminus \{v\}$ , while  $v$  contributes another (operator-valued) symbolic component, the so-called conormal symbol  $\sigma_c(A)$ . In the splitting of variables  $(r, x) \in X^\wedge$  locally near  $v$  we have

$$\sigma_c(A)(w) := \sum_{j=0}^{\mu} a_j(0) w^j : H^s(X) \rightarrow H^{s-\mu}(X), \quad (1.4)$$

a family of operators between standard Sobolev spaces  $H^s(X)$ , parametrised by  $w \in \mathbb{C}$  (or, later on, by  $w$  on a weight line  $\Gamma_{\frac{n+1}{2}-\gamma}$  with  $\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}$ ).

Let us now pass to the typical differential operators on a manifold  $M$  with edge  $Y$ .

A manifold  $M$  with edge  $Y$  is a topological space containing  $Y$  as a subspace, such that  $M \setminus Y$  and  $Y$  are  $C^\infty$  manifolds, and  $M$  is locally near any point of  $Y$  modelled on a wedge

$$X^\Delta \times \Omega \quad (1.5)$$

with a  $C^\infty$  manifold  $X$  and open  $\Omega \subseteq \mathbb{R}^q$ , with  $q = \dim Y$ .

An operator  $A \in \operatorname{Diff}^\mu(M \setminus Y)$  is said to be edge-degenerate, if it has locally near  $Y$  in a splitting of variables  $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$  the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, x) \left(-r \frac{\partial}{\partial r}\right)^j (r D_y)^\alpha \quad (1.6)$$

with coefficients  $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}_+} \times \Omega, \operatorname{Diff}^{\mu-(j+|\alpha|)}(X))$ . Let  $\operatorname{Diff}_{\operatorname{deg}}^\mu(M)$  denote the set of all such  $A$ .

From the embedding  $\operatorname{Diff}_{\operatorname{deg}}^\mu(M) \subset L_{\operatorname{cl}}^\mu(M \setminus Y)$  we have the standard principal symbol  $\sigma_\psi(A)$  of any  $A \in \operatorname{Diff}_{\operatorname{deg}}^\mu(M)$ . Moreover, locally near a point of  $Y$  in the variables  $(r, x, y)$  with covariables  $(\varrho, \xi, \eta)$  we have a reduced symbol

$$\tilde{\sigma}_\psi(A)(r, x, y, \varrho, \xi, \eta) := r^\mu \sigma_\psi(A)(r, x, r^{-1}\varrho, \xi, r^{-1}\eta) \quad (1.7)$$

which is smooth up to  $r = 0$ , as we see from (1.6). The edge  $\Omega$  contributes another principal symbolic component, the so-called edge symbol which is locally in the variables  $(r, x, y)$  defined by the expression

$$\sigma_\wedge(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) \left(-r \frac{\partial}{\partial r}\right)^j (r\eta)^\alpha, \quad (1.8)$$

$(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ .

The operators (1.8) are  $(y, \eta)$ -wise of the form (1.2) and act in weighted distribution spaces  $K^{s,\gamma}(X^\wedge)$  on the infinite stretched cone  $X^\wedge = \mathbb{R}_+ \times X$

$$\sigma_\wedge(A)(y, \eta) : K^{s,\gamma}(X^\wedge) \rightarrow K^{s-\mu,\gamma-\mu}(X^\wedge) \quad (1.9)$$

Those spaces are defined in terms of a more general class of spaces, namely,

$$\mathcal{K}^{s,\gamma;g}(X^\wedge) := \omega \mathcal{H}^{s,\gamma}(X^\wedge) + (1 - \omega) H_{\operatorname{cone}}^{s;g}(X^\wedge), \quad (1.10)$$

by setting  $K^{s,\gamma}(X^\wedge) := \mathcal{K}^{s,\gamma;s-\gamma}(X^\wedge)$ . Here  $\omega(r)$  is a cut-off function (i.e.,  $\omega \in C_0^\infty(\overline{\mathbb{R}_+})$ ,  $\omega = 1$  near  $r = 0$ ). Moreover,  $\mathcal{H}^{s,\gamma}(X^\wedge)$  and  $H_{\operatorname{cone}}^{s;g}(X^\wedge)$  are formulated in terms of order reducing families on  $X$ .

Recall that for every  $\mu \in \mathbb{R}$  there exists a parameter-dependent elliptic family  $R^\mu(\lambda) \in L_{\operatorname{cl}}^\mu(X; \mathbb{R}^l)$  such that

$$R^\mu(\lambda) : H^s(X) \rightarrow H^{s-\mu}(X)$$

is a family of isomorphisms for all  $\lambda \in \mathbb{R}^l$  and  $s \in \mathbb{R}$ .

Applying that for  $l = 1$ , the space  $\mathcal{H}^{s,\gamma}(X^\wedge)$  is defined to be the completion of  $C_0^\infty(\mathbb{R}_+, C^\infty(X))$  with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} := \left\{ \frac{1}{2\pi i} \int_{\frac{n+1}{2}-\gamma} \|R^s(\operatorname{Im} w)Mu(w)\|_{L^2(X)}^2 dw \right\}^{\frac{1}{2}},$$

with  $Mu(w) = \int_0^\infty r^{w-1}u(r)dr$  being the Mellin transform,  $w \in \mathbb{C}$ .

Taking an order reducing family  $R^s(\tilde{\varrho}, \tilde{\eta}) \in L_{\text{cl}}^s(X; \mathbb{R}_{\tilde{\varrho}, \tilde{\eta}}^{1+q})$  we first define  $H_{\text{cone}}^{s;0}(\mathbb{R} \times X)$  to be the completion of  $C_0^\infty(\mathbb{R}, C^\infty(X))$  with respect to the norm

$$\|u\|_{H_{\text{cone}}^{s;0}(\mathbb{R} \times X)}^2 := \left\{ \int \|\langle r \rangle^{-s} R^s(r\varrho, r\eta^1)(Fu)(\varrho)\|_{L^2(X)}^2 d\varrho \right\}^{1/2} \quad (1.11)$$

with  $F$  being the Fourier transform on  $\mathbb{R}$  and any fixed  $\eta^1 \in \mathbb{R}^q \setminus \{0\}$ ,  $\langle r \rangle := (1 + |r|^2)^{1/2}$ , and then we set

$$H_{\text{cone}}^{s;g}(X^\wedge) := \langle r \rangle^{-g} H_{\text{cone}}^{s;0}(\mathbb{R} \times X)|_{\mathbb{R}_+ \times X}.$$

In the formula (1.11) the dimension of the extra covariable  $\eta$  is arbitrary, but  $\neq 0$ .

If we endow the space  $K^{s,\gamma}(X^\wedge)$  with a strongly continuous group of isomorphisms

$$\kappa_\lambda : u(r, x) \rightarrow \lambda^{\frac{n+1}{2}+s-\gamma} u(\lambda r, x),$$

$\lambda \in \mathbb{R}_+$ , then we have twisted homogeneity in the sense

$$\sigma_\wedge(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\wedge(A)(y, \eta) \kappa_\lambda^{-1}$$

for all  $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ ,  $\lambda \in \mathbb{R}_+$ .

**1.2. Operators with edge conditions.** An aspect of our calculus is the following. Given an algebra of differential operators on a configuration  $M$ , say  $\bigcup_{\mu \in \mathbb{N}} \text{Diff}_{\text{deg}}^\mu(M)$  when  $M$  is a manifold with edge  $Y$ , pass to a pseudo-differential calculus on  $M \setminus Y$  which contains the parametrices of elliptic elements. There are then two basic questions: What is ellipticity, and what are the specific features of such a calculus. In this section we give an impression on what we understand by ellipticity.

The case of edge singularities has much in common with boundary value problems.

For instance, consider the half-space  $\overline{\mathbb{R}}_+^{1+q} \ni (r, y)$  with boundary  $\mathbb{R}^q$  and a differential operator  $A$  with smooth coefficients

$$A = \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) D_r^j D_y^\alpha. \quad (1.12)$$

Then the standard ellipticity means  $\sigma_\psi(A)(r, y, \varrho, \eta) = \sum_{j+|\alpha|=\mu} a_{j\alpha}(r, y) \varrho^j \eta^\alpha \neq 0$  for  $(\varrho, \eta) \in \mathbb{R}^{1+q} \setminus \{0\}$ . However, analogously as (1.8) we have an extra symbol, namely, the principal boundary symbol

$$\sigma_\partial(A)(y, \eta) := \sigma_\psi(A)(0, y, D_r, \eta) : H^s(\mathbb{R}_+) \rightarrow H^{s-\mu}(\mathbb{R}_+) \quad (1.13)$$

which plays a role for the ellipticity of a boundary value problem. In fact, the  $\sigma_\psi$ -ellipticity entails the Fredholm property of (1.13) for  $\eta \neq 0$ ,  $s - \mu > -\frac{1}{2}$  (the operators (1.13) are surjective in this case). Then extra boundary operators

$$u \rightarrow B_1 u|_{r=0}, \dots, B_N u|_{r=0}$$

with differential operators  $B_j \in \text{Diff}^{\mu_j}(\overline{\mathbb{R}}_+^{1+q})$ ,  $j = 1, \dots, N$ , give rise to a vector of mappings

$$\sigma_\partial(|_{r=0} \circ B_j)(y, \eta) := |_{r=0} \circ \sigma_\psi(B_j)(0, y, D_r, \eta) : H^s(\mathbb{R}_+) \rightarrow \mathbb{C}, \quad j = 1, \dots, N,$$

that fills up (1.13) to a family of isomorphisms

$${}^t(\sigma_\partial(A), \sigma_\partial(|_{r=0} \circ B_1), \dots, \sigma_\partial(|_{r=0} \circ B_N))(y, \eta) : H^s(\mathbb{R}_+) \rightarrow H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C}^N \quad (1.14)$$

for all  $(y, \eta) \in T^*\overline{\mathbb{R}}^{1+q} \setminus 0$ . The isomorphism (1.14) is just what characterises the ellipticity of the boundary value problem

$$Au = f \text{ in } \mathbb{R}^{1+q}, B_j u|_{r=0} = g_j \text{ on } \mathbb{R}^q, j = 1, \dots, N. \quad (1.15)$$

Classical examples of ellipticity of this kind are the Dirichlet or Neumann problem for the Laplacian  $A = \Delta$ .

Let us set  $T_j := |_{r=0} \circ B_j$  and  $T := {}^t(T_1, \dots, T_N)$ ,  $\sigma_\partial(T)(y, \eta) := {}^t(\sigma_\partial(T_1), \dots, \sigma_\partial(T_N))$ . The boundary value problem (1.15) can be identified with a column matrix of operators

$$\mathcal{A} := {}^t(A \quad T) \quad (1.16)$$

with the principal symbol

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A})),$$

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(A), \sigma_\partial(\mathcal{A}) := {}^t(\sigma_\partial(A) \quad \sigma_\partial(T)).$$

Analogous definitions make sense on an arbitrary  $C^\infty$  manifold with boundary. The problem to complete the system of such operators  $\mathcal{A}$  to a pseudo-differential calculus has been solved in [3] in terms of pseudo-differential operators with the transmission property at the boundary. This calculus consists of operator block matrices

$$\mathcal{A} = \begin{pmatrix} A + G & K \\ T & Q \end{pmatrix} \quad (1.17)$$

with  $A$  having the transmission property at the boundary, a so-called Green operator  $G$ , a trace operator  $T$ , a potential operator  $K$ , and a pseudo-differential operator  $Q$  on the boundary. The role of  $G$  is to complete parametrices of a given  $\sigma_\psi$ -elliptic upper left corner to an analogue of Green's function. The role of potential operators is quite evident. Parametrices of elliptic operators (1.16) are certainly of row matrix form  $(P \quad K)$  with such a  $K$  as the second component, while  $P$  is just 'Green's' function of the boundary value problem.

It is obvious that the half-space  $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$  is a special wedge, where the base  $X$  of the model cone  $X^\Delta$  is of dimension zero, i.e.  $X^\Delta = \overline{\mathbb{R}}_+$ , and the boundary  $\mathbb{R}^q$  is the edge. However, the operators in such an edge calculus are much more general than the ones with the transmission property. They contain at least the pseudo-differential operators without the transmission property, such as

operators with principal symbols  $|\xi|$  ( $\xi := (\varrho, \eta)$ ). Operators with symbols of the latter kind are well-known in the context of the Zaremba problem when we reduce Neumann conditions  $T_N$  in terms of the potential  $K_D$  of the Dirichlet problem to the boundary, i.e., form  $T_N \circ K_D$ . Such operators have been studied by many authors, mostly without any connection with the edge calculus, see the work of Vishik and Eskin [33], Eskin [10]. In particular, Vishik and Eskin observed and systematically studied the role of additional entries of trace and potential type, similarly as in (1.17). Note that an algebra of such operators was constructed in [22]. An interpretation within the edge calculus of [23] is given in [30].

Considering now an edge-degenerate differential operator (1.6) we can ask to what extent the scheme of boundary value problems as outlined for (1.12) can be realised on a manifold  $M$  with edge  $Y$ . A calculus of that kind is developed in [23], see also [25], or [9]. Note that when  $X$  is  $C^\infty$  with boundary, then such an edge calculus is to be combined with the calculus of boundary value problems, see [15] or [13]. This theory contains, in particular, an operator algebra that solves the Zaremba problem and other mixed problems in terms of parametrix constructions and yields elliptic regularity with asymptotics, see [7], or [13].

In the present chapter for convenience we assume  $X$  to be closed and compact.

The substitute of the half-space  $\overline{\mathbb{R}}_+^{1+q}$  is now a wedge (1.12), and the analogue of the boundary symbol (1.13) is the edge symbol (1.8). In contrast to elliptic boundary value problems for differential operators the edge symbol (1.8) will not always be a family of Fredholm operators (when  $A$  is elliptic as usual and (1.7) does not vanish for all  $(\varrho, \xi, \eta) \neq 0$ , up to  $r = 0$ ). The Fredholm property would be desirable to fill up the family (1.8) to a family of isomorphisms by the edge symbols of extra conditions, similarly as explained in the context of (1.14).

The Fredholm property of (1.8) requires the ellipticity of the operators  $(y, \eta)$ -wise in the cone algebra on the infinite (stretched) cone  $X^\wedge$ . This refers to the symbols  $\sigma_\psi$  and  $\sigma_c$  from the cone calculus, where  $\sigma_\psi \sigma_\wedge(A)$  is automatically non-vanishing as well as  $\tilde{\sigma}_\psi(A)$  up to  $r = 0$  (see (1.3)). However, (1.4) has the form

$$\sigma_c \sigma_\wedge(A)(y, w) = \sum_{j=0}^{\mu} a_{j0}(0, y) w^j : H^s(X) \rightarrow H^{s-\mu}(X), \quad (1.18)$$

and ellipticity in the cone calculus requires the bijectivity of (1.18) for all  $w \in \Gamma_{\frac{n+1}{2}-\gamma}$  for some chosen weight  $\gamma$  and all  $y$ .

**Theorem 1.2.1.** *Let  $A$  be an edge-degenerate differential operator which is  $(\sigma_\psi, \tilde{\sigma}_\psi)$ -elliptic and where (1.18) is a family of isomorphisms. Then (1.9) is a family of Fredholm operators for all  $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ .*

**Remark 1.2.2.** *A general feeling for ellipticity on a non-compact manifold would suggest that the conical exit of  $X^\wedge$  to  $r = \infty$  also needs an ellipticity condition. However, this is automatically satisfied in this case.*

To illustrate Remark 1.2.2 we consider an example, namely, an operator

$$\tilde{A}(\tilde{x}, D_{\tilde{x}}) = \sum_{|\alpha| \leq \mu} a_{\tilde{\alpha}}(\tilde{x}) D_{\tilde{x}}^{\tilde{\alpha}}$$

in  $\mathbb{R}^{n+1} \cong X^\Delta$  for  $X := S^n$ . Let the coefficients be classical symbols in the ‘covariable’  $\tilde{x}$  of order zero, and let  $a_{\tilde{\alpha}}(\tilde{x})_{(0)}$  denote the homogeneous principal components in  $\tilde{x} \neq 0$ . The Fredholm property of

$$\tilde{A} : \langle \tilde{x} \rangle^{-g} H^s(\mathbb{R}^{n+1}) \rightarrow \langle \tilde{x} \rangle^{-g} H^{s-\mu}(\mathbb{R}^{n+1})$$

for any  $s, g \in \mathbb{R}$  is equivalent to the usual ellipticity of  $\tilde{A}$ , i.e.,  $\sigma_\psi(\tilde{A})(\tilde{x}, \tilde{\xi}) \neq 0$ ,  $(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ , together with the conditions

$$\sigma_e(\tilde{A})(\tilde{x}, \tilde{\xi}) := \sum_{|\alpha| \leq \mu} a_{\tilde{\alpha}}(\tilde{x})_{(0)} \tilde{\xi}^\alpha \neq 0, (\tilde{x}, \tilde{\xi}) \in (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{R}^{n+1}, \quad (1.19)$$

and

$$\sigma_{\psi,e}(\tilde{A})(\tilde{x}, \tilde{\xi}) := \sum_{|\alpha|=\mu} a_{\tilde{\alpha}}(\tilde{x})_{(0)} \tilde{\xi}^\alpha \neq 0, (\tilde{x}, \tilde{\xi}) \in (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}). \quad (1.20)$$

General information on the structure of operators globally in  $\mathbb{R}^n$  in standard Sobolev spaces may be found in [21], [5]; concerning (1.19), (1.20) see [25]. Applying this to an operator family

$$\tilde{A}(\tilde{x}, D_{\tilde{x}}, \eta) := \sum_{|\tilde{\alpha}|+|\beta| \leq \mu} a_{\tilde{\alpha},\beta}(\tilde{x}) D_{\tilde{x}}^{\tilde{\alpha}} \eta^\beta, \quad \eta \in \mathbb{R}^q,$$

which is parameter-dependent elliptic in the sense  $\sum_{|\alpha|+|\beta|=\mu} a_{\tilde{\alpha},\beta}(\tilde{x}) \tilde{\xi}^{\tilde{\alpha}} \eta^\beta \neq 0$  for all  $(\tilde{x}, \tilde{\xi}, \eta) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1+q} \setminus \{0\})$ , then for every fixed  $\eta \neq 0$  the symbolic conditions (1.19), (1.20) are automatically satisfied. Thus, if we introduce in  $\tilde{A}(\tilde{x}, D_{\tilde{x}}, \eta)$  polar coordinates  $\mathbb{R}_{\tilde{x}}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}_+ \times S^n$ ,  $\tilde{x} \rightarrow (r, x)$ , then the situation is quite similar to (1.8) what concerns the behaviour for  $|\tilde{x}| = r \rightarrow \infty$ , and  $\eta \neq 0$ ; for  $X^\Delta = \mathbb{R}_{\tilde{x}}^{n+1}$ , i.e.,  $X = S^n$ , we have  $(1 - \omega) \langle \tilde{x} \rangle^{\gamma-s} H^s(\mathbb{R}^{n+1}) = (1 - \omega) H_{\text{cone}}^{s;s-\gamma}(X^\Delta) = (1 - \omega) K^{s,\gamma}(X^\Delta)$ .

**Remark 1.2.3.** *An essential difference between the boundary symbol (1.13) of a  $\sigma_\psi$ -elliptic differential operator (1.12) and the edge symbol (1.8) of a  $(\sigma_\psi, \tilde{\sigma}_\psi)$ -elliptic edge-degenerate operator (1.6) is that the conormal symbol (1.18) is required to be a family of isomorphisms which depend on the chosen weight  $\gamma$ . If this is the case, the operators (1.8) are not necessarily surjective, and  $\text{ind } \sigma_\wedge(A)(y, \eta)$  may also depend on  $\gamma$ .*

*Therefore, in general, to carry out a program of elliptic edge conditions, we need to take into account analogues of  $2 \times 2$  block matrices (1.17) that contain trace and potential operators at the same time.*

In other words the calculus on a (say, compact) manifold  $M$  with edge  $Y$  consists of operators

$$\mathcal{A} : H^{s,\gamma}(M) \oplus H^s(Y, J_-) \rightarrow H^{s-\mu,\gamma-\mu}(M) \oplus H^s(Y, J_+) \quad (1.21)$$

between weighted spaces  $H^{s,\gamma}(M)$  on  $M \setminus Y$  of smoothness  $s \in \mathbb{R}$  and weight  $\gamma \in \mathbb{R}$ , and standard Sobolev spaces  $H^s(Y, J_\pm)$  of distributional sections in (smooth complex) vector bundles  $J_\pm$  on  $Y$ . Such operators have a principal symbolic hierarchy

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$$

with  $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$  as usual, and the edge symbol

$$\sigma_\wedge(\mathcal{A})(y, \eta) : K^{s,\gamma}(X^\wedge) \oplus J_{-,y} \rightarrow K^{s-\mu,\gamma-\mu}(X^\wedge) \oplus J_{+,y} \quad (1.22)$$

(with subscript ‘ $y$ ’ indicating the fibre of the respective bundle over  $y$ ),  $(y, \eta) \in T^*Y \setminus 0$ .

**Theorem 1.2.4.** *The operator (1.21) is Fredholm for some  $s = s_0 \in \mathbb{R}$  if and only if  $A$  is  $(\sigma_\psi, \tilde{\sigma}_\psi)$ -elliptic, and if (1.22) a family of isomorphisms.*

**Remark 1.2.5.** (i) *The ellipticity of  $\mathcal{A}$  in the sense of the conditions of Theorem 1.2.4 is equivalent to the Fredholm property of (1.21) for all  $s \in \mathbb{R}$ .*

(ii) *The edge calculus, developed in [23] (see also [25], [9]) contains the parametrices of elliptic elements.*

**Remark 1.2.6.** *Given an edge-degenerate operator  $A$  (say, a differential operator) such that  $\sigma_\wedge(A)(y, \eta)$  defines a family of Fredholm operators, it is not always guaranteed that it can be filled up to a family of isomorphisms (1.22) with vector bundles  $J_\pm$  over the edge. For that there is a similar topological obstruction as for elliptic boundary value problems (see Atiyah and Bott [2], Boutet de Monvel [3], or [26], [31]). The problem is similar to that for Dirac operators in even dimensions that do not allow Shapiro-Lopatinskij elliptic boundary conditions; instead of that one can always pose global projection conditions as is done in [26], [31], see also Nazaikinskij, Savin, Schulze, and Sternin [19], and the bibliographies in these papers.*

## 2. OPERATORS ON SPACES WITH HIGHER SINGULARITIES

**2.1. Manifolds with higher singularities.** Manifolds with  $C^\infty$  structure form a category denoted by  $\mathfrak{M}_0$ , with morphisms (isomorphisms) being the differentiable mappings (diffeomorphisms). In a similar manner manifolds with conical singularities or edges form a category  $\mathfrak{M}_1$ , where the morphisms (isomorphisms) over the main strata agree with the ones from the smooth category, while close to the singularities they respect the local conical or wedge structures.

It is very important for the definition of manifolds with higher singularities that

$$M \in \mathfrak{M}_k \Rightarrow M \times \Omega \in \mathfrak{M}_k \text{ for every } \Omega \in \mathfrak{M}_0, \quad (2.1)$$

$k = 0, 1$ . Such a property is quite natural, and, of course, it will hold also for arbitrary  $k \in \mathbb{N}$ . The definition of higher (regular) singularities of order  $k$  is inductive and reduces things to the order  $k - 1$ ,  $k \geq 1$ .

**Definition 2.1.1.** *A topological space  $M$  is said to be a manifold of singularity order  $k \geq 1$ , written  $M \in \mathfrak{M}_k$ , if*

- (i) there is chosen a subspace  $Y \subset M$ ,  $Y \in \mathfrak{M}_0$ , such that  $M \setminus Y \in \mathfrak{M}_{k-1}$ ;
- (ii)  $Y$  has a neighbourhood  $U$  in  $M$  which has the structure of a (locally trivial) cone bundle over  $Y$  with fibre  $X^\Delta$ , with some  $X \in \mathfrak{M}_{k-1}$ .

We call  $Y$  the minimal stratum of  $M$ .

Denoting  $Y$  in the latter definition for the moment by  $Y^k$ , the space  $M \setminus Y^k \in \mathfrak{M}_{k-1}$  has again a minimal stratum  $Y^{k-1} \in \mathfrak{M}_0$  with  $(M \setminus Y^k) \setminus Y^{k-1} \in \mathfrak{M}_{k-2}$ , and so on. In this way we obtain a sequence of  $C^\infty$  manifolds  $Y^{k-j} \in \mathfrak{M}_0$ ,  $j = 0, \dots, k$ , such that

$$M = \bigcup_{j=0}^k Y^j \quad (\text{disjoint union}).$$

Let us call  $Y^0$  the maximal stratum of  $M$ , and set  $\dim M := \dim Y^0$ .

**Example 2.1.2.** *If  $M$  is the unit cube in  $\mathbb{R}^3$ , the maximal stratum  $Y^0$  is the open interior,  $Y^1$  consists of the 6 open boundary faces,  $Y^2$  of the 12 open one-dimensional edges, and the minimal stratum  $Y^3$  of the 8 corner points. We then have  $M \in \mathfrak{M}_3$  and  $\partial M = M \setminus Y^0 \in \mathfrak{M}_2$ .*

Also  $\mathfrak{M}_k$  for arbitrary  $k \in \mathbb{N}$  is a category in a natural way, with morphisms and isomorphisms.

From Definition 2.1.1 it immediately follows that every  $y \in Y$  has a neighbourhood  $V$  of  $y$  in  $M$  which is isomorphic to a wedge, i.e., there is a mapping

$$\chi : V \rightarrow X^\Delta \times \Omega$$

for an  $\Omega \subseteq \mathbb{R}^q$  open ( $\dim Y = q$ ) and some  $X \in \mathfrak{M}_{k-1}$  that restricts to isomorphisms

$$\chi_{\text{int}} : V \setminus Y \rightarrow X^\Delta \times \Omega = \mathbb{R}_+ \times X \times \Omega \text{ in } \mathfrak{M}_{k-1}, \quad (2.2)$$

$$\chi_0 : V \cap Y \rightarrow \Omega \text{ in } \mathfrak{M}_0. \quad (2.3)$$

The mapping (2.3) is nothing other than a chart on  $Y$ . The case  $\dim Y = 0$  is also admitted, then  $Y$  consists of corner points (a generalisation of conical singularities with base  $X$  in  $\mathfrak{M}_{k-1}$ ). From (2.2) we obtain local splittings of variables  $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ .

Assuming by induction that the (Fréchet) space  $\text{Diff}_{\text{deg}}^\mu(N)$  of typical differential operators on any  $N \in \mathfrak{M}_{k-1}$  is already defined, by  $\text{Diff}_{\text{deg}}^\mu(M)$ ,  $M \in \mathfrak{M}_k$ , we denote the set of all  $A \in \text{Diff}_{\text{deg}}^\mu(M \setminus Y)$  that are locally near  $Y$  in the variables  $(r, x, y)$  of the form

$$A = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) \left(-r \frac{\partial}{\partial r}\right)^j (r D_y)^\alpha \quad (2.4)$$

with coefficients  $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}_+} \times \Omega, \text{Diff}_{\text{deg}}^{\mu-(j+|\alpha|)}(X))$ . For  $\dim Y = 0$  instead of (2.4) we assume

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left(-r \frac{\partial}{\partial r}\right)^j, \quad (2.5)$$

$a_j \in C^\infty(\overline{\mathbb{R}}_+, \text{Diff}^{\mu-j}(X))$ .

In the following we assume that  $X \in \mathfrak{M}_{k-1}$  is compact.

Let us give a definition of the principal symbolic hierarchy of an operator (2.4) or (2.5) (see Calvo, Martin and Schulze [4]).

Having defined  $\sigma(B) = (\sigma_0(B), \dots, \sigma_{k-1}(B))$  for every  $B \in \text{Diff}_{\text{deg}}^\mu(N)$ ,  $N \in \mathfrak{M}_{k-1}$ , with  $\sigma_0(B)$  being the standard homogeneous principal symbol of  $B$  on the main stratum  $\in \mathfrak{M}_0$  of  $N$ , for  $A \in \text{Diff}_{\text{deg}}^\mu(M)$ ,  $M \in \mathfrak{M}_k$ , we inductively set

$$\sigma(A) := (\sigma_0(A), \dots, \sigma_{k-1}(A), \sigma_k(A)) \quad (2.6)$$

with  $\sigma(A|_{M \setminus Y}) =: (\sigma_0(A), \dots, \sigma_{k-1}(A))$  from the step before, and

$$\sigma_k(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) \left(-r \frac{\partial}{\partial r}\right)^j (r\eta)^\alpha \quad (2.7)$$

for  $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ , when  $q \geq 1$ , or

$$\sigma_k(A)(w) := \sum_{j=0}^{\mu} a_j(0) w^j \quad (2.8)$$

for  $w \in \Gamma_{\frac{n+1}{2}-\gamma}$ ,  $n = \dim X$ , when  $q = 0$ .

The symbols (2.7) and (2.8) take values in the spaces

$$\mathcal{L}(\mathcal{K}^{s, \gamma'; g}(X^\wedge), \mathcal{K}^{s-\mu, \gamma'-\mu; g}(X^\wedge))$$

and  $\mathcal{L}(H^{s, \gamma'}(X), H^{s-\mu, \gamma'-\mu}(X))$ , respectively, with weighted spaces of smoothness  $s \in \mathbb{R}$  and weights  $\gamma' = (\gamma_1, \dots, \gamma_{k-1}) \in \mathbb{R}^{k-1}$ ,  $\gamma' - \mu := (\gamma_j - \mu)_{1 \leq j \leq k-1}$  (for  $k = 0$  those weights are meaningless and omitted). Similarly as in (1.10) the weight  $g \in \mathbb{R}$  is responsible for the conical exit of  $X^\wedge$  to infinity.

**2.2. Elements of the higher corner calculus.** Similarly as in Chapter 1 where we employed the parameter-dependent calculus of pseudo-differential operators on a  $C^\infty$  manifold  $X$ , we assume by induction that the parameter-dependent calculus of operators on any  $N \in \mathfrak{M}_{k-1}$  is already established, with a list of properties, also to be postulated, which play the role of theorems in the calculus of next higher singularity order  $k$ . As noted in the context of boundary value or edge problems we have to expect block matrices of operators, in the present case,  $k \times k$  block matrices  $\mathcal{A}$ , with trace, potential and Green operators with respect to all strata of  $N \in \mathfrak{M}_{k-1}$ . Let us content ourselves for the moment with spaces of operators of type of upper left corners.

Let  $\mathfrak{A}^\mu(N; \mathbb{R}^l)$  denote the space of such operators with parameter  $\lambda \in \mathbb{R}^l$ ,  $l \in \mathbb{N}$ . Moreover, by induction we assume that the weighted spaces  $H^{s, \gamma'}(X)$  for compact  $X \in \mathfrak{M}_{k-1}$  are already defined, moreover  $H_{[\text{comp}]}^{s, \gamma'}(X)$ ,  $H_{[\text{loc}]}^{s, \gamma'}(X)$  when  $X$  is not necessarily compact (the use of '[' and ')' is motivated by supports 'up to the singularities', similarly as  $H_{[\text{comp}]/[\text{loc}]}^s(\overline{\mathbb{R}}_+^{1+q}) := H_{\text{comp}/\text{loc}}^s(\mathbb{R}^{1+q})|_{\overline{\mathbb{R}}_+^{1+q}}$ ). We also feed in the information that for every  $\mu \in \mathbb{R}$  and weights  $\gamma' \in \mathbb{R}^{k-1}$

the space  $\mathfrak{A}^\mu(X; \mathbb{R}^l)$ ,  $X \in \mathfrak{M}_{k-1}$  compact, contains an element  $R^\mu(\lambda)$  that induces isomorphisms

$$R^\mu(\lambda) : H^{s, \gamma'}(X) \rightarrow H^{s-\mu, \gamma'-\mu}(X)$$

for all  $s \in \mathbb{R}$  (the weights  $\gamma' \in \mathbb{R}^{k-1}$  are fixed; clearly  $R^\mu(\lambda)$  depends on  $\gamma'$ ).

One of the properties of the spaces  $H^{s, \gamma'}(X)$  (and similarly of the ‘[comp]/[loc]’ variants of these spaces) is that

$$H^{s, \gamma'}(X) = h^{\gamma'} H^{s, 0}(X)$$

for all  $s \in \mathbb{R}$ , with a strictly positive weight function  $h^{\gamma'}$  which is locally in the variables  $(r_1, \dots, r_{k-1})$  of the form  $h^{\gamma'} = r_1^{\gamma_1} \dots r_{k-1}^{\gamma_{k-1}} \varphi$  with a  $\varphi \in C^\infty(X)$  ( $:= \text{Diff}_{\text{deg}}^0(X)$ ). Moreover, the space  $H^{0, 0}(X)$  is a weighted  $L^2$ -space on the main stratum of  $X$ , for instance, when  $X$  locally near its minimal stratum is modelled on a wedge  $\mathbb{R}_+ \times X_{k-1} \times \mathbb{R}^{q_{k-1}} \ni (r_{k-1}, \cdot, y_{k-1})$ , with  $X_{k-2} \in \mathfrak{M}_{k-2}$  then  $H^{0, 0}(\mathbb{R}_+ \times X_{k-2} \times \mathbb{R}^{q_{k-1}}) = r^{-\frac{n_{k-2}}{2}} L^2(\mathbb{R}_+ \times \mathbb{R}^{q_{k-1}}, H^{0, 0}(X_{k-2}))$ , etc.

Alternatively, a weight function  $h^{\gamma'}$  can be iteratively defined as a function which is locally in the variables  $(r_{k-1}, \cdot, y_{k-1})$  of the form  $h^{\gamma'} = r_{k-1}^{\gamma_{k-1}} h^{\gamma''} \varphi$  with a weight function  $h^{\gamma''}$  on  $X_{k-2} \times \mathbb{R}^{q_{k-1}}$ , and a  $\varphi \in C^\infty(\mathbb{R}_+ \times X_{k-2} \times \mathbb{R}^{q_{k-1}})$ ,  $\gamma'' := (\gamma_1, \dots, \gamma_{k-2}) \in \mathbb{R}^{k-2}$ .

We define  $\mathcal{H}^{s, \gamma}(X^\wedge)$  for  $s \in \mathbb{R}$ ,  $\gamma := (\gamma', \gamma_k) \in \mathbb{R}^k$ , to be the completion of  $C_0^\infty(\mathbb{R}_+, H^{\infty, \gamma'}(X))$  with respect to the norm

$$\|u\|_{\mathcal{H}^{s, \gamma}(X^\wedge)} := \left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2} - \gamma_k}} \|R^s(\text{Im } w) M u(r)\|_{H^{0, \gamma'-s}(X)}^2 dw \right\}^{1/2},$$

$n = \dim X$ ; here  $R^s(\lambda)$  is an order reducing family in  $\mathfrak{A}^s(X; \mathbb{R})$ .

Moreover, if  $R^s(\tilde{\varrho}, \tilde{\eta}) \in \mathfrak{A}^s(X; \mathbb{R}_{\tilde{\varrho}, \tilde{\eta}}^{1+q})$ ,  $q \geq 1$ , is an order reducing family, we define the space  $H_{\text{cone}}^{s, \gamma'; 0}(\mathbb{R} \times X)$  to be the completion of  $C_0^\infty(\mathbb{R}, H^{\infty, \gamma'}(X))$  with respect to the norm

$$\|u\|_{H_{\text{cone}}^{s, \gamma'; 0}(\mathbb{R} \times X)} := \left\{ \int \|R^s(r \varrho, r \eta^1)(F u)(\varrho)\|_{\langle r \rangle^{\frac{n}{2}} L^2(\mathbb{R}, H^{0, \gamma'-s}(X))}^2 \right\}^{1/2}$$

for any fixed  $\eta^1 \in \mathbb{R}^q \setminus \{0\}$ . We then pass to

$$H_{\text{cone}}^{s, \gamma'; g}(X^\wedge) := \langle r \rangle^{-g} H_{\text{cone}}^{s, \gamma'; 0}(\mathbb{R} \times X)|_{X^\wedge}$$

and set

$$\mathcal{K}^{s, \gamma; g}(X^\wedge) := \omega \mathcal{H}^{s, \gamma}(X^\wedge) + (1 - \omega) H_{\text{cone}}^{s, \gamma'; g}(X^\wedge),$$

with any fixed cut-off function  $\omega$ , and  $K^{s, \gamma}(X^\wedge) := \mathcal{K}^{s, \gamma; s - \gamma_k}(X^\wedge)$ .

**Remark 2.2.1.** *Setting  $(\kappa_\lambda u)(r, x) := \lambda^m u(\lambda r, x)$ ,  $\lambda \in \mathbb{R}_+$ , for any  $m \in \mathbb{R}$  we obtain a strongly continuous group of isomorphisms,*

$$\kappa_\lambda : \mathcal{K}^{s, \gamma; g}(X^\wedge) \rightarrow \mathcal{K}^{s, \gamma; g}(X^\wedge)$$

for every  $s, \gamma, g \in \mathbb{R}$ . Usually we set  $m = \frac{n+1}{2} + s - \gamma_k$ .

The expected length of this overview does not admit to develop all essential properties of our scales of spaces. Let us only observe that the specific choice of the order reducing families is unimportant, as well as of  $\eta^1 \neq 0$  or the cut-off function.

The proper content of the spaces is, of course, hidden in the nature of the parameter-dependent calculus. This is not completely straightforward. The following considerations give an idea of how they locally look like.

To sketch elements of the calculus, consider, for instance, the case  $q > 0$ . The main ingredients are operator functions

$$\tilde{p}(r, y, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathfrak{A}^\mu(X; \mathbb{R}_{\tilde{\varrho}, \tilde{\eta}}^{1+q})),$$

$X \in \mathfrak{M}_{k-1}$ , and associated edge-degenerate families

$$p(r, y, \varrho, \eta) := \tilde{p}(r, y, r\varrho, r\eta). \quad (2.9)$$

A crucial theorem of the ‘higher edge quantisation’ is the following theorem, which transforms operators

$$\text{Op}_r(p)(y, \eta)u = \iint e^{i(r-r')\varrho} p(r, y, \varrho, \eta)u(r')dr' d\varrho$$

based on the Fourier transform on  $\mathbb{R}_+$  into operators

$$\text{op}_M^\delta(h)(y, \eta) := M_{\delta, w \rightarrow r}^{-1} h(r, y, w, \eta) M_{\delta, r' \rightarrow w}$$

based on the Mellin transform, modulo some smoothing operators. Here  $M_\delta$  is the weighted Mellin transform, i.e.,  $(M_\delta u)(w) := M(r^{-\delta}u)(w + \delta)$ ,  $\delta \in \mathbb{R}$ , and  $h(r, y, w, \eta)$  is an operator-valued Mellin amplitude function of the form

$$\begin{aligned} h(r, y, w, \eta) &:= \tilde{h}(r, y, w, r\eta), \\ \tilde{h}(r, y, w, \tilde{\eta}) &\in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathfrak{A}^\mu(X; \mathbb{C} \times \mathbb{R}^q)); \end{aligned} \quad (2.10)$$

$\mathbb{C}$  indicates holomorphic dependence on  $w \in \mathbb{C}$ , and the parameter-dependence refers to  $\text{Im } w \in \mathbb{R}$  on the lines  $\Gamma_\beta$  for every  $\beta \in \mathbb{R}$ , uniformly in compact  $\beta$ -intervals.

**Theorem 2.2.2.** *To every edge-degenerate family (2.2.2) there exists a holomorphic Mellin symbol (2.10) such that*

$$\text{Op}_r(p)(y, \eta) = \text{op}_M^\delta(h)(y, \eta)$$

mod  $C^\infty(\Omega, \mathfrak{A}^{-\infty}(\mathbb{R}_+ \times X; \mathbb{R}^q))$  for every  $\delta \in \mathbb{R}$ .

Edge-degenerate operators  $\text{Op}_y(p)$  are then quantised near  $Y$  by forming operators  $\text{Op}_y(a)$  with edge amplitude functions of the form

$$\begin{aligned} a(y, \eta) &:= \sigma r^{-\mu} \{ \omega(r[\eta]) \text{op}_M^{\gamma - \frac{n}{2}}(h)(y, \eta) \tilde{\omega}(r'[\eta]) \\ &\quad + (1 - \omega(r[\eta])) \omega_0(r[\eta], r'[\eta]) \text{Op}_r(p)(y, \eta) (1 - \tilde{\omega}(r'[\eta])) \} \tilde{\sigma} \end{aligned} \quad (2.11)$$

with cut-off functions  $\omega, \tilde{\omega}, \tilde{\omega}, \sigma, \tilde{\sigma}$ , such that  $\tilde{\omega} \equiv 1$  on  $\text{supp } \omega$ ,  $\omega \equiv 1$  on  $\text{supp } \tilde{\omega}$ ,  $\tilde{\sigma} \equiv 1$  on  $\text{supp } \sigma$ . Moreover,  $\omega_0(r, r') := \psi((r - r')^2 / (1 + (r - r')^2))$  with any  $\psi(t) \in C_0^\infty(\mathbb{R}_+)$  such that  $\psi(t) = 1$  for  $t < 1/2$ ,  $\psi(t) = 0$  for  $t > \frac{3}{2}$ .

The ‘non-smoothing’ part of operators in  $\mathfrak{A}^\mu(M)$ ,  $M \in \mathfrak{M}_k$ ,  $k \geq 1$ , close to the higher singularity  $Y$ , is of the form  $\text{Op}_y(a)$  with an amplitude functions (2.11). Far from  $Y$  those operators are simply elements of  $\mathfrak{A}^\mu(M \setminus Y)$  constructed before, using  $M \setminus Y \in \mathfrak{M}_{k-1}$ .

For the step from  $k$  to  $k+1$  we can produce  $\mathfrak{A}^\mu(M; \mathbb{R}^l)$  with parameters  $\lambda \in \mathbb{R}^l$ ,  $l \in \mathbb{N}$ , by formally replacing the covariable  $\eta \in \mathbb{R}^q$  in (2.11) by  $(\eta, \lambda) \in \mathbb{R}^{q+l}$ . Together with  $\mathfrak{A}^\mu(M \setminus Y; \mathbb{R}^l)$ , known from the preceding step, we then obtain the non-smoothing part of the parameter-dependent calculus on  $M$ .

The operator function is (2.11) an operator-valued symbol in  $S^\infty(\Omega \times \mathbb{R}^q; E, \tilde{E})$  for

$$E := \mathcal{K}^{s, \gamma; g}(X^\wedge), \tilde{E} := \mathcal{K}^{s-\mu, \gamma-\mu; g}(X^\wedge) \quad (2.12)$$

for every  $s, g \in \mathbb{R}$ . Here  $S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$  is the space of all  $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(E, \tilde{E}))$  such that

$$\|\kappa_{\langle \eta \rangle}^{-1} \{D_y^\alpha D_\eta^\beta a(y, \eta)\} \kappa_{\langle \eta \rangle}\|_{\mathcal{L}(E, \tilde{E})} \leq c \langle \eta \rangle^{\mu-|\beta|}$$

for all  $(y, \eta) \in K \times \mathbb{R}^q$ ,  $K \Subset \Omega$ , and all  $\alpha, \beta \in \mathbb{N}^q$ , with constants  $c(\alpha, \beta, K) > 0$ . The space  $H^{s, \gamma}(M)$  on  $M \in \mathfrak{M}_k$ ,  $s \in \mathbb{R}$ ,  $\gamma \in \mathbb{R}^k$ , are locally along the minimal stratum  $Y$ , identified via a chart to  $\mathbb{R}^q$ , modelled on edge spaces  $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s, \gamma; s-\gamma k}(X^\wedge))$ . Here  $\mathcal{W}^s(\mathbb{R}^q, E)$  is the completion of  $\mathcal{S}(\mathbb{R}^q, E)$  with respect to the norm  $\|\langle \eta \rangle^s \kappa_{\langle \eta \rangle}^{-1} \hat{u}(\eta)\|_{\mathcal{L}^2(\mathbb{R}^q, E)}$ . In a similar manner we define ‘comp’- and ‘loc’-spaces in  $y \in \Omega \subseteq \mathbb{R}^q$ . Then, taking (2.11), we obtain continuous operators

$$\text{Op}_y(a) : \mathcal{W}_{\text{comp}}^s(\Omega, E) \rightarrow \mathcal{W}_{\text{loc}}^{s-\mu}(\Omega, \tilde{E})$$

with the spaces (2.12), for all  $s \in \mathbb{R}$ .

Another element are so-called Mellin operators in the upper left corners and block matrix Green operators; the latter ones encode edge conditions of trace and potential type. This gives beyond the scope of the present overview.

Those operators, together with global smoothing operators give rise to the  $k$ th generation of calculus, i.e.,  $\mathfrak{A}^\mu(M)$ ,  $M \in \mathfrak{M}_k$ . Similarly as (2.6) they have a principal symbolic hierarchy, with  $\sigma_0(A)$  being the standard principal symbol of the upper left corner as a classical pseudo-differential operator on the main stratum  $Y^0$  of  $M$ . As in the edge calculus (see, e.g., [18]) one can show that an operator is compact in weighted spaces when all components of its symbol vanish. This is then a new starting point for the Fredholm property of elliptic operators and other expected features of the calculus.

Let us finally note that already the ‘usual’ edge calculus of [23] which was refined and completed later on by many other authors (see [14], [8], [32], [34], [17], [15], [6]), and also the subsequent corner theory belonging to the singularity order  $k=2$  with its applications (see [24], [28], [27], [29], [13], [12]), contains many theories as substructures that are often known under different notation, namely (apart from the standard pseudo-differential calculus), the theory of elliptic boundary problems, both for smooth domains, as well as with conical, edge, and corner singularities, and operators on other kinds of non-compact configurations, especially, with conical exits to infinity.

There remain many tasks for future activities. In respect of the high complexity of the theory it is still important to make the approach more and more transparent, in order to pass later on to further applications, e.g., in the context of geometric operators, or index theory.

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