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Isochronous Fractional PDEs

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In this paper we study a number of nonlinear fractional equations, involving Caputo derivative in space or/and in time, admitting explicit solution in separating variable form. Some of these equations are particularly interesting because they admit completely periodic solutions. When time-fractional derivatives are introduced, this property is lost, but in the space-fractional case we can obtain new interesting equations admitting these solutions. This can be the starting point for a more general analysis about fractional isochronous partial differential equations.

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1. Introduction

Twenty years ago, Calogero developed a simple transformation (often called "the trick") modifying a dynamical system so that the new system yielded by it is isochronous [1]. This transformation is applicable to a quite large class of dynamical systems and it yields $\omega - modified$ autonomous systems which are isochronous, with period $T = \frac{2\pi}{\omega}$. The trick was applied mainly on dynamical systems described by systems of nonlinear Ordinary Differential Equations.

Recently, the same approach was extended to treat time evolutions described by nonlinear Partial Differential Equations (PDEs), displaying, list of $\omega - modified$ isochronous PDEs, characterized by the important property to possess lots of completely periodic solutions, living in open regions of their phase space (see chapter 7 of [1]).

In particular, in [2] and [3], a number of nonlinear evolution PDEs are modified so that they possess isochronous solutions, i.e. completely periodic solutions with a period that does not depend on the initial data. In some of the cases, considered in [2], the nonlinear PDEs admit explicit solutions, in a separating variable form. In the general case, it is not possible to construct space or time-fractional versions of these ω-modified PDEs admitting explicit solutions. On the other hand, the particular cases admitting separating variable solutions can be generalized to the fractional case. This leads to new interesting methods and results in the field of nonlinear fractional PDEs. This can be the starting point for the analysis of

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fractional isochronous PDEs, i.e. nonlinear fractional evolutive equations, admitting completely periodic solutions. We will prove that this property is lost in the time-fractional case (and this is quite intuitive) while it is preserved in the space-fractional case. In any cases we are able to construct explicit exact solutions for the considered equations, in terms of Mittag-Leffler functions. We also consider generalizations of these equations, based on the application of Laguerre derivatives.

2. Isochronous space-fractional PDEs

In this Section we consider some simple cases of nonlinear space-fractional partial differential equations (PDEs) admitting completely periodic solution, inspired by [2]. Here we consider space fractional derivatives in the sense of Caputo of order $\nu \in (0,1)$ (see [4]) that is

$$\left({}_a\partial_x^\nu f\right)(x) = \frac{1}{\Gamma(1-\nu)} \int_a^x \frac{\partial_{x'} f(x')}{(x-x')^\nu} dx', \tag{1}$$

Hereafter we will consider $a = 0$ for simplicity.
We here observe that the classical Leibniz and chain rules are not valid for the fractional derivatives in the sense of Caputo, as can be simply observed by definition. This is the reason why, in most cases it is not possible to apply classical methods to solve nonlinear fractional PDEs. On the other hand, many recent studies have been devoted to find particular exact solutions for nonlinear fractional equations by means of Lie symmetry group and invariant subspace method (see e.g. [5–7] and the references therein). These methods in many cases permit to find separating variable solutions.

As a first interesting candidate, inspired by [2], we consider the following equation

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha} u = a u^\alpha \left(\frac{\partial^\nu u}{\partial x^\nu} + \frac{ik}{\alpha} u \right), \tag{2}$$

that admits the completely periodic separating variable solution

$$u(x,t) = \exp\left(\frac{i\omega}{\alpha} t\right) E_\nu\left(-\frac{ik}{\alpha} x^\nu\right), \tag{3}$$

where

$$E_\nu(-x^\nu) = \sum_{k=0}^{\infty} \frac{(-x)^{\nu k}}{\Gamma(\nu k + 1)}, \tag{4}$$

is the Mittag-Leffler function (see [8]). This result can be proved by direct substitution.
Observe that the fractional derivative appearing in (2) is in the sense of Caputo (see [4]) and we consider the problem in the semi-line $x \geq 0$. Indeed in this case, it

is well-known that the Mittag-Leffler function is an eigenfunction of the fractional derivative in the sense of Caputo.

In the case $\nu = 1$, we recover the simple solution

$$u(x,t) = \exp\left(\frac{i\omega}{\alpha}t - \frac{ik}{\alpha}x\right). \tag{5}$$

Another interesting example is given by the equation

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha}u - u\frac{\partial^\nu u}{\partial x^\nu} = -u^2, \tag{6}$$

that admits as a solution the function (3). Equation (6) can be interpreted as a nonlinear-nonlocal advection-reaction equation and the possible physical applications should be deepened. Starting from (6) a sort of simple hierarchy of space-fractional equations, admitting solutions of the form (6) can be built. Indeed, it is simple to prove that the general family of nonlinear space-fractional PDEs

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha}u = u^{n-1}\frac{\partial^\nu u}{\partial x^\nu} - u^n, \tag{7}$$

admits solution of the form (3).
We can conclude this section, observing that space-fractional generalizations of isochronous PDEs, admitting separating variable solutions still admit completely periodic solutions. More general cases should be an object of further analysis.

2.1. *Isochronous fractional Burgers-Hopf equation*

From the ω-modified version of the "generalized Burgers-Hopf" PDE (see [2])

$$\frac{\partial u}{\partial t} + i\frac{1}{\alpha-2}\omega u + i\frac{1-\alpha}{\alpha-2}\omega x\frac{\partial u}{\partial x} = au\frac{\partial u}{\partial x} + b\frac{\partial}{\partial x}\left(u^\alpha\frac{\partial u}{\partial x}\right) \tag{8}$$

we consider an homogeneous nonlinear space-fractional standard Burgers-Hopf equation ($\alpha = 0$)

$$\frac{\partial u}{\partial t} - i\omega u = \frac{\partial^\nu}{\partial x^\nu}\left[\frac{\tilde{a}}{2}\left(\frac{\partial^{1-\nu}u}{\partial x^{1-\nu}}\right)^2 + \tilde{b}\frac{\partial^2}{\partial x^2}J_x^\nu u + \frac{i\omega}{2}x^\nu u\right] \tag{9}$$

where

$$J_x^\nu u(x,t) = \int_0^x \frac{(x-x')^{\nu-1}}{\Gamma(\nu)}u(x',t)dx',$$

is the Riemann-Liouville integral of order $\nu \in (0,1)$ and $\partial^\nu/\partial x^\nu$ is the Caputo fractional derivative with respect to x-variable.

This equation is an homogeneous isochronous modification of a class of non-homogeneous nonlinear nonlocal diffusive equation, recently investigated in ([9]).

We consider the following Cole-Hopf transformation:

$$u(x,t) = -\frac{\partial^\nu}{\partial x^\nu} \ln \psi(x,t). \tag{10}$$

By using the Caputo fractional derivatives, the following equality holds

$$\frac{\partial^{1-\nu}}{\partial x^{1-\nu}} \frac{\partial^\nu}{\partial x^\nu} u = \frac{\partial}{\partial x} u \tag{11}$$

and it can be proved that

$$J_x^\nu \frac{\partial^\nu}{\partial x^\nu} u(x,t) = u(x,t) - u(0,t) \;\; \nu \in (0,1]\,,\; x > 0. \tag{12}$$

Considering $u(0,t) = 0$ as a boundary condition, we have

$$\frac{\partial^\nu}{\partial x^\nu}\left[-\frac{\partial}{\partial t}\ln\psi + i\omega \ln\psi + \tilde{b}\frac{\partial^2}{\partial x^2}\ln\psi + \tilde{b}\left(\frac{\partial}{\partial x}\ln\psi\right)^2 - i\omega x^\nu \frac{\partial^\nu}{\partial x^\nu}\ln\psi \right] = 0. \tag{13}$$

This means that

$$-\frac{\partial}{\partial t}\ln\psi + i\omega \ln\psi + \tilde{b}\frac{\partial^2}{\partial x^2}\ln\psi + \tilde{b}\left(\frac{\partial}{\partial x}\ln\psi\right)^2 - i\omega x^\nu \frac{\partial^\nu}{\partial x^\nu}\ln\psi = f(t). \tag{14}$$

We suppose that $f(t) = 0$, $\tilde{a} = 2\tilde{b}$ and we consider the Gaussian *ansatz*

$$\psi(x,t) = \exp\left[a(t)x^2 + b(t)x + c(t)\right] \tag{15}$$

as a trial solution of (13). Then, by using Cole-Hopf transformation (10) we obtain the following solution of (9)

$$u(x,t) = -\frac{\partial^\nu}{\partial x^\nu}\left(a(t)x^2 + b(t)x + c(t)\right) = -\left(\frac{b(t)x^{1-\nu}}{\Gamma(2-\nu)} + \frac{2a(t)x^{2-\nu}}{\Gamma(3-\nu)}\right) \tag{16}$$

where the two functions $a(t)$ and $b(t)$ satisfy the two Riccati equations, special case 1 and 2 respectively,

$$\dot{a}(t) = 4\tilde{b}a(t)^2 + i\omega\left(1 - \frac{1}{\Gamma(3-\nu)}\right)a(t), \tag{17}$$

$$\dot{b}(t) = 4\tilde{b}a(t)b(t) + i\omega\left(1 - \frac{1}{2\Gamma(2-\nu)}\right)b(t), \tag{18}$$

whose solutions are

$$a(t) = e^{i\omega\left(1-\frac{1}{\Gamma(3-\nu)}\right)t}\left[C - \frac{4\tilde{b}}{i\omega\left(1-\frac{1}{\Gamma(3-\nu)}\right)}e^{i\omega\left(1-\frac{1}{\Gamma(3-\nu)}\right)t}\right]^{-1}, \tag{19}$$

$$b(t) = \tilde{C}\phi(t)e^{i\omega\left(1-\frac{1}{2\Gamma(2-\nu)}\right)t}, \tag{20}$$

where $\phi(t) = \frac{1}{4\tilde{b}}\left[C - \frac{4\tilde{b}}{i\omega\left(1-\frac{1}{\Gamma(3-\nu)}\right)}e^{i\omega\left(1-\frac{1}{\Gamma(3-\nu)}\right)t}\right]$, which are completely periodic in t with period $T = \frac{2\pi}{\omega}$.

3. The time-fractional case

Inspired by the simplest cases of [2], considered in the previous section, we show that, by using time-fractional derivative it is not possible to obtain completely periodic solutions. Moreover, we underline again that we are forced to consider the time-fractional generalization of the equations, admitting separating variable solutions since the analysis of the more general cases does not lead to analytical solutions in the closed form (that is the main interest of this paper).

We first consider the equation

$$\frac{\partial^\nu u}{\partial t^\nu} - \frac{i\omega}{2\alpha-1}u = a\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - u\frac{\partial^2 u}{\partial x\partial y}\right)^\alpha. \tag{21}$$

Also in this case we consider Caputo time-fractional derivatives of order $\nu \in (0,1)$ and $t \geq 0$. More general cases can be obtained from this one by simple calculations.

Ansatz. We first observe that this equation admits the polynomial solution in the form

$$u(x,y,t) = m(t) + n(t)x + l(t)y + p(t)x^2 + q(t)y^2 + s(t)xy. \tag{22}$$

This ansatz can be justified by means of the theory of invariant subspace method (see e.g. [7]) and essentially permits us to reduce the original problem to the solution of a system of fractional ODEs. Let us consider, in particular, the case

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - u\frac{\partial^2 u}{\partial x\partial y} = 0,$$

using the previous ansatz we have the following conditions:

$$nl = ms;\; pl = 0;\; qn = 0;\; ps = 0;\; qs = 0;\; pq = 0.$$

Therefore, we have $p=0,\ q=0$ and

$$\frac{\partial^\nu m}{\partial t^\nu}+x\,\frac{\partial^\nu n}{\partial t^\nu}+y\,\frac{\partial^\nu l}{\partial t^\nu}+xy\,\frac{\partial^\nu s}{\partial t^\nu}-\frac{i\omega}{2\alpha-1}\,(m+x\,n+y\,l+xy\,s)=0. \tag{23}$$

We obtain four decoupled equations:

$$\begin{cases}\dfrac{\partial^\nu m}{\partial t^\nu}=\dfrac{i\omega}{2\alpha-1}m,\\ \dfrac{\partial^\nu n}{\partial t^\nu}=\dfrac{i\omega}{2\alpha-1}n,\\ \dfrac{\partial^\nu l}{\partial t^\nu}=\dfrac{i\omega}{2\alpha-1}l,\\ \dfrac{\partial^\nu s}{\partial t^\nu}=\dfrac{i\omega}{2\alpha-1}s.\end{cases} \tag{24}$$

Recalling again that the solution of the fractional linear differential equation

$$\frac{\partial^\nu u}{\partial t^\nu}=\lambda u(t),$$

is given by the Mittag-Leffler function, i.e.

$$u(t)=E_\nu(\lambda t^\nu)=\sum_{k=0}^{\infty}\frac{(\lambda t^\nu)^k}{\Gamma(\nu k+1)},$$

we have that

$$m(t)=n(t)=l(t)=s(t)=E_\nu\left(\frac{i\omega}{2\alpha-1}t^\nu\right). \tag{25}$$

Hence the *ansatz* (22) leads to the following explicit separating variable solution

$$u(x,y,t)=(1+x+y+xy)\,E_\nu\left(\frac{i\omega}{2\alpha-1}t^\nu\right)=(1+x)\,(1+y)\,E_\nu\left(\frac{i\omega}{2\alpha-1}t^\nu\right). \tag{26}$$

We underline that this polynomial solution represents an interesting explicit solution but it is just a particular case of the more general class of separating variable solution

$$u(x,y,t)=f(x)g(y)E_\nu\left(\frac{i\omega}{2\alpha-1}t^\nu\right). \tag{27}$$

Indeed, it is simple to prove, by direct substitution, that (27) satisfies equation (21) for all the choices of sufficiently smooth functions $f(x)$ and $g(y)$. This general class of solutions corresponds, *a posteriori*, to the solution of a problem with initial condition $u(x,y,0)=f(x)g(y)$ (recalling that $E_\nu(t=0)=1$). Moreover, observe that we obtain completely periodic solutions only in the limit-case $\nu=1$, leading to the solution (see [2])

$$u(x,y,t) = f(x)g(y)\exp\left(\frac{i\omega}{2\alpha-1}t\right). \tag{28}$$

Therefore, as expected, the introduction of fractional derivatives in time leads to essential changes on the structure of the solution of the PDEs that still can be represented in separating variable form but are not anymore isochronous.

A second model equation that can be treated by similar methods is the following one

$$\frac{\partial^{2\beta}u}{\partial t^{2\beta}} - i\omega\frac{(2\alpha+3)}{(2\alpha-1)}\frac{\partial^{\beta}u}{\partial t^{\beta}} - \frac{2\,(2\alpha+1)}{(2\alpha-1)^2}\omega^2 u = a\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - u\frac{\partial^2 u}{\partial x\partial y}\right)^{\alpha} \tag{29}$$

with $0 < \beta < 1$.

Using the same *ansatz* (22), taking p=q=0, we have to solve the following four decoupled equations:

$$\begin{cases} \frac{\partial^{2\beta}m}{\partial t^{2\beta}} - i\omega\frac{(2\alpha+3)}{(2\alpha-1)}\frac{\partial^{\beta}m}{\partial t^{\beta}} - \frac{2(2\alpha+1)}{(2\alpha-1)^2}\omega^2 m = 0 \\ \frac{\partial^{2\beta}n}{\partial t^{2\beta}} - i\omega\frac{(2\alpha+3)}{(2\alpha-1)}\frac{\partial^{\beta}n}{\partial t^{\beta}} - \frac{2(2\alpha+1)}{(2\alpha-1)^2}\omega^2 n = 0 \\ \frac{\partial^{2\beta}l}{\partial t^{2\beta}} - i\omega\frac{(2\alpha+3)}{(2\alpha-1)}\frac{\partial^{\beta}l}{\partial t^{\beta}} - \frac{2(2\alpha+1)}{(2\alpha-1)^2}\omega^2 l = 0 \\ \frac{\partial^{2\beta}s}{\partial t^{2\beta}} - i\omega\frac{(2\alpha+3)}{(2\alpha-1)}\frac{\partial^{\beta}s}{\partial t^{\beta}} - \frac{2(2\alpha+1)}{(2\alpha-1)^2}\omega^2 s = 0 \end{cases} \tag{30}$$

And since the linear second order fractional differential equation

$$\frac{\partial^{2\beta}y}{\partial t^{2\beta}} - (a+b)\frac{\partial^{\beta}y}{\partial t^{\beta}} + aby = 0, \tag{31}$$

has the solution

$$y(t) = A\sum_{k=0}^{\infty}\frac{(at^{\beta})^k}{\Gamma(\beta k+1)} + B\sum_{k=0}^{\infty}\frac{(bt^{\beta})^k}{\Gamma(\beta k+1)}, \tag{32}$$

we conclude that the fractional PDE (29) admits the following solution

$$u(x,y,t) = \left[AE_{\beta}\left(i\omega\frac{2\alpha+1}{2\alpha-1}t^{\beta}\right) + BE_{\beta}\left(\frac{2i\omega}{2\alpha-1}t^{\beta}\right)\right](1+x)(1+y), \tag{33}$$

where the real coefficients A and B depends by the initial conditions.

4. Isochronous Laguerre evolution equations

In this last section, we consider another generalization of the isochronous PDE considered before involving Laguerre derivatives in space (see [10] and [11] about the definitions and main applications). Let us consider the nonlinear PDE

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha} u = a u^{\alpha} \left(\frac{\partial}{\partial x} x \frac{\partial u}{\partial x} + \frac{ik}{\alpha} u \right). \tag{34}$$

We recall that the so-called Laguerre derivative is defined as follows

$$D_L = \frac{\partial}{\partial x} x \frac{\partial}{\partial x}.$$

The equation (34) admits the completely periodic separating variable solution

$$u(x,t) = \exp\left(\frac{i\omega}{\alpha} t \right) C_0 \left(-\frac{ik}{\alpha} x \right), \tag{35}$$

where

$$C_0(-x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{(k!)^2}, \tag{36}$$

is the so-called Tricomi function, that is an eigenfunction of the Laguerre derivative (see [10]) and it is directly related to the Bessel function. Also in this case, the time-fractional generalization admits and explicit exact solution but the property of completely periodicity is lost. Indeed, in this case we have

$$\frac{\partial^{\nu} u}{\partial t^{\nu}} - \frac{i\omega}{\alpha} u = a u^{\alpha} \left(\frac{\partial}{\partial x} x \frac{\partial u}{\partial x} + \frac{ik}{\alpha} u \right), \tag{37}$$

admits the solution

$$u(x,t) = E_{\nu} \left(\frac{i\omega}{\alpha} t^{\nu} \right) C_0 \left(-\frac{ik}{\alpha} x \right). \tag{38}$$

Considering the space-fractional extension of the nonlinear PDE (34):

$$\frac{\partial u}{\partial t} - \frac{i\omega}{\alpha} u = a u^{\alpha} \left(\frac{\partial^{\beta}}{\partial x^{\beta}} x^{\nu} \frac{\partial u}{\partial x} + \frac{ik}{\alpha} x^{\nu-1} u \right), \tag{39}$$

we can find an explicit exact isochronous solution

$$u(x,t) = \exp\left(\frac{i\omega}{\alpha} t \right) W_{\beta,\nu} \left(-\frac{ik}{\alpha\beta} x^{\beta} \right), \tag{40}$$

where $W_{\beta,\nu}(z)$ denotes the classical Wright function which is defined by the series representation, convergent in the whole complex plane (see [12]),

$$W_{\beta,\nu}(z) := \sum_{k=0}^{\infty} \frac{(z)^k}{k!\Gamma(\beta n + \nu)}, \quad \beta > -1, \nu \in \mathbb{C} \tag{41}$$

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