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Shear Deformations for *Weakly*-Nonlinear Elastic Materials

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We consider a simple boundary value problems in the context of non-linear elasticity: the rectilinear shear deformation driven by a gradient of pressure in a slab made of an incompressible hyperelastic material confined between two rigid plates. Basic conditions for the existence, uniqueness and regularity of the solution are provided by following two different approaches. Exact solutions are found for some models of the strain-energy function.

Keywords: nonlinear elasticity, rectilinear shear, axial shear, generalized neo-Hookean materials.

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1. Introduction and Basic Equations

Let $\boldsymbol{X} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ be the position vector (relative to an origin O) of a particle P of a body $\mathcal{B}$ at the initial time $t = 0$, and $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector (relative to the same origin O) of the same particle at time $t > 0$. For convenience we choose the configuration occupied by $\mathcal{B}$ at the initial time as the reference configuration and denote it $\mathcal{B}_r$. A motion of the body $\mathcal{B}$ in the time interval $[0, T]$ is a mapping $\boldsymbol{\chi}$ which assigns to $(\boldsymbol{X}, t) \in \mathcal{B}_r \times [0, T]$ a point $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ of the three-dimensional Euclidean point space and is such that for any $t \in [0, T]$ $\boldsymbol{\chi}_t \equiv \boldsymbol{\chi}(\cdot, t)$ is one-to-one. The configuration of the solid at time t, $\mathcal{B}_t = \boldsymbol{\chi}_t(\mathcal{B}_r) = \boldsymbol{\chi}(\mathcal{B}_r, t)$, is called current configuration.

In many situations, as the situations we shall study, one wish to consider only two configurations of the body, the initial configuration $\mathcal{B}_r$ and the final configuration $\mathcal{B}_T$. The mapping $\boldsymbol{\chi}_T : \boldsymbol{X} \in \mathcal{B}_r \mapsto \boldsymbol{x} = \boldsymbol{\chi}_T(\boldsymbol{X}) \in \mathcal{B}_T$ is then referred to as a deformation of $\mathcal{B}$. The deformation gradient $\boldsymbol{F}$ and the left Cauchy-Green deformation tensor $\boldsymbol{B}$ are the second-order Cartesian tensors

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}_T}{\partial \boldsymbol{X}}, \quad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T. \tag{1}$$

The mathematical model for the material behaviour of an incompressible hyperelastic solid is characterized by a strain-energy density (measured per unit volume in the undeformed state)

$$W = W(I_1, I_2), \tag{2}$$

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where

$$I_1 = \mathrm{tr}\boldsymbol{B} \quad \text{and} \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{B})^2 - \mathrm{tr}\boldsymbol{B}^2\right] = \mathrm{tr}\boldsymbol{B}^{-1} \tag{3}$$

are the first and second principal invariants of $\boldsymbol{B}$. (The third principal invariant $I_3 = \det\boldsymbol{B} = (\det\boldsymbol{F})^2$ is equal to unity due to the incompressibility of the material.)

For consistency of the model (2) with linear elasticity in the limit of infinitesimal strains, it is necessary that

$$W_1(3,3) + W_2(3,3) = \frac{\mu}{2}, \tag{4}$$

where the subscript i ($i = 1, 2$) denotes differentiation with respect to I_i and μ is the infinitesimal shear modulus.

Since later we shall be interested also in regimes in which the strains are small but not infinitesimal, it is convenient to report here some basics of the weakly non-linear theory of elasticity. Within this theory the elastic stored energy W is expanded in terms of the invariants of the Green-Lagrange strain tensor $\boldsymbol{E} = (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})/2$ as

$$\mathcal{J}_1 = \mathrm{tr}(\boldsymbol{E}), \quad \mathcal{J}_2 = \mathrm{tr}(\boldsymbol{E}^2), \quad \mathcal{J}_3 = \mathrm{tr}(\boldsymbol{E}^3). \tag{5}$$

For incompressible solids, Ogden [11] showed that the expansion of the strain energy function W up to terms of order four involves only three material constants. In the notation of Hamilton et al. [7], it is written as

$$W = \mu\mathcal{J}_2 + \frac{\mathcal{A}}{3}\mathcal{J}_3 + \mathcal{D}\mathcal{J}_2^2, \tag{6}$$

where $\mathcal{A}$ and $\mathcal{D}$ are non-linear Landau elastic constants. Because of the incompressibility constraint the invariants $\mathcal{J}$'s are not independent as they must satisfy the equation [3]

$$\mathcal{J}_1 + \mathcal{J}_1^2 - \mathcal{J}_2 + \frac{2}{3}\mathcal{J}_1^3 - 2\mathcal{J}_1\mathcal{J}_2 + \frac{4}{3}\mathcal{J}_3 = 0. \tag{7}$$

On using such a restriction and the connections

$$I_1 = 2\mathcal{J}_1 + 3, \quad I_2 = 2\mathcal{J}_1^2 - 2\mathcal{J}_2 + 4\mathcal{J}_1 + 3, \tag{8}$$

one can verify that, for consistency of (2) with (6), it is necessary that (4) and the following two identities

$$\begin{gathered} W_1(3,3) + 2W_2(3,3) = -\frac{\mathcal{A}}{8}, \\ W_1(3,3) + 3W_2(3,3) + W_{11}(3,3) + 2W_{12}(3,3) + W_{22}(3,3) = \frac{\mathcal{D}}{2} \end{gathered} \tag{9}$$

hold true.

For an incompressible isotropic hyperelastic material the Cauchy stress tensor $\boldsymbol{T}$ is derived from the strain-energy density (2) through the constitutive equation

$$\boldsymbol{T} = -p\boldsymbol{I} + 2W_1\boldsymbol{B} - 2W_2\boldsymbol{B}^{-1}, \tag{10}$$

where p is a Lagrange multiplier associated with the constraint of incompressibility.

Subject to the analyticity of the strain-energy (2) in a neighbourhood of the reference configuration $I_1 = I_2 = 3$, one can write W as an infinite series in powers of $I_1 - 3$ and $I_2 - 3$. Thus,

$$W(I_1, I_2) = \frac{1}{2}\sum_{p,q=0}^{+\infty} C_{pq}(I_1 - 3)^p(I_2 - 3)^q, \tag{11}$$

where the coefficients C_{pq} do not depend on the deformation. Since the energy is defined up to an additive constant, W may be assumed to vanish in the reference configuration and hence it is usual to require that $C_{00} = 0$. An isotropic strain-energy function can be approximated as closely as desired by an expansion of the form (11) containing a finite number of terms. The neo-Hookean strain-energy function

$$W_{nH} = \frac{\mu}{2}(I_1 - 3), \tag{12}$$

with $C_{10} = \mu$ for consistency with the linear theory, and the Mooney-Rivlin model

$$W_{MR} = \frac{C_{10}}{2}(I_1 - 3) + \frac{C_{01}}{2}(I_2 - 3), \tag{13}$$

with $C_{10} + C_{01} = \mu$, are approximations of the strain-energy function W that have been most widely adopted in the development of the non-linear theory of elasticity and its early applications. In this note, to illustrate our analytic results, we shall consider the following approximation of W:

$$W_{qMR} = W_{MR} + \frac{1}{2}\left[C_{20}(I_2 - 3)^2 + C_{11}(I_1 - 3)(I_2 - 3) + C_{02}(I_2 - 3)^2\right], \tag{14}$$

where the elastic moduli C_{ij} $(i, j = 0, 1, 2)$ satisfy the relations

$$C_{10} = 2\mu + \frac{\mathcal{A}}{4}, \quad C_{01} = -\mu - \frac{\mathcal{A}}{4}, \quad C_{20} + C_{11} + C_{02} = \frac{1}{2}\left(\mu + \frac{\mathcal{A}}{2} + \mathcal{D}\right) \tag{15}$$

in order to meet (4) and (9). Since W_{qMR} contains terms that are quadratic in the invariants $I_1 - 3$ and $I_2 - 3$ we shall henceforth refer to (15) as the quadratic model.

In the literature relevant to the applications of non-linear elasticity to soft tissues large use of the so-called generalized neo-Hookean models, i.e. strain-energy functions in the form $W = W_{gnH}(I_1)$, is made. The interest for generalized neo-Hookean models is motivated by reasons of mathematical feasibility and by the fact that they catch the typical J-shaped curves in uniaxial tension tests on biological tissues [9]. Elastic materials for which the adoption of a neo-Hookean model for the strain energy yield theoretical results in good agreement with the experimental evidences are called generalized neo-Hookean materials. As an immediate consequence of (4), for any generalized neo-Hookean material one has

$$\frac{\mathrm{d}W_{gnH}}{\mathrm{d}I_1}(3) = \frac{\mu}{2}. \tag{16}$$

Three notable examples of generalized neo-Hookean models are

- the Fung-Demiray model [2, 5]

$$W_{FD} = \frac{\mu}{2\alpha}\left\{\exp[\alpha(I_1 - 3)] - 1\right\}, \tag{17}$$

  where α is a positive constant accounting for effects due to the stiffening of the material;
- the Gent strain-energy density [6]

$$W_G = -\frac{\mu}{2}J_m \log\left(1 - \frac{I_1 - 3}{J_m}\right), \tag{18}$$

  where the positive parameter J_m provides a measure of the limiting chain extensibility of the elastomers;
- the Knowles power-law model

$$W_K = \frac{\mu}{2b}\left\{\left[1 + \frac{b}{n}(I_1 - 3)\right]^n - 1\right\}, \tag{19}$$

  where b and k are two positive constitutive parameters.

Observe that the Fung-Demiray, Gent and Knowles models tend to the neo-Hookean strain-energy for small values of the constitutive parameters α, $1/J_m$ and b, respectively.

The aim of this note is to asses some constitutive peculiarities of the models for the strain-energy density reported above by solving a simple boundary value problem: the rectilinear shear deformation of a slab hinged to two rigid plates and subjected to a uniform gradient of pressure in a direction parallel to the boundaries. This deformation has been studied by several authors and a detailed review is given in [4]. Here, we propose two different approaches to study the existence, uniqueness and regularity of the solution. The former is based on the strong ellipticity of the strain-energy function and the introduction of the generalized shear compliance which play a fundamental role in the inversion of the shear stress-shear strain relation. The latter is instead based on classical results in calculus of variations. The variational problem we shall consider consists in the minimization of the energy functional resulting from the sum of the elastic stored energy and the potential energy due to the action of a uniform gradient of pressure.

The plan of the paper is as follows. In section 2 we derive the equations governing the rectilinear shear deformation and introduce the boundary value problem (BVP) the solvability of which is discussed in sections 3 and 4. In particular, we show that the strong ellipticity or, equivalently, the strict convexity of the strain energy function may guarantee the existence and uniqueness of the solution of the BVP governing rectilinear shear deformations. Finally, we provide some exact solutions in section 5 and conclude with some remarks (section 6).

2. Rectilinear shear

Let us consider the *rectilinear shear* deformation

$$x = X + u(Z), \quad y = Y, \quad z = Z, \tag{20}$$

where $u(Z)$ is an unknown function to be determined. This is an isochoric inhomogeneous deformation which reduces to simple shear when u is linear in Z, i.e. $u(Z) = kZ$. The left Cauchy-Green deformation tensor associated with the rectilinear shear deformation and its inverse are as follows

$$[\boldsymbol{B}] = \begin{pmatrix} 1+u_Z^2 & 0 & u_Z \\ 0 & 1 & 0 \\ u_Z & 0 & 1 \end{pmatrix}, \quad [\boldsymbol{B}^{-1}] = \begin{pmatrix} 1 & 0 & -u_Z \\ 0 & 1 & 0 \\ -u_Z & 0 & 1+u_Z^2 \end{pmatrix}, \tag{21}$$

by which the first and second principal scalar invariants of $\boldsymbol{B}$ are found to be $I_1 = I_2 = 3 + u_Z^2$.

The Cauchy stress tensor necessary to support the deformation (20) has components

$$\begin{gathered} T_{11} = -p + 2W_1(1+u_Z^2) - 2W_2, \quad T_{13} = 2(W_1 + W_2)u_Z, \\ T_{22} = -p - 2W_2, \quad T_{33} = -p + 2W_1 - 2W_2(1+u_Z^2), \quad T_{12} = T_{23} = 0, \end{gathered} \tag{22}$$

where the derivatives of the strain energy function are evaluated at $I_1 = I_2 = 3+u_Z^2$.

In this case the equilibrium equations reduce to the system

$$-p_X + \frac{\mathrm{d}}{\mathrm{d}Z}[Q(u_Z^2)u_Z] = 0, \quad p_Y = 0, \quad \frac{\mathrm{d}}{\mathrm{d}Z}\left[-p + 2W_1 - 2W_2(1+u_Z^2)\right] = 0, \tag{23}$$

where

$$Q(u_Z^2) = 2\left[W_1(3+u_Z^2, 3+u_Z^2) + W_2(3+u_Z^2, 3+u_Z^2)\right] \tag{24}$$

is the generalized shear modulus.

System (23) is an overdetermined system of three partial differential equations in the two unknowns $p = p(X, Y, Z)$ and $f = f(Z)$. This overdetermined system is similar to the one introduced by Zhang and Rajagopal [12] for studying Poisuille-type motions in nonlinear elastic solids. It is easy to show that system (23) is compatible if and only if the Lagrange multiplier is in the form

$$p = A_0 X + 2W_1 - 2W_2(1+u_Z^2), \tag{25}$$

with A_0 being a constant which in what follows, with abuse of terminology, we will refer to as the (uniform) gradient of pressure in the X-direction, and the displacement field u satisfies the ordinary differential equation

$$\frac{d}{dZ}\left[Q(u_Z)u_Z\right] = A_0. \tag{26}$$

It is clear that a *formal* solution of (26) has not to be confused with the solution to a specific *physical* problem associated to the deformation field (20). Here, will not discuss the existence of formal solutions to (26). We will instead look for rectilinear shear deformations of a slab with thickness $2H$ that is clamped at the boundaries $Z = \pm H$. We shall then solve (26) supplemented by the Dirichlet boundary conditions

$$u(\pm H) = 0. \tag{27}$$

To simplify the sequent analysis we introduce the dimensionless variables

$$Z^\star = \frac{Z}{H}, \quad u^\star = \frac{u}{U}, \quad Q^\star = \frac{Q}{\mu}, \tag{28}$$

where $U = |A_0|H^2/\mu$ is the most appropriate reference value for the displacement field. This is motivated by the fact that since the boundaries of the slab are fixed, the rectilinear shear deformation is caused only by the gradient of pressure in the X-direction and, on the other hand, the infinitesimal shear modulus provides a measure of the rigidity (and hence resistance to deformation) of the material. Introducing the non-dimensional quantities (28) into (26) and (27) and omitting the asterisks (for simplicity of notation) produce the dimensionless BVP

$$\frac{\mathrm{d}}{\mathrm{d}Z}[Q(\varepsilon^2 u_Z^2)u_Z] = \mathrm{sign}(A_0), \qquad u(\pm 1) = 0, \tag{29}$$

where the parameter

$$\varepsilon = \frac{U}{H} = \frac{|A_0|H}{\mu} \tag{30}$$

gives a measure of the magnitude of the shear strain.

In view of the invariance of the BVP (29) under the transformation $Z \to -Z$ we deduce that its solutions (if any) are symmetric around $Z = 0$. Then, the strain u_Z vanishes at $Z = 0$ and integration of $(29)_1$ yields

$$Q\left(\varepsilon^2 u_Z^2\right) u_Z = \mathrm{sign}(A_0) Z. \tag{31}$$

From (31) we deduce that if the generalized shear modulus is positive (as we shall soon see, this is the case when the strain energy function is strongly elliptic), then the solution to the BVP (29) is positive in the interval $]-1,1[$ if the gradient of pressure A_0 is negative, whereas it is negative if $A_0 > 0$. Since the sign of the gradient of pressure A_0 depends exclusively on the choice of direction of the X-axis, without loss of generality we shall henceforth assume that A_0 is negative. In other words, in the following sections we shall study the existence, uniqueness and regularity of the solutions of equation (31) with $\mathrm{sign}(A_0) = -1$.

From (31) it is easy to realize that the existence and uniqueness of the solution of the BVP (29) depends on the invertibility of the shear stress ($\tau = T_{13}$) - shear strain ($\gamma = u_Z$) relation

$$\tau = Q(\gamma^2)\gamma \equiv \mathcal{F}(\gamma). \tag{32}$$

In fact, should the interval $[-\varepsilon, \varepsilon]$ be contained in the codomain of $\mathcal{F}$ and $\mathcal{F}$ be invertible with inverse $\mathcal{F}^{-1}$, then (29) could be solved uniquely to obtain $u_Z = \mathcal{F}^{-1}(-\varepsilon Z)$ and hence determine the displacement field u by quadrature.

3. Invertibility of the stress-strain relation

We now show that if the strain energy function (2) satisfies the strong ellipticity condition then the stress-strain relation (32) is invertible. The spectrum of the left Cauchy-Green deformation tensor associated with the rectilinear shear deformation (20), i.e. the set of principal stretches, is $\{\lambda^2, 1, \lambda^{-2}\}$ with

$$\lambda = \sqrt{\frac{\gamma^2 + 2 - \sqrt{\gamma^2(\gamma^2+4)}}{2}}, \tag{33}$$

where, for brevity of notation, we have set $\gamma = u_Z$. In terms of the principal stretches the scalar invariants I_1 and I_2 read

$$I_1 = I_2 = \lambda^2 + \lambda^{-2} + 1. \tag{34}$$

In view of (34), we can express the strain energy function in terms of the stretch λ by setting $\widetilde{W}(\lambda) = W(\lambda^2 + \lambda^{-2} + 1, \lambda^2 + \lambda^{-2} + 1)$. As proven by Ogden [10], W satisfies the strong ellipticity condition if and only if $\widetilde{W}$ satisfies the inequalities

$$\frac{\lambda \widetilde{W}'(\lambda)}{\lambda^2 - 1} > 0, \quad \lambda^2 \widetilde{W}''(\lambda) + \frac{2\lambda \widetilde{W}'(\lambda)}{\lambda^2 + 1} > 0, \tag{35}$$

where the prime denotes differentiation of with respect to λ.

With the aid of (33) and (34), these inequalities can be rewritten as

$$\begin{aligned} &\frac{\lambda^2+1}{\lambda} Q(\gamma^2) > 0, \\ &\frac{(\lambda^2+1)^2}{\lambda^2} \underbrace{\left[Q(\gamma^2) + 2\gamma^2 \frac{\mathrm{d}Q}{\mathrm{d}\gamma^2} \right]}_{= \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\gamma}} > 0, \end{aligned} \tag{36}$$

Inequality $(36)_1$ implies the positivity of the generalized shear modulus Q, while $(36)_2$ yields that $\mathcal{F}$ is an increasing function of the strain. $\mathcal{F}$ is then invertible.

Finally, since in terms of γ the principal scalar invariants read $I_1 = I_2 = 3 + \gamma^2$, the elastic energy stored in a rectilinear shear deformation depends solely on the shear strain γ according to

$$\widehat{W}(\gamma) = W(\gamma^2 + 3, \gamma^2 + 3). \tag{37}$$

From (37) it is easy to check that

$$\mathcal{F} = \frac{\mathrm{d}\widehat{W}}{\mathrm{d}\gamma} \quad \text{and} \quad Q = \frac{1}{\gamma}\frac{\mathrm{d}\widehat{W}}{\mathrm{d}\gamma}, \tag{38}$$

by which one can prove that the strong ellipticity condition is equivalent to the requirement that $\widehat{W}$ is strictly convex.

3.1. *Generalized shear compliance*

Under the assumption that $\widehat{W}$ is a strictly convex function of strain, the stress-strain relation (32) can be inverted to give the shear strain in terms of the shear stress according to

$$\gamma = \frac{\tau}{Q([\mathcal{F}^{-1}(\tau)]^2)} \equiv \nu(\tau^2)\tau, \tag{39}$$

where the function $\nu = \nu(\tau^2)$ generalizes the concept of shear compliance (i.e. the inverse of the infinitesimal shear modulus) in linear elasticity to a non-linear setting. For this reason we call ν the generalized shear compliance.

Obviously, within the theory of non-linear elasticity the functional form of the shear compliance is strictly related to the particular model adopted for the strain-energy function. Here are the shear compliances corresponding to some models for the strain-energy reported in section 1:

- Fung-Demiray model (17)

$$\nu_{FD}(\tau^2) = \exp\left[-\frac{1}{2}\mathrm{W}(2\alpha\tau^2)\right], \tag{40}$$

  where W is the Lambert W function;
- Gent model (18)

$$\nu_G(\tau^2) = \frac{\sqrt{4J_m\tau^2 + J_m^2} - J_m}{2\tau^2}; \tag{41}$$

- Knowles model (19) with $n = 1/2$

$$\nu_K(\tau^2) = \frac{1}{\sqrt{1 - 2b\tau^2}}; \tag{42}$$

- quadratic model (14), with constitutive parameters such that $C_{10} + C_{01} > 0$ and

$C_{20} + C_{11} + C_{02} > 0$ for guaranteeing the strictly convexity of $\widehat{W}$,

$$\nu_{qMR}(\tau^2) = \sqrt[3]{\frac{\sqrt{1 + \frac{4}{27\kappa\tau^2}} + 1}{2\kappa\tau^2}} - \sqrt[3]{\frac{\sqrt{1 + \frac{4}{27\kappa\tau^2}} - 1}{2\kappa\tau^2}}, \tag{43}$$

where, in view of (15),

$$\kappa = 1 + \frac{\mathcal{A} + 2\mathcal{D}}{2\mu}. \tag{44}$$

As a direct consequence of the strictly convexity of $\widehat{W}$, the parameter κ is positive.

3.2. *Weakly non-linear theory of the fourth order*

The introduction of the generalized shear compliance allows us to give an estimation of the magnitude of the shear strain u_Z. Indeed, by inverting (31) and taking into account the positiveness of the generalized shear compliance we obtain

$$\|u_Z\|_\infty \equiv \max_{Z\in[-1,1]} |u_Z| = \max_{\xi\in[0,\varepsilon]} \nu(\xi^2). \tag{45}$$

From (45) we deduce that $\|u_Z\|_\infty \to \nu(0) = 1$ as $\varepsilon \to 0$. Consequently, for small values of ε it makes sense to study the rectilinear shear deformations in a slab within the fourth order theory of elasticity. In this framework the (dimensionless) generalized shear modulus reads

$$Q(\gamma^2) = 1 + \kappa\gamma^2, \tag{46}$$

with κ as in (44), while the (dimensionless) generalized shear compliance takes the form

$$\nu(\tau^2) = 1 - \kappa\tau^2, \tag{47}$$

thanks to which equation (31) can be inverted to give u_Z in terms of the spatial variable Z as

$$u_Z = -(1 - \varepsilon^2\kappa Z^2)Z. \tag{48}$$

On integrating (48) we find that to this order of approximation the solution to the BVP (29) is given by

$$u = \frac{1}{2}(1 - Z^2) - \frac{\kappa}{4}\varepsilon^2(1 - Z^4). \tag{49}$$

It is worth noting that (46), (47) and (49) hold true also for negative values of κ; in particular, without assuming that $\widehat{W}$ is strictly convex. This is due to the fact within the fourth order theory of elasticity there exists a neighbourhood of $\gamma = 0$

in which the stress-strain relation (32) can be inverted to give the shear strain as a function of the shear stress.

4. Variational approach

We now observe that the existence and uniqueness of the solution to the BVP (29) can be proven by means of standard results in calculus of variations. To this aim we observe that the BVP (29) is the Euler-Lagrange equation associated with the energy functional

$$\mathcal{E} : u \in H_0^1[-1,1] \mapsto \int_{-1}^{1} \left[\widehat{W}(\varepsilon u_Z) + \operatorname{sign}(A_0)\varepsilon u\right] \mathrm{d}Z \tag{50}$$

supplemented with homogeneous Dirichlet boundary conditions.

The integrand in (50),

$$f(\gamma,\eta) = \widehat{W}(\gamma) + \operatorname{sign}(A_0)\eta \quad (\gamma = \varepsilon u_Z,\ \eta = \varepsilon u), \tag{51}$$

represents the sum of the elastic stored energy density and the potential energy density due to the action of a constant pressure gradient field.

A general theorem in calculus of variations (see [1] for details) states that if f is strictly convex then the energy functional $\mathcal{E}$ admits a unique minimizer. In addition, if $f \in \mathrm{C}^\infty(\mathbb{R}\times\mathbb{R})$ and

$$\frac{\partial^2 f}{\partial \gamma^2} > 0 \tag{52}$$

for all $(\gamma,\eta) \in \mathbb{R}\times\mathbb{R}$, then the minimizer is in $\mathrm{C}^\infty[-1,1]$.

Form (51) we deduce that requiring that f is strictly convex is equivalent to require the strict convexity of $\widehat{W}$. On the other hand, the strict convexity of $\widehat{W}$ implies also that inequality (52) is satisfied. Therefore, if we limit our analysis to infinitely many times differentiable strain energy functions (2) (as we have thus far assumed tacitly) there exists the minimizer of the energy functional (50) (and hence the solution to the BVP (29)), and it is unique and smooth.

In the previous section we have proven how the strict convexity of $\widehat{W}$ implies existence and uniqueness of the solution to (29) by resorting to the strong ellipticity condition. The advantage of that approach stands in the fact that it provides a theoretical method to determining the solution to (29). The variational approach provides instead conditions under which the solution is smooth.

5. Exact solutions

The results in section 2 and 3 gives us guidance on how to determining the solution of the BVP (29). Should the interval $[-\varepsilon,\varepsilon]$ be contained in the codomain of $\mathcal{F}$, then the solution of (31) (and hence of (29)) is

$$u = -\int_{-1}^{z} \nu(\varepsilon^2\zeta^2)\zeta \mathrm{d}\zeta = \frac{1}{2\varepsilon^2}\left[\Gamma(\varepsilon^2) - \Gamma(\varepsilon^2 z^2)\right], \tag{53}$$

where Γ is the antiderivative of ν.

For illustration, we report below some exact solutions in closed form corresponding to different models for the strain-energy (2).

For the Fung-Demiray model, the codomain of $\mathcal{F}$ is the whole real axis and for any $\varepsilon > 0$ the unique solution of (31) (see figure 1(a)) reads

$$u_{FD} = \frac{1}{2\alpha\varepsilon^2}\left\{[\mathrm{W}(2\alpha\varepsilon^2) - 1]\exp\left[\frac{\mathrm{W}(2\alpha\varepsilon^2)}{2}\right] - [\mathrm{W}(2\alpha\varepsilon^2 Z^2) - 1]\exp\left[\frac{\mathrm{W}(2\alpha\varepsilon^2 Z^2)}{2}\right]\right\}. \tag{54}$$

For the Gent model (18), the codomain of $\mathcal{F}$ is $\mathbb{R}$ and for any $\varepsilon > 0$ the solution of (31) (figure 1(b)) is given by

$$u_G = \frac{1}{2\varepsilon^2}\left[\sqrt{4J_m\varepsilon^2 + J_m^2} + \frac{J_m}{2}\ln\left(\frac{\sqrt{4J_m\varepsilon^2 + J_m^2} - J_m}{\sqrt{4J_m\varepsilon^2 + J_m^2} + J_m}\right) + J_m \ln|Z| - \sqrt{4J_m\varepsilon^2 Z^2 + J_m^2} - \frac{J_m}{2}\ln\left(\frac{\sqrt{4J_m\varepsilon^2 Z^2 + J_m^2} - J_m}{\sqrt{4J_m\varepsilon^2 Z^2 + J_m^2} + J_m}\right)\right]. \tag{55}$$

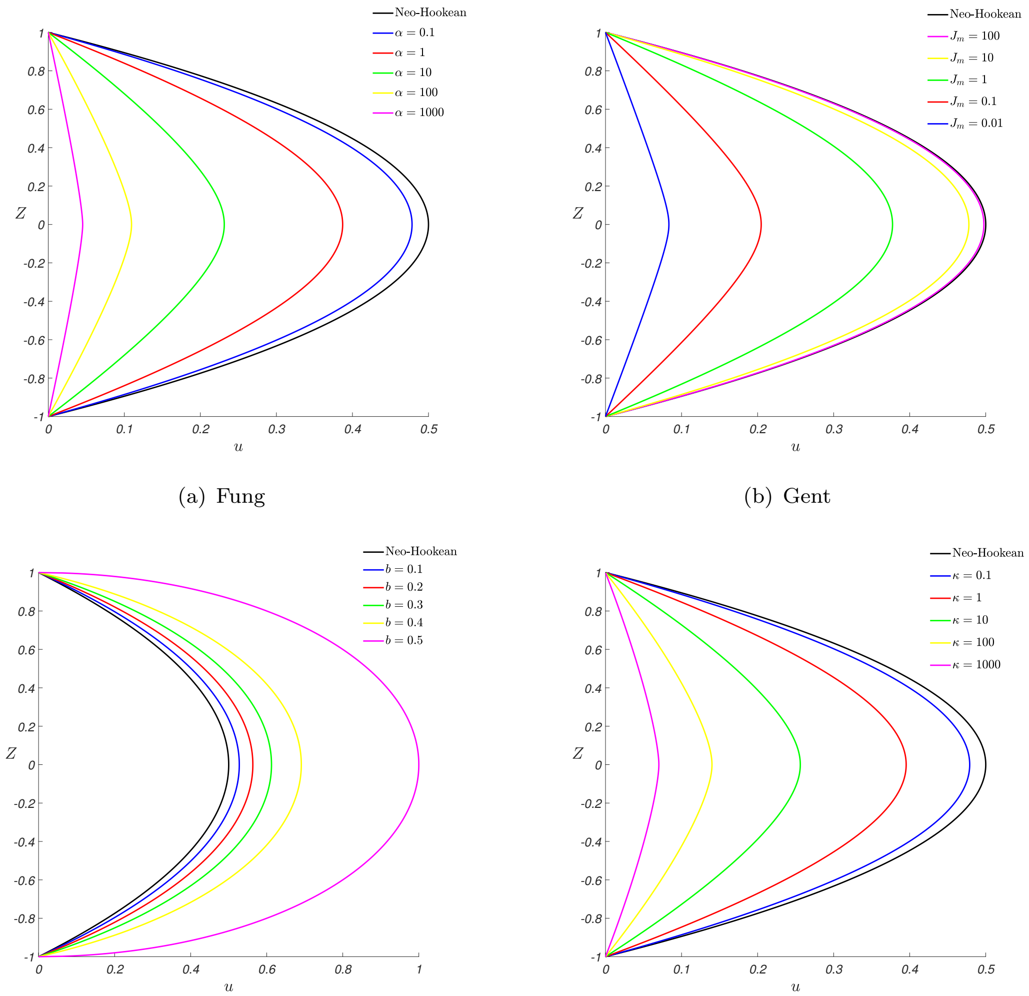
For the Knowles model (19) with $n = 1/2$, the codomain of $\mathcal{F}$ is the interval $[-1/\sqrt{2b}, 1/\sqrt{2b}]$. Hence, the BVP (29) admits a solution only if $\varepsilon \leq 1/\sqrt{2b}$. For these values of ε the solution of (31) (figure 1(c)) is given by

$$u_K = \frac{1}{2b\varepsilon^2}\left[\sqrt{1 - 2b\varepsilon^2 Z^2} - \sqrt{1 - 2b\varepsilon^2}\right]. \tag{56}$$

For the quadratic model (14) there exists the solution of (31) for all $\varepsilon > 0$, but it is not possible to express it in a closed form. Figure 1(d) displays the solutions to the BVP (29) for different values of κ that have been obtained from (43) and (53) via numerical quadrature.

6. Concluding Remarks

In this note we have assessed some constitutive peculiarities of different models for the strain-energy function of an incompressible isotropic material by considering the rectilinear shear deformations in an elastic slab. Particular attention has been given to generalized neo-Hookean materials which are widely used in biological applications of non-linear elasticity. The problem of existence, uniqueness and regularity of the solution to the BVP governing the rectilinear deformations due to the action of a uniform gradient of pressure in a slab hinged at its boundaries to



(a) Fung (b) Gent

(c) Knowles with $n = 1/2$ (d) Quadratic model

Figure 1. Solutions to the BVP (29) for different models for the strain-energy function W. In each case the solution tends to the solution of (29) corresponding to the neo-Hookean model as the the parameter accounting for the non-linearity of the material tends to zero. Such a parameter is given by α in (a), $1/J_m$ in (b), b in (c) and κ in (d).

two fixed plates has been solved by using two different approaches. One is based on the strong ellipticity condition of the strain-energy function, the other consists in classical methods in calculus of variations. Exact solutions have been determined.

What is emerged from our analysis is that the strict convexity of the elastic stored energy $\widehat{W}$ implies the existence uniqueness and, providing that $\widehat{W}$ is infinitely many times differentiable, the smoothness of the solution. Should $\widehat{W}$ be only convex and not strictly convex, then it could be proven that the solution might be not smooth. To justify this assertion consider $\widehat{W}(\gamma) = \kappa\gamma^4/4$, with $\kappa > 0$, which is convex but not strictly convex. For this particular choice of $\widehat{W}$ the solution to the BVP (29) is

$$u = \frac{3}{4\sqrt[3]{\kappa\varepsilon^2}}\left(1 - Z^{4/3}\right), \tag{57}$$

which fails to be in $\mathrm{C}^2[-1, 1]$ as it has a cusp in $Z = 0$.

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