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Numerical Solution of a Nonlinear Problem of Deformation of a Thermoelastic Beam with a Variable Cross-Section

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The paper discusses the nonlinear problem of a beam's planar deformation in situations of large displacements. It presents the study of the strain state of an elastic beam with a linearly varying rectangular cross-section under thermomechanical loading. A nonlinear system of ordinary differential equations has been obtained for the components of the displacement vector—axial and transverse displacements, the angle of rotation of the normal, and the components of internal force factors (axial and transverse forces, bending moment). Boundary conditions are stated. The temperature field is stationary and varies along the axis of the beam at a given time. The nonlinear system of equations has been solved numerically using the built-in function of the mathematical editor Mathcad.

Keywords and phrases: Nonlinear problem, beam, variable cross-section, displacement, differential equation, numerical solution.

AMS subject classification: 74K10, 74S30.

1 Introduction

Geometrically nonlinear problems of planar deformation of the beams are discussed in the following publications [1, 2, 3, 4]. A fundamental system of equations has been obtained for the beams ($y/\rho_0 < 1$, ρ_0-is the initial curvature of the axis of the beam; y is a coordinate in the cross-section thickness) under thermomechanical loading. A nonlinear boundary value problem is formulated. For a given one-dimensional stationary temperature field, the fundamental system consists of six nonlinear ordinary differential equations, solved using the shooting method. Numerical calculations have been performed for the straight thermally elastic beams with a constant cross-section. Longitudinal and transverse displacements, rotation angles of the cross-sections, as well as internal force factors, have been determined. The stability problem has been analyzed, and the critical force value has been calculated numerically.

The mathematical model of an elastic beam with a variable cross-section is studied in the works of G. Jaiani [5, 6]. Using the three-dimensional theory, the fields of displacements, deformations, and stresses are expanded into an orthogonal double Fourier-Legendre series with respect to the variables of the cross-sectional thickness and width. All terms in the series, except the first, are neglected. The case where the variable cross-section degenerates into a linearly varying segment or a point is considered. The initial boundary value problem is analyzed, and the existence and uniqueness of solutions are demonstrated.

The deformation of a long thermally elastic beam with a rectangular cross-section under mixed boundary conditions is examined in [7]. Mixed boundary conditions involve a combination of prescribed temperature and displacement at the beam's ends. The beam's deformation

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is analyzed using the boundary integral method, which accounts for the combined effects of thermal and mechanical loads. The analysis also considers the thermally elastic properties of the material, including temperature-dependent thermal conductivity. The boundary integral formulation involves dividing the beam's surface into elements, representing the boundary conditions using integral equations, and solving the resulting system of equations to determine the beam's deformation. This study provides insights into the deformation analysis of long thermally elastic beams under mixed boundary conditions.

In [8], researchers developed a three-dimensional elastic beam model for bending analysis to determine and optimize the shapes of active membrane structures. This model is beneficial for the design and optimization of membrane constructions used as molds for concrete shells. The proposed model represents membrane elements as interconnected 3D elastic beams, enabling the analysis of large deformations. Residual forces at the nodes are computed using an energy-based approach within the framework of the dynamic relaxation method. This method iteratively refines the mesh shape until a self-equilibrated configuration is achieved. The results of the proposed model were validated by comparing them with those obtained using the finite element method (FEM), based on the minimization of total potential energy, demonstrating the model's accuracy and applicability for shape analysis and optimization.

Article [9] investigates the equilibrium of compressed beams with variable cross-sections. The study focuses on beams with varying thicknesses, where a fourth-order differential equation with variable coefficients describes deformation under longitudinal compression. The solution to this problem provides insights into the equilibrium behavior of the beam, accounting for variations in its transverse dimensions. It is shown that the variability of the beam's cross-sectional width significantly affects its equilibrium. The differential equation captures the complex interactions between the beam's geometry and its mechanical response during compression. This research contributes to a deeper understanding of the equilibrium of beams with non-uniform cross-sections. The findings have potential applications in various engineering fields, including structural mechanics and materials science.

Monographs [10, 11] discuss one-dimensional nonlinear problems of deformation for elastic beams. The work in [10] provides a comprehensive analysis of beam deformation, where the displacements of the points along the beam's axis are parallel to a single plane. The geometry of the deflected line (the beam's axis) is considered within the elastic limits during the bending of the beams. Both linear and nonlinear problems assume that the length of the elastic line remains unchanged [10, 11]. Under thermomechanical loading, all layers of the beam undergo deformation due to thermal expansion (compression). In some cases, displacements associated with stretching and compression and those related to changes in curvature may be of the same order.

2 The fundamental equations of beam deformation

Let us consider the deformation of curved beams in the yz-plane under thermomechanical loading with large displacements. The layer, concerning which we analyze the deformation geometry, is called the thermally elastic layer, and its projection onto the yz-plane is called the thermally elastic line. The thermally elastic line represents the curve passing through the centroids of the beam's cross-sections projected onto the yz-plane [12].

Let us study the displacements and deformation based on the change in the geometry of the thermally elastic line of the beam. We assume that the temperature field is one-dimensional and varies along the beam's axis. Let the curvature of the thermally elastic line before and after deformation be denoted by ρ_0 and ρ, respectively. The angle of inclination of the sides of the thermally elastic line relative to the z-axis before and after deformation will be denoted by θ_0 and θ, respectively. The displacement along the z-axis is denoted by w, and the displacement along the y-axis is denoted by $-v$. It is clear that

$$w = w(l),\ v = v(l),\ \rho = \rho(l),\ \theta = \theta(l),$$

where l-is the arc length of the thermally elastic line in the deformed configuration (corresponding coordinate), or $w = w(l_0),\ v = v(l_0),\ \rho = \rho(l_0),\ \theta = \theta(l_0)$, with l_0 being the arc length of the thermally elastic line in the undeformed configuration.

The system of fundamental equations for the deformation of a thermally elastic beam in the plane has the following form [1]:

The geometry equations:

$$\begin{aligned} \frac{dv}{dl_0} &= (1+\varepsilon_0)\sin\theta - \sin\theta_0 \\ \frac{dw}{dl_0} &= (1+\varepsilon_0)\cos\theta - \cos\theta_0 \\ \frac{d\theta}{dl_0} &= \chi_x + \frac{1+\varepsilon_0}{\rho_0} \end{aligned} \tag{1}$$

The equations of statics:

$$\begin{aligned} \frac{dM}{dl_0} &= (1+\varepsilon_0)\,(H\sin\theta - R\cos\theta - m) \\ \frac{dR}{dl_0} &= -\,(1+\varepsilon_0)\,q_y \\ \frac{dH}{dl_0} &= -\,(1+\varepsilon_0)\,q_z \end{aligned} \tag{2}$$

where ε_0 is the deformation of the thermally elastic line, $\chi_x$is the characteristic of the change in curvature, M is the bending moment, R and H are the internal force components relative to the y and z axes, respectively, q_y and q_z are the distributed external forces relative to the y and z axes, respectively, and m is the intensity of the bending moment.

If the material's elasticity modulus E is constant, then the deformation ε_0 of the thermally elastic line and the characteristic parameter of the change in the curvature χ_x are determined from the following relationships:

$$\varepsilon_0 = \frac{N}{A^*} + \frac{1}{A^*}\cdot E\int \varepsilon^T dA \tag{3}$$

$$\chi_x = \frac{\mathrm{M}}{\mathrm{I_x^*}} + \frac{\mathrm{E}}{\mathrm{I_x^*}}\int \varepsilon^{\mathrm{T}}\cdot \mathrm{y}\cdot \mathrm{dA}, \tag{4}$$

where A^* is the generalized cross-sectional area of the beam, I_x^* is the generalized moment of inertia of the cross-section, N is the normal force in the cross-section, and ε^T is the thermal deformation.

The normal force in the cross-section is determined as [1,12]:

$$N = H\cos\theta + R\sin\theta \tag{5}$$

The generalized cross-sectional area A^* and the generalized moment of inertia $\mathrm{I_x^*}$ are calculated using the following formulas [12]: $A^*=\int E\,(T)\,dA$, $I_x^* = \int y^2 E\,(T)\,dA$,

The thermal deformation is given by $\varepsilon^T = \alpha\,(T - T_0)$, where α is the coefficient of linear expansion, T_0 is the initial temperature, and T is the temperature at the given moment in time.

From equations (1) and (2), when $\theta_0 = 0$ and $\rho_0 \to \infty$ we have $dl_0 = dz$, we obtain the equations for the bending of straight beams. For a rectangular linearly variable cross-section of a straight beam, see Figure 1, the system of fundamental equations takes the form:

$$\begin{aligned}
&\frac{dv}{dz} = (1+\varepsilon_0)\sin\theta \\
&\frac{dw}{dz} = (1+\varepsilon_0)\cos\theta\ - 1 \\
&\frac{d\theta}{dz} = \chi_x \\
&\frac{dM}{dz} = (1+\varepsilon_0)(H\sin\,\theta\ - R\cos\,\theta\ - m) \\
&\frac{dR}{dz} = -(1+\varepsilon_0)q_y \\
&\frac{dH}{dz} = -(1+\varepsilon_0)q_z
\end{aligned} \tag{6}$$

The deformation of the thermally elastic line and the radius of curvature after deformation are determined by the following formulas [13]:

$$\varepsilon_0 = \frac{N}{EA\,(z)} + \varepsilon^T \tag{7}$$

$$\rho = \frac{1+\varepsilon_0}{\chi_x}, \tag{8}$$

where the cross-sectional area of the beam is determined according to Figure 1:

$$A\,(z) = ab\left[\left(\frac{b_0}{b} - 1\right)\frac{z}{l} + 1\right]$$

According to formula (4), the characteristic parameter of the curvature change is determined as follows [13]:

$$\chi_x = \frac{12M}{Eab^3\left[\left(\frac{b_0}{b} - 1\right)\frac{z}{l} + 1\right]^3} + \frac{E}{I_x^*}\int \varepsilon^{\mathrm{T}}\cdot\mathrm{y}\cdot\mathrm{dA} \tag{9}$$

The temperature deformation ε^T does not change with respect to the y coordinate. Accordingly, in relationship (9), the second term on the right-hand side is zero relative to the axis passing through the centroid of the section.

The nonlinear differential equations (7), along with formulas (5), (7), and (9), form a closed system of equations. For integrating the system of differential equations there are boundary conditions depending on the type of fixation at the ends of the beam: fixed support $v = 0,\ w = 0,\ M = 0$; rigid support $v = 0,\ w = 0,\ \theta = 0$, and so on.

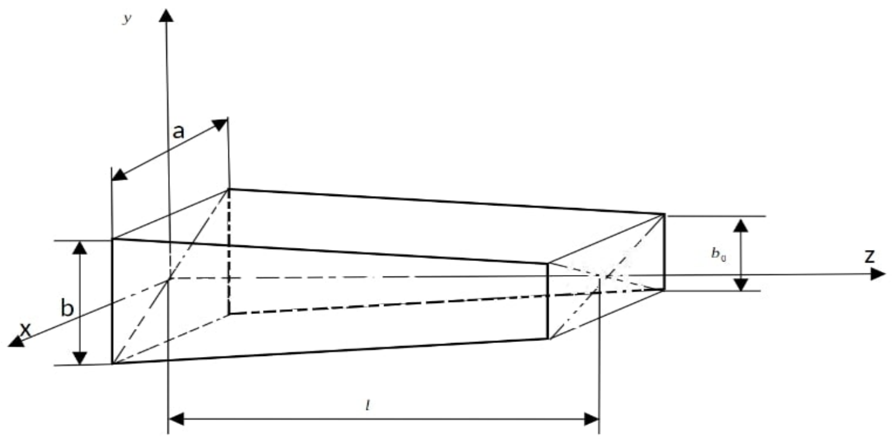

Figure 1: Straight beam with a variable rectangular cross-section.

3 Numerical calculation

The nonlinear boundary problem can be effectively solved using numerical methods. For numerical calculations, we enter dimensionless quantities. [13]:

$$\hat{z} = \frac{z}{l};\ \hat{q}_{y,z} = \frac{q_{y,z}}{q_*};\ \widehat{\chi}_x, = \chi_x l;\ \hat{v} = \frac{v}{l};\ \hat{w} = \frac{w}{l};\ \widehat{\rho} = \frac{\rho}{l};\ \hat{N} = \frac{N}{q_* l};\ \hat{y} = \frac{y}{b};$$

$$\hat{M} = \frac{M}{q_* l^2};\ \hat{H} = \frac{H}{q_* l};\ \hat{R} = \frac{R}{q_* l};$$

$$\hat{m} = \frac{m}{q_* l};\ \hat{T} = \frac{T}{T_0}$$

$$\hat{A}\,(z) = \tfrac{A(z)}{A_0};\ \lambda_1 = \tfrac{q_* l}{Eab};\ \lambda_2 = \alpha T_0;\ \lambda_3 = \tfrac{12 q_* l^3}{Eab^3},$$

where l is the length of the beam before deformation, and q_*- is the maximum value of the distributed load.

The system of equations (7), (5), (3), and (4) can be written in dimensionless quantities as follows:

$$\begin{aligned}
&\frac{d\hat{v}}{d\hat{z}} = (1+\varepsilon_0)\sin\theta\ ,\\
&\frac{d\hat{w}}{d\hat{z}} = (1+\varepsilon_0)\cos\theta\ - 1,\\
&\frac{d\theta}{d\hat{z}} = \hat{\chi}_x,\\
&\frac{d\hat{R}}{d\hat{z}} = -(1+\varepsilon_0)\hat{q}_y,\\
&\frac{d\hat{H}}{d\hat{z}} = -(1+\varepsilon_0)\hat{q}_z,\\
&\frac{d\hat{M}}{d\hat{z}} = (1+\varepsilon_0)(\hat{H}\sin\theta\ - \hat{R}\cos\theta\ - \hat{m})
\end{aligned} \tag{10}$$

$$\hat{N} = \hat{H}\cos\theta \ + \hat{R}\sin\theta \tag{11}$$

$$\varepsilon_0 = \frac{\lambda_1 \hat{N}}{\left(\frac{b_0}{b} - 1\right)\overline{z} + 1} + \lambda_2(\hat{T} - 1) \tag{12}$$

$$\widehat{\chi}_x = \frac{\lambda_3 \hat{M}}{\left[\left(\frac{b_0}{b} - 1\right)\overline{z} + 1\right]^3} \tag{13}$$

The system of equations (10) can be written in a vector-matrix form as follows:

$$Y' = F(y,\hat{q}), \qquad where \qquad y = (\hat{v}, \hat{w}, \theta, \hat{R}, \hat{H}, \hat{M})^T,$$

For performing numerical calculations in the Mathcad system, the following designations have been introduced: $y_1 = \hat{v}$; $y_2 = \hat{w}$; $y_3 = \theta$; $y_4 = \hat{R}$; $y_5 = \hat{H}$; $y_6 = \hat{M}$.

Then, the system of equations will have the following form:

$$y_1' = (1 + \varepsilon_0)\sin y_3,$$
$$y_2' = (1 + \varepsilon_0)\cos y_3 \ - 1,$$

$$y_3' = \frac{\lambda_3 y_6}{\left[\left(\frac{b_0}{b} - 1\right)\overline{z} + 1\right]^3},$$
$$y_4' = -(1 + \varepsilon_0)\hat{q}_y \ ,$$
$$y_5' = -(1 + \varepsilon_0)\hat{q}_z,$$
$$y_6' = (1 + \varepsilon_0)(y_5 \sin y_3 \ - y_4 \cos y_3 \ - \hat{m}),$$
$$\varepsilon_0 = \lambda_1 \frac{y_5 \cos y_3 + y_4 \sin y_3}{\left(\frac{b_0}{b} - 1\right)\hat{z} + 1} + \lambda_2(\hat{T} - 1),$$
$$\widehat{\chi}_x = \frac{\lambda_3 y_6}{\left[\left(\frac{b_0}{b} - 1\right)\hat{z} + 1\right]^3}.$$

4 Calculation results

As a test item, the transverse bending of a cantilever beam with a consistent cross-section with a load distributed with a constant intensity was calculated. In this case, the boundary conditions have the form: $\overline{z}$=0; y_1 =0, y_2 =0 y_3 =0; $\overline{z}$=1; y_4 =0, y_5 =0 y_6 =0. The results of the numerical calculation are exactly in line with the known analytical solution.

Figure 2 and Figure 3 illustrate the results of numerical calculation for components of displacements, components of internal force, and bending moment under thermomechanical loading. The calculation was made for the following data: a = 5cm; b = 10 cm; l = 1m; $E = 2 \cdot 10^{11}$ n/m^2; q_y = 100 n/m; $q_z = 0$; $m = 0$; $\alpha = 2 \cdot 10^{-6}$ 1/℃; T_0 = 20℃; $\hat{T} = 3$. The initial values of shooting parameters (y_4, y_5, y_6) were taken according to the solution to

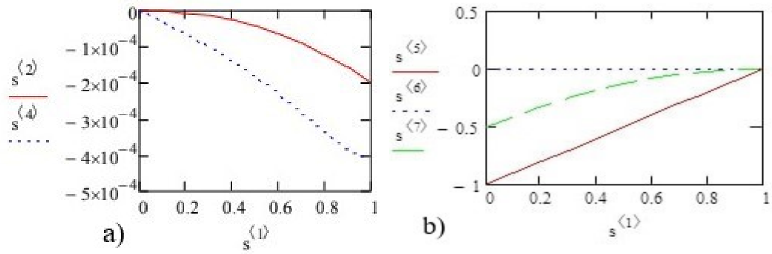

Figure 2: Calculation results for the relationship b_0/b=0.1 a) Displacement components, b) Internal force components and bending moment $s^{\langle 1\rangle} \equiv \bar{z},\ s^{\langle 2\rangle} \equiv \bar{v},\ s^{\langle 4\rangle} \equiv \theta,\ s^{\langle 5\rangle} \equiv \bar{R},\ s^{\langle 6\rangle} \equiv \bar{H},\ s^{\langle 7\rangle} \equiv \bar{M}$

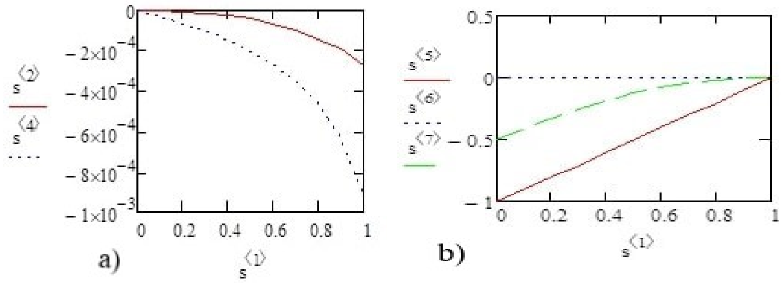

Figure 3: Calculation results for the relationship b_0/b=0.01 a) Displacement components, b) Internal force components and bending moment $s^{\langle 1\rangle} \equiv \bar{z},\ s^{\langle 2\rangle} \equiv \bar{v},\ s^{\langle 4\rangle} \equiv \theta,\ s^{\langle 5\rangle} \equiv \bar{R},\ s^{\langle 6\rangle} \equiv \bar{H},\ s^{\langle 7\rangle} \equiv \bar{M}$

the linear problem. The Mathcad's built-in function *sbval* allows us to effectively define the shooting parameters: $y_4 \equiv \bar{R}$, $y_5 \equiv \bar{H}$, $y_6 \equiv \bar{M}$.

Figure 2 and Figure 3 illustrate that the components of the displacements increase significantly with the reduction in the relationship b_0/b. It is valid to say that the mathematical editor Mathcad effectively solves the nonlinear boundary problem.

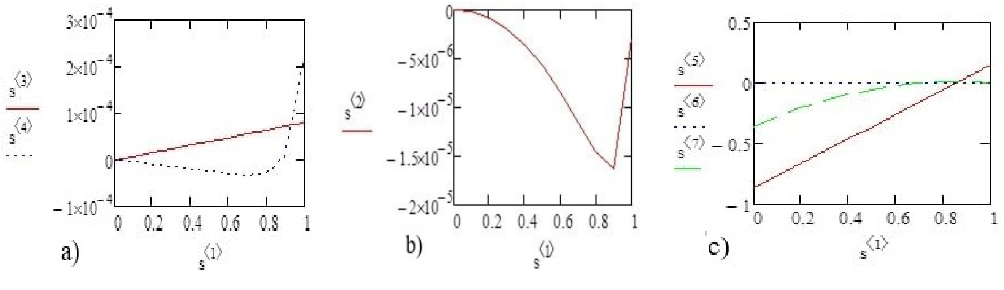

Figure 4: Calculation results for a cantilever beam with a movable hinge at the right end for the relationship b_0/b=0.01. a),b) Displacement components; c) Internal force components and bending moment $s^{\langle 1\rangle} \equiv \bar{z},\ s^{\langle 2\rangle} \equiv \bar{v},\ s^{\langle 4\rangle} \equiv \theta,\ s^{\langle 5\rangle} \equiv \bar{R},\ s^{\langle 6\rangle} \equiv \bar{H},\ s^{\langle 7\rangle} \equiv \bar{M}$

A statically indeterminate cantilever beam with a movable hinge at the other end was calculated. Boundary conditions have the following form: $\hat{z} = 0,\ y_1 = 0,\ y_2 = 0,\ y_3 = 0;\quad \bar{z} = 1,\ y_1 = 0,\ y_5 = 0,\ y_6 = 0$ For the relationship $b_0/b = 0.01$, the shooting parameters found by the *sbval* function are equal to: $y_4(0) = -0.858$, $y_5(0) = 0$, $y_6(0) = -0.358$. Figure 4 illustrates the results of the numerical calculation, particularly, displacement components and internal force components for the relationship $b_0/b = 0.01$. Figure 5 illustrates the results of numerical calculation, components of displacement and components of internal forces

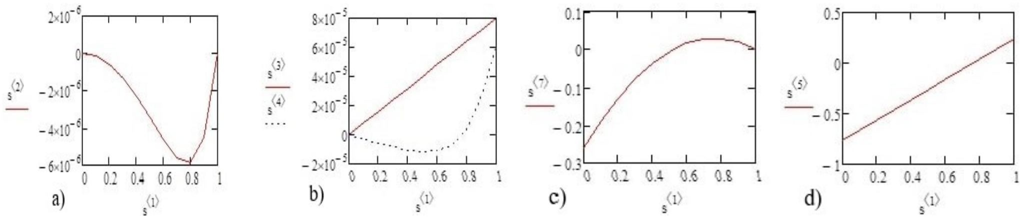

Figure 5: Calculation results for a cantilever beam with a movable hinge at the right end for the relationship $b_0/b=0.1$. a),b) Displacement components; c) Bending moment d) Transverse force. $s^{\langle 1\rangle} \equiv \bar{z}$, $s^{\langle 2\rangle} \equiv \bar{v}$, $s^{\langle 4\rangle} \equiv \theta$, $s^{\langle 5\rangle} \equiv \bar{R}$, $s^{\langle 6\rangle} \equiv \bar{H}$, $s^{\langle 7\rangle} \equiv \bar{M}$

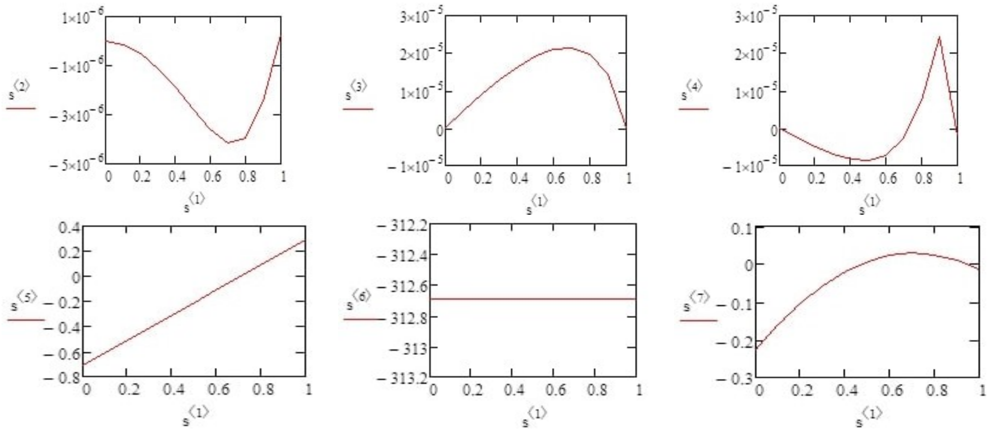

Figure 6: Calculation results of a beam rigidly fixed at both ends for the relationship $b_0/b=0.1$. $s^{\langle 1\rangle} \equiv \bar{z}$, $s^{\langle 2\rangle} \equiv \bar{v}$, $s^{\langle 4\rangle} \equiv \theta$, $s^{\langle 5\rangle} \equiv \bar{R}$, $s^{\langle 6\rangle} \equiv \bar{H}$, $s^{\langle 7\rangle} \equiv \bar{M}$

Figure 6 illustrate the calculation results of a beam rigidly fixed at both ends for different values of the relationship b_0/b. The shooting parameters for the value of $b_0/b = 0.1$are equal to:$y_4(0) = -0.712$, $y_5(0) = -312.692$, $y_6(0) = -0.226$. Figure 6 illustrates the results of the numerical calculation: displacement components and internal force components. The values of the shooting parameters for the relationship $b_0/b = 0.01$ are equal to:$y_4(0) = -0.855$, $y_5(0) = -171.976$, $y_6(0) = -0.357$. It should be noted that the number of iterations for finding the shooting parameters increases significantly when solving a statically uncertain problem.

5 Conclusion

a) The convergence of the iterative process essentially depends on the selection of the initial values of the shooting parameters. As an initial approximation for the values of shooting parameters, we can choose the solution of a linear problem for a beam with a constant cross-section;

b) reduction in the relationship b_0/b essentially affects the values of the transverse displacements and the angle of rotation of the cross-section;

c) An applied program for solving the nonlinear boundary value problem with the mathematical editor Mathcad has been drawn up.

It is valid to say that nonlinear problems of transverse bending of beams of variable cross-section under thermomechanical loading can be effectively solved numerically by Mathcad software.

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