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Hierarchical Models for the Investigation of Problems in Angular 3D Domains

George Jaiani*

I. Vekua Institute of Applied Mathematics & Faculty of Exact and Natural Sciences of I. Javakhishvili Tbilisi State University, 11 University St., 0186 Tbilisi, Georgia

The present paper deals with the state of art in application of Ilia Vekua's dimension reduction method of constructing the hierarchical models for different physical models. The special attention is paid to the study of peculiarities of setting the boundary conditions caused by three-dimensional (3D) angularity of domains.

Keywords and phrases: Deformable solids and fluids; elastic, viscoelastic, piezoelectric, Kelvin-Voigt materials; isotropic, transversaly isotropic, anisotropic materials; materials with voids; Thermoelastic deformation of chiral porous prismatic shells; ideal and viscous fluids, prismatic shell-like and bar-like containers, angular pipes, wedge-shaped canals.

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*Corresponding author. Email: george. jaiani@gmail.com

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1 Introduction

Hierarchical models constructed by I. Vekua's dimension reduction method is a powerful tool for investigation of Problems considered in angular 3D domains (see Appendix). The advantage is threefold:

1. 3D problem is reduced to 2D problems;

2. 3D geometric singularity is transferred into singularity of governing differential equations in 2D domains, therefore, into investigation of BVPs and IBVPs for singular ordinary and partial differential equations and systems. Ilia Vekua's method is especially fitting this case;

3. The hierarchy models obtained in that way can be discretized by the p-version of the finite elements (In this connection see [19], [20]).

While constructing hierarchical models for elastic prismatic and standard shells of variable thickness, Ilia Vekua suggested and developed a dimension reduction method [1-3] (for the state of art in this direction see [4], [5]) which in [6,7] is formalized in such a form that it is applicable directly for constructing the hierarchical models corresponding to physical models containing the thickness as a physical variable.

The present paper deals with the state of art in application of Ilia Vekua's method of dimension reduction for constructing and investigating the hierarchical models for different physical models. The special attention is paid to the case of 3D angular domains, namely, to the study of peculiarities of posing the boundary conditions caused by it within the framework of the corresponding 2D boundary value (BV) and initial BV (IBV) problems. In other words the special attention is paid to the case of 3D angular (such as that dihedral and polyhedral) domains and the peculiarities of setting 2D BVPs caused by them.

Section 2 deals with the hierarchical models for elastic and piezoelectric, viscoelastic Kelvin-Voight with voids prismatic and standard shells and bars (see Appendix).

Section 3 deals with the hierarchical models for thermoelastic deformation of chiral porous prismatic shells.

Section 4 is devoted to fluids in prismatic and standard shell-like, bar-like, and canal-like domains.

In Section 5 we indicate how the above techniques may be used for different materials and summarise conclusions.

Each section is as much as possible selfcontained.

Throughout the paper we use Einstein's summation convention on repeated indices (that Latin and Greek run values 1,2,3 and 1,2, respectively) and the simplified notation for the partial derivative $(\cdots)_{,i}$ and $\overset{\cdot}{\overbrace{(\cdots)}}$ mean differentiation with respect to variable x_i and time t, respectively. Further, $\overset{N}{v}_{kr}$ and $\overset{N}{u}_{kr}$ $k=\overline{1,3}$, $r=0,1,2,...,N$, $N=0,1,2,...$ mean solutions of the governing systems of the Nth order approximation with respect to that unknowns, while u_{kr} and v_{kr} mean the Fourier- Legendre coefficients up to the fector $(r+1/2)^{1/2}a^{1/2}$ (i.e. the mathematical moments of the unknown displacements u_k, $k=\overline{1,3}$) and that weighted ones (i.e.

s.c. weighted moments given by $v_{kr} := h^{-r-1}u_{kr}$). Sometimes the Fourier -Legendre coefficient itself is called the mathematical moment of order k (see [3]). Clearly,

$$\varphi_r := \Big(r+\frac{1}{2}\Big)^{\frac{1}{2}} a^{\frac{1}{2}} P_r(ax_3-b), \quad r=0,1,2,...$$

is the orthonormal system since

$$\int_{\overset{(-)}{h}}^{\overset{(+)}{h}} \Big(k+\frac{1}{2}\Big)^{1/2}\Big(l+\frac{1}{2}\Big)^{1/2} P_k(ax_3-b)P_l(ax_3-b)a\,dx_3 = \delta_{kl}$$

(see [18] p. 219 (the first edition), 258 (the second addition)) and the Fourier-Legandre coefficient

$$\alpha_r := (u_k,\varphi_r) := \int_{\overset{(-)}{h}}^{\overset{(-)}{h}} \Big(r+\frac{1}{2}\Big)^{1/2} a^{1/2} u_k P_r(ax_3-b)dx_3 = \Big(r+\frac{1}{r}\Big)^{1/2} a^{1/2} u_{kr},$$

where

$$u_{kr} := \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(-)}{h}(x_1,x_2)} u_k(x_1,x_2,x_3)P_r\Big(\frac{x_3}{h(x_1,x_2)} - \frac{\widetilde{h}}{h}\Big)dx_3,$$

is the rth order mathematical moment.

$$v_{kr} := h^{-r-1}(x_1,x_2)u_{kr} = h^{-r-1}(x_1,x_2)\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(-)}{h}(x_1,x_2)} u_k(x_1,x_2,x_3)P_r\Big(\frac{x_3}{h(x_1,x_2)} - \frac{\widetilde{h}}{h}\Big)dx_3.$$

Evidently, in the sense of mean convergence [i.e., in $L_2(\overset{(-)}{h}(x_1,x_2), \overset{(+)}{h}(x_1,x_2))$]

$$u_k = \sum_{r=0}^{\infty}\alpha_r\varphi_r \equiv \sum_{r=0}^{\infty}\Big(r+\frac{1}{r}\Big)a u_{kr}P_r(ax_3-b) \equiv \sum_{r=0}^{\infty}\Big(r+\frac{1}{r}\Big)h^r v_{kr}P_r(ax_3-b).$$

Note that (see [13] and for that of cusped prismatic shells [14])

$$u_{kr} = \lim_{N\to\infty}\overset{N}{u}_{kr}, \quad k=\overline{1,3}.$$

Mainly, in the literature the upper index $N=0,1,2,...$, indicating the order of the approximation, for the sake of simplicity of notion, is omitted, and the reader should be carefull not to be confused.

Ilia Vekua's approximated solution

$$\overset{N}{u}_k = \sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)a\overset{N}{u}_{kr}P_r(ax_3-b), \quad k=\overline{1,3}, \quad a := \frac{1}{h(x_1,x_2)}, \quad b(x_1,x_2) := \frac{\overset{(+)}{h} + \overset{(-)}{h}}{\overset{(+)}{h} - \overset{(-)}{h}},$$

and the partial sum of the Fourier-Legendre series

$$\sum_{r=0}^{N}\Big(r+\frac{1}{2}\Big)au_{kr}P_r(ax_3-b),\quad k=\overline{1,3},$$

are different but the both tend to the unique exact solution u of the corresponding $3D$ BVP as $N\to\infty$ (see [13] and [14]).

It is also remarkable that I. Vekua (see [1] pp. 401-405) in the $N=1$ approximation besides classical normal, tangential, and transversal (intersecting) forces in other words, according to I. Vekua, the zero order weighted mathematical moments and the first - order mathematical moments according to I. Vekua, defined the additional first order mathematical moment called by him as the splitting couple of forces which is nothing more than the equilibrated stress vector that can be identified with singularities in classical linear elasticity known as double force systems without physical moments equivalent to two oppositely directed forces at the same point (see [15], p. 127). Singularities of this type were first discussed by Love [16] (see p.56).

One thing more, in some practical (enginering) models displacements are represented as polynomials of order $\leq n$ but they may represented as some linear combinations of Legenre polynomials (see [17], p. 529), in particular,

$$x_3^n=a_{0n}P_n(x_3)+a_{1n}P_1(x_3)+\cdots+a_{nn}P_n(x_3),$$

therefore models of such type are contained as particular cases in I. Vekua's hierarchical models.

2 Elastic prismatic and standard shells and bars

For the sake of simplicity we restrict ourselves to prismatic shells.

The first version of Vekua's hierarchical models for cusped, in general, homogeneous elastic prismatic shells in the Nth approximation has the form (see [4], page 19, we refrain from giving the proof here)

$$\mu\left[\Big(h^{2r+1}\overset{N}{v}_{\alpha r,j}\Big)_{,\alpha}+\Big(h^{2r+1}\overset{N}{v}_{jr,\alpha}\Big)_{,\alpha}\right]+\lambda\delta_{\alpha j}\Big(h^{2r+1}\overset{N}{v}_{\gamma r,\gamma}\Big)_{,\alpha} \tag{2.1}$$

$$+\sum_{s=r+1}^{N}\Big(\overset{r}{B}_{\alpha jks}\,h^{r+s+1}\,\overset{N}{v}_{ks}\Big)_{,\alpha}+\sum_{l=0}^{r-1}\overset{r}{a}_{il}\Big[\lambda\delta_{ij}h^{r+l+1}\overset{N}{v}_{\gamma l,\gamma}+\mu h^{r+l+1}\Big(\overset{N}{v}_{il,j}+\overset{N}{v}_{jl,i}\Big)$$

$$+\sum_{s=l+1}^{N}\overset{l}{B}_{ijks}\,h^{r+s+1}\overset{N}{v}_{ks}\Big]+h^r\overset{r}{X}_j=\rho h^r\frac{\partial^2 h^{r+1}\,\overset{N}{v}_{jr}}{\partial\, t^2},$$

$$r=\overline{0,N},\quad j=\overline{1,3},\quad \sum_{q}^{q-1}(\cdots)\equiv 0,$$

where

$$\overset{r}{B}_{ijks}:=\lambda\delta_{ij}\overset{r}{b}_{ks}+\mu\delta_{kj}\overset{r}{b}_{is}+\mu\delta_{ik}\overset{r}{b}_{js},$$

$$\overset{N}{v}_{kr}:=\frac{\overset{N}{u}_{kr}}{h^{r+1}},\quad k=\overline{1,3},\quad r=\overline{0,N}. \tag{2.2}$$

For $j=3$ from the three groups of terms of the main part of (2.1) remains $(\mu h^{2r+1}\overset{N}{v}_{3r,\alpha})_{,\alpha}$ which will give integral criteria of well-posedness of Dirichlet ($I:=\int\limits_0^{\varepsilon}\frac{dx_2}{h^{2r+1}}<+\infty$) and Keldysh type ($I=+\infty$) problems (for the case $N=0$ see Appendix).

For bars see [4] p. 54 (for more in details see [11]).

2.1 Field equations for piezoelectric Kelvin-Voigt materials with voids

Let a piezoelectric solid occupy a reference configuration $\Omega\in\mathbb{R}^3$. Under the quasi-static conditions, when the rate of change of the magnetic field is small and there is no electric current, i.e., the electric field $\mathbf{E}$ and magnetic field $\mathbf{M}$ are curl free, the 3D governing equations have the following form

Motion Equations

$$X_{ji,j}+\Phi_i=\rho\ddot{u}_i(x_1,x_2,x_3,t),\quad (x_1,x_2,x_3)\in\Omega\subset\mathbb{R}^3,\ \ t>t_0,\ \ i=\overline{1,3}; \tag{2.3}$$

$$H_{j,j}+H_0+\mathcal{F}=k\ddot{\varphi}, \tag{2.4}$$

$$D_{j,j}=f_e,\quad B_{j,j}=0,\quad \Omega\times]0,T[, \tag{2.5}$$

where $X_{ij}\in C^1(\Omega)$ is the stress tensor; Φ_i are the volume force components; k is the equilibrated inertia per volume unit, ρ is the mass density; $\varphi:=\nu_0-\nu\in C^2(\Omega)$ is the change of the volume fraction from the matrix reference volume fraction ν_0 (clearly, the bulk density $\rho=\nu\gamma$, $0<\nu\leq 1$, here γ is the matrix reference density); $u_i\in C^2(\Omega)$ are the displacements; $H_j\in C^1(\Omega)$ is the component of the equilibrated stress vector, $H_0\equiv g$ and $\mathcal{F}\equiv l$ are the intrinsic and extrinsic equilibrated volume forces; we remind that Einstein's summation convention is used; indices after comma mean differentiation with respect to the corresponding variables of the Cartesian frame $Ox_1x_2x_3$ (throughout the work we assume existence of the indicated (continuous) derivatives unless otherwise stated); dots as superscripts of the symbols mean derivatives with respect to time t; $\chi:\Omega\times]0,T[\to\mathbb{R}^1$ and $\eta:\Omega\times]0,T[\to\mathbb{R}^1$ are electric and magnetic potentials, respectively, i.e., $\mathbf{E}=-grad\chi$, $\mathbf{M}=-grad\eta$, $f_e:\Omega\times]0,T[\to\mathbb{R}^1$ is electric charge density. $\mathbf{D}:=(D_1,D_2,D_3):\ \Omega\times]0,T[\to\mathbb{R}^3$ is the electrical displacement vector, $\mathbf{B}:=(B_1,B_2,B_3):\ \Omega\times]0,T[\to\mathbb{R}^3$ is the magnetic induction vector.

Kinematic Relations

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad i,j=\overline{1,3}, \tag{2.6}$$

$e_{ij}\in C^1(\Omega)$ is the strain tensor;

Constitutive Equations

$$\begin{aligned}X_{ji}=X_{ij}=E_{ijkl}e_{kl}+E^*_{ijkl}\dot{e}_{kl}+\tilde{b}_{ij}\varphi+b^*_{ij}\dot{\varphi}+d_{ijk\varphi,k}+d^*_{ijk}\dot{\varphi}_{,k}\\+p_{kij}\chi_{,k}+p^*_{kij}\dot{\chi}_{,k}+q_{kij}\eta_{,k}+q^*_{kij}\dot{\eta}_{,k}\,,\ \ i,j=\overline{1.3},\end{aligned} \tag{2.7}$$

$$H_j=d_{klj}e_{kl}+d^*_{klj}\dot{e}_{kl}+d_j\varphi+d^*_j\dot{\varphi}+\tilde{a}_{ji}\varphi_{,i}+a^*_{ji}\dot{\varphi}_{,i},\ \ j=\overline{1,3}, \tag{2.8}$$

$$H_0=-\tilde{b}_{ij}e_{ij}-\tilde{\xi}_{\varphi}-d_i\varphi_{,i}-b^*_{ij}\dot{e}_{ij}-\xi^*\dot{\varphi}-d^*_i\dot{\varphi}_{,i}, \tag{2.9}$$

$$D_j=p_{jkl}e_{kl}+p^*_{jkl}\dot{e}_{kl}-\zeta_{jl}\chi_{,l}-\tilde{a}_{jk}\eta_{,l},\ \ j=\overline{1,3}, \tag{2.10}$$

$$B_j=q_{jkl}e_{kl}+q^*_{jkl}\dot{e}_{kl}-\tilde{a}_{jl}\chi_{,l}-\xi_{jl}\eta_{,l}\ \ \ j=\overline{1,3}, \tag{2.11}$$

where the constitutive coefficients E_{ijkl} (elasticity), E^*_{ijkl} (viscosity), $\tilde{b}_{ij}$, d_i, d_{klj}, $\tilde{\alpha}_{ji}$, $\tilde{\xi}$ (porosity), b^*_{ij}, d^*_i, d^*_{klj}, α^*_{ji}, ξ^* (viscoporosity), p_{kij}, p^*_{jkl}, q_{kij}, q^*_{jkl}, ζ_{jl}, $\tilde{\alpha}_{jl}$, ξ_{jl}, p_{kij} are the piezoelectric coefficients, q_{kij} are the piezomagnetic coefficients, ς_{jl} and ξ_{jl} are the dielectric (permittivity) and magnetic permeability coefficients, respectively, $\tilde{a}_{jl}$ are the coupling coefficients connecting electric and magnetic fields, satisfy the following relations

$$
\begin{aligned}
&E_{ijkl} = E^*_{jikl} = E_{jilk} = E_{klij}; \;\; E^*_{ijkl} = E^*_{jikl} = E^*_{jilk} = E^*_{klij}\\
&\tilde{b}_{ij} = \tilde{b}_{ji}, \;\; d_{ijk} = d_{jik}, \;\; \tilde{\alpha}_{ij} = \tilde{\alpha}_{ji};\\
&b^*_{ij} = b^*_{ji}, \; d^*_{ijk} = d^*_{jik}, \; \alpha^*_{ij} = \alpha^*_{ji}, \; p_{jkl} = p_{jlk}, \; q_{jkl} = q_{jlk}, \; \zeta_{jl} = \zeta_{lj},\\
&\tilde{a}_{jl} = \tilde{a}_{lj}, \; \xi_{jl} = \xi_{lj}, \; p^*_{jkl} = p^*_{jlk}, \; q^*_{jkl} = q^*_{jlk}.
\end{aligned}
$$

The constitutive coefficients also meet some other conditions, following from physical considerations, with a view to apply I. Vekua's dimension reduction method, we require the constitutive coefficients to be independent of x_3.

Let us consider the general BVPs and IBVPs with the following mixed BCs

$$u_i = f_i \quad on \quad \Gamma_0, \quad X_{ij}n_j = g_i \quad on \quad \Gamma_1 = \partial\Omega\backslash\overline{\Gamma_0}, \;\; i = \overline{1,3}, \tag{2.12}$$

$$\varphi = f^{\varphi} \quad on \quad \Gamma^{\varphi}_0, \quad H_j n_j = g^{\varphi} \quad on \quad \Gamma^{\varphi}_1 = \partial\Omega\backslash\overline{\Gamma^{\varphi}_0}, \;\; i = \overline{1,3}, \tag{2.13}$$

$$\chi = f^{\chi} \quad on \quad \Gamma^{\chi}_0, \quad D_j n_j = g^{\chi} \quad on \quad \Gamma^{\chi}_1 = \partial\Omega\backslash\overline{\Gamma^{\chi}_0}, \;\; i = \overline{1,3}, \tag{2.14}$$

$$\eta = f^{\eta} \quad on \quad \Gamma^{\eta}_0, \quad B_j n_j = g^{\eta} \quad on \quad \Gamma^{\eta}_1 = \partial\Omega\backslash\overline{\Gamma^{\eta}_0}, \;\; i = \overline{1,3}, \tag{2.15}$$

and the standard ICs in the case of dynamical problems

$$
\begin{gathered}
\mathbf{u}(x,0) = \mathbf{u}^0(x), \;\; \dot{\mathbf{u}}(x,0) = \mathbf{u}^1(x), \;\; \varphi(x,0) = \varphi^0(x), \;\; \dot{\varphi}(x,0) = \varphi^1(x),\\
x \in \Omega;
\end{gathered} \tag{2.16}
$$

here $\mathbf{n} := (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$, (f_1, f_2, f_3), f^{φ}, f^{χ}, f^{η} are the given displacement vector, volume fraction, electric and magnetic potentials, respectively, (g_1, g_2, g_3), g^{φ}, g^{χ} and g^{η} are the given stress vector, normal components of the equilibrated stress, electric displacement and magnetic induction vectors, respectively, while $\mathbf{u}^0$ and $\mathbf{u}^1$ are the initial mechanical displacement and velocity vectors, whereas φ^0 and φ^1 are the initial volume fraction distribution and its rate. Note that the sub-manifolds Γ_0, Γ^{φ}_0, Γ^{χ}_0, and Γ^{η}_0, of the boundary $\partial\Omega$ in boundary conditions (2.12)-(2.15) are different, in general, from each other and depending on the physical problem some of them may be empty.

2.1.1 Construction of hierarchical models. Nth approximation

Below we use formulas (for proof see [6], Section 10).

$$
\begin{aligned}
&\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,\alpha}\, dx_3\\
&= f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,\alpha}, \;\; \alpha = 1, 2,
\end{aligned} \tag{2.17}
$$

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s + \overset{(+)}{f} - (-1)^r \overset{(-)}{f}, \tag{2.18}$$

or in the unified form

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,j}\,dx_3 = f_{r,j} + \sum_{s=r}^{\infty} \overset{r}{b}_{js} f_s, \quad j=\alpha,3, \quad \alpha=1,2, \tag{2.19}$$

provided $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are not prescribed on the face surfaces.

$$\begin{aligned} \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,j}\,dx_3 &= f_{r,j} + \sum_{s=0}^{r} \overset{r}{a}_{js} f_s \\ &+ \overset{(+)}{f}\overset{(+)}{n}_j \sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2} \\ +(-1)^r \overset{(-)}{f}\overset{(-)}{n}_j &\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2}, \quad j=\alpha,3, \quad \alpha=1,2, \end{aligned} \tag{2.20}$$

provided $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are prescribed on the face surfaces. Here

$$\overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r; \quad \overset{r}{a}_{\alpha \underline{r}} = r\frac{h_{,\alpha}}{h},$$

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^r}{2h}, \quad 2h := \overset{(+)}{h} - \overset{(-)}{h},$$

$$\overset{r}{b}_{js} := -\overset{r}{a}_{js}, \quad j \neq r, \quad \overset{r}{b}_{\alpha \underline{r}} := -(r+1)\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2h}, \quad \overset{r}{b}_{3\underline{r}} = 0.$$

From (2.3)-(2.11), after multiplying them by $P_r(ax_3-b)$ for $r=0,1,\cdots$, and then integrating within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$ with respect to the thickness variable x_3, we obtain the following formulas in ω:
(i) from (2.3)-(2.5), correspondingly, it follows that

$$X_{\alpha ir,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{js} X_{jis} + \overset{r}{X}_i = \rho\frac{\partial^2 u_{ir}}{\partial t^2}, \quad i=\overline{1,3}, \quad r=0,1,\cdots, \tag{2.21}$$

$$H_{\alpha r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} H_{\alpha s} + H_{0r} + \overset{r}{H} = k\ddot{\varphi}_r - \mathcal{F}_r, \quad r=0,1,\cdots, \tag{2.22}$$

$$D_{\gamma r,\gamma} + \sum_{s=0}^{r} \overset{r}{a}_{is} D_{is} + \overset{r}{D} = f_{er}, \quad r=0,1,\cdots, \tag{2.23}$$

$$B_{\gamma r,\gamma} + \sum_{s=0}^{r} \overset{r}{a}_{is} B_{is} + \overset{r}{B} = 0, \quad r=0,1,\cdots; \tag{2.24}$$

where

$$\overset{r}{X}_i := \overset{(+)}{X}_{3i} - \overset{(+)}{X}_{\alpha i}\overset{(+)}{h}_{,\alpha} + (-1)^r\Big[- \overset{(-)}{X}_{3i} + \overset{(-)}{X}_{\alpha i}\overset{(-)}{h}_{,\alpha}\Big] + \Phi_{ir}$$

$$= X_{\overset{(+)}{n} i}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2} + (-1)^r X_{\overset{(-)}{n} i}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2} + \Phi_{ir},$$

$$i=\overline{1,3},\quad r=0,1,2,\cdots;$$

since

$$x_3 = \overset{(\pm)}{h}(x_1,x_2) \Rightarrow F(x_1,x_2,x_3) := x_3 - \overset{(\pm)}{h}(x_1,x_2) = 0$$

$$\overset{(\pm)}{n}_{,i} = \frac{\overset{(\pm)}{F}_{,i}}{\pm\sqrt{\overset{(\pm)}{F}{}^2_{,1}+\overset{(\pm)}{F}{}^2_{,2}+\overset{(\pm)}{F}{}^2_{,3}}} \Rightarrow \overset{(\pm)}{n}_{,\alpha} = \frac{-\overset{(\pm)}{h}_{,\alpha}}{\pm\sqrt{\overset{(\pm)}{F}{}^2_{,1}+\overset{(\pm)}{F}{}^2_{,2}+\overset{(\pm)}{F}{}^2_{,3}}},\quad \overset{(\pm)}{n}_{,3} = \frac{1}{\pm\sqrt{\overset{(\pm)}{F}{}^2_{,1}+\overset{(\pm)}{F}{}^2_{,2}+\overset{(\pm)}{F}{}^2_{,3}}}$$

here $X_{\overset{(+)}{n} i}$ and $X_{\overset{(-)}{n} i}$ are components of the stress vectors $X_{\overset{(+)}{n}}$ and $X_{\overset{(-)}{n}}$ acting on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively,

$$\overset{r}{H} := \overset{(+)}{H}_3 - \overset{(+)}{H}_{\alpha}\overset{(+)}{h}_{,\alpha} + (-1)^r\Big[- \overset{(-)}{H}_3 + \overset{(-)}{H}_{\alpha}\overset{(-)}{h}_{,\alpha}\Big] + \mathcal{F}_r$$

$$\overset{(+)}{H}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^2+\Big(\overset{(+)}{h}_{,2}\Big)^2} + (-1)^r\overset{(-)}{H}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^2+\Big(\overset{(-)}{h}_{,2}\Big)^2} + \mathcal{F}_r \quad r=0,1,\cdots,$$

$\overset{(+)}{H}_j$ and $\overset{(-)}{H}_j$ are components of the equilibrated stress vectors on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively.

$$\begin{aligned}
\overset{r}{D} &:= \overset{(+)}{D}_3 - \overset{(+)}{D}_{\gamma}\overset{(+)}{h}_{,\gamma} + (-1)^r\Big[- \overset{(-)}{D}_3 + \overset{(-)}{D}_{\gamma}\overset{(-)}{h}_{\gamma}\Big] \\
&= \overset{(+)}{D}_i\overset{(+)}{n}_i\sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2} + \overset{(-)}{D}_i\overset{(-)}{n}_i\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2}, \\
\overset{r}{B} &:= \overset{(+)}{B}_3 - \overset{(+)}{B}_{\gamma}\overset{(+)}{h}_{,\gamma} + (-1)^r\Big[- \overset{(-)}{B}_3 + \overset{(-)}{B}_{\gamma}\overset{(-)}{h}_{\gamma}\Big] \\
&= \overset{(+)}{B}_i\overset{(+)}{n}_i\sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2} + \overset{(-)}{B}_i\overset{(-)}{n}_i\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2};
\end{aligned}$$

(ii) from (2.6), using (2.19), it follows that

$$e_{ijr} = \frac{1}{2}\Big(u_{ir,j}+u_{jr,i}\Big) + \frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{is}u_{js} + \frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{js}u_{is}, i,j=\overline{1,3},\quad r=0,1,\cdots, \tag{2.25}$$

by virtue of

$$v_{ir} := h^{-r-1}u_{ir},$$

we have

$$e_{ijr} = \frac{1}{2}h^{r+1}\Big(v_{ir,j} + v_{jr,i}\Big) + \frac{1}{2}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{is}v_{js} + \overset{r}{b}_{js}v_{is}\Big), \tag{2.26}$$
$$i,j=\overline{1,3},\ r=0,1,\cdots.$$

(iii) from (2.7), taking into account (2.19) for $j=\gamma$ and $j=3$, it follows that

$$\begin{aligned}
X_{ijr} &= E_{ijkl}e_{klr} + E^*_{ijkl}\dot{e}_{klr} + \tilde{b}_{ij}\varphi_r + b^*_{ij}\dot{\varphi}_r + d_{ij\gamma}\Big(\varphi_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\varphi_s\Big)\\
&-d_{ij3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{\gamma s}\varphi_s + d^*_{ij\gamma}\Big(\dot{\varphi}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\varphi}_s\Big) - d^*_{ij3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\varphi}_s\\
&+p_{\gamma ij}(\chi_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s) - p_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s + p^*_{\gamma ij}\Big(\dot{\chi}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\chi}_s\Big)\\
&-p^*_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\chi}_s + q_{\gamma ij}(\eta_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s) - q_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s\\
&+q^*_{\gamma ij}(\dot{\eta}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\eta}_s) - q^*_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\eta}_s, \quad i,j=\overline{1,3}, \quad r=0,1,\cdots.
\end{aligned} \tag{2.27}$$

Therefore, by virtue of (2.25),

$$\begin{aligned}
X_{ijr} &= \frac{1}{2}E_{ijkl}\Big(u_{kr,l} + u_{lr,k}\Big) + \frac{1}{2}E_{ijkl}\sum_{s=r}^{\infty}\Big(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks}\Big)\\
&+\frac{1}{2}E^*_{ijkl}\Big(\dot{u}_{kr,l} + \dot{u}_{lr,k}\Big) + \frac{1}{2}E^*_{ijkl}\sum_{s=r}^{\infty}\Big(\overset{r}{b}_{ks}\dot{u}_{ls} + \overset{r}{b}_{ls}\dot{u}_{ks}\Big)\\
&+\tilde{b}_{ij}\varphi_r + b^*_{ij}\dot{\varphi}_r + d_{ij\gamma}\Big(\varphi_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\varphi_s\Big)\\
&-d_{ij3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{\gamma s}\varphi_s + d^*_{ij\gamma}\Big(\dot{\varphi}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\varphi}_s\Big) - d^*_{ij3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\varphi}_s\\
&+p_{\gamma ij}\Big(\chi_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big) - p_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s + p^*_{\gamma ij}\Big(\dot{\chi}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\chi}_s\Big)\\
&-p^*_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\chi}_s + q_{\gamma ij}\Big(\eta_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big) - q_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s\\
&+q^*_{\gamma ij}(\dot{\eta}_{r,\gamma} + \sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\dot{\eta}_s) - q^*_{3ij}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\dot{\eta}_s, \quad i,j=\overline{1,3}, \quad r=0,1,\cdots.
\end{aligned} \tag{2.28}$$

Let

$$v_{kr} := \frac{u_{kr}}{h^{r+1}}, \quad \psi_r := \frac{\varphi_r}{h^{r+1}}, \quad \widetilde{\chi}_r := \frac{\chi_r}{h^{r+1}}, \quad \widetilde{\eta}_r := \frac{\eta_r}{h^{r+1}}. \tag{2.29}$$

Substituting (2.26) into (2.27), and taking into account (2.29), it follows that

$$X_{ijr} = \frac{1}{2}E_{ijkl}h^{r+1}\Big(v_{kr,l} + v_{lr,k}\Big) + \frac{1}{2}E_{ijkl}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{ks}v_{ls} + \overset{r}{b}_{ls}v_{ks}\Big)$$
$$+\frac{1}{2}E^*_{ijkl}h^{r+1}\Big(\dot{v}_{kr,l} + \dot{v}_{lr,k}\Big) + \frac{1}{2}E^*_{ijkl}\sum_{s=r+1}^{\infty} h^{s+1}\Big(\overset{r}{b}_{ks}\dot{v}_{ls} + \overset{r}{b}_{ls}\dot{v}_{ks}\Big)$$
$$+\tilde{b}_{ij}h^{r+1}\psi_r + b^*_{ij}h^{r+1}\dot{\psi}_r + d_{ij\gamma}h^{r+1}\psi_{r,\gamma} + d_{ijk}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\psi_s + d^*_{ij\gamma}h^{r+1}\dot{\psi}_{r,\gamma}$$
$$+d^*_{ijk}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\dot{\psi}_s + p_{\gamma ij}h^{r+1}\widetilde{\chi}_{r,\gamma} + p_{kij}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\widetilde{\chi}_s$$
$$+p^*_{\gamma ij}h^{r+1}\dot{\widetilde{\chi}}_{r,\gamma} + p^*_{kij}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\dot{\widetilde{\chi}}_s + q_{\gamma ij}h^{r+1}\widetilde{\eta}_{r,\gamma} + q_{kij}\sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{ks}\widetilde{\eta}_s$$
$$+q^*_{\gamma ij}h^{r+1}\dot{\widetilde{\eta}}_{r,\gamma} + q^*_{kij}\sum_{s=r+1}^{\infty} h^{s+1}\,\overset{r}{b}_{ks}\dot{\widetilde{\eta}}_s, \quad i,j=\overline{1,3}, \quad r=0,1,\cdots \tag{2.30}$$

(because of

$$(h^{r+1}\tilde{\chi}_r)_{,\gamma} - h^{r+1}(r+1)\frac{h_{,\alpha}}{h}\tilde{\chi}_r = h^{r+1}\tilde{\chi}_{r,\gamma},$$

and the similar formulas for ψ and $\tilde{\eta}$).

Analogously, from (2.8) we have

$$H_{jr} = \frac{1}{2}d_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}d_{klj}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks})$$
$$+\frac{1}{2}d^*_{klj}(u_{kr,l} + u_{lr,k}) + \frac{1}{2}d^*_{klj}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls} + \overset{r}{b}_{ls}u_{ks}) + d_j\varphi_r + d^*_j\dot{\varphi}_r$$
$$+\tilde{\alpha}_{j\beta}\Big[\varphi_{r,\beta} + \sum_{s=0}^{r}\overset{r}{a}_{\beta s}\varphi_s - \overset{(+)}{\varphi}\overset{(+)}{h}_{,\beta} + (-1)^r\overset{(-)}{\varphi}\overset{(-)}{h}_{,\beta}\Big]$$
$$+\tilde{\alpha}_{j3}\Big[\sum_{s=0}^{r}\overset{r}{a}_{3s}\varphi_s + \overset{(+)}{\varphi} - (-1)^r\overset{(-)}{\varphi}\Big]$$
$$+\alpha^*_{j\beta}\Big[\dot{\varphi}_{r,\beta} + \sum_{s=0}^{r}\overset{r}{a}_{\beta s}\dot{\varphi}_s - \overset{(+)}{\dot{\varphi}}\overset{(+)}{h}_{,\alpha} - (-1)^r\overset{(-)}{\dot{\varphi}}\overset{(-)}{h}_{,\alpha}\Big]$$
$$+\alpha^*_{j3}\Big[\sum_{s=0}^{r}\overset{r}{a}_{3s}\dot{\varphi}_s + \overset{(+)}{\dot{\varphi}} - (-1)^r\overset{(-)}{\dot{\varphi}}\Big], \quad j=\overline{1,3},$$

and substituting here the corresponding Fourier-Legendre expansions of φ on the upper and lower face surfaces

$$\overset{(\pm)}{\varphi} = \sum_{s=0}^{\infty}\frac{(\pm 1)^s(2s+1)}{2h}\varphi_s,$$

we get

$$H_{jr}=\frac{1}{2}d_{klj}(u_{kr,l}+u_{lr,k})+\frac{1}{2}d_{klj}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls}+\overset{r}{b}_{ls}u_{ks})$$

$$+\frac{1}{2}d^*_{klj}(u_{kr,l}+u_{lr,k})+\frac{1}{2}d^*_{klj}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls}+\overset{r}{b}_{ls}u_{ks})+d_j\varphi_r+d^*_j\dot{\varphi}_r$$

$$+\tilde{\alpha}_{jk}\Big(\varphi_{r,k}+\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\varphi_s\Big)+\alpha^*_{jk}\Big(\dot{\varphi}_{r,k}+\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\dot{\varphi}_s\Big),\ j=\overline{1,3},\ r=0,1,\cdots, \tag{2.31}$$

i.e. (see (2.29))

$$H_{jr}=\frac{1}{2}d_{klj}h^{r+1}(v_{kr,l}+v_{lr,k})+\frac{1}{2}d_{klj}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}v_{hs}+\overset{r}{b}_{ls}v_{ks})$$

$$+\frac{1}{2}d^*_{klj}h^{r+1}(\dot{v}_{kr,l}+\dot{v}_{lr,k})+\frac{1}{2}d^*_{klj}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{hs}+\overset{r}{b}_{ls}\dot{v}_{ks})d_jh^{r+1}\psi_r+d^*_jh^{r+1}\dot{\psi}_r$$

$$+\tilde{\alpha}_{ji}\Big(h^{r+1}\psi_{r,i}+\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{is}\psi_s\Big)+\alpha^*_{ji}\Big(h^{r+1}\dot{\psi}_{r,i}+\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{is}\dot{\psi}_s\Big), \tag{2.32}$$

$$j=\overline{1,3},\ r=0,1\cdots.$$

From (2.9), on account of combined (2.19), (2.20), evidently, it follows that

$$H_{0r}=-d_i(\varphi_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\varphi_s)-\tilde{b}_{ij}e_{ijr}-\tilde{\xi}\varphi_r$$

$$-d^*(\dot{\varphi}_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\dot{\varphi}_s)-b^*_{ij}\dot{e}_{ijr}-\xi^*\dot{\varphi}_r,\quad r=0,1,\cdots$$

and, in view of (2.22),

$$H_{0r}=-d_i(\varphi_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\varphi_s)$$

$$-\tilde{b}_{ij}\Big[\frac{1}{2}\Big(u_{ir,j}+u_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{is}u_{js}+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{js}u_{is}\Big]$$

$$-\tilde{\xi}\varphi_r-d^*(\dot{\varphi}_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\dot{\varphi}_s)$$

$$-b^*_{ij}\Big[\frac{1}{2}\Big(\dot{u}_{ir,j}+\dot{u}_{jr,i}\Big)+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{is}\dot{u}_{js}+\frac{1}{2}\sum_{s=r}^{\infty}\overset{r}{b}_{js}\dot{u}_{is}\Big]-\xi^*\dot{\varphi}_r, \tag{2.33}$$

while, by virtue of (2.26) and (2.29),

$$H_{0r}=-\tilde{b}_{ij}\Big[\frac{1}{2}h^{r+1}\Big(v_{ir,j}+v_{jr,i}\Big)+\frac{1}{2}\sum_{s=r+1}^{\infty}h^{s+1}\Big(\overset{r}{b}_{is}v_{js}+\overset{r}{b}_{js}v_{is}\Big)\Big]$$
$$-\tilde{\xi}h^{r+1}\psi_r-d_i\Big(h^{r+1}\psi_{r,i}+\sum_{s=r+1}^{\infty}\overset{r}{b}_{is}\psi_s\Big)$$
$$-b^*_{ij}\Big[\frac{1}{2}h^{r+1}\Big(\dot{v}_{ir,j}+\dot{v}_{jr,i}\Big)+\frac{1}{2}\sum_{s=r+1}^{\infty}h^{s+1}\Big(\overset{r}{b}_{is}\dot{v}_{js}+\overset{r}{b}_{js}\dot{v}_{is}\Big)\Big]$$
$$-\xi^*h^{r+1}\dot{\psi}_r-d^*_i\Big(h^{r+1}\dot{\psi}_{r,i}+\sum_{s=r+1}^{\infty}\overset{r}{b}_{is}\dot{\psi}_s\Big),\quad r=0,1,\cdots. \tag{2.34}$$

Similarly, from (2.10) it follows

$$D_{jr}=p_{jkl}e_{klr}+p^*_{jkl}\dot{e}_{klr}-\varsigma_{j\gamma}\Big(\chi_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big)+\varsigma_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s$$
$$-\tilde{a}_{j\gamma}\Big(\eta_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big)+\tilde{a}_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s,\quad j=\overline{1,3},\quad r=0,1,\cdots,$$

i.e., in view of (2.25),

$$D_{jr}=\frac{1}{2}p_{jkl}(u_{kr,l}+u_{lr,k})+\frac{1}{2}p_{jkl}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls}+\overset{r}{b}_{ls}u_{ks})$$
$$+\frac{1}{2}p^*_{jkl}(\dot{u}_{kr,l}+\dot{u}_{lr,k})+\frac{1}{2}p^*_{jkl}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}\dot{u}_{ls}+\overset{r}{b}_{ls}\dot{u}_{ks})$$
$$-\varsigma_{j\gamma}\Big(\chi_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big)+\varsigma_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s$$
$$-\tilde{a}_{j\gamma}\Big(\eta_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big)+\tilde{a}_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s,\quad j=\overline{1,3},\quad r=0,1,\cdots, \tag{2.35}$$

while by virtue of (2.26) and (2.29), we have

$$D_{jr}=\frac{1}{2}p_{jkl}h^{r+1}(v_{kr,l}+v_{lr,k})+\frac{1}{2}p_{jkl}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}v_{ls}+\overset{r}{b}_{ls}v_{ks})$$
$$+\frac{1}{2}p^*_{jkl}h^{r+1}(\dot{v}_{kr,l}+\dot{v}_{lr,k})+\frac{1}{2}p^*_{jkl}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{ls}+\overset{r}{b}_{ls}\dot{v}_{ks})$$
$$-\varsigma_{j\gamma}\Big(h^{r+1}\widetilde{\chi}_{r,\gamma}+\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\chi}_s\Big)+\varsigma_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\chi}_s$$
$$-\tilde{a}_{j\gamma}\Big(h^{r+1}\widetilde{\eta}_{r,\gamma}+\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\eta}_s\Big)+\tilde{a}_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\eta}_s, \tag{2.36}$$
$$j=\overline{1,3},\quad r=0,1,\cdots.$$

In the same way from (2.11) we get

$$B_{jr}=\frac{1}{2}q_{jkl}(u_{kr,l}+u_{lr,k})+\frac{1}{2}q_{jkl}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}u_{ls}+\overset{r}{b}_{ls}u_{ks})$$
$$+\frac{1}{2}q^{*}_{jkl}(\dot{u}_{kr,l}+\dot{u}_{lr,k})+\frac{1}{2}q^{*}_{jkl}\sum_{s=r}^{\infty}(\overset{r}{b}_{ks}\dot{u}_{ls}+\overset{r}{b}_{ls}\dot{u}_{ks})$$
$$-\tilde{a}_{j\gamma}\Big(\chi_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\chi_s\Big)+\tilde{a}_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\chi_s$$
$$-\xi_{j\gamma}\Big(\eta_{r,\gamma}+\sum_{s=r}^{\infty}\overset{r}{b}_{\gamma s}\eta_s\Big)+\xi_{j3}\sum_{s=r+1}^{\infty}\overset{r}{a}_{3s}\eta_s,\quad j=\overline{1,3},\quad r=0,1,\cdots, \tag{2.37}$$

i.e.

$$B_{jr}=\frac{1}{2}q_{jkl}h^{r+1}(v_{kr,l}+v_{lr,k})+\frac{1}{2}q_{jkl}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}v_{ls}+\overset{r}{b}_{ls}v_{ks})$$
$$+\frac{1}{2}q^{*}_{jkl}h^{r+1}(\dot{v}_{kr,l}+\dot{v}_{lr,k})+\frac{1}{2}q^{*}_{jkl}\sum_{s=r+1}^{\infty}h^{s+1}(\overset{r}{b}_{ks}\dot{v}_{ls}+\overset{r}{b}_{ls}\dot{v}_{ks})$$
$$-\tilde{a}_{j\gamma}\Big(h^{r+1}\widetilde{\chi}_{r,\gamma}+\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\chi}_s\Big)+\tilde{a}_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\chi}_s$$
$$-\xi_{j\gamma}\Big(h^{r+1}\widetilde{\eta}_{r,\gamma}+\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{b}_{\gamma s}\widetilde{\eta}_s\Big)+\xi_{j3}\sum_{s=r+1}^{\infty}h^{s+1}\overset{r}{a}_{3s}\widetilde{\eta}_s, \tag{2.38}$$
$$j=\overline{1,3},\ r=0,1,\cdots;$$

Substituting (2.30) into (2.21); (2.32) and (2.34) into (2.22); (2.36) into (2.23); (2.38) into (2.24) we get an infinity system that after truncation it gives the governing system of the Nth approximation with respect to

$$v_{kr},\ \psi_r,\ \widetilde{\chi}_r,\ \widetilde{\eta}_r,\quad k=\overline{1,3},\ r=\overline{0,N}:$$
$$\frac{1}{2}\Big(E_{\alpha ik\delta}h^{2r+1}v_{kr,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha i\gamma l}h^{2r+1}v_{lr,\gamma}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$
$$+\frac{1}{2}\Big(E_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(E^{*}_{\alpha ik\delta}h^{2r+1}\dot{v}_{kr,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(E^{*}_{\alpha i\gamma l}h^{2r+1}\dot{v}_{lr,\gamma}\Big)_{,\alpha}$$
$$+\frac{1}{2}\Big(E^{*}_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(E^{*}_{\alpha ikl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}+\Big(\tilde{b}_{\alpha i}h^{2r+1}\psi_r\Big)_{,\alpha}$$
$$+\Big(b^{*}_{\alpha i}h^{2r+1}\dot{\psi}_r\Big)_{,\alpha}+\Big(d_{\alpha i\gamma}h^{2r+1}\psi_{r,\gamma}\Big)_{,\alpha}+\Big(d_{\alpha ik}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\psi_s\Big)_{,\alpha}$$
$$+\Big(d^{*}_{\alpha i\gamma}h^{2r+1}\dot{\psi}_{r,\gamma}\Big)_{,\alpha}+\Big(d^{*}_{\alpha ik}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\psi}_s\Big)_{,\alpha}$$

$$+\Big(p_{\gamma\alpha i}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha}+\Big(p_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_{s}\Big)_{,\alpha}$$

$$+\Big(p^{*}_{\gamma\alpha i}h^{2r+1}\dot{\tilde{\chi}}_{r,\gamma}\Big)_{,\alpha}+\Big(p^{*}_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\tilde{\chi}}_{s}\Big)_{,\alpha}$$

$$+\Big(q_{\gamma\alpha i}h^{2r+1}\tilde{\eta}_{r,\gamma}\Big)_{,\alpha}+\Big(q_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_{s}\Big)_{,\alpha}$$

$$+\Big(q^{*}_{\gamma\alpha i}h^{2r+1}\dot{\tilde{\eta}}_{r,\gamma}\Big)_{,\alpha}+\Big(q^{*}_{k\alpha i}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\tilde{\eta}}_{s}\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{js}\Big[\frac{1}{2}E_{jikl}h^{r+s+1}\Big(v_{ks,l}+v_{ls,k}\Big)+\frac{1}{2}E_{jikl}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'}\Big)$$

$$+\frac{1}{2}E^{*}_{jikl}h^{r+s+1}\Big(\dot{v}_{ks,l}+\dot{v}_{ls,k}\Big)+\frac{1}{2}E^{*}_{jikl}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'}\Big)$$

$$+\tilde{b}_{ij}h^{r+s+1}\psi_{s}+b^{*}_{ij}h^{r+s+1}\dot{\psi}_{s}+d_{ji\gamma}h^{r+s+1}\psi_{s,\gamma}+d_{jik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\psi_{s'}$$

$$+d^{*}_{ji\gamma}h^{r+s+1}\dot{\psi}_{s,\gamma}+d^{*}_{jik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\psi}_{s'}+p_{\gamma ji}h^{r+s+1}\tilde{\chi}_{s,\gamma}$$

$$+p_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}+p^{*}_{\gamma ji}h^{r+s+1}\dot{\tilde{\chi}}_{s,\gamma}+p^{*}_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\tilde{\chi}}_{s'}$$

$$+q_{\gamma ji}h^{r+s+1}\tilde{\eta}_{s,\gamma}+q_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}+q^{*}_{\gamma ji}h^{r+s+1}\dot{\tilde{\eta}}_{s,\gamma}$$

$$+q^{*}_{kji}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\tilde{\eta}}_{s'}\Big]+h^{r}\overset{r}{X}_{i}=\rho h^{r}\frac{\partial^{2}h^{r+1}v_{ir}}{\partial t^{2}}, \tag{2.39}$$

$$i=\overline{1,3},\quad r=\overline{0,N},$$

$$\frac{1}{2}\Big(d_{kl\alpha}h^{2r+1}v_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}h^{2r+1}v_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(d_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(d^{*}_{kl\alpha}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(d^{*}_{kl\alpha}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(d^{*}_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(d^{*}_{kl\alpha}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}$$

$$
\begin{aligned}
&+\Big(d_\alpha h^{2r+1}\psi_r\Big)_{,\alpha}+\Big(d^*_\alpha h^{2r+1}\dot{\psi}_r\Big)_{,\alpha}+\Big(\tilde{\alpha}_{\alpha k}h^{2r+1}\psi_{r,k}\Big)_{,\alpha}\\
&+\Big(\tilde{\alpha}_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\psi_s\Big)_{,\alpha}+\Big(\alpha^*_{\alpha k}h^{2r+1}\dot{\psi}_{r,k}\Big)_{,\alpha}+\Big(\alpha^*_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{\psi}_s\Big)_{,\alpha}\\
&+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}d_{kli}h^{r+s+1}\Big(v_{ks,l}+v_{ls,k}\Big)+\frac{1}{2}d_{kli}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'}\Big)\\
&+\frac{1}{2}d^*_{kli}h^{r+s+1}\Big(\dot{v}_{ks,l}+\dot{v}_{ls,k}\Big)+\frac{1}{2}d^*_{kli}\sum_{s'=s+1}^{N}h^{r+s'+1}\Big(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'}\Big)\\
&+d_ih^{r+s+1}\psi_s+d^*_ih^{r+s+1}\dot{\psi}_s+\tilde{\alpha}_{ik}\Big(h^{r+s+1}\psi_{s,k}+\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\psi_{s'}\Big)\\
&+\alpha^*_{ik}\Big(h^{r+s+1}\dot{\psi}_{s,k}+\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\dot{\psi}_{s'}\Big)\Big]\\
&-\tilde{b}_{ij}\Big[\frac{1}{2}h^{2r+1}\Big(v_{ir,j}+v_{jr,i}\Big)+\frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}v_{js}+\frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{js}h^{r+s+1}v_{is}\Big]\\
&-\tilde{\xi}h^{2r+1}\psi_r-d_i\Big(h^{2r+1}\psi_{r,i}+\sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}\psi_s\Big)\\
&-b^*_{ij}\Big[\frac{1}{2}h^{2r+1}\Big(\dot{v}_{ir,j}+\dot{v}_{jr,i}\Big)+\frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}\dot{v}_{js}+\frac{1}{2}\sum_{s=r+1}^{N}\overset{r}{b}_{js}h^{r+s+1}\dot{v}_{is}\Big]\\
&-\xi^*h^{2r+1}\dot{\psi}_r-d^*_i\Big(h^{2r+1}\dot{\psi}_{r,i}+\sum_{s=r+1}^{N}\overset{r}{b}_{is}h^{r+s+1}\dot{\psi}_s\Big)\\
&+h^r\overset{r}{H}=\rho k h^r\frac{\partial^2 h^{r+1}\psi_r}{\partial t^2},\quad r=\overline{0,N},
\end{aligned}
\tag{2.40}
$$

$$
\begin{aligned}
&\frac{1}{2}\Big(p_{\alpha kl}h^{2r+1}v_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}h^{2r+1}v_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}\\
&+\frac{1}{2}\Big(p_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha kl}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha kl}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}\\
&+\frac{1}{2}\Big(p^*_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks}\Big)_{,\alpha}\\
&-\Big(\varsigma_{\alpha\gamma}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha}-\Big(\varsigma_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_s\Big)_{,\alpha}\\
&-\Big(\tilde{a}_{\alpha\gamma}h^{2r+1}\tilde{\eta}_{r,\gamma}\Big)_{,\alpha}-\Big(\tilde{a}_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_s\Big)_{,\alpha}\\
&+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}h^{r+s+1}p_{ikl}(v_{ks,l}+v_{ls,k})+\frac{1}{2}p_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'})
\end{aligned}
$$

$$+\frac{1}{2}p^*_{ikl}h^{r+s+1}(\dot{v}_{ks,l}+\dot{v}_{ls,k})+\frac{1}{2}p^*_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'})$$

$$-\varsigma_{i\gamma}h^{r+s+1}\tilde{\chi}_{s,\gamma}-\varsigma_{ik}\textstyle\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}$$

$$+\tilde{a}_{i\gamma}h^{r+s+1}\tilde{\eta}_{s,\gamma}+\tilde{a}_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}\Big]+h^r\overset{r}{D}=h^r f_{er},\quad r=\overline{0,N}. \tag{2.41}$$

$$\frac{1}{2}\Big(q_{\alpha kl}h^{2r+1}v_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}h^{2r+1}v_{lr,k}\Big)_{,\alpha}+\frac{1}{2}\Big(q_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}v_{ls}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(q_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}v_{ks})\Big)_{,\alpha}+\frac{1}{2}\Big(q^*_{\alpha kl}h^{2r+1}\dot{v}_{kr,l}\Big)_{,\alpha}+\frac{1}{2}\Big(q^*_{\alpha kl}h^{2r+1}\dot{v}_{lr,k}\Big)_{,\alpha}$$

$$+\frac{1}{2}\Big(q^*_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\dot{v}_{ls}\Big)_{,\alpha}+\frac{1}{2}\Big(q^*_{\alpha kl}\sum_{s=r+1}^{N}\overset{r}{b}_{ls}h^{r+s+1}\dot{v}_{ks})\Big)_{,\alpha}$$

$$-\Big(\tilde{a}_{\alpha\gamma}h^{2r+1}\tilde{\chi}_{r,\gamma}\Big)_{,\alpha}-\Big(\tilde{a}_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\chi}_{s}\Big)_{,\alpha}$$

$$-\Big(\xi_{\alpha\gamma}h^{2r+1}\tilde{\eta}_{r,\gamma}\Big)_{,\alpha}-\Big(\xi_{\alpha k}\sum_{s=r+1}^{N}\overset{r}{b}_{ks}h^{r+s+1}\tilde{\eta}_{s}\Big)_{,\alpha}$$

$$+\sum_{s=0}^{r}\overset{r}{a}_{is}\Big[\frac{1}{2}q_{ikl}h^{r+s+1}(v_{ks,l}+v_{ls,k})+\frac{1}{2}q_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}v_{ls'}+\overset{s}{b}_{ls'}v_{ks'})$$

$$+\frac{1}{2}q^*_{ikl}h^{r+s+1}(\dot{v}_{ks,l}+\dot{v}_{ls,k})+\frac{1}{2}q^*_{ikl}\sum_{s'=s+1}^{N}h^{r+s'+1}(\overset{s}{b}_{ks'}\dot{v}_{ls'}+\overset{s}{b}_{ls'}\dot{v}_{ks'})$$

$$-\tilde{a}_{i\gamma}h^{r+s+1}\tilde{\chi}_{s,\gamma}-\tilde{a}_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\chi}_{s'}$$

$$-\xi_{i\gamma}h^{r+s+1}\tilde{\eta}_{s,\gamma}-\xi_{ik}\sum_{s'=s+1}^{N}\overset{s}{b}_{ks'}h^{r+s'+1}\tilde{\eta}_{s'}]+h^r\overset{r}{B}=0,\quad r=\overline{0,N}. \tag{2.42}$$

In the Nth approximation (hierarchical model):

$$(u_i,\varphi,\chi,\eta)(x_1,x_2,x_3,t)\cong\sum_{r=0}^{N}\frac{1}{h}\Big(r+\frac{1}{2}\Big)h^r\Big(\overset{N}{v}_{ir},\overset{N}{\psi}_r,\overset{N}{\tilde{\chi}}_r,\overset{N}{\tilde{\eta}}_r\Big)(x_1,x_2,x_3,t)P_r(ax_3-b),$$

where $\Big(\overset{N}{v}_{ir},\overset{N}{\psi}_r,\overset{N}{\tilde{\chi}}_r,\overset{N}{\tilde{\eta}}_r\Big)$ is a solution of the above system (2.39)-(2.42).

2.1.2 $N=0$ *approximation*

The governing system in $N=0$ approximations has the form (see also [6], page 42), as it directly follows from Subsection 2.1.1 for $N=0$,

$$X_{\alpha j0,\alpha}+\overset{0}{X}_j=\rho\frac{\partial^2 hv_{j0}}{\partial t^2},\quad j=\overline{1,3}, \tag{2.43}$$

$$H_{\alpha 0,\alpha}+H_{00}+\overset{0}{H}=\rho k\frac{\partial^2 h\psi_0}{\partial t^2}, \tag{2.44}$$

$$D_{\alpha 0,\alpha}+\overset{0}{D}=f_{e0}, \tag{2.45}$$

$$B_{\alpha 0,\alpha}+\overset{0}{B}=0, \tag{2.46}$$

$$v_{j0}=\frac{u_{j0}}{h},\ \ \psi_0=\frac{\varphi_0}{h};$$

$$X_{ij0}=\frac{1}{2}E_{ijkl}h\Big(v_{k0,l}+v_{l0,k}\Big)+\frac{1}{2}E^*_{ijkl}h\Big(\dot v_{k0,l}+\dot v_{l0,k}\Big)$$

$$+\tilde b_{ij}h\psi_0+b^*_{ij}h\dot\psi_0+d_{ij\gamma}h\psi_{0,\gamma}+d^*_{ij\gamma}h\dot\psi_{0,\gamma}+p_{\gamma ij}h\widetilde{\chi}_{r,\gamma}$$

$$+p^*_{\gamma ij}h\dot{\tilde\chi}_{0,\gamma}+q_{\gamma ij}h\widetilde{\eta}_{0,\gamma}+q^*_{\gamma ij}h\dot{\tilde\eta}_{0,\gamma},\quad i,j=\overline{1,3}, \tag{2.47}$$

$$\tilde\chi_0:=\frac{\chi_0}{h},\qquad \tilde\eta_0:=\frac{\eta_0}{h};$$

$$H_{j0}=\frac{1}{2}d_{klj}h(v_{k0,l}+v_{l0,k})+\frac{1}{2}d^*_{klj}h(\dot v_{k0,l}+\dot v_{l0,k})+d_jh\psi_0+d^*_jh\dot\psi_0$$

$$+\tilde\alpha_{ji}h\psi_{0,i}+\alpha^*_{ji}h\dot\psi_{0,i},\ \ j=\overline{1,3}, \tag{2.48}$$

$$H_{00}=-\frac{1}{2}\tilde b_{ij}h\Big(v_{i0,j}+v_{j0,i}\Big)-\tilde\xi h\psi_0-d_ih\psi_{0,i}-\frac{1}{2}b^*_{ij}h\Big(\dot v_{i0,j}+\dot v_{j0,i}\Big)-\xi^*h\dot\psi_0-d^*_ih\dot\psi_{0,i}; \tag{2.49}$$

$$D_{j0}=\frac{1}{2}p_{jkl}h(v_{k0,l}+v_{l0,k})+\frac{1}{2}p^*_{jkl}h(\dot v_{k0,l}+\dot v_{l0,k})+\varsigma_{j\gamma}h\widetilde{\chi}_{0,\gamma}+\tilde a_{j\gamma}h\widetilde{\eta}_{0,\gamma},\ \ j=\overline{1,3}; \tag{2.50}$$

$$B_{j0}=\frac{1}{2}q_{jkl}h(v_{k0,l}+v_{l0,k})+\frac{1}{2}q^*_{jkl}h(\dot v_{k0,l}+\dot v_{l0,k})+\tilde a_{j\gamma}h\widetilde{\chi}_{0,\gamma}+\xi_{j\gamma}h\widetilde{\eta}_{0,\gamma},\ \ j=\overline{1,3}; \tag{2.51}$$

$$e_{ij0}=\frac{1}{2}h\Big(v_{i0,j}+v_{j0,i}\Big),\ \ i,j=\overline{1,3}, \tag{2.52}$$

Substituting (2.47)-(2.51) into (2.43)-(2.46), respectively, we obtain the governing system of equations with respect to v_{i0}, ψ_0, $\widetilde{\alpha}_0$, $\widetilde{\eta}_0$:

$$\frac{1}{2}\Big(E_{\alpha ik\delta}hv_{k0,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(E_{\alpha i\gamma l}hv_{l0,\gamma}\Big)_{,\alpha}+\frac{1}{2}\Big(E^*_{\alpha ik\delta}h\dot v_{k0,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(E^*_{\alpha i\gamma l}h\dot v_{l0,\gamma}\Big)_{,\alpha}$$

$$+\tilde b_{\alpha i}h\psi_{0,\alpha}+\Big(\tilde b_{\alpha i}h\Big)_{,\alpha}\psi_0+b^*_{\alpha i}h\dot\psi_{0,\alpha}+\Big(b^*_{\alpha i}h\Big)_{,\alpha}\dot\psi_0+\Big(d_{\alpha i\gamma}h\psi_{0,\gamma}\Big)_{,\alpha}$$

$$+\Big(d^*_{\alpha i\gamma}h\dot\psi_{0,\gamma}\Big)_{,\alpha}+\Big(p_{\gamma\alpha i}h\tilde\chi_{0,\gamma}\Big)_{,\alpha}+\Big(p^*_{\gamma\alpha i}h\dot{\tilde\chi}_{0,\gamma}\Big)_{,\alpha}+\Big(q_{\gamma\alpha i}h\tilde\eta_{0,\gamma}\Big)_{,\alpha}$$

$$+\Big(q^*_{\gamma\alpha i}h\dot{\tilde\eta}_{0,\gamma}\Big)_{,\alpha}+\overset{0}{X}_i=\rho\frac{\partial^2 hv_{i0}}{\partial t^2},\quad i=\overline{1,3}, \tag{2.53}$$

$$\overset{0}{X}_i:=\overset{(+)}{X}_{3i}-\overset{(+)}{X}_{\alpha i}\overset{(+)}{h}_{,\alpha}-\overset{(-)}{X}_{3i}+\overset{(-)}{X}_{\alpha i}\overset{(+)}{h}_{,\alpha}+\Phi_{i0}=Q_{\overset{(+)}{n}i}$$

$$\frac{1}{2}\Big(d_{k\delta\alpha}hv_{k0,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(d_{\gamma l\alpha}hv_{l0,\gamma}\Big)_{,\alpha}+\frac{1}{2}\Big(d^*_{k\delta\alpha}h\dot{v}_{k0,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(d^*_{\gamma l\alpha}h\dot{v}_{l0,\gamma}\Big)_{,\alpha}$$
$$+\Big(d_\alpha h\Big)_{,\alpha}\psi_0+\Big(d^*_\alpha h\Big)_{,\alpha}\dot{\psi}_0+\Big(\tilde{\alpha}_{\alpha\delta}h\psi_{0,\delta}\Big)_{,\alpha}+\Big(\alpha^*_{\alpha\delta}h\dot{\psi}_{0,\delta}\Big)_{,\alpha}-\tilde{b}_{i\alpha}hv_{i0,\alpha}$$
$$-\tilde{\xi}h\psi_0-b^*_{i\alpha}h\dot{v}_{i0,\alpha}-\xi^*h\dot{\psi}_0+\overset{0}{H}=k\frac{\partial^2 h\psi_0}{\partial t^2}, \tag{2.54}$$

$$\frac{1}{2}\Big(p_{\alpha k\delta}hv_{k0,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(p_{\alpha\gamma l}hv_{l0,\gamma}\Big)_{,\alpha}+\frac{1}{2}\Big(p^*_{\alpha k\delta}h\dot{v}_{k0,\delta}\Big)_{,\alpha}$$
$$+\frac{1}{2}\Big(p^*_{\alpha\gamma l}h\dot{v}_{l0,\gamma}\Big)_{,\alpha}+\Big(\varsigma_{\alpha\gamma}h\tilde{X}_{0,\gamma}\Big)_{,\alpha}+\Big(\tilde{a}_{\alpha\gamma}h\tilde{\eta}_{0,\gamma}\Big)_{,\alpha}+\overset{0}{D}=f_{e0}, \tag{2.55}$$
$$\frac{1}{2}\Big(q_{\alpha k\delta}hv_{k0,\delta}\Big)_{,\alpha}+\frac{1}{2}\Big(q_{\alpha\gamma l}hv_{l0,\gamma}\Big)_{,\alpha}+\frac{1}{2}\Big(q^*_{\alpha k\delta}h\dot{v}_{k0,\delta}\Big)_{,\alpha}$$
$$+\frac{1}{2}\Big(q^*_{\alpha\gamma l}h\dot{v}_{l0,\gamma}\Big)_{,\alpha}+\Big(\tilde{a}_{\alpha\gamma}h\tilde{X}_{0,\gamma}\Big)_{,\alpha}+\Big(\xi_{\alpha\gamma}h\tilde{\eta}_{0,\gamma}\Big)_{,\alpha}+\overset{0}{B}=0. \tag{2.56}$$

Similarly, we may conctruct the governing system for the Nth approximation with respect to v_{ir}, ψ_r, $\widetilde{\xi}_r$, $\widetilde{\eta}_r$, $r=\overline{0,N}$, $i=\overline{1,3}$ (see [6], pp. 31-42 for details).

2.2 $N=0$ *approximation for porous isotropic elastic prismatic shells*

In the case under consideration, assuming the constitutive coefficients $\lambda := E_{1122} = E_{1133}$ and $\mu := \frac{1}{2}(E_{1111} - E_{1122})$ (the Lamé constants), $\tilde{\alpha}$, $\tilde{b}$, and $\tilde{\xi}$ to be constants[i] from (2.53)-(2.56) we get the following governing system (see also [6], pages 66, 67)

$$\mu\Big[(hv_{\alpha0,\beta})_{,\alpha}+(hv_{\beta0,\alpha})_{,\alpha}\Big]+\lambda(hv_{\gamma0,\gamma})_{,\beta}+\tilde{b}(h\psi_0)_{,\beta}+\overset{0}{X}_\beta=\rho h\ddot{v}_{\beta0},\ \beta=1,2; \tag{2.57}$$
$$\mu(hv_{30,\alpha})_{,\alpha}+\overset{0}{X}_3=\rho h\ddot{v}_{30}; \tag{2.58}$$
$$\tilde{\alpha}(h\psi_{0,\alpha})_{,\alpha}-\tilde{b}hv_{\gamma0,\gamma}-\tilde{\xi}h\psi_0+\overset{0}{H}=\rho h\ddot{\psi}_0-\mathcal{F}_0. \tag{2.59}$$

BCs for the weighted displacements and the weighted volume fraction are non-classical in the case of cusped prismatic shells. Namely, we are not always able to prescribe them at cusped edges.

Let ω be a domain bounded by a sufficiently smooth arc $(\partial\omega \setminus \overline{\gamma^0})$ lying in the half-plane $x_2 > 0$ and a segment $\overline{\gamma^0}$ of the x_1-axis $(x_2 = 0)$.

If the thickness looks like

$$2h(x_1,x_2)=h_0x_2^\kappa, \qquad h_0,\ \kappa=const>0, \tag{2.60}$$

then we can prescribe the displacements and volume fraction at the cusped edge $\overline{\gamma}_0$ if $\kappa < 1$, while we cannot do it if $\kappa \geq 1$.

Let us show it for the particular case of deformation when

$$v_{\alpha0}\equiv 0,\ \ \alpha=1,2;\ \ v_{30}\not\equiv 0.$$

[i]Clearly, $E_{1122} \equiv E_{1133} = \lambda$, $E_{1111} = \lambda + 2\mu$. Other elastic coefficients are equal to zero.

Then in the static case, taking into account (2.60), from (2.58), (2.59) we get

$$x_2 v_{30,\alpha\alpha} + \kappa v_{30,2} = 2(\mu h_0)^{-1} x_2^{1-\kappa} \overset{\circ}{X}_3, \tag{2.61}$$

$$x_2 \psi_{0,\alpha\alpha} + \kappa \psi_{0,2} - \xi \alpha^{-1} x_2 \psi_0 = -2(\tilde{\alpha} h_0)^{-1} x_2^{1-\kappa} \Big(\overset{\circ}{H} + \mathcal{F}_0 \Big), \tag{2.62}$$

respectively.

Problem D (Dirichlet Problem: Find solutions

$$v_{30}, \psi_0 \in C^2(\omega) \cap C(\bar{\omega})$$

of (2.61), (2.62) by their values prescribed on $\partial\omega$)

and

Problem E (Keldysh Problem: Find bounded solutions

$$v_{30}, \psi_0 \in C^2(\omega) \cap C(\omega \cup (\partial\omega \setminus \overline{\gamma^0}))$$

of (2.61), (2.62) by their values prescribed only on the arc $\partial\omega \setminus \overline{\gamma^0}$)

are uniquely solvable for equations (2.61), (2.62) by $\kappa < 1$ and $\kappa \geq 1$, correspondingly. It follows from

Theorem 2.1. (Jaiani, see [7], Section 3.9) *If the coefficients a_α, $\alpha = 1, 2$, and c of the equation*

$$x_2^{\kappa_\alpha} u,_{\alpha\alpha} + a_\alpha(x_1, x_2) u,_\alpha + c(x_1, x_2) u = 0, \quad c \leq 0, \quad \kappa_\alpha = const \geq 0, \quad \alpha = 1, 2,$$

are analytic in $\overline{\omega}$, then

(i) if either $\kappa_2 < 1$, or $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) < x_2^{\kappa_2 - 1} \tag{2.63}$$

in $\overline{\omega}_\delta$ for some $\delta = const > 0$, where

$$\omega_\delta := \{(x_1, x_2) \in \omega \ : \ 0 < x_2 < \delta\},$$

the Dirichlet problem (Problem D, $u \in C^2(\omega) \cap C(\bar{\omega})$) is well-posed;

(ii) if $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) \geq x_2^{\kappa_2 - 1} \tag{2.64}$$

in ω_δ and $a_1(x_1, x_2) = O(x_2^{\kappa_1})$, $x_2 \to 0_+$ (O is the Landau symbol), the Keldysh problem (Problem E, bounded $u \in C^2(\omega) \cap C(\bar{\omega} \setminus \gamma^0)$) is well-posed.

Indeed, from (2.63) and (2.64), it follows $a_2(x_1, x_2) = \kappa < 1$ for Problem D and $a_2(x_1, x_2) = \kappa \geq 1$ for Problem E, respectively, since $\kappa_1 = \kappa_2 = 1$, $a_1 \equiv 0$; in addition for (2.62) $c = -\xi \alpha^{-1} x_2 < 0$.

2.3 Transversely isotropic solids

Let us now consider the transversely isotropic elastic piezoelectric material in the case when the poling axis coincides with one of the material symmetry axes [22]. A material behavior is said to be transversely isotropic if it is invariant with respect to an arbitrary rotation about a given axis. This material behavior is of special importance in the modelling of fibre-reinforced composite materials with a coordinate axis in the fibre direction and assumed isotropic in cross-sections orthogonal to fibre direction [23] (in our case to poling axis as well, since in the case under consideration they coincide). The transverse isotropic model is also suitable for biological applications because it adequately describes the elastic properties of bundled fibers aligned in one direction (see [24], [25]).

It is well-known [22] that the electric field that develops in piezoelectrics can be assumed to be quasi-static because the velocity of the elastic waves is much smaller than the velocity of electromagnetic waves. Therefore, the magnetic field due to the elastic waves is negligible $\mathbf{B} \approx 0$. This fact implies that

$$\frac{\partial \mathbf{B}}{\partial t} \approx 0.$$

So one of Maxwell's equations of electrodynamics becomes

$$rot\, \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \approx 0$$

and, as it was already assumed,

$$\mathbf{E} = -grad\, \chi.$$

Consequently, considering transversely isotropic piezoelectric continuum, it will be based on the governing equations of elastodynamics in the care of small deformations and quasi-electrostatic fields. Note that piezoelectric materials show in most cases a crystal structure with a symmetry of hexagonal 6 mm class. In the case when the poling axis coincides with one of the material symmetry axes these materials become transversely isotropic.

Restricting to the case of time-harmonic motion with frequency o, i.e., all the sought quantities, s.c. free members of governing equations, and boundary data are represented as the products of e^{iot} and of the same quantities (to avoid redundant indices and symbols we leave the same notation) depending only on the space variables, from the governing equations of dynamics (2.3), (2.5), (2.6), (2.7), (2.8) we get the following governing equations

$$X_{ij,j} + \rho o^2 u_i = -\Phi_i, \quad i = \overline{1,3}; \tag{2.65}$$

$$D_{j,j} = f_e; \tag{2.66}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i,j = \overline{1,3};$$

$$E_i = -\chi_{,i}, \quad i = \overline{1,3};$$

$$\begin{pmatrix} X_{11} \\ X_{22} \\ X_{33} \\ X_{23} \\ X_{31} \\ X_{12} \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = C \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{2.67}$$

where (see [22])

$$C := \begin{pmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\ E_{1122} & E_{1111} & E_{1133} & 0 & 0 & 0 & 0 & 0 & p_{311} \\ E_{1133} & E_{1133} & E_{3333} & 0 & 0 & 0 & 0 & 0 & p_{333} \\ 0 & 0 & 0 & E_{2323} & 0 & 0 & 0 & p_{113} & 0 \\ 0 & 0 & 0 & 0 & E_{2323} & 0 & p_{113} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(E_{1111}-E_{1122}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{113} & 0 & -\varsigma_{11} & 0 & 0 \\ 0 & 0 & 0 & p_{113} & 0 & 0 & 0 & -\varsigma_{11} & 0 \\ p_{311} & p_{311} & p_{333} & 0 & 0 & 0 & 0 & 0 & -\varsigma_{33} \end{pmatrix} \tag{2.68}$$

From (2.67), (2.68) we have

$$\begin{gathered} X_{11} = E_{1111}e_{11} + E_{1122}e_{22} + E_{1133}e_{33} - p_{311}E_3, \\ X_{22} = E_{1122}e_{11} + E_{1111}e_{22} + E_{1133}e_{33} - p_{311}E_3, \\ X_{33} = E_{1133}e_{11} + E_{1133}e_{22} + E_{3333}e_{33} - p_{333}E_3, \\ X_{23} = 2E_{2323}e_{23} - p_{113}E_2, \quad X_{31} = 2E_{2323}e_{31} - p_{113}E_1, \\ X_{12} = (E_{1111} - E_{1122})e_{12}, \\ D_1 = 2p_{113}e_{13} + \varsigma_{11}E_1, \quad D_2 = 2p_{113}e_{23} + \varsigma_{11}E_2, \\ D_3 = p_{311}e_{11} + p_{311}e_{22} + p_{333}e_{33} + \varsigma_{33}E_3, \end{gathered}$$

i.e.,

$$\begin{gathered} X_{11} = E_{1111}u_{1,1} + E_{1122}u_{2,2} + E_{1133}u_{3,3} - p_{311}E_3, \\ X_{22} = E_{1122}u_{1,1} + E_{1111}u_{2,2} + E_{1133}u_{3,3} - p_{311}E_3, \\ X_{33} = E_{1133}u_{1,1} + E_{1133}u_{2,2} + E_{3333}u_{3,3} - p_{333}E_3, \\ X_{23} = E_{2323}(u_{2,3} + u_{3,2}) - p_{113}E_2, \quad X_{31} = E_{2323}(u_{3,1} + u_{1,3}) - p_{113}E_1, \\ X_{12} = \frac{1}{2}(E_{1111} - E_{1122})(u_{1,2} + u_{2,1}), \\ D_1 = p_{113}(u_{3,1} + u_{1,3}) + \varsigma_{11}E_1, \quad D_2 = p_{113}(u_{2,3} + u_{3,2}) + \varsigma_{11}E_2, \\ D_3 = p_{311}u_{1,1} + p_{311}u_{2,2} + p_{333}u_{3,3} + \varsigma_{33}E_3. \end{gathered} \tag{2.69}$$

Conditions of Anti-plane Piezoelectric State [22] have the form

$$\begin{array}{ll} 1. & u_1 \equiv 0, \quad u_2 \equiv 0, \quad u_3 \not\equiv 0; \\ 2. & X_{13} \not\equiv 0, \quad X_{23} \not\equiv 0; \quad X_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad X_{33} \equiv 0; \\ 3. & e_{13} \not\equiv 0, \quad e_{23} \not\equiv 0; \quad e_{\alpha\beta} \equiv 0, \quad \alpha, \beta = 1, 2; \quad e_{33} \equiv 0; \\ 4. & E_1 \not\equiv 0, \quad E_2 \not\equiv 0, \quad E_3 \equiv 0; \\ 5. & D_1 \not\equiv 0, \quad D_2 \not\equiv 0, \quad D_3 \equiv 0. \end{array} \tag{2.70}$$

Taking into account (2.70), from the first three relations of (2.69) we have

$$u_{3,3} \equiv 0, \quad u_3 = u_3(x_1, x_2);$$

the fourth and fifth relations of (2.69) give

$$X_{23} = E_{2323} u_{3,2} - p_{113} E_2, \tag{2.71}$$
$$X_{31} = E_{2323} u_{3,1} - p_{113} E_1, \tag{2.72}$$

respectively;
the sixth of (2.69) is identically fulfilled;
the seventh and eighth relations of (2.69) give

$$D_1 = p_{113} u_{3,1} + \varsigma_{11} E_1, \tag{2.73}$$
$$D_2 = p_{113} u_{3,2} + \varsigma_{11} E_2. \tag{2.74}$$

respectively;
the ninth of (2.69) is identically fulfilled.
From the first two of (2.65) it follows that

$$\Phi_\alpha \equiv 0, \ \ \alpha = 1, 2, \tag{2.75}$$

the third of (2.65) will have the form

$$X_{31,1} + X_{32,2} + \rho o^2 u_3 = -\Phi_3; \tag{2.76}$$

while (2.66) will have the form

$$D_{1,1} + D_{2,2} = f_e. \tag{2.77}$$

Substituting (2.71) and (2.72) into (2.76) and (2.73) and (2.74) into (2.77) we get

$$(E_{2323} u_{3,1})_{,1} + (E_{2323} u_{3,2})_{,2} - (p_{113} E_1)_{,1} - (p_{113} E_2)_{,2} + \rho o^2 u_3 = -\Phi_3,$$

and

$$(p_{113} u_{3,1})_{,1} + (p_{113} u_{3,2})_{,2} + (\varsigma_{11} E_1)_{,1} + (\varsigma_{11} E_2)_{,2} = f_e,$$

respectively.
Taking into account

$$E_\alpha = -\chi_{,\alpha}, \quad \alpha = 1, 2.$$

We obtain the following governing equations in the anti-plane piezoelectric state

$$(E_{2323} u_{3,1})_{,1} + (E_{2323} u_{3,2})_{,2} + (p_{113} \chi_{,1})_{,1} + (p_{113} \chi_{,2})_{,2} + \rho o^2 u_3 = -\Phi_3,$$
$$(p_{113} u_{3,1})_{,1} + (p_{113} u_{3,2})_{,2} - (\varsigma_{11} \chi_{,1})_{,1} - (\varsigma_{11} \chi_{,2})_{,2} = f_e,$$

i.e.,

$$(E_{2323} u_{3,\alpha})_{,\alpha} + (p_{113} \chi_{,\alpha})_{,\alpha} + \rho o^2 u_3 = -\Phi_3, \tag{2.78}$$
$$(p_{113} u_{3,\alpha})_{,\alpha} - (\varsigma_{11} \chi_{,\alpha})_{,\alpha} = f_e. \tag{2.79}$$

Let the plane domain of interest have the form given in subsection 2.2 and let

$$E_{2323} = E_0 x_2^{\kappa_1}, \quad E_0 = const > 0, \quad \kappa_1 = const \geq 0;$$
$$p_{113} = p_0 x_2^{\kappa_2}, \quad p_0 = const > 0, \quad \kappa_2 = const \geq 0;$$
$$\varsigma_{11} = \varsigma_0 x_2^{\kappa_3}, \quad \varsigma_0 = const > 0, \quad \kappa_3 = const \geq 0,$$

then (2.78) and (2.79) take the forms

$$E_0(x_2^{\kappa_1} u_{3,\alpha})_{,\alpha} + p_0(x_2^{\kappa_2} \chi_{,\alpha})_{,\alpha} + \rho o^2 u_3 = -\Phi_3, \tag{2.80}$$

and

$$p_0(x_2^{\kappa_2} u_{3,\alpha})_{,\alpha} - \varsigma_0(x_2^{\kappa_3} \chi_{,\alpha})_{,\alpha} = f_e, \tag{2.81}$$

respectively.

Case 1. $\kappa_i = \kappa = const \geq 0$, $i = \overline{1,3}$.

After some actions, from (2.80) and (2.81) we get

$$(\varsigma_0 E_0 + p_0^2)(x_2^{\kappa} u_{3,\alpha})_{,\alpha} + \varsigma_0 \rho o^2 u_3 = -\varsigma_0 \Phi_3 + p_0 f_e, \tag{2.82}$$

and

$$(p_0^2 + \varsigma_0 E_0)(x_2^{\kappa} \chi_{,\alpha})_{,\alpha} + p_0 \rho o^2 u_3 = -p_0 \Phi_3 - E_0 f_e. \tag{2.83}$$

(2.82) and (2.83) we rewrite as

$$\begin{aligned} &x_2 u_{3,\alpha\alpha} + \kappa u_{3,2} + \varsigma_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho o^2 u_3 \\ &= (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-\varsigma_0 \Phi_3 + p_0 f_e), \end{aligned} \tag{2.84}$$

and

$$\begin{aligned} &x_2 \chi_{,\alpha\alpha} + \kappa \chi_{,2} + p_0(\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa} \rho o^2 u_3 \\ &= (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \end{aligned} \tag{2.85}$$

respectively.

In the static case $o = 0$ and from (2.84), (2.85) we obtain separate equations

$$x_2 u_{3,\alpha\alpha} + \kappa u_{3,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-\varsigma_0 \Phi_3 + p_0 f_e) \tag{2.86}$$
$$x_2 \chi_{,\alpha\alpha} + \kappa \chi_{,2} = (\varsigma_0 E_0 + p_0^2)^{-1} x_2^{1-\kappa}(-p_0 \Phi_3 - E_0 f_e), \tag{2.87}$$

with respect to u_3 and χ, correspondingly.

Theorem 2.2. *The values of u_3 and χ should be prescribed on the entire boundary (Problem D) for $\kappa < 1$, while on the part of the boundary, where $x_2 = 0$, should be freed at all of boundary conditions (Problem E) for $\kappa \geq 1$. Both problems are uniquely solvable in the classical sense.*

Proof. Indeed, for $\kappa < 1$ and $\kappa \geq 1$, correspondingly, (2.63) and (2.64) are fulfilled, which proves the theorem. □

Remark 2.3. If $\bar{\omega}$ is a stripe $\{-\infty < x_1 < +\infty,\ 0 \leq x_2 \leq L = const\}$ and all the quantities depend only on x_2 (it means that we consider cylindrical strain) then in the static case ($o = 0$) from (2.82) and (2.83) we obtain

$$(x_2^{\kappa} u_{3,2})_{,2} = (\varsigma_0 E_0 + p_0^2)^{-1}(-\varsigma_0 \Phi_3 + p_0 f_e)$$

and

$$(x_2^{\kappa}\chi_{,2})_{,2}=(\varsigma_0 E_0+p_0^2)^{-1}(-p_0\Phi_3-E_0 f_e),$$

respectively. Their general solutions have the forms

$$u_3(x_2)=(\varsigma_0 E_0+p_0^2)^{-1}\int_L^{x_2}\frac{d\tau}{\tau^{\kappa}}\int_L^{\xi}[-\varsigma_0\Phi_3(t)+p_0 f_e(t)]dt+c_2^1$$

$$+c_1^1\begin{cases}(1-\kappa)^{-1}(x_2^{1-\kappa}-L^{1-\kappa}) & \text{for} \quad \kappa\neq 1,\\ \ln x_2-\ln L & \text{for} \quad \kappa=1\end{cases}$$

and

$$\chi(x_2)=(\varsigma_0 E_0+p_0^2)^{-1}\int_L^{x_2}\frac{d\tau}{\tau^{\kappa}}\int_L^{\xi}[-p_0\Phi_3(t)-E_0 f_e(t)]dt+c_2^2$$

$$+c_1^2\begin{cases}(1-\kappa)^{-1}(x_2^{1-\kappa}-L^{1-\kappa}) & \text{for} \quad \kappa\neq 1,\\ \ln x_2-\ln L & \text{for} \quad \kappa=1.\end{cases}$$

In the case under consideration BCs look like

$$u_3(0)=c_0^1,\ \chi(0)=c_0^2;\ \ u_3(L)=c_L^1,\ \chi(L)=c_L^2 \text{ (Problem D)};$$
$$u_3(x_2)=O(1),\ \chi(x_2)=O(1),\ x_2\to 0+;\ \ u_3(L)=c_L^1,\ \chi(L)=c_L^2 \text{ (Problem E)}.$$

From these BCs we easily calculate constants

$$c_{\beta}^{\alpha},\ \ \alpha,\beta=1,2,\ \ \text{for}\ \ \kappa<1\ \ \text{(Problem D)}$$

and

$$c_2^{\alpha},\ \ \alpha=1,2,\ \ \text{for}\ \ \kappa\geq 1\ \ \text{(Problem E)},$$

in the last case $c_1^{\alpha}=0$, $\alpha=1,2$, (otherwise solutions will be unbounded) and some restrictions on $\Phi_3(x_2)$, $f_e(x_2)$ are required as well.

Case 2. $\kappa_2=\kappa_3=\kappa=const\geq 0$.

After some actions, from (2.80) and (2.81) we get

$$((p_0^2 x_2^{\kappa}+\varsigma_0 E_0 x_2^{\kappa_1})u_{3,\alpha})_{,\alpha}+\varsigma_0\rho o^2 u_3=-\varsigma_0\Phi_3+p_0 f_e, \tag{2.88}$$

$$\varsigma_0(x_2^{\kappa}\chi_{,\alpha})_{,\alpha}=p_0(x_2^{\kappa}u_{3,\alpha})_{,\alpha}-f_e. \tag{2.89}$$

So, for $\kappa_1=0$ and any $\kappa\geq 0$, equation (2.88) is not a degenerate one, while equation (2.89) is a degenerate one. If $o=0$, i.e., we deal with the static case and from (2.88), (2.89) we arrive at the system

$$((p_0^2 x_2^{\kappa}+\varsigma_0 E_0)u_{3,\alpha})_{,\alpha}=-\varsigma_0\Phi_3+p_0 f_e, \tag{2.90}$$

$$\varsigma_0(x_2^{\kappa}\chi_{,\alpha})_{,\alpha}=p_0(x_2^{\kappa}u_{3,\alpha})_{,\alpha}-f_e. \tag{2.91}$$

As (2.90) is not a degenerate equation, the values of u_3 should be prescribed on the entire boundary (Problem D), while, according to Theorem 2.1, the values of χ should be prescribed on the entire boundary (Problem D) for $0\leq\kappa<1$ and the part where $x_2=0$ should be freed of BCs (Problem E) for $\kappa\geq 1$. It will be clear if we rewrite (2.91) in the following form

$$x_2\chi_{,\alpha\alpha}+\kappa\chi_{,2}=\varsigma_0^{-1}[p_0(x_2^{\kappa}u_{3,\alpha})_{,\alpha}-f_e]x_2^{1-\kappa}.$$

Indeed, for $\kappa<1$ and $\kappa\geq 1$, correspondingly, (2.63) and (2.64) are realized. So we have proved the following

Theorem 2.4. *Problem D for equation (2.90) for all $\kappa \geq 0$ is uniquely solvable in the classical sense. Problem D for $0 \leq \kappa < 1$ and Problem E for $\kappa \geq 1$ for equation (2.91) are uniquele solvable in the classical sense. In other words Problem D for system (2.90), (2.91) has a unique classical solution, while Problem E has a unique classical solution for $\kappa \geq 1$.*

Remark 2.5. Similarly to Case 1 we solve BVPs in the explicit form in the case of cylindrical strain.

2.4 Conclusions

1. Differential hierarchical models for piezoelectric nonhomogeneous viscoelastic Kelvin-Voigt prismatic shells with voids are constructed. The ways of investigation of boundary value problems and initial boundary value problems, including the case of cusped prismatic shells are indicated and some preliminary results are presented.

2. It is shown that in the case of hierarchical models of cusped prismatic shells, depending on the character of vanishing of the thickness at the lateral boundary of the prismatic shell, for well-posedness of the boundary value and initial boundary value problems the setting of boundary conditions is nonclassical, in general. Namely, in the case of nonclassical setting of boundary conditions they should be either weighted ones or the cusped edge should be freed from boundary conditions. In other words, at cusped edges: in the case of piezoelectric viscoelastic materials the displacements, volume fraction, and electric potential cannot always be prescribed.

3. If either elastic, piezoelectric, and dielectric constitutive coefficients are independent of the space points while the thickness of the prismatic shell vanishes in some way at some part of the boundary of the prismatic shell or the thickness of the prismatic shell is constant while the elastic, piezoelectric, and dielectric constitutive coefficients vanish in the same way at the same part of the boundary of the prismatic shell, then peculiarities of setting the boundary conditions for the displacement in the first case and those arising for the volume fraction function and the electric potential in the second case coincide. The stress-strain states coincide as well.

4. Antiplane deformation of piezoelectric nonhomogeneous transversely isotropic materials in the three-dimensional formulation and in $N = 0$ approximation is analysed. Some boundary value problems are solved in explicit forms in concrete cases.

3 Hierarchical models for the thermoelastic deformation of chiral porous prismatic shells

Applying I. Vekua's dimension reduction method, the present paper is devoted to construction of hierarchical models for thermoelastic deformation of chiral porous prismatic shells. Special attention is paid to the case, when the prismatic shell considered as a 3D body occupies a spatial angular domain and to the study of consequent mathematical and physical peculiarities, since by dimension reduction the geometrical 3D singularity will be transferred to the BVPs for governing singular partial differential equations and exclusiveness of well-posedeness of BVPs will be needed to be investigated. For field equations we use the strain gradient theory (see [9], chapter 14). Note that, in contrast to the case of chiral materials, the thermal field in chiral cylinders produces torsional effects.

The field equations of 3D model

i) geometric (kinematic) equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varkappa_{ijk} = u_{k,ij}; \tag{3.1}$$

ii) constitutive equations (relations)

$$\begin{aligned} X_{ij} &= \lambda e_{nn}\delta_{ij} + 2\mu e_{ij} + d\varphi\delta_{ij} + f(\mathcal{E}_{ikm}\varkappa_{jkm} + \mathcal{E}_{jkm}\varkappa_{ikm}) - bT\delta_{ij} \\ \mu_{ijk} &= \frac{1}{2}\alpha_1(\varkappa_{nni}\delta_{jk} + \varkappa_{nnj}\delta_{ik} + 2\varkappa_{knn}\delta_{ij}) + \alpha_2(\varkappa_{inn}\delta_{jk} + \varkappa_{jnn}\delta_{ik}) \\ &\quad + 2\alpha_3\varkappa_{nnk}\delta_{ij} + \beta_1\delta_{ij}\varphi_{,k} + \beta_2(\delta_{ik}\varphi_{,j} + \delta_{jk}\varphi_{,i}) + 2\alpha_4\varkappa_{ijk} \\ &\quad + \alpha_5(\varkappa_{kji} + \varkappa_{kij}) + f(\mathcal{E}_{ikn}e_{jn} + \mathcal{E}_{jkn}e_{in}); \\ H_i &= \beta_1\varkappa_{nni} + 2\beta_2\varkappa_{inn} + \alpha_0\varphi_{,i}, \quad g = de_{nn} + \xi\varphi - \beta T; \end{aligned} \tag{3.2}$$

iii) equilibrium (motion) equations

$$X_{ji,j} - \mu_{kji,kj} + \Phi_i = 0\Big(\frac{du}{dt}\Big) \quad H_{i,j} + g + l = 0(k\ddot{\varphi}), \quad i = \overline{1,3}, \tag{3.3}$$

where e_{ij} is the strain tensor and $\varkappa_{ijk}$ is the strain gradient tensor, Φ_i are the volume force vector components, X_{ij} is the stress tensor, μ_{ijk} is the dipolar stress tensor, φ is the change of the volume fraction function from the matrix reference volume fraction, k is the equilibrated inertia, H_i are the equilibrated stress vector components g, and l are the intrinsic and extrinsic equilibrated volume forces, T is temperature change, δ_{ik} is the Kroneker delta $\mathcal{E}_{ijk}$ is the Levy-Civita symbol, λ, μ and b are constitutive constants of the classical theory of thermoelasticity; α_i, $i = \overline{1,5}$ and β_j, $j = 1,2$, are constitutive constants associated with the gradient terms, d, α_0, ξ and β are the constitutive constants linked to porosity and f is the constant associated with chiral behavior

On the lateral boundary Σ_L either the fractions or displacements are given:

$$X_{ni}(x_1, x_2, x_3)|_{\Sigma_L} = F_i(x_1, x_2, x_3), \tag{3.4}$$

$$u_i(x_1, x_2, x_3)|_{\Sigma_L} = \varphi_i(x_1, x_2, x_3). \tag{3.5}$$

This boundary conditions on Σ_L can be replaced by ripermissible mixed triples of components of $\vec{X}_n$ and $\vec{u}$.

On the face surfaces $\overset{(\pm)}{\Sigma}_F$ either stress or displacement vectors are prescribed:

$$\vec{X}_n(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2)) = \overset{(\pm)}{X}_n(x_1, x_2), \tag{3.6}$$

$$\vec{u}_i(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2)) = \overset{(\pm)}{u}(x_1, x_2), \quad (x_1, x_2) \in \omega, \tag{3.7}$$

n being the unit vector of external normal to $\partial\Omega$ and $F \equiv X_n$ the surface density of the surface forces applied at the point $x \in \partial\Omega$ considered (more precisely exerted on a surface element passing through x normal to n).

Herewith on $\overset{\pm}{\sum}_F$ we replace, correspondingly values of $\vec{u}_i(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2))$ in the case (3.6) and $X_n(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2))$ in the case (3.7) by the values of the Fourier-Legendre expansions of $\vec{u}_i(x_1,x_2,x_3)$) and $X_n(x_1,x_2,x_3)$ for $x_3 \in [\overset{(-)}{h}(x_1,x_2),\ \overset{(+)}{h}(x_1,x_2)$, on upper and lower face surfaces $\overset{(\pm)}{\Sigma}_F$

$$x_3 = \overset{(+)}{h}(x_1,x_2),\ and\ x_3 = \overset{(-)}{h}(x_1,x_2)], \tag{3.8}$$

respectively.

3.1 2D problem

Now, we reformulate BVP (3.1)-(3.4), (3.6) in terms of the mathematical moments:

$$(u_{ir}, X_{ijr} \equiv \tau_{ijr}, e_{ijr}, \varkappa_{ijkr}, \mu_{ijkr}, g_r, \varphi_r, T_r, v_{ir}, \phi_{ir} \equiv X_{ir}, l_r, X_{nir}, H_{ir}) := \Big(r + \frac{1}{2}\Big)a$$

$$\times \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} (u_i, X_{ij} \equiv \tau_{ij}, e_{ij}, \varkappa_{ijk}, \mu_{ijk}, g, \varphi, T, v_i, \phi_i \equiv X_i, l, X_{ni}, H_i)\, P_r(ax_3 - b)dx_3, \tag{3.9}$$

$$a(x_1,x_2) := \frac{1}{h(x_1,x_2)},\ \ b(x_1,x_2) := \frac{\widetilde{h}(x_1,x_2)}{h(x_1,x_2)},$$
$$i,j,k = \overline{1,3},\ r = 0,1,2,\cdots,N,\cdots.$$

Under the well-known restrictions (see e.g. [1]) the following Fourier-Legendre series

$$\begin{aligned}(u_i, X_{ij} \equiv \tau_{ij}, e_{ij}, \varkappa_{ijk}, \mu_{ijk}, g, \varphi, T, v_i, \phi_i \equiv X_i, l, X_{ni}, H_i)\,(x_1,x_2,x_3,t)\\ = \sum_{r=0}^{\infty} a(r+\frac{1}{2})\,(u_{ir}, X_{ijr} \equiv \tau_{ijr}, e_{ij}, \varkappa_{ijkr}, \mu_{ijkr},\\ v_{ir}, \phi_{ir} \equiv X_{ir}, l_r, X_{nir}, H_{ir})\,(x_1,x_2,t)P_r(ax_3 - b)\end{aligned} \tag{3.10}$$

are convergent. The Nth, $N = 0,1,2,\cdots$ approximation of hierarchical models means that in equations (relations) all the moments of the order greater than N equal zero and we get $4N+4$ governing (basic) equations (relations) with respect to $4N+4$ unknown moments $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ of the order $r \le N$. Hence, in the formulas derived below in the infinite sums the limit "∞" should be replaced by "N".

In the Nth approximation, e.g.,

$$(\varphi, u_i)(x_1,x_2,x_3,t) \cong \sum_{s=0}^{N} a\left(s + \frac{1}{2}\right)(\overset{N}{\varphi}_s, \overset{N}{u}_{is})(x_1,x_2,t)P_s(ax_3 - b),\ i = \overline{1,3}. \tag{3.11}$$

In particular, in the $N = 0$ (zeroth) approximation

$$(\varphi, u_i)(x_1,x_2,x_3,t) \cong \frac{1}{2h(x_1,x_2)}(\overset{0}{\varphi}_0, \overset{0}{u}_{i0})(x_1,x_2,t) =: \frac{1}{2}(\overset{0}{\psi}_0, \overset{0}{v}_{i0})(x_1,x_2,t), \tag{3.12}$$

$$\overset{N}{\psi}_r := \frac{\overset{r}{\varphi}_N}{h^{r+1}} \tag{3.13}$$

and in the $N=1$ i.e. first approximation

$$\begin{aligned}
&(\phi, u_i(x_1,x_2,x_3,t))\\
&\cong \frac{1}{2h(x_1,x_2)}(\overset{1}{\varphi}_0,\overset{1}{u}_{i0})(x_1,x_2,t)) + \frac{3}{2h^2(x_1,x_2)}(\overset{1}{\varphi}_1,\overset{1}{u}_{i1})(x_1,x_2,t)(x_3-\tilde{h})\\
&=: \frac{1}{2}(\overset{1}{\psi}_0,\overset{1}{v}_{i0})(x_1,x_2,t) + \frac{3}{2}(x_3-\tilde{h})(\overset{1}{\psi}_1,\overset{1}{v}_{i0})(x_1,x_2,t). \qquad (3.14)
\end{aligned}$$

To this end we multiply (3.1)-(3.4), (3.6) by $P_r(ax_3-b)$ and then integrate the obtained with respect to the thickness variable x_3 within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$. In this way:

From $(3.1)_1$ we get (see [6] formula (3.8) and also (3.25) below)

$$e_{jir} = \frac{1}{2}(u_{ir,j}+u_{jr,i}) + \frac{1}{2}\sum_{i=r}^{\infty}(\overset{r}{b}_{js}u_{is}+\overset{r}{b}_{is}u_{js}),\ i,j=\overline{1,3},\ \ r=0,1,\cdots, \qquad (3.15)$$

where

$$\overset{r}{b}_{\alpha s} := -\overset{r}{a}_{\alpha s},\ \ s\neq r,\ \alpha=1,2,\ s,r=0,1,2,\cdots; \qquad (3.16)$$

$$\underset{\alpha s}{\overset{r}{a}} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha}-(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h},\ \ s\neq r, \qquad (3.17)$$

$$\overset{r}{a}_{\alpha r} := r\frac{h,a}{h},\ \ \overset{r*}{a}_{\alpha r} := (2r+1)\frac{h,a}{h}; \qquad (3.18)$$

$$\overset{r}{b}_{\alpha r} := a^r_{\alpha r} - \overset{r*}{a}_{\alpha r} = -(r+1)\frac{h,a}{h}; \qquad (3.19)$$

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h} \Rightarrow \overset{r}{a}_{3r}=0, \qquad (3.20)$$

$$\overset{r}{b}_{3s} := -\overset{r}{a}_{3s} \Rightarrow \overset{r}{b}_{3r}=0, \qquad (3.21)$$

i.e. {see [6], deduction of Formula (3.9)} from (3.15) it follows

$$e_{jir} = \frac{1}{2}h^{r+1}(v_{jr,i}+v_{ir,j}) + \frac{1}{2}\sum_{s=r+1}^{\infty} h^{s+1}\left(\overset{r}{b}_{js}v_{is}+\overset{r}{b}_{is}v_{js}\right), \qquad (3.22)$$

$$i,j=\overline{1,3},\ r=0,1,\ldots,$$

$$\Theta_r = e_{iir} = h^{r+1}v_{ir,i} + \sum_{s=r+1}^{\infty} h^{s+1}\overset{r}{b}_{is}v_{is}, \qquad (3.23)$$

where

$$v_{ir} := h^{-r-1}u_{ir},\ \ i=\overline{1,3},\ r=0,1,\ldots \qquad (3.24)$$

which are derived, when $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are not known (prescribed) on the face surfaces, $\overset{(+)}{\Sigma}_F$ and $\overset{(-)}{\Sigma}_F$.

The formulas (10.11) and (10.12) of [6], i.e. (2.90) and (2.91) of [7] which are derived, when $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are not known (prescribed) on the face surfaces $\overset{(+)}{\Sigma}_F$ and $\overset{(-)}{\Sigma}_F$. we may rewrite in the unified form as follows

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} f_{,j} P_r(ax_3-b)dx_3 = f_{r,j} + \sum_{s=r}^{\infty} \overset{r}{b}_{js} f_s. \tag{3.25}$$

If $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are known (prescribed) on the face surfaces $\overset{(+)}{\Sigma}_F$ and $\overset{(-)}{\Sigma}_F$, the formulas (10.14), (10.13) of [6] and (2.92), (2.93) of [7]) we may rewrite in the unified form as follows

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} f_{,j} P_r(ax_3-b)dx_3 = f_{r,j} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s + \overset{(+)}{f}\overset{(+)}{n}_j \overset{(+)}{\sqrt{\ }} + \overset{(-)}{f}\overset{(-)}{n}_j \overset{(-)}{\sqrt{\ }}, \tag{3.26}$$

$$\overset{(\pm)}{n}_\alpha = -\overset{(\pm)}{h}_{,\alpha} \left(\overset{(\pm)}{\sqrt{\ }}\right)^{-1},$$

$$\overset{(\pm)}{n}_3 = \left(\overset{(\pm)}{\sqrt{\ }}\right)^{-1}, \overset{(\pm)}{\sqrt{\ }} := \pm\sqrt{1 + \left(\overset{(\pm)}{h}_{,1}\right)^2 + \left(\overset{(\pm)}{h}_{,2}\right)^2}. \tag{3.27}$$

Similarly, using twice (3.25), From $(3.1)_2$ we get

$$\varkappa_{ijkr} := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \varkappa_{ijk} P_r(ax_3-b|dx_3) = u_{kr,ij} + \sum_{s=r}^{\infty} \left(\overset{r}{b}_{is} u_{ks,j} + \overset{r}{b}_{js} u_{ks,i}\right)$$

$$+ \sum_{s=r}^{\infty} \left[\overset{r}{b}_{is,j} u_{ks} + \overset{r}{b}_{js} \sum_{s'=s}^{\infty} \overset{s}{b}_{is'} u_{ks'}\right], \ i,j,k = \overline{1,3}. \tag{3.28}$$

Taking into account (3.24), from (3.28) we get

$$\varkappa_{jikr} \equiv \varkappa_{ijkr} = (h^{r+1} \overset{r+1}{v}_{kr})_{,ij} + \sum_{s=r}^{\infty} \left[\overset{r}{b}_{is}(h^{s+1} v_{ks})_{,j} + \overset{r}{b}_{js}(h^{s+1} v_{ks})_{,i}\right]$$

$$+ \sum_{s=r}^{\infty} \overset{r}{b}_{is,j} h^{s+1} v_{ks} + \sum_{s=r}^{\infty} \left[\overset{r}{b}_{js} \sum_{s'=s}^{\infty} \overset{s}{b}_{is'} h^{s'+1} v_{ks'}\right]. \tag{3.29}$$

Proof of (3.28) Clearly, using (3.25) more precisely, substituting there f by $u_{k,i}$, from

$$\varkappa_{ijk} = u_{k,ij}$$

we have

$$\varkappa_{ijkr} := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \varkappa_{ijk} P_r(ax_3-b|dx_3) = \int\limits_{\overset{(-)}{h}(x_1 x_2)}^{\overset{(+)}{h}(x_1 x_2)} u_{k,ij} P_r(ax_3-b)dx_3$$

$$= \int\limits_{\overset{(-)}{h}(x_1x_2)}^{\overset{(+)}{h}(x_1x_2)} (u_{k,i})_{,j}\, P_r(ax_3-b)dx_3 = (u_{k,i})_{r,j} + \sum_{s=r}^{\infty} \overset{r}{b}_{js}(u_{k,i})_s. \tag{3.30}$$

Clearly, applying (3.10) for $u_{k,i}$and bearing in mind $P_r(ax_3-b)\Big|_{x_3=\overset{(\pm)}{h}(x_1x_2)} = P_s(\pm 1) = (\pm 1)^s$, we obtain

$$\overset{(\pm)}{u}_{k,l} = \sum_{s=0}^{\infty} \left(s+\frac{1}{2}\right)(u_{k,i})_s(\pm 1)^s = \sum_{s=0}^{\infty} \frac{(\pm)^s(2s+1)}{2h}(u_k,i)_s.$$

The last formula had been taken into account by us while deriving (3.30).

Now, using once more (3.25), replacing there f by u_k, we get

$$(u_{k,i})_r := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} u_{k,i}P_r(ax_3-b)dx_3, = u_{kr,i} + \sum_{s=r}^{\infty} \overset{r}{b}_{is}u_{ks}, \tag{3.31}$$

clearly, here we have used

$$\overset{(\pm)}{u_k} = u_k(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)) = \sum_{s=0}^{\infty} \frac{(\pm 1)^s(2s+1)}{2h}\psi_{ks}.$$

Hence, if we substitute (3.31) into (3.30), we obtain

$$\varkappa_{ijkr} = u_{kr,ij} + \sum_{s=r}^{\infty}(\overset{r}{b}_{is,j}u_{ks} + \overset{r}{b}_{is}u_{ks,j}) + \sum_{s=r}^{\infty}[\overset{r}{b}_{js}(u_{ks,i} + \sum_{s'=s}^{\infty} \overset{s}{b}_{is'}u_{k,s'})]. \tag{3.32}$$

From (3.32) it follows (3.28), when we have in the sum with the summation index s the limit r to replace by s, in practical use we consider useful summation index s to replace by s' it was easy for us in order to avoid confusing tremendous variety of letters as indices.

Further, from (3.2) we get [see (3.22)]

$$X_{jir} = \lambda e_{nnr}\delta_{ij} + 2\mu e_{ijr} + d\varphi_r\delta_{ij} + f(\mathcal{E}_{ikm}\varkappa_{jkmr} + \mathcal{E}_{jkm}\varkappa_{ikmr}) - bT_r\delta_{ij}, \quad i,j=\overline{1,3}, \quad r=0,1,2,\cdots, \tag{3.33}$$

$$\mu_{jikr} = \frac{1}{2}\alpha_1(\varkappa_{nnir}\delta_{jk} + \varkappa_{nnjr}\delta_{ik} + 2\varkappa_{knnr}\delta_{ij}) + \alpha_2(\varkappa_{innr}\delta_{jk} + \varkappa_{jnnr}\delta_{ik})^{\text{ii}}$$

ii where

$$\begin{aligned}\mathcal{E}_{ikm}\varkappa_{jkmr} &= \mathcal{E}_{123}\varkappa_{j23} + \mathcal{E}_{132}\varkappa_{j32} + \mathcal{E}_{213}\varkappa_{j13} + \mathcal{E}_{231}\varkappa_{j31} + \mathcal{E}_{312}\varkappa_{j12} + \mathcal{E}_{321}\varkappa_{j21} \\ &= \varkappa_{j23} - \varkappa_{j32} - \varkappa_{j13} + \varkappa_{j31} + \varkappa_{j12} - \varkappa_{j21},\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{jkm}\varkappa_{ikmr} &= \mathcal{E}_{123}\varkappa_{i23} - \mathcal{E}_{132}\varkappa_{i32} - \mathcal{E}_{213}\varkappa_{i13} + \mathcal{E}_{231}\varkappa_{i31} + \mathcal{E}_{312}\varkappa_{i12} - \mathcal{E}_{321}\varkappa_{i21} \\ &= \varkappa_{i23} - \varkappa_{i32} + \varkappa_{j31} - \varkappa_{i13} + \varkappa_{i12} - \varkappa_{j21}.\end{aligned}$$

$$+2\alpha_3\varkappa_{nnkr}\delta_{ij}+\beta_1\delta_{ij}(\varphi_{r,k}+\sum_{s=r}^{\infty}\overset{r}{b}_{ks}\varphi_s)+\beta_2\left[\delta_{ik}\left(\varphi_{r,j}+\sum_{s=r}^{\infty}\overset{r}{b}_{js}\varphi_s\right)\right.$$

$$\left.+\delta_{jk}\left(\varphi_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\varphi_s\right)\right]+2\alpha_4\varkappa_{ijkr}+\alpha_5(\varkappa_{kjir}+\varkappa_{kijr})$$

$$+f(\mathcal{E}_{ikn}e_{jnr}+\mathcal{E}_{jkn}e_{inr}),\quad i,j,k=\overline{1,3},\quad r=0,1,2,\cdots, \tag{3.34}$$

$$H_{ir}=\beta_1\varkappa_{nnir}+2\beta_2\varkappa_{innr}+\alpha_0\left(\varphi_{r,i}+\sum_{s=r}^{\infty}\overset{r}{b}_{is}\varphi_s,\right),\ i=\overline{1,3},\ r=0,1,2,\cdots, \tag{3.35}$$

$$g_r=de_{nnr}+\xi\varphi_r-\beta T_r,\quad r=0,1,2,\cdots. \tag{3.36}$$

From (3.3) we have [see (3.28) and (3.29)]

$$X_{\alpha ir,\alpha}+\sum_{s=0}^{r}\overset{r}{a}_{js}X_{jis}+\overset{r}{X}_i-\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}\mu_{kji,kj}P_r(ax_3-b)dx_3=0\ \left(\rho\frac{\partial^2u_ir}{dt^2}\right), \tag{3.37}$$

$$i=\overline{1,3},\quad r=0,1,2,\cdots,$$

$$\overset{r}{X}_i:=X_{\overset{(+)}{n}i}\overset{(+)}{\surd}+(-1)^rX_{\overset{(-)}{n}i}\overset{(-)}{\surd}\ \ [\text{see }\ (3.27)],\quad i=\overline{1,3},\quad r=0,1,2,\cdots, \tag{3.38}$$

$$H_{\alpha r,\alpha}+\sum_{s=r}^{\infty}\overset{r}{b}_{js}H_{js}+g_r+l_r=0\ \ (k\ddot{\varphi}_r),\quad r=0,1,2,\cdots. \tag{3.39}$$

From (3.4) we obtain

$$X_{nir}|_{\partial\omega}=F_{ir},\quad i=\overline{1,3},\quad r=0,1,2\cdots, \tag{3.40}$$

now, we prove

$$(\mu_{kji,kj})_r:=\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}\mu_{kji,kj}P_r(ax_3-b)dx_3=(\mu_{kji})_{r,kj}+\overset{r}{M}_i, \tag{3.41}$$

$$i=\overline{1,3},\quad r=0,1,2,\cdots,$$

where

$$\overset{r}{M}_i:=\sum_{s=r}^{\infty}[(\overset{r}{b}_{ks})_{,j}(\mu_{kji})_s+\overset{r}{b}_{ks}(\mu_{kji})_{s,j}]+\sum_{s=r}^{\infty}\overset{r}{b}_{js}[(\mu_{kji})_{s,k}+\sum_{s'=r}^{\infty}\overset{r}{b}_{ks'}(\mu_{kji})_{s'}], \tag{3.42}$$

$$i=\overline{1,3},\quad r=0,1,2,\cdots.$$

Indeed, using (3.25), for $f = \mu_{kji,k}$,

$$(\mu_{kji,kj})_r := \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \mu_{kji,kj} P_r(ax_3 - b)dx_3$$

$$= \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} (\mu_{kji,k}),_J P_r(ax_3 - b)dx_3 = (\mu_{kji,k})_{r,j} + \sum_{s=r}^{\infty} \overset{r}{b}_{js}(\mu_{kji,k})_s, \tag{3.43}$$

$$i = \overline{1,3}, \quad r = 0,1,2,\cdots,$$

provided on face surfaces $\mu_{kji,k}$ are not prescribed and we apply (see (3.10))

$$\overset{(\pm)}{\mu}_{kji,k} = \sum_{s=0}^{\infty} a\left(s + \frac{1}{2}\right)(\mu_{kj,i,k})_s(\pm 1)^s, \quad i,j = \overline{1,3},$$

similarly, we obtain for $f = \mu_{kji}$

$$(\mu_{kji,k})_r := \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} (\mu_{kji,k})P_r(ax_3 - b)dx_3 \tag{3.44}$$

$$= \int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} (\mu_{kji}),_k P_r(ax_3 - b)dx_3 = (\mu_{kji})_{r,k} + \sum_{s=r}^{\infty} \overset{r}{b}_{ks}(\mu_{kji})_s,$$

$$i,j = \overline{1,3}, \quad r = 0,1,2,\cdots,$$

provided on face surfaces $f = \mu_{kji}$, are not prescribed and we apply

$$\overset{(\pm)}{\mu}_{kji} = \sum_{s=0}^{\infty} a\left(s + \frac{1}{2}\right)(\pm 1)^s(\mu_{kji})_s.$$

If we substitute (3.44) into (3.43) we get

$$(\mu_{kji,kj})_r := (\mu_{kji})_{r,kj} + \sum_{s=r}^{\infty}\left[\left(\overset{r}{b}_{ks}\right),_j(\mu_{kji})_s + \overset{r}{b}_{ks}(\mu_{kji})_{s,j}\right]$$

$$+ \sum_{s=r}^{\infty} \overset{r}{b}_{js}\left[(\mu_{kji})_{s,k} + \sum_{s'=s}^{\infty} \overset{s}{b}_{ks'}(\mu_{kji})_{s'}\right], i = \overline{1,3}, \quad r = 0,1,2,\cdots. \tag{3.45}$$

So we have proved (3.41). Q.E.D

Substituting (3.41) into (3.37), we derive

$$X_{\alpha ir,\alpha} + \sum_{j=0}^{r} \overset{r}{a}_{js} X_{jis} + \overset{r}{X}_i - (\mu_{\gamma\alpha i})_{r,\gamma\alpha} - \overset{r}{M}_i = 0 \, \left(\rho \frac{\partial^2 u_{ir}}{\partial t^2}\right), \tag{3.46}$$
$$i = \overline{1,3}, \; r = 0,1,2,\cdots .$$

Substituting (3.22), (3.23) and (3.29) into (3.33)-(3.35), (3.36) and then the obtained into (3.46). (3.39) we arrive at the infinite governing system from which we easily derive the governing systems of hierarchical models as it was explained at the beginning of Section 3.

For the sake of transparency we restict ourselve to the zeroth approximation in the next section.

3.2 The zeroth (N=0) aproximation

Since the moments are independent x_3, their derivatives with respect to x_3 are equal to zero and, consequently, in the zeroth approximation $N = 0, r = 0, \; s = 0, s = 0, \; \sum_1^0 (\cdots) \equiv 0$ [iii], bearing in mind (3.21), (3.12), i.e., $\overset{0}{b}_{30} = 0, \; \overset{0}{b}_{\alpha 0} = -\frac{h_{,\alpha}}{h}$ from (3.15), and (3.28), we obtain

$$\begin{aligned}
\overset{0}{e}_{\alpha\beta 0} &= \frac{1}{2}\left(\overset{0}{u}_{\alpha 0,\beta} + \overset{0}{u}_{\beta 0,\alpha}\right) - \frac{1}{2h}\left(h_{,\alpha} \overset{0}{u}_{\beta,0} + h_{,\beta} \overset{0}{u}_{\alpha 0}\right) \\
&= \frac{1}{2}h\left(\overset{0}{v}_{\alpha 0,\beta} + \overset{0}{v}_{\beta 0,\alpha}\right), \quad \alpha,\beta = 1,2; \\
\overset{0}{e}_{3\beta 0} &= \frac{1}{2}\left(\overset{0}{u}_{30,\beta} - \frac{h_{,\beta}}{h}\overset{0}{u}_{\alpha 0}\right) = \frac{1}{2}h\overset{0}{v}_{30,\beta}, \; \beta = 1,2; \\
e_{\alpha 30} &= \frac{1}{2}\left(\overset{0}{u}_{30,\alpha} - \frac{h_{,\alpha}}{h}\overset{0}{u}_{\beta 0}\right) = \frac{1}{2}h\overset{0}{v}_{30,\alpha}, \; \alpha = 1,2; \\
\overset{0}{e}_{330} &= 0
\end{aligned} \tag{3.47}$$

and

$$\varkappa_{\alpha\beta k0} = u_{k0,\alpha\beta} - \frac{h_{,\beta}}{h}u_{k0,\alpha} - \frac{h_{,\alpha}}{h}u_{k0,\beta} - \left(\frac{h_{,\alpha}}{h}\right)_{,\beta} u_{k0} + \frac{h_{,\beta}}{h}\left(\frac{h_{,\alpha}}{h}\right)u_{k0}, \tag{3.48}$$
$$\alpha,\beta = 1,2; \quad k = \overline{1,3};$$
$$\varkappa_{33k0} = 0, \; \varkappa_{\alpha 3k0} = \left(-\frac{h_{,\alpha}}{h}\right)_{,3} u_{k0} = 0, \; \varkappa_{3\beta k0} = 0, \; \alpha,\beta = 1,2, \; k = \overline{1,3}. \tag{3.49}$$

In the zero approximation see (3.11) and (3.12) from (3.33) we derive

$$\begin{aligned}
\overset{0}{X}_{ji0} = \quad & \lambda e_{nn0}\delta_{ij} + 2\mu e_{ji0} + d\varphi_0\delta_{ij} + f(\mathcal{E}_{ikm}\varkappa_{jkm0} + \mathcal{E}_{jkm}\varkappa_{ikm0}) \\
& -bT_0\delta_{ij} \quad i,j = \overline{1,3}.
\end{aligned} \tag{3.50}$$

[iii]since in the zeroth approximation in (3.15) and (3.12)

$$\sum_{s=0}^{\infty} = \overset{0}{b}_{j0}u_{i0} + \overset{0}{b}_{i0}u_{j0} \text{ and } (\varphi, u_i) \approx \sum_{s=0}^{\infty}(\cdots) = a\frac{1}{2}(\varphi_0, u_{i0}),$$

respectively, and subsums $\sum_{s=1}^{\infty}(\cdots) \equiv 0$ there and in what follows, in the zeroth approximation

From (3.47) and (3.49) in terms of v_{k0}[see (3.24)] it follows:

$$\overset{0}{e}_{\alpha\beta 0} = \frac{1}{2}h\left(v_{\beta 0,\alpha} + v_{\alpha 0,\beta}\right), \quad \overset{0}{e}_{3\beta 0} = \frac{1}{2h}hv_{\beta 0,\alpha},$$
$$\overset{0}{e}_{\beta 30} = \frac{1}{2h}hv_{30,\beta}, \quad \alpha,\beta = 1,2; \ \overset{0}{e}_{330} = 0, \tag{3.51}$$

$$\varkappa_{\alpha\beta k0} = (hv_{k0})_{,\alpha\beta} - \frac{h_{,\alpha}}{h}(hv_{k0})_{,\beta} - \frac{h_{,\beta}}{h}(hv_{k0})_{,\alpha} - \left(\frac{h_{,\alpha}}{h}\right)_{,\beta} hv_{k0} + \frac{h_{,\beta}h_{,\alpha}}{h}v_{k0}$$
$$= (h_{,\alpha}v_{k0} + hv_{k0,\alpha})_{,\beta} - \frac{h_{,\alpha}h_{,\beta}}{h}v_{k0} - h_{,\alpha}v_{k0,\beta} - \left(\frac{h_{,\alpha}}{h}\right)_{,\beta} hv_{k0} + h_{,\beta}$$
$$= h_{,\alpha\beta}v_{k0} + h_{,\alpha}v_{k0,\beta} + h_{,\beta}v_{k0,\alpha} + h_{,\alpha}v_{k0,\alpha\beta} - h_{,\beta}v_{k0,\alpha} - \frac{h_{,\alpha}h_{,\beta}}{h}v_{k0} - h_{,\alpha}v_{k0,\beta}$$
$$- \left(\frac{h_{,\alpha}}{h}\right)_{,\beta} hv_{k0} = \left[h_{,\alpha\beta} - \frac{h_{,\alpha}h_{,\beta}}{h} - \left(\frac{h_{,\alpha}}{h}\right)_{,\beta}\right] v_{k0} + hv_{k0,\alpha,\beta}, \tag{3.52}$$
$$\alpha,\beta = 1,2, \ k = \overline{1,3}$$

[see also (3.22), (3.23), and (3.29) for $r = 0$].

From (3.34), (3.35), (3.36), (3.46), (3.39) for $r = 0$, bearing in mind (3.49), (3.23), (3.18)-(3.21), we obtain

$$\mu_{jik0} = \frac{1}{2}\alpha_1(\varkappa_{nni0}\delta_{jk} + \varkappa_{nnj0}\delta_{ik} + 2\varkappa_{knn0}\delta_{ji}) \tag{3.53}$$
$$+\alpha_2(\varkappa_{inn0}\delta_{jk} + \varkappa_{jnn0}\delta_{ik}) + 2\alpha_3\varkappa_{nnk0}\delta_{ji} + \beta_1\delta_{ij}\left[(h\psi_0)_{,k} + \overset{0}{b}_{k0}h\psi_s\right]$$
$$+\beta_2\left[\delta_{ik}\left((h\psi_0)_{,j} - \overset{0}{b}_{j0}h\psi_0\right) + \delta_{jk}\left((h\psi_0)_{,i} - \overset{0}{b}_{i0}h\psi_0\right)\right]$$
$$+2\alpha_4\varkappa_{ijk0} + \alpha_5(\varkappa_{kji0} + \varkappa_{kij0}) + f\left(\mathcal{E}_{ikn}e_{jn0} + \mathcal{E}_{jkn}e_{in0}\right), \ i,j,k = \overline{1,3};$$

$$H_{i0} = \beta_1\varkappa_{\gamma\gamma i0} + 2\beta_2\varkappa_{inn0} + \alpha\left[(h\psi_0)_{,i} + \overset{0}{b}_{i0}h\psi_0\right], \quad i = \overline{1,3}, \tag{3.54}$$

$$g_0 = dhv_{\gamma 0,\gamma} + \xi h\psi_0 - \beta T_0, \tag{3.55}$$

$$X_{\alpha i0,\alpha} - (\mu_{\gamma ji})_{0,\gamma j} + \overset{0}{X}_i - \overset{0}{M}_i = 0 \ \left(\rho\frac{\partial^2 hv_{ir}}{\partial t^2}\right), \quad i = \overline{1,3}, \tag{3.56}$$

$$H_{\alpha 0,\alpha} - \frac{h_{,\gamma}}{h}H_{\gamma 0} + g_0 + l_0 = 0 \ \ (kh\psi_0). \tag{3.57}$$

Substituting (3.50), (3.53)-(3.55) into (3.56), (3.57) we arrive at the governing system of the $N = 0$ approximation which we investigate for the case

$$h(x_1,x_2) = h_0x_2^\kappa, \quad h_0 = \text{const} > 0 \ \ \kappa = \text{const} \geq 0,$$

applying the approach developed in [6], we prove the following expected result:
in the case of the cusped (i.e. $\kappa > 0$) prismatic shell the cusped edge i.e., where the thickness vanishes ($2h(x_1,0) = 0$) the edge may be fixed only if

$$0 < \kappa < 1,$$

in other words, the Dirichlet type problem, when desired displacement may be prescribed on the entire lateral boundary of the prismatic shell under consideration, is well-posed, while when

$$\kappa \geq 1$$

it is not the case, the cusped edge cannot be fixed and boundary condition should be replaced by the demand of boundedness of the displacement near the cusped edge, in other words, the Keldysh type boundary value problem is well-posed. Note that the vertical displacement is not affected by chirality, since in (3.16) $\varkappa_{\alpha 3k0} = 0$, $\varkappa_{3\beta k0} = \nu$ (see (3.15)) and, therefore, $f \cdot (\mathcal{E}_{3km}\varkappa_{3km0} + \mathcal{E}_{3km}\varkappa_{3km0}) = 0$, but it is not the case in other approximations.

3.3 Conclusion

Applying I. Vekua's dimension reduction method, hierarchical models (approximations) for thermoelastic chiral porous prismatic shells have been constructed. In the $N = 0$ approximation, using the approach developed in [6] we have proved for the case

$$h(x_1, x_2) = h_0 x_2^{\kappa}, \quad h_0 = \text{const} > 0 \quad \kappa = \text{const} \geq 0,$$

the following expected result: in the case of the cusped (i.e. $\kappa > 0$) prismatic shell the cusped edge i.e., where the thickness vanishes $(2h(x_1, 0) = 0)$ the edge may be fixed only if

$$0 < \kappa < 1,$$

in other words the Dirichlet type problem, when desired displacement may be prescribed on the entire lateral boundary of the prismatic shell under consideration, is well-posed, while when

$$\kappa \geq 1$$

it is not the case, the cusped edge cannot be fixed and boundary condition should be replaced by the demand of boundedness of the displacement near the cusped edge (or should be posed a weighted boundary condition).

In the Nth approximation as it follows from the note at the end of the preface of Section 2, bearing in mind that

$$h^{2r+1} = h_0 x_2^{\kappa(2r+1)}$$

in the case under consideration the Dirichlet problem is well-posed when

$$\kappa(2r+1) < 1, \quad r = \overline{0, N}, \quad \text{i.e.} \quad \kappa \leq \frac{1}{2N+1},$$

while the Keldysh problem is well-posed when

$$\kappa(2r+1) \geq 1, \quad r = \overline{0, N}, \quad \text{i.e.} \quad \kappa \geq 1.$$

As $N \to +\infty$, i.e. for the infinite system, i.e. $3D$ problem we get $\kappa = 0$.

4 The hierarchical models for fluids

Let us recall the governing equations of the Newtonian viscous fluid (see e.g., [26], Ch. 2 Conservation of mass and Momentum, Ch. 6 Viscosity and the Navier-Stokes equations and [27] Ch 1, §1 Classical fluids and Navier-Stokes system):

As it is well known, motion of the Newtonian fluid is characterized by the following equations

$$\rho\frac{dv_i(x_1,x_2,x_3,t)}{dt}=\sigma_{ji,j}(x_1,x_2,x_3,t)+\Phi_i(x_1,x_2,x_3,t),\quad i=\overline{1,3}, \tag{4.1}$$

$$\sigma_{ji}=-\delta_{ji}p+\lambda\delta_{ji}\theta(v)+2\mu\epsilon_{ji}(v),\quad i,j=\overline{1,3}, \tag{4.2}$$

$$\epsilon_{ji}(v):=\frac{1}{2}\Big(v_{j,i}+v_{i,j}\Big),\quad i,j=\overline{1,3}, \tag{4.3}$$

$$\theta:=\epsilon_{ii}=v_{k,k}=:\mathrm{div}v, \tag{4.4}$$

$$\lambda:=\mu'-\frac{2}{3}\mu$$

where $v:=(v_1,v_2,v_3)$ is a strain velocity vector, σ_{ij} is a stress tensor, $\epsilon_{ij}(v)$ is a velocity (rate) tensor, p is a pressure, Φ_i, $i=\overline{1,3}$, are components of the volume force, μ' and μ are the second viscosity and the viscosity respectively, ρ is a density of the fluid. Throughout the paper we use, on the one hand Einstein's summation convention on repeated indices, bar under one of the repeated indices means that we do not sum. Latin indices run values 1,2,3, while Greek indices run values 1,2 and on the other hand, the simplified notation for the partial derivative e.g.,

$$\frac{\partial\sigma_{ji}}{\partial x_j}=:\sigma_{ji,j}.$$

As well-known an incompressible fluid is defined as the fluid whose volume or density doesn't change with pressure (see e.g. [25], p. 6 and p. 17). In reality, rigorous incompressible fluid doesn't exist.

In the case of incompressible barotropic fluids, to the system (4.1)-(4.3) we add the equation

$$\mathrm{div}v=0, \tag{4.5}$$

which expresses the fact that the velocity of change of cubical dilatation of each parcel of moving fluid is unchangeable (constant) during moving.

In general, for compressible fluids the continuity equation has the form

$$\frac{d\rho}{dt}+\rho\,\mathrm{div}v=0^{\mathrm{iv}} \tag{4.6}$$

[this last equation can also be written as

$$\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho v)=0]$$

clearly, for $\rho=const$ from (4.6) we get again (4.5), and the state equation

$$\chi(\rho,p)=0, \tag{4.7}$$

[iv] When $\rho=\rho(x_1,x_2)$ then (4.6) has the form

$$\frac{\partial\rho}{\partial t}+\rho_{,\gamma}v_\gamma+\rho(x_1,x_2)v_{k,k}=0$$

where χ is a certain function defining the state equation.

In the prismatic and standard shell-like, bar-like, and canal- like domains two type of hierarchical models are constructed (for details see [10]):

(i) when on the face surfaces of the fluids container stresses are prescribed;

(ii) when on the face surfaces velocities are prescribed.

4.1 Mathematical moments

Here we follow Section 10 of [6] [v].

Let $f(x_1, x_2, x_3)$ be a given function on $\overline{\Omega}$ having integrable partial derivatives, let f_r denote its r-th order moment, defined as follows

$$f_r(x_1, x_2) := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} f(x_1, x_2, x_3) P_r(ax_3 - b) dx_3, \tag{4.8}$$

where (see also Subsection 2.1 of [10] and Section 2.3 of [7])

$$a(x_1, x_2) := \frac{1}{h(x_1, x_2)}, \quad b(x_1, x_2) := \frac{\widetilde{h}(x_1, x_2)}{h(x_1, x_2)},$$
$$2h(x_1, x_2) = \overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2) > 0,$$
$$2\widetilde{h}(x_1, x_2) = \overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2) > 0,$$

and

$$P_r(\tau) = \frac{1}{2^r r!} \frac{d^r (\tau^2 - 1)^r}{d\tau^r}, \quad r = 0, 1, \cdots,$$

are the r-th order Legendre polynomials with the orthogonality property

$$\int\limits_{-1}^{+1} P_m(\tau) P_n(\tau) d\tau = \frac{2}{2m+1} \delta_{mn}.$$

From here, substituting

$$\tau = ax_3 - b = \frac{2}{\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)} x_3 - \frac{\overset{(+)}{h}(x_1,\, x_2) + \overset{(-)}{h}(x_1,\, x_2)}{\overset{(+)}{h}(x_1,\, x_2) - \overset{(-)}{h}(x_1,\, x_2)},$$

we have

$$\Big(m + \frac{1}{2}\Big) a \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_m(ax_3 - b) P_n(ax_3 - b) dx_3 = \delta_{mn}.$$

[v] where I. Vekua's dimension reduction method is reformulated and presented in the unified form, that formulas for arbitrary functions independent of physical meaning, allow to construct easily for any physical model containing the thickness (or something like that) the Vekua-Babushka and Schwab type [20]. Hierarchical models which are suitable for use of the (p-version of the finite the element method as it is indicated in Douge at. al. [19]

Using the well-known formulas of integration by parts (with respect to x_3) and differentiation with respect to a parameter of integrals depending on parameters (x_α), taking into account $P_r(1) = 1$, $P_r(-1) = (-1)^r$, we deduce

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,\alpha}\, dx_3 = f_{r,\alpha} - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,\alpha}$$

$$- \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P'_r(ax_3 - b)(a_{,\alpha}\, x_3 - b_{,\alpha}) f dx_3, \quad \alpha = 1, 2, \tag{4.9}$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,3}\, dx_3 = -a \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P'_r(ax_3 - b) f dx_3 + \overset{(+)}{f} - (-1)^r \overset{(-)}{f}, \tag{4.10}$$

where superscript prime means differentiation with respect to the argument $ax_3 - b$, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables, $\overset{(\pm)}{f} := f[x_1, x_2, \overset{(\pm)}{h}(x_1, x_2)]$. Applying the following relations from the theory of the Legendre polynomials (see [1] p. 197, [3] p. 27, [2] pp. 22, 23 and [18], p. 299 or p. 338, 339 of the second edition)

$$P'_r(\tau) = \sum_{s=0}^{r} (2s + 1) \frac{1 - (-1)^{r+s}}{2} P_s(\tau)^{\text{vi}}, \tag{4.11}$$

$$\tau P'_r(\tau) = rP_r(\tau) + P'_{r-1}(\tau) = rP_r(\tau) + \sum_{s=0}^{r-1} (2s + 1) \frac{1 + (-1)^{r+s}}{2} P_s(\tau)^{\text{vii}} \tag{4.12}$$

and, in view of $\frac{a_{,\alpha}}{a} = (\ln a)_{,\alpha} = -\frac{h_{,\alpha}}{h}$, $\frac{a_{,\alpha}}{a} b = \widetilde{h} a_{,\alpha}$, $b_{,\alpha} = (\widetilde{h} a)_{,\alpha}$, it is easily seen that

$$P'_r(ax_3 - b)(a_{,\alpha}\, x_3 - b_{,\alpha})^{\text{ viii}} = \frac{a_{,\alpha}}{a}(ax_3 - b) P'_r(ax_3 - b) + (\frac{a_{,\alpha}}{a} b - b_{,\alpha}) P'_r(ax_3 - b)$$

$$= -h_{,\alpha}\, h^{-1}(ax_3 - b) P'_r(ax_3 - b) - \widetilde{h}_{,\alpha}\, h^{-1} P'_r(ax_3 - b)$$

$$= -\overset{r}{a}_{\alpha \underline{r}}\, P_r(ax_3 - b) - \sum_{s=0}^{r-1} \overset{r}{a}_{\alpha s}\, P_s(ax_3 - b) = -\sum_{s=0}^{r} \overset{r}{a}_{\alpha s} P_s(ax_3 - b), \tag{4.13}$$

[vi] on the top of the symbol $\sum$ both $r - 1$ and r are true since the last term equals zero.
[vii] on the top of the symbol $\sum$ both $r - 2$ and $r - 1$ are true since the last term equals zero.
[viii] Clearly,

$$a_{,\alpha}\, x_3 - b_{,\alpha} = \frac{a_{,\alpha}}{a} a x_3 - b_{,\alpha} = \frac{a_{,\alpha}}{a} a x_3 - \frac{a_{,\alpha}}{a} b + \frac{a_{,\alpha}}{a} b - b_{,\alpha} = \frac{a_{,\alpha}}{a}(ax_3 - b) - \widetilde{h}_{,\alpha}\, \frac{1}{h}$$

because of

$$\frac{a_{,\alpha}}{a} b - b_{,\alpha} = a_{,\alpha}\, \widetilde{h} - (\widetilde{h} a)_{,\alpha} = \widetilde{h} a_{,\alpha} - \widetilde{h} a_{,\alpha} - \widetilde{h}_{,\alpha}\, a = -\widetilde{h}_{,\alpha} a.$$

where

$$\overset{r}{a}_{\alpha\underline{r}} := r\frac{h_{,\alpha}}{h}, \quad \overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r, \tag{4.14}$$

in deed, taking into account (4.12) and (4.11),

$$-h_{,\alpha}h^{-1}(ax_3-b)P_r'(ax_3-b) - \widetilde{h}_{,\alpha}h^{-1}P_r'(ax_3-b) = -rh_{,\alpha}h^{-1}P_r(ax_3-b)$$

$$= -h_{,\alpha}h^{-1}\sum_{s=0}^{r-1}(2s+1)\frac{1+(-1)^{r+s}}{2}P_s(ax_3-b) - \widetilde{h}_{,\alpha}h^{-1}\sum_{s=0}^{r}(2s+1)\frac{1-(-1)^{r+s}}{2}P_s(ax_3-b)$$

$$= -r\frac{h_{,\alpha}}{h}P_r(ax_3-b) - \sum_{s=0}^{r-1}(2s+1)\left[\frac{h_{,\alpha}+(-1)^{r+s}h_{,\alpha}}{2h} + \frac{\widetilde{h}_{,\alpha}-(-1)^{r+s}\widetilde{h}_{,\alpha}}{2h}\right]P_s(ax_3-b)$$

$$= -r\frac{h_{,\alpha}}{h}P_r(ax_3-b) - \sum_{s=0}^{r-1}\frac{(2s+1)}{2h}\left(\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha} + \overset{(+)}{h}_{,\alpha}(-1)^{r+s} - \overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\right.$$

$$\left.+\frac{\overset{(+)}{h}_{,\alpha} + \overset{(-)}{h}_{,\alpha} - \overset{(+)}{h}_{,\alpha}(-1)^{r+s} - \overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\right)P_s(ax_3-b)$$

$$= -r\frac{h_{,\alpha}}{h}P_r(ax_3-b) - \sum_{s=0}^{r-1}(2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}P_s(ax_3-b) = -\sum_{s=0}^{r}\overset{s}{a}_{\alpha s}P_s(ax_3-b)$$

because of

$$h_{,\alpha} = \frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2}, \quad \widetilde{h}_{,\alpha} = \frac{\overset{(+)}{h}_{,\alpha} + \overset{(-)}{h}_{,\alpha}}{2}.$$

Now, by substituting (4.13) and (4.11) into (4.9) and (4.10), respectively, we obtain

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 = f_{r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{\alpha s}\,f_s - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha}, \quad \alpha = 1,2, \tag{4.15}$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r}\overset{r}{a}_{3s}\,f_s + \overset{(+)}{f} - (-1)^r\overset{(-)}{f}, \tag{4.16}$$

respectively. Here

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}, \tag{4.17}$$

clearly,

$$\overset{r}{a}_{3r} = 0. \tag{4.18}$$

Remark 4.1. We may take down formulas (4.15) and (4.16) in the following unified form

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,j}\,dx_3 = f_{r,j} + \sum_{s=0}^{r} \overset{r}{a}_{js}\, f_s - \overset{(+)}{f}\overset{(+)}{h}_{,j} + (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,j}, \quad j=\overline{1,3},$$

provided we formally read $\overset{(\pm)}{h}_{,3}$ as -1 and vice versa, since moments are independent of x_3, i.e. $f_{r,3}\equiv 0$. A justification of this convention looks like as follows using equations of the upper and lower face surfaces in the implicit form

$$\overset{(\pm)}{F}(x_1,x_2,x_3) := x_3 - \overset{(\pm)}{h}(x_1,x_2) = 0,$$

the last formula we may rewrite as

$$\int_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,j}\,dx_3 = f_{r,j} + \sum_{s=0}^{r} \overset{r}{a}_{js} f_s - \overset{(+)}{f}\overset{(+)}{F}_{,j} + (-1)^r \overset{(-)}{f}\overset{(-)}{F}_{,J}$$

because of

$$\overset{(\pm)}{n}_{,i} = \frac{\overset{(\pm)}{F}_{,i}(x_1,x_2,x_3)}{\pm\sqrt{\overset{(\pm)}{F}{}_{,1}^2 + \overset{(\pm)}{F}{}_{,2}^2 + \overset{(\pm)}{F}{}_{,3}^2}} \Rightarrow \overset{(\pm)}{n}_{,\alpha} = \frac{-\overset{(\pm)}{h}_{,\alpha}(x_1,x_2)}{\pm\sqrt{\overset{(\pm)}{F}{}_{,1}^2 + \overset{(\pm)}{F}{}_{,2}^2 + \overset{(\pm)}{F}{}_{,3}^2}}, \quad \alpha = 1,2;$$
$$\overset{(\pm)}{n}_{,3} = \frac{1}{\pm\sqrt{\overset{(\pm)}{F}{}_{,1}^2 + \overset{(\pm)}{F}{}_{,2}^2 + \overset{(\pm)}{F}{}_{,3}^2}}.$$

Here $(+)$ and $(-)$ before root we take for the upper (a normal forms an acute angle with the x_3 -axis) and lower (a normal forms a blunt angle with the x_3 -axis) sides, respectively.

Let the Fourier-Legendre expansion of $f(\cdot,\cdot,x_3)\in C^2(\overset{(-)}{h},\overset{(+)}{h})$ be

$$\begin{aligned} f(x_1,x_2,x_3) &= \sum_{s=0}^{\infty} a\Big(s+\frac{1}{2}\Big) f_s(x_1,x_2)P_s(ax_3-b) \\ &= \sum_{s=0}^{\infty}\Big(s+\frac{1}{2}\Big)h^s \widetilde{f}_s(x_1,x_2)P_s(ax_3-b), \\ \widetilde{f}_s(x_1,x_2) &:= \frac{f_s(x_1,x_2)}{h^{s+1}(x_1,x_2)}, \end{aligned} \tag{4.19}$$

then, if $\overset{(\pm)}{f}$ are not known, using (4.19), we calculate them as follows

$$\overset{(\pm)}{f} := f(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)) = \sum_{s=0}^{\infty} a\Big(s+\frac{1}{2}\Big) f_s(\pm 1)^s = \sum_{s=0}^{\infty}\frac{(\pm 1)^s(2s+1)}{2h} f_s, \tag{4.20}$$

whence, denoting $\overset{r}{a}{}^*_{3s} := -\overset{r}{a}_{3s}$, bearing in mind (4.20), (4.17),

$$\overset{(+)}{f} - (-1)^r \overset{(-)}{f} = \sum_{s=0}^{\infty} \overset{r}{a}{}^*_{3s} f_s, \tag{4.21}$$

and, by virtue of (4.20), (4.14),

$$\overset{(+)}{f}\overset{(+)}{h}{}_{,\alpha} - (-1)^r \overset{(-)}{f}\overset{(-)}{h}{}_{,\alpha} = \sum_{s=0}^{\infty} \overset{r}{a}{}^*_{3s} f_s, \quad \alpha = 1, 2, \tag{4.22}$$

where, taking into account (4.18),

$$\overset{r}{a}{}^*_{3r} = -\overset{r}{a}_{3r} = 0, \quad \overset{r}{a}{}^*_{3s} := -\overset{r}{a}_{3s}; \quad \overset{r}{a}{}^*_{\alpha s} = \overset{r}{a}_{\alpha s}, \ s \neq r, \ \overset{r}{a}{}^*_{\alpha \underline{r}} = (2r+1)\frac{h_{,\alpha}}{h}. \tag{4.23}$$

Substituting (4.22) and (4.21) into (4.15) and (4.16), respectively, we get

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,\alpha}\, dx_3 = f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s - \sum_{s=0}^{\infty} \overset{r}{a}{}^*_{\alpha s} f_s = f_{r,\alpha} + \sum_{s=r}^{\infty} \overset{r}{b}_{\alpha s} f_s, \quad \alpha = 1, 2, \tag{4.24}$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,3}\, dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s - \sum_{s=0}^{\infty} \overset{r}{a}_{3s} f_s = -\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s} f_s = \sum_{s=r+1}^{\infty} \overset{r}{b}_{3s} f_s = \sum_{s=r}^{\infty} \overset{r}{b}_{3s} f_s, \tag{4.25}$$

respectively, where

$$\overset{r}{b}_{js} := -\overset{r}{a}{}^*_{js}, \quad s > r; \quad \overset{r}{b}_{js} = 0, \quad j = 1, 2, 3, \quad s < r; \tag{4.26}$$

from (4.14), (4.23), (4.18), (4.21) it follows that

$$\overset{r}{b}_{\alpha \underline{r}} := \overset{r}{a}_{\alpha \underline{r}} - \overset{r}{a}{}^*_{\alpha \underline{r}} = r\frac{h_{,\alpha}}{h} - (2r+1)\frac{h_{,\alpha}}{h} = -(r+1)\frac{h_{,\alpha}}{h}, \quad \overset{r}{b}_{3\underline{r}} := \overset{r}{a}_{3\underline{r}} - \overset{r}{a}_{3\underline{r}} = 0. \tag{4.27}$$

We may take down (4.24) and (4.25) in the unified form as follows (here $\overset{(\pm)}{f}$ are not known and clearly (4.20), more precisely (4.21) and (4.22), are applied)

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,j}\, dx_3 \ = \ f_{r,j} + \sum_{s=r}^{\infty} \overset{r}{b}_{js} f_s, \quad j = \overline{1,3}. \tag{4.28}$$

If $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are known (prescribed), then from (4.15) and (4.16), correspondingly, we obtain

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 = f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s}\, f_s$$

$$+\overset{(+)}{f}\overset{(+)}{n}_\alpha\sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2}+(-1)^r\overset{(-)}{f}\overset{(-)}{n}_\alpha\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2} \quad (4.29)$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s}\, f_s$$

$$+\overset{(+)}{f}\overset{(+)}{n}_3\sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2}+(-1)^r\overset{(-)}{f}\overset{(-)}{n}_3\sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2}, \quad (4.30)$$

since $\overset{(\pm)}{n}_\alpha = \frac{\mp \overset{(\pm)}{h}_{,\alpha}}{\sqrt{1+(\overset{(\pm)}{h}_{,1})^2+(\overset{(\pm)}{h}_{,2})^2}}$, $\overset{(\pm)}{n}_3 = \frac{\pm 1}{\sqrt{1+(\overset{(\pm)}{h}_{,1})^2+(\overset{(\pm)}{h}_{,2})^2}}$.

Remark 4.2. Evidently, (4.29), (4.30) we may rewrite in the unified form as (here $\overset{(\pm)}{f}$ are known) [compare with (4.28)]

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,j}dx_3 = f_{r,j} + \sum_{s=0}^{r} \overset{r}{a}_{js} f_s + \overset{(+)}{f}\overset{(+)}{n}_j\sqrt{1+\left(\overset{(+)}{h}_{,1}\right)^2+\left(\overset{(+)}{h}_{,2}\right)^2}$$

$$+(-1)^r\overset{(-)}{f}\overset{(-)}{n}_j\sqrt{1+\left(\overset{(-)}{h}_{,1}\right)^2+\left(\overset{(-)}{h}_{,2}\right)^2}, \quad j=\overline{1,3}, \quad (4.31)$$

because of $f_{r,3}(x_1,x_2)\equiv 0$, where $\overset{(+)}{n}$ and $\overset{(-)}{n}$ are the outward normals to the surfaces $x_3 = \overset{(+)}{h}(x_1,x_2)$ and $x_3 = \overset{(-)}{h}(x_1,x_2)$, respectively (see the end of Remark 4.1).

4.2 The first type hierarchical model

The basic relations of the N-th approximation in term of mathematical moments have the following forms (see Subsection 3.1 of [10]

$$(h^{2r+1}\widetilde{\sigma}_{jir})_{,j} + \sum_{l=0}^{r-1} \overset{r}{a}_{kl}\, h^{r+l+1}\widetilde{\sigma}_{kil} + h^r\,\overset{r}{X}_i = \int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}} \rho\frac{dv_i}{dt}P_r(ax_3-b)dx_3,\ i=\overline{1,3}. \quad (4.32)$$

$$\varepsilon_{jir} = \frac{1}{2}h^{r+1}\left(\widetilde{v}_{jr,i}+\widetilde{v}_{ir,j}\right)+\frac{1}{2}\sum_{s=r+1}^{\infty} h^{s+1}\left(\overset{r}{b}_{is}\,\widetilde{v}_{js}+\overset{r}{b}_{js}\,\widetilde{v}_{is}\right), \quad (4.33)$$

$$\theta_r \equiv \varepsilon_{k'k'r} = h^{r+1}\widetilde{v}_{\gamma r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1}\,\overset{r}{b}_{k's}\,\widetilde{v}_{k's}, \quad r=0,1,,N. \quad (4.34)$$

$$\sigma_{jir} = -\delta_{ji}p_r(x_1,x_2,t) + \lambda\delta_{ji}h^{r+1}\widetilde{v}_{\gamma r,\gamma} + \mu h^{r+1}(\widetilde{v}_{jr,i} + \widetilde{v}_{ir,j}) + \sum_{s=r+1}^{\infty}\overset{r}{B}_{jik's}h^{s+1}\widetilde{v}_{k's}, \tag{4.35}$$

$$\widetilde{\sigma}_{jir} = -\delta_{ji}\tilde{p}_r + \lambda\delta_{ji}\widetilde{v}_{\gamma r,\gamma} + \mu(\widetilde{v}_{jr,i} + \widetilde{v}_{ir,j}) + \sum_{s=r+1}^{\infty}\overset{r}{B}_{jik's}h^{s-r}\widetilde{v}_{k's}, \tag{4.36}$$

$$-\Big(h^r p_r\Big)_{,i} + \Big[\lambda h^{2r+1}\widetilde{v}_{\gamma r,\gamma}\Big]_{,i} + \Big[\mu h^{2r+1}(\widetilde{v}_{jr,i} + \widetilde{v}_{ir,j})\Big]_{,j}$$
$$+ \sum_{s=r+1}^{\infty}\Big(\overset{r}{B}_{jik's}h^{r+s+1}\widetilde{v}_{k's}\Big)_{,j} + \sum_{l=0}^{r-1}\overset{r}{a}_{kl}\Big[-\delta_{ki}p_l h^r + \lambda\delta_{ki}h^{r+l+1}\widetilde{v}_{\gamma l,\gamma} \tag{4.37}$$
$$+\mu h^{r+l+1}(\widetilde{v}_{kl,i} + \widetilde{v}_{il,k}) + \sum_{s=l+1}^{\infty}\overset{l}{B}_{kik's}h^{r+s+1}\widetilde{v}_{k's}\Big] + h^r\overset{r}{X}_i = \int_{\overset{(-)}{h}}^{\overset{(+)}{h}}\rho\frac{dv_i}{dt}P_r(ax_3-b)dx_3,$$

$$i=\beta,3,\quad \beta=1,2,\quad r=0,1,2,...,\sum_{q}^{q-1}(\cdots)=0,$$

$$-\Big(h^{2r+1}\tilde{p}_r\Big)_{,\beta} + \Big[\lambda h^{2r+1}\widetilde{v}_{\gamma r,\gamma}\Big]_{,\beta} + \Big[\mu h^{2r+1}(\widetilde{v}_{\alpha r,\beta} + \widetilde{v}_{\beta r,\alpha})\Big]_{,\alpha}$$
$$+ \sum_{s=r+1}^{\infty}\Big(\overset{r}{B}_{\alpha\beta k's}h^{r+s+1}\widetilde{v}_{k's}\Big)_{,\alpha} + \sum_{l=0}^{r-1}\overset{r}{a}_{\beta l}\Big[-\tilde{p}_l h^{r+l+1} + \lambda h^{r+l+1}\widetilde{v}_{\gamma l,\gamma}\Big]$$
$$+\sum_{l=0}^{r-1}\overset{r}{a}_{kl}\ \mu h^{r+l+1}\widetilde{v}_{kl,\beta} + \sum_{l=0}^{r-1}\overset{r}{a}_{\alpha l}\ \mu h^{r+s+1}\widetilde{v}_{\beta l,\alpha} + \sum_{l=0}^{r-1}\overset{r}{a}_{kl}\sum_{s=l+1}^{\infty}\overset{l}{B}_{k\beta k's}h^{r+s+1}\widetilde{v}_{k's}$$
$$+h^r\overset{r}{X}_\beta = \int_{\overset{(-)}{h}}^{\overset{(+)}{h}}\rho\frac{dv_\beta}{dt}P_r(ax_3-b)dx_3,\quad r=0,1,2,...,\ \sum_{q}^{q-1}(\cdots)\equiv 0; \tag{4.38}$$

$$\Big[\mu h^{2r+1}\widetilde{v}_{3r,\alpha}\Big]_{,\alpha} + \sum_{s=r+1}^{\infty}\Big(\overset{r}{B}_{\alpha 3ks}h^{r+s+1}\widetilde{v}_{ks}\Big)_{,\alpha} + \sum_{l=0}^{r-1}\overset{r}{a}_{3l}\Big[-p_l h^r + \lambda h^{r+l+1}\widetilde{v}_{\gamma l,\gamma}\Big]$$
$$+\sum_{l=0}^{r-1}\overset{r}{a}_{\alpha l}\ \mu h^{r+l+1}\widetilde{v}_{3l,\alpha} + \sum_{l=0}^{r-1}\overset{r}{a}_{kl}\sum_{s=l+1}^{\infty}\overset{l}{B}_{k3k's}h^{r+s+1}\widetilde{v}_{k's} + h^r\overset{r}{X}_3$$
$$= \int_{\overset{(-)}{h}}^{\overset{(+)}{h}}\rho\frac{dv_3}{dt}P_r(ax_3-b)dx_3,,\quad r=0,1,2,...,\ \sum_{q}^{q-1}(\cdots)\equiv 0; \tag{4.39}$$

for $i=\beta=1,2$,

$$-(h^{2r+1}\overset{N}{\widetilde{p}}_r)_{,\beta} + \Big(\lambda h^{2r+1}\overset{N}{\widetilde{v}}_{\gamma r,\gamma}\Big)_{,\beta} + \Big(\mu h^{2r+1}\overset{N}{\widetilde{v}}_{\alpha r,\beta}\Big)_{,\alpha} + \Big(\mu h^{2r+1}\overset{N}{\widetilde{v}}_{\beta r,\alpha}\Big)_{,\alpha}$$
$$+\sum_{s=r+1}^{N}\Big(\overset{r}{B}_{\alpha\beta k's}h^{r+s+1}\overset{N}{\widetilde{v}}_{k's}\Big)_{,\alpha} + \sum_{l=0}^{r-1}\overset{r}{a}_{\beta l}\Big[-\overset{N}{p}_l h^r + \lambda h^{r+l+1}\overset{N}{\widetilde{v}}_{\gamma l,\gamma}\Big]$$

$$+\sum_{l=0}^{r-1}\overset{r}{a}_{kl}\,\mu h^{r+l+1}\overset{N}{\widetilde{v}}_{kl,\beta}+\sum_{l=0}^{r-1}\overset{r}{a}_{\alpha l}\,\mu h^{r+l+1}\overset{N}{\widetilde{v}}_{\beta l,\alpha}+\sum_{l=0}^{r-1}\overset{r}{a}_{kl}\sum_{s=l+1}^{N}\overset{l}{B}_{k\beta k's}h^{r+s+1}\overset{N}{\widetilde{v}}_{k's}$$

$$+h^r\overset{N}{X}_\beta=\int_{\overset{(-)}{h}}^{\overset{(+)}{h}}\rho\frac{dv_\beta}{dt}P_r(ax_3-b)dx_3,\quad r=\overline{0,N},\quad \sum_q^{q-1}(\cdots)\equiv 0; \tag{4.40}$$

for $i=3$

$$\Big(\mu h^{2r+1}\overset{N}{\widetilde{v}}_{3r,\alpha}\Big)_{,\alpha}+\sum_{s=r+1}^{N}\Big(\overset{r}{B}_{\alpha 3k's}h^{r+s+1}\overset{N}{\widetilde{v}}_{k's}\Big)_{,\alpha}+\sum_{l=0}^{r-1}\overset{r}{a}_{3l}\Big[-\overset{N}{p}_l h^r+\lambda h^{r+l+1}\overset{N}{\widetilde{v}}_{\gamma l,\gamma}\Big]$$

$$+\sum_{l=0}^{r-1}\overset{r}{a}_{\alpha l}\Big[\mu h^{r+l+1}\overset{N}{\widetilde{v}}_{3l,\alpha}\Big]+\sum_{l=0}^{r-1}\overset{r}{a}_{kl}\sum_{s=l+1}^{N}\overset{l}{B}_{k3k's}h^{r+s+1}\overset{N}{\widetilde{v}}_{k's}+h^r\overset{N}{X}_3$$

$$=\int_{\overset{(-)}{h}}^{\overset{(+)}{h}}\rho\frac{dv_3}{dt}P_r(ax_3-b)dx_3,\quad r=\overline{0,N},\quad \sum_q^{q-1}(\cdots)\equiv 0. \tag{4.41}$$

In the $N=0$ approximation $r=0$, therefore

$$\sum_{s=1}^{0}(\cdots)\equiv 0,\quad \sum_{l=0}^{-1}(\cdots)\equiv 0$$

and, thus the governing system as it follows correspondingly from (4.40) and (4.41):

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta}+\Big[\lambda h\overset{0}{\widetilde{v}}_{\gamma 0,\gamma}\Big]_{,\beta}+\Big[\mu h\Big(\overset{0}{\widetilde{v}}_{\alpha 0,\beta}+\overset{0}{\widetilde{v}}_{\beta 0,\alpha}\Big)\Big]_{,\alpha}+\overset{0}{X}_\beta=\rho h\frac{\partial\overset{0}{\widetilde{v}}_{\beta 0}}{\partial t},\quad \beta=1,2;\text{[ix]} \tag{4.42}$$

$$\Big(\mu h\overset{0}{\widetilde{v}}_{30,\alpha}\Big)_{,\alpha}+\overset{0}{X}_3=\rho h\frac{\partial\overset{0}{\widetilde{v}}_{30}}{\partial t}, \tag{4.43}$$

provided $\rho=\rho(x_1,x_2)$ and we consider Stoke's approximation.

We add to system (4.42), (4.43) the additional equation

$$v_{\gamma 0,\gamma}-\frac{h_{,\gamma}}{h}v_{\gamma 0}=0, \tag{4.44}$$

according to (4.5), (4.4), (4.24), (4.25), i.e., in terms of weighted moments

$$(h\overset{0}{\widetilde{v}}_{\gamma 0})_{,\gamma}-h_{,\gamma}\overset{0}{\widetilde{v}}_{\gamma 0}=0,$$

[ix] In terms of $\overset{0}{p}_0$ moment for presure the first term in (4.42) looks like

$$\overset{0}{p}_{0,\beta}.$$

whence

$$\overset{0}{\widetilde{v}}_{\gamma 0,\gamma} = 0. \tag{4.45}$$

In the stationary case, bearing in mind (4.45), from (4.42) we obtain

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta} + (\mu h)_{,\alpha}\,(\overset{0}{\widetilde{v}}_{\alpha 0,\beta} + \overset{0}{\widetilde{v}}_{\beta 0,\alpha}) + \mu h \overset{0}{\widetilde{v}}_{\beta 0,\alpha\alpha} + \overset{0}{X}_\beta = 0, \quad \beta = 1,2.$$

Differentiating the last with respect to x_β and then summing with respect to β we get

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta\beta} + (\mu h)_{,\alpha\beta}\,(\overset{0}{\widetilde{v}}_{\alpha 0,\beta} + \overset{0}{\widetilde{v}}_{\beta 0,\alpha}) + (\mu h)_{,\alpha}\,\overset{0}{\widetilde{v}}_{\alpha 0,\beta\beta} + (\mu h)_{,\beta}\,\overset{0}{\widetilde{v}}_{\beta 0,\alpha\alpha} + \overset{0}{X}_{\beta,\beta} = 0, \quad \beta = 1,2. \tag{4.46}$$

Therefore, if $\mu h = const$ we have

$$(h\overset{0}{\widetilde{p}}_0)_{,\beta\beta} = -\overset{0}{X}_{\beta,\beta}, \quad \beta = 1,2.$$

Further,

$$\begin{aligned}
\overset{0}{\widetilde{v}}_{\beta 0,\alpha\alpha} &= -\frac{1}{\mu h}\Big[\overset{0}{X}_\beta + (h\overset{0}{\widetilde{p}}_0)_{,\beta}\Big], \quad \beta = 1,2,\\
\overset{0}{\widetilde{v}}_{30,\alpha\alpha} &= -\frac{\overset{0}{X}_3}{\mu h}.
\end{aligned}$$

Remark 4.3. If $h=$ const by virtue of (4.45), clearly, (4.42) takes the form

$$\overset{0}{\widetilde{p}}_{0,\beta} + \overset{0}{\widetilde{v}}_{\beta 0,\alpha\alpha} + h^{-1}\overset{0}{X}_\beta = \rho\frac{\partial \overset{0}{\widetilde{v}}_{\beta 0}}{\partial t}, \quad \beta = 1,2,$$

differentiating the last with respect to x_β and then summing with respect to β we obtain

$$\overset{0}{\widetilde{p}}_{0,\beta\beta} + h^{-1}\overset{0}{X}_{\beta,\beta} = \Big[\rho\frac{\partial \overset{0}{\widetilde{v}}_{\beta 0}}{\partial t}\Big]_{,\beta} = \rho_{,\beta}\,\frac{\partial \overset{0}{\widetilde{v}}_{\beta 0}}{\partial t}.$$

4.3 The second type hierarchical model

In the $N=0$ approximation we have (see Subsection 3.2 of [10]

$$\begin{aligned}
&-q_{0,\beta} + \Big\{\lambda\Big[(\ln h)_{,\gamma}\,\overset{0}{w}_{\gamma 0} + \frac{1}{2}\overset{0}{\Psi}_{l'l'}\Big]\Big\}_{,\beta} + \Big\{\mu\Big[(\overset{0}{w}_{\alpha 0,\beta} + \overset{0}{w}_{\beta 0,\alpha}) + (\ln h)_{,\beta}\,\overset{0}{w}_{\alpha 0}\\
&+(\ln h)_{,\alpha}\,\overset{0}{w}_{\beta 0} + h^{-1}\overset{0}{\Psi}_{\alpha\beta}\Big]\Big\}_{,\alpha} + \overset{0}{Y}_\beta = \rho h^{-1}\frac{\partial v_{\beta 0}}{\partial t} = \rho\frac{\partial w_{\beta 0}}{\partial t}, \quad \beta = 1,2 \qquad (4.47)\\
&\Big\{\mu\Big[\overset{0}{w}_{30,\alpha} + (\ln h)_{,\alpha}\,\overset{0}{w}_{30}\Big]\Big\}_{,\alpha} + h^{-1}\overset{0}{\Psi}_{\alpha 3,\alpha} + \overset{0}{Y}_3 = \rho h^{-1}\frac{\partial v_{30}}{\partial t} = \rho\frac{\partial w_{30}}{\partial t}, \qquad (4.48)\\
&(h\overset{0}{w}_{\gamma 0})_{,\gamma} + \overset{0}{V} = 0, \qquad (4.49)
\end{aligned}$$

where

$$
\begin{aligned}
&\overset{r}{\Psi}_{ji} := -\overset{(+)}{v}_j \overset{(+)}{h}_{,i} + (-1)^r \overset{(-)}{v}_j \overset{(-)}{h}_{,i} - \overset{(+)}{v}_i \overset{(+)}{h}_{,j} + (-1)^r \overset{(-)}{v}_i \overset{(-)}{h}_{,j},\\
&\overset{r}{\Psi}_{\alpha\beta} := -\overset{(+)}{v}_\alpha \overset{(+)}{h}_{,\beta} + (-1)^r \overset{(-)}{v}_\alpha \overset{(-)}{h}_{,\beta} - \overset{(+)}{v}_\beta \overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{v}_\beta \overset{(-)}{h}_{,\alpha},\\
&\overset{r}{\Psi}_{3\alpha} \equiv \overset{r}{\Psi}_{\alpha 3} := \overset{(+)}{v}_\alpha - (-1)^r \overset{(-)}{v}_\alpha - \overset{(+)}{v}_3 \overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{v}_3 \overset{(-)}{h}_{,\alpha},\\
&\overset{r}{\Psi}_{33} := 2[\overset{(+)}{v}_3 - (-1)^r \overset{(-)}{v}_3],
\end{aligned}
\tag{4.50}
$$

are known functions, since $\overset{(+)}{v}_i$, $\overset{(-)}{v}_i$, $i = 1, 2, 3$ are prescribed on the face surfaces. Let them be equal to zero in Subsection 4.4 for the sake of simplicity.

We may rewrite (4.50), according to the mentioned at the end of Remark 1, as follows

$$
\begin{aligned}
&\overset{r}{\Psi}_{ji} := \overset{(+)}{v}_j \overset{(+)}{F}_{,i} + (-1)^r \overset{(-)}{v}_j (-\overset{(-)}{F}_{,i}) + \overset{(+)}{v}_i \overset{(+)}{F}_{,j} + (-1)^r \overset{(-)}{v}_i (-\overset{(-)}{F}_{,j}),\\
&\overset{r}{\Psi}_{\alpha 3} := \overset{(+)}{v}_\alpha + (-1)^r \overset{(-)}{v}_\alpha (-1) + \overset{(+)}{v}_3 (-\overset{(+)}{h}_{,\alpha}) + (-1)^r \overset{(-)}{v}_3 \overset{(-)}{h}_{,\alpha}\\
&= \overset{(+)}{v}_\alpha - (-1)^r \overset{(-)}{v}_\alpha - \overset{(+)}{v}_3 \overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{v}_3 \overset{(-)}{h}_{,\alpha},\\
&\overset{r}{\Psi}_{3\alpha} = \overset{(+)}{v}_3 (-\overset{(+)}{h}_{,\alpha}) + (-1)^r \overset{(-)}{v}_3 \overset{(-)}{h}_{,\alpha} + \overset{(+)}{v}_\alpha + (-1)^r \overset{(-)}{v}_\alpha (-1)\\
&= -\overset{(+)}{v}_3 (\overset{(+)}{h}_{,\alpha}) + (-1)^r \overset{(-)}{v}_3 \overset{(-)}{h}_{,\alpha} + \overset{(+)}{v}_\alpha - (-1)^r \overset{(-)}{v}_\alpha,\\
&\overset{r}{\Psi}_{33} := \overset{(+)}{v}_3 - (-1)^r \overset{(-)}{v}_3 + \overset{(+)}{v}_3 - (-1)^r \overset{(-)}{v}_3 = 2[\overset{(+)}{v}_3 - (-1)^r \overset{(-)}{v}_3],
\end{aligned}
$$

because of

$$
\begin{aligned}
&\overset{(\pm)}{F}_{,\alpha} = -\overset{(\pm)}{h}_{,\alpha}, \quad \overset{(+)}{F}_{,3} = 1,\\
&-\overset{(\pm)}{F}_{,\alpha} = \overset{(\pm)}{h}_{,\alpha}, \quad -\overset{(+)}{F}_{,3} = -1.
\end{aligned}
$$

For the sake of transparency in Subsection 4.4 we give analysis of the the governing system in the N=0 approximation.

4.4 Discussion of peculiarities of well-posedness of boundary conditions for $D3$ angular domain Ω in $N = 0$ approximation

In the same manner we can construct hierarchical models when on one face surface either the surface forces or neither the surface forces nor velocities are prescribed, while on the another one the velocities are prescribed.

Hierarchical models in Eulerian coordinates for Newtonian viscous fluid in prismatic shell-like domains (as container) are constructed. As is already clear, when the effects of viscosity may be supposed to be negligible, we get hierarchical models for perfect fluids. Initial, contact, and boundary value conditions from classical ones should be rewritten in the explained in the present paper (see Subsubsection 2.1.1) way of passage to the moments. The governing equations are the singular equations, in the case of angular $3D$ domains. On transparent examples it is shown that by investigating well-posedness of BVPs, boundary conditions may be nonclassical, in general.

In order to illustrate it we analyse two concrete examples when geometry of angular 3D domain is defined by

$$h(x_1, x_2) = h_0 x_2^{\kappa}, \ \ \kappa \geq 0, \ \ 0 \leq x_2 \leq L, \ \ L_1 < x_1 < L_2. \tag{4.51}$$

$L_1 = -\infty$, $L_2 = +\infty$ are admissible as well.

In this case we have to do with the following two equations

$$x_2^2\left(\overset{0}{w}_{30,11} + \overset{0}{w}_{30,22}\right) + \kappa x_2 \overset{0}{w}_{30,2} - \kappa \overset{0}{w}_{30} = 0 \tag{4.52}$$

which it follows from (4.48) and from (4.43) it follows

$$x_2\left(\overset{0}{\widetilde{v}}_{30,11} + \overset{0}{\widetilde{v}}_{30,22}\right) + \kappa \overset{0}{\widetilde{v}}_{30,2} = 0 \tag{4.53}$$

in the case when either velocities or stresses are known on the face surfaces, respectively.

Equations (4.52) and (4.53) are singular PDEs, in other words, PDEs with the order and type degeneracy on the degeneracy line $x_2 = 0$.

For equation (4.52) only the Keldysh Problem is well-posed and it's only, when $\kappa \geq 1$.

For equation (4.53) when $\kappa < 1$ the Dirichlet and when $\kappa \geq 1$ the Keldysh BVPs are well-posed.

We consider fluid flow in prismatic shell-like 3D domain when at the edge of the domain tangent half-planes to the face surfaces create a dihedral angle with a line angle φ. It will be observed that considering viscous flow near the fixed dihedral angle, replacing the boundary condition velocity $v = 0$ on the edge by boundedness of velocity v in a neighborhood of the edge for $\kappa \geq 1$ i.e., $\varphi \in [0, \pi[$, in particular, of the mathematical cusp it means $\kappa > 1$, i.e., $\varphi = 0$, as it is in the case of the Keldysh problem. When the face surfaces smoothly merge each into another through the cusped edge, it means for $\kappa < 1$ i.e., $\varphi = \pi$ the Dirichlet problem is well-posed and the boundary condition should be $v = 0$. These results are in a good accordance with the viscous boundary layer concept, according to experimental results of J. Nikuradse.

A case of non-homogeneous viscosity is discussed as well.

5 Conclusions

For different materials: homogeneous, non-homogeneous, isotropic, anisotropic, piezoelectric, viscoelastic, nanostructures, etc., the differential hierarchical models are constructed and peculiarities caused by singularity of geometry of solid bodies and containers of fluids are discussed.

To this end I. Vekua's dimension reduction method formalized in [6], [7] in an unified form, which is directly applicable to any physical problem (model) containing the thickness as a parametre, is utilized.

I. Vekua's dimension reduction method for prismatic shells, i.e., in symmetric case for plates of variable thickness, is generalized for prismatic bars (see [11]). A fluid contained in such bar-like containers is considered in [12], [14].

By means of Vekua's dimension reduction method and of his modifications two type hierarchical models are constructed:

- the first type models, when on the face surfaces stresses are prescribed, while displacements for solids or velocities for fluids are calculated from values on the face surfaces of their Fourier-Legandre series;

- the second type models, when on the face surfaces displacements for solids or velocities for fluids are prescribed, while stresses are calculated from values of their the Fourier-Legandre series on the face surfaces.

During considering non-homogeneous materials constitutive coefficients should be either independent of x_3 or they may depend on x_3 but peculiarities arising by that will depend on the kind of vanishing of constitutive coefficients on the boundary of projection for the solid body and that of container boundary in the case of the constant thickness for homogeneous cases, while for the case of the variable thickness it will be depending on the kind of vanishing of the product of the constitutive coefficient and the thickness.

Here it should be noted that $\overset{N}{v}_{kr}$ and $\overset{N}{u}_{kr}$ $k=\overline{1,3}$, $r=0,1,2,...,N$, $N=0,1,2,...$ mean solutions of the governing system of the Nth approximation with respect to that unknowns, while u_{kr} and v_{kr} mean the Fourier- Legendre coefficients up to the fector $(\tau+\frac{1}{2})^{\frac{1}{2}}h^{-\frac{1}{2}}$ (i.e. the mathematical moments of the unknown displacements u_k, $k=\overline{1,3}$), respectively and that weighted ones (i.e. s.c. weighted moments given by (2.2)).

Note that (see [13] and for that of cusped prismatic shells see [14])

$$u_{kr}=\lim_{N\to\infty}\overset{N}{u}_{kr},\quad k=\overline{1,3}.$$

Sometimes (mainly) in the literature the upper index $N=0,1,2,...$, indicating the order of the approximation, for the sake of simplicity of notion, is omitted, and the reader should be careful not to be confused.

I. Vekua's approximated solution

$$\overset{N}{u}_k=\sum_{r=0}^{N}a\Big(r+\frac{1}{2}\Big)\overset{N}{u}_{kr}P_r(ax_3-b),\quad k=\overline{1,3},$$

and the partial sum

$$\sum_{r=0}^{N}a\Big(r+\frac{1}{2}\Big)u_{kr}P_r(ax_3-b)$$

of the Fourier-Legendre series are different but the both tend to the exact solution u as $N\to\infty$.

It is also remarkable that I. Vekua (see [1] pp. 401-405) in the $N=1$ approximation besides classical normal, tangential, and transversal (intersecting) forces in other words, according to I. Vekua, the zero order weighted mathematical moments and the first order mathematical moments according I. Vekua, defined the additional first order mathematical moment called by him as the splitting couple of forces which is nothing more then the equilibrated stress vector that can be identified with singularities in classical linear clasticity known as double force systems without physical moments equivalent to two oppositely directed forces at the same point (see [15], p. 127). Singularities of this type were first discussed by Love [16].

One more thing, in some practical (enginering) models displacements are represented as polynomials of order $\leq n$ but they may represented as some linear combinations of Legenre polynomials (see [17], p. 529), in particular,

$$x_3^n=a_{0n}P_n(x_3)+a_{1n}P_1(x_3)+\cdots+a_{nn}P_n(x_3),$$

therefore models of such type are contained as particular cases in I. Vekua's hierarchical models.

Applying I. Vekua's dimension reduction method, hierarchical models (approximations) for thermoelastic chiral porous prismatic shells have been constructed. In the $N = 0$ approximation, using the approach developed in [6, 7] we have proved for the case

$$h(x_1, x_2) = h_0 x_2^{\kappa}, \quad h_0 = \text{const} > 0 \;\; \kappa = \text{const} \geq 0,$$

the following expected result: in the case of the cusped (i.e. $\kappa > 0$) prismatic shell the cusped edge i.e., where the thickness vanishes $(2h(x_1, 0) = 0)$ the edge may be fixed only if

$$0 < \kappa < 1,$$

in other words the Dirichlet type problem, when desired displacement may be prescribed on the entire lateral boundary of the prismatic shell under consideration, is well-posed, while when

$$\kappa \geq 1$$

it is not the case, the cusped edge cannot be fixed and boundary condition should be replaced by the demand of boundedness of the displacement near the cusped edge, in other words, the Keldysh type boundary value problem is well-posed. Note that the vertical displacement is not affected by chirality, since in (3.16) $\varkappa_{\alpha 3k0} = 0$, $\varkappa_{3\beta k0} = \nu$ (see (3.15)) and, therefore, $f \cdot (\mathcal{E}_{3km}\varkappa_{3km0} + \mathcal{E}_{3km}\varkappa_{3km0}) = 0$, but it is not the case in other approximations.

6 Appendix. Prismatic shell-like and bar-like 3D domains

First a few words about prismatic shells (see also [1], [3], [4]).

Let us consider prismatic shells (see, Figures 6.1 and etc., and also [1], [3], [4]), occupying 3D domain Ω with the projection ω (on the plane $x_3 = 0$) and the face surfaces

$$x_3 = \overset{(+)}{h}(x_1, x_2) \in C^2(\omega) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2) \in C^2(\omega), \ (x_1, x_2) \in \omega.$$

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) > 0, \ (x_1, x_2) \in \omega, \tag{1}$$

is the thickness of the prismatic shell. A part of $\partial\omega$, where the thickness vanishes, i.e., $2h = 0$, is said to be a cusped edge. We shall call it a blunt edge, if in the symmetric case (see below) $\partial\Omega$ contains it smoothly, otherwise, i.e., the points of the cusped edge are points of nonsmoothness of $\partial\Omega$, we shall call it a sharp edge (see Figures 6.2, 6.3). In the nonsymmetric case the cusp edge we shall call blunt provided at least one tangent to a profile is orthogonal to the shell projection (see Figures 6.6-6.12).

Let

$$2\tilde{h}(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2), \ (x_1, x_2) \in \omega. \tag{2}$$

In the case of the symmetric prismatic shell, i.e., when

$$\overset{(-)}{h}(x_1, x_2) = -\overset{(+)}{h}(x_1, x_2),$$

evidently

$$2\tilde{h}(x_1, x_2) \equiv 0, \ (x_1, x_2) \in \omega.$$

If

$$h(0) = 0 \ \& \ 0 \le \frac{\partial h}{\partial \nu} < \infty \ \text{ or } \ \int_0^{\varepsilon} \frac{d\nu}{h(\nu)} = \infty$$

the cusped edge is sharp, in the $N = 0$ approximation (model) Problem E (Keldysh) is well-posed,

in particular, if

$$h = h_0 x_2^{\varkappa}, \ h_0, \ \varkappa = const > 0, \text{ then } 1 \le \varkappa, \ \int_0^{\varepsilon} \frac{dx_2}{h(x_2)} = \infty;$$

If

$$h(0) = 0, \ \& \ \frac{\partial h}{\partial \nu} = \infty, \ \text{ or } \ \int_0^{\varepsilon} \frac{d\nu}{h(\nu)} < \infty$$

the cusped edge is blunt, in the $N = 0$ approximation (model) Problem D is well-posed, in particular, if

$$h = h_0 x_2^{\varkappa}, \ h_0, \ \varkappa = const > 0, \ \varkappa < 1, \ \text{ then } \int_0^{\varepsilon} \frac{dx_2}{h(x_2)} < \infty.$$

Distinctions between the prismatic shell of constant thickness and the standard shell of constant thickness are shown in Figures 6.4, 6.5, where cross-sections of the prismatic shell of constant thickness with its projection and of the standard shell of constant thickness with its middle surface are given in red and green colors, respectively, with common parts in blue. In other words, the lateral boundary of the standard shell is orthogonal to the "middle surface" of the shell, while the lateral boundary of the prismatic shell is orthogonal to the prismatic shell's projection on $x_3 = 0$ (see [4]).

In particular, let ω be a domain bounded by a sufficiently smooth arc $(\partial\omega \setminus \overline{\gamma^0})$ lying in the half -plane $x_2 > 0$ and a segment $\overline{\gamma^0}$ of the x_1-axis ($x_2 = 0$). Let the thickness look like (see Figures 6.2, 6.3)

$$2h(x_1, x_2) = 2h_0 x_2^{\kappa}, \qquad h_0,\ \kappa = const > 0, \tag{3}$$

which corresponds to the case

$$\overset{(\pm)}{h}(x_1, x_2) = \overset{(\pm)}{h_0} x_2^{\kappa}, \quad \overset{(\pm)}{h_0} = const, \quad \overset{(+)}{h_0} > \overset{(-)}{h_0}, \quad 2h_0 := \overset{(+)}{h_0} - \overset{(-)}{h_0}.$$

In this case we have to do with a blunt edge for $\kappa < 1$ and with a sharp edge for $\kappa \geq 1$, respectively.

In Figures 6.6-6.20 ($\hat{\varphi}$ is the angle at the cusp between tangents $\overset{(+)}{T}$ and $\overset{(-)}{T}$, ν is an inward normal at O to $\partial\omega$) we show some characteristic (typical) profiles (cross-sections) of cusped prismatic shells.

Let

$$2h(x_1, x_2, t) = 2h_0 t^{\kappa},$$

then we will have a time dependent reverse thinning (i.e. blunting) with respect to t with the above described geometry in the Cartesian frame $0tx_3$.

In this case we have to do with singular hyperbolic equations and systems, in the principal part of which it plays a crucial ole a member $\frac{\partial^2}{\partial t^2}[t^{\kappa} v_{30}(t)]$. Supposing unknown functions depending only on time we arrive at

$$\frac{\partial^2}{\partial t^2}\Big[t^{\kappa} v_{30}(t)\Big] = 0$$

$$v_{30}(t) = c_1 t^{1-\kappa} + c_2 t^{-\kappa}; \quad v_{30}(t) = c_1 t^{1-\kappa}, \ \ c_2 = 0; \quad \dot{v}_{30}(t) = c_1(1-\kappa)t^{-\kappa}$$

$\kappa < 1:$

$$v_{30}(0) = 0$$

either

$$\lim_{t\to 0} t^{\kappa} \dot{v}_{30}(t) = c_1(1-\kappa) = m, \ \ c_1 = \frac{m}{1-\kappa}$$

or

$$\dot{v}_{30}(t) \ \text{ is bounded, i.e., } \ \mathrm{m} = 0$$

$\kappa \neq 1, \ \ c_2 = 0:$ either

$$\lim_{t\to 0} t^{\kappa-1} v_{30}(t) = c_1 = m,$$

or

$$v_{30}(t) \ \text{ is bounded, i.e., } \ \mathrm{m} = 0$$

either

$$\lim_{t\to 0} t^{\kappa} v_{30}(t) = c_1(1-\kappa) = m,$$

or

$$\dot{v}_{30}(t) \ \text{ is bounded, i.e., } \ \mathrm{m} = 0.$$

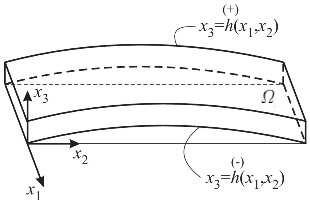

Figure 6.1: A prismatic shell of constant thickness. $\partial\Omega$ is a Lipschitz boundary

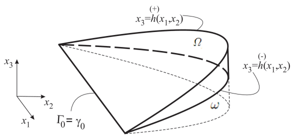

Figure 6.2: A sharp cusped prismatic shell with a semicircle projection. $\partial\Omega$ is a Lipschitz boundary

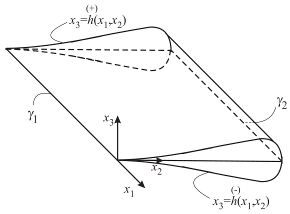

Figure 6.3: A cusped plate with sharp γ_1 and blunt γ_2 edges, $\gamma^0 := \gamma_1 \cup \gamma_2$. $\partial\Omega$ is a non-Lipschitz boundary

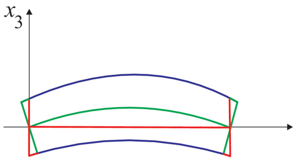

Figure 6.4: Comparison of cross-sections of prismatic and standard shells

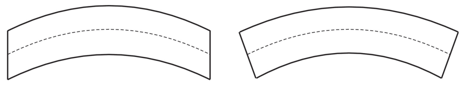

Figure 6.5: Cross-sections of a prismatic (left) and a standard shell with the same mid-surface

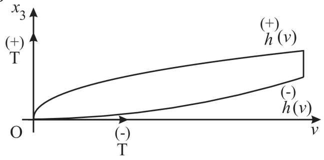

Figure 6.6: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} = \frac{\pi}{2}$). It has a Lipschitz boundary

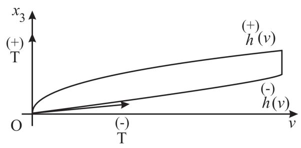

Figure 6.7: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} \in]0, \frac{\pi}{2}[$). It has a Lipschitz boundary

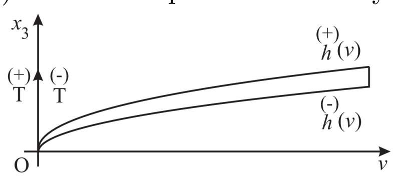

Figure 6.8: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} = 0$). It has a non-Lipschitz boundary

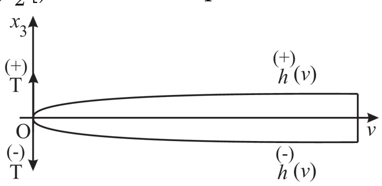

Figure 6.9: A cross-section of a blunt cusped plate ($\hat{\varphi} = \pi$). It has a Lipschitz boundary

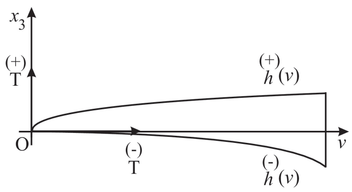

Figure 6.10: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} = \frac{\pi}{2}$). It has a Lipschitz boundary

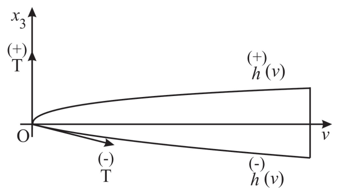

Figure 6.11: A cross-section of a blunt cusped prismatic shell ($\hat{\varphi} \in]\frac{\pi}{2}, \pi[$). It has a Lipschitz boundary

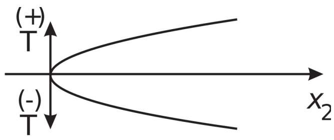

Figure 6.12: $\hat{\varphi} = \pi$

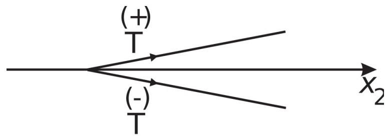

Figure 6.13: Wedge, $\hat{\varphi} \in]0, \pi[$

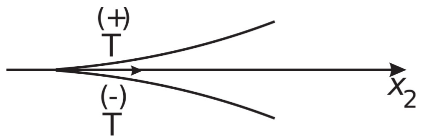

Figure 6.14: $\hat{\varphi} = 0$

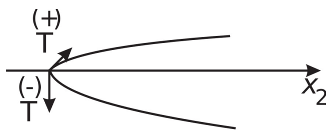

Figure 6.15: $\frac{\pi}{2} < \hat{\varphi} < \pi$

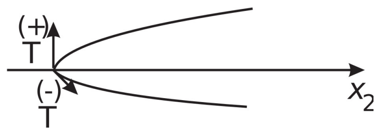

Figure 6.16: $\frac{\pi}{2} < \hat{\varphi} < \pi$

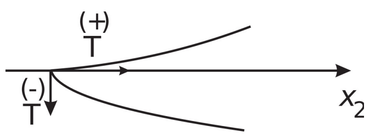

Figure 6.17: $\hat{\varphi} = \frac{\pi}{2}$

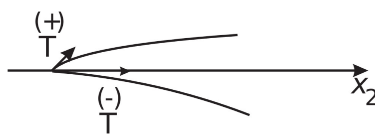

Figure 6.18: $0 < \hat{\varphi} < \frac{\pi}{2}$

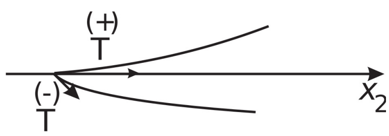

Figure 6.19: $0 < \hat{\varphi} < \frac{\pi}{2}$

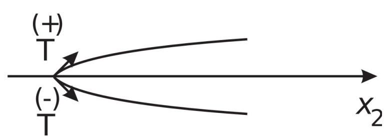

Figure 6.20: $0 < \hat{\varphi} < \pi$

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