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DUALITY FOR SIMPLE ω-CATEGORIES AND DISKS

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A young schizophrenic named Struther
When told of the death of his mother,
Said, 'Yes, it's too bad,
But I can't feel too sad.
After all, I *still* have each other.' The Pan Book of Limericks

ABSTRACT. A. Joyal [J] has introduced the category $\mathcal{D}$ of the so-called *finite disks*, and used it to define the concept of θ*-category*, a notion of weak ω-category. We introduce the notion of an ω-graph being *composable* (meaning roughly that 'it has a unique composite'), and call an ω-category *simple* if it is freely generated by a composable ω-graph. The category $\mathcal{S}$ of simple ω-categories is a full subcategory of the category, with strict ω-functors as morphisms, of all ω-categories. The category $\mathcal{S}$ is a key ingredient in another concept of weak ω-category, called protocategory [MM1], [MZ]. We prove that $\mathcal{D}$ and $\mathcal{S}$ are contravariantly equivalent, by a duality induced by a suitable schizophrenic object living in both categories. In [MZ], this result is one of the tools used to show that the concept of θ-category and that of protocategory are equivalent in a suitable sense. We also prove that composable ω-graphs coincide with the ω-graphs of the form T^* considered by M.Batanin [B], which were characterized by R. Street (as announced in [S]) and called 'globular cardinals'. Batanin's construction, using globular cardinals, of the free ω-category on a globular set plays an important role in our paper. We give a self-contained presentation of Batanin's construction that suits our purposes.

Contents

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1. Introduction

This paper is a contribution to recent studies aimed at clarifying the concept of weak higher dimensional category.

In each of certain recent proposals of a precise notion of weak n-category, for $n \leq \omega$, a specific small category of 'shapes of cells' (briefly, a *shape-category*) is introduced, and a weak n-category is defined as a presheaf on the shape-category having certain additional *properties* (let us emphasize: *not* as a presheaf with *additional structure*). Thus, in particular, a weak n-category $\mathbf{W}$ of the given kind has *cells* of various *shapes*: for an object A of the shape-category, the elements of the set $\mathbf{W}(A)$ are the cells of shape A. The various arrows of the shape category are interpreted in $\mathbf{W}$ as *face* and *degeneracy operators*, by extending the terminology used for simplicial sets.

In [BD], the *opetopic* weak n-categories (for finite n) are in fact defined without first describing the shape-category, but the latter is implicit: it is the *category of opetopes.* In [HMP], the shape-category is made explicit: it is the *category of multitopes.* In [MM2], where the definition of multitopic ω-category is completed, and, also, the ambient structure comprising all multitopic ω-categories is clarified, the shape-category (the category of multitopes) plays an active role.

In Joyal's concept of θ-category [J], the shape-category is the opposite of a certain category $\mathcal{D}$, called the category of *finite disks.* Joyal denotes $\mathcal{D}^{op}$ by Θ, and calls it the category of *Batanin cells. Cellular sets* are set-valued functors on $\mathcal{D}$; a θ-category is a cellular set satisfying certain conditions that are analogs of Daniel Kan's horn-filling conditions (see [K], and many later sources). In fact, the opposite of the ('non-augmented') simplicial category: the category Δ^+ of non-empty finite linear orders is equivalent to a full subcategory of $\mathcal{D}$, and thus, every cellular set has, as a part, an underlying simplicial set 'in it'. The underlying simplicial set of an ω-category satisfies the so-called restricted Kan-condition [BV]. We may regard the passage from simplicial sets to cellular sets as the result of extending the range of 'shapes of cells' under consideration.

The concept of protocategory was arrived at, by the first author of this paper, independently of Joyal's work on θ-categories; however, the two concepts are closely related. In the talk [MM1], the first author described only a certain part, the subcategory $\mathcal{L}$, of the shape-category for protocategories; the whole shape-category will be spelled out in [MZ]. The description of $\mathcal{L}$ will bring us to one of the two main concepts treated in this paper: *simple categories.*

By an ω*-graph*, we mean the same as, for instance, [B] means by a *globular set.* Consider $\mathcal{S}_{\omega gr}$, the following category-sketch (graph with composition relations; *category-presentation* could be another name):

$$\begin{tikzcd} \cdots & G_n \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_{n-1} & \cdots & G_2 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_0 \end{tikzcd}$$

subject to $d \circ d = d \circ c$ and $c \circ d = c \circ c$, (of course, the various d's and c's have suitable subscripts distinguishing them; and the 'globular' identities are understood as all the meaningful ones of the forms given). A model of $\mathcal{S}_{\omega gr}$ (a graph-map, that is, diagram on $\mathcal{S}_{\omega gr}$, obeying the identities) is an ω-graph.

The category of small ω-graphs, with morphism all the natural transformations (of diagrams), is denoted by ωGr.

It is well-known how to present the notion of ω-category equationally over ωGr. Writing ωCat for the category, with ordinary ω-functors as morphisms, of all small ω-categories, we have the forgetful functor $\omega\mathrm{Cat} \longrightarrow \omega\mathrm{Gr}$, and its left adjoint $[-] : \omega\mathrm{Gr} \longrightarrow \omega\mathrm{Cat}$; for an ω-graph G, $[G]$ is freely generated by G.

It is easy to see that if $m : G \to H$ is a monomorphism of ω-graphs, then the induced arrow $[m] : [G] \to [H]$ is also a monomorphism.

Let G be an ω-graph. Let us call an element (cell) a of $[G]$ *maximal* if it is *proper*, that is, not an identity cell, and if the only monomorphisms $m : H \to G$ for which a belongs to the image of $[m]$ are isomorphisms. Intuitively, an element is maximal if it is proper, and the whole graph G is needed to generate it. We call G *composable* if $[G]$ has a *unique* maximal element; in that case, the maximal element may be called the *composite* of the graph.

(Some remarks. It is easy to see that the proper arrows in any ω-category of the form $[G]$ form a sub-ω-graph of $[G]$: in other words, the domain and the codomain of a proper cell is proper. Moreover, the generating ω-graph G is uniquely recoverable from $[G]$ as consisting of those proper cells that are indecomposable in the sense that they are not composites of two proper cells.)

The graphs

are not composable; for the first as G, $[G]$ has no maximal element, for the second, $[G]$ has infinitely many. (As we will see, this is the general situation: there are either 0, or exactly 1, or else infinitely many maximal elements.) The examples that the reader would think are composable, in an intuitive sense, are indeed such, as experimentation shows. Indeed, the definition is a rigorous formulation of the idea of having a well-defined composite, the latter being the unique maximal element. Note that the definition immediately implies that a composable ω-graph is finite.

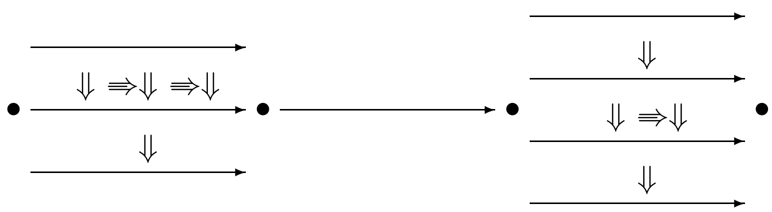
An ω-category is *simple* if it is of the form $[G]$ for a *composable* ω-graph. The category $\mathcal{S}$ is defined as the full subcategory of ωCat on the simple ω-categories as objects.

The category $\mathcal{L}$, mentioned above as part of the shape-category for protocategories, is the skeleton of the subcategory of $\mathcal{S}$ with the same objects as $\mathcal{S}$, but with only the monomorphisms as arrows. $\mathcal{L}$ has the distinguishing property of being *one-way*: the endo-monoids of all the objects are trivial. This still holds for the full shape-category for protocategories, and indeed, this is a basic fact about it, making the specification of the concept of protocategory to be one in FOLDS (First Order Logic with Dependent Sorts); see [MM3], [MZ]. We are not going to have to do anything with FOLDS here; however, let us remark that the 'one-way' condition on the shape-category amounts to the fact that, speaking in reference to the second paragraph of this Introduction, the cells of a protocategory do not have *degeneracy* operators on them, only *face* operators. Note that $\mathcal{S}$ is not a one-way category, and the concept of θ-category is not specified within FOLDS.

As a part of our work, we give an explicit combinatorial description of the composable ω-graphs. More precisely, we introduce the combinatorial concept of 'simple' ω-graph, and prove that 'composable' coincides with 'simple'. The concept of 'simple' ω-graph is due to R. Street [S], where he called the concept 'globular cardinal'. Our description of the notion differs only inessentially from his.

Let G be an ω-graph. We call two cells a and b in G parallel if either they are both of dimension 0, or $d(a) = d(b)$ and $c(a) = c(b)$. By $hom(a, b)$, we mean the set of all cells e for which $d(e) = a$ and $c(e) = b$. Let us fix the parallel cells a and b, and define the binary relation $F = F_{a,b}$ on the set $hom(a, b)$ by saying that eFf ('f *follows* e') holds iff there is g (of dimension exactly 1 higher than e and f) such that $d(g) = e$ and $c(e) = f$. The ω-graph G is called *simple* if for any parallel pair (a, b) the transitive closure of $F = F_{a,b}$ is an irreflexive total order R on $hom(a, b)$, and eFf iff f is the immediate successor of e in R: eRf and there is no h such that $eRhRf$. We will prove that an ω-graph is composable iff it is simple. Let us remark here that the interesting direction of this equivalence is that 'composable' implies 'simple'; the other direction is easy.

Here is an example of a simple ω-graph:



Note that the 1-dimensional simple (composable) ω-graphs are the chains of arrows.

A *disk*, according to [J], is a sequence

$$\cdots \xrightarrow{\;p\;} D^n \xrightarrow{\;p\;} D^{n-1} \quad \cdots \quad D^1 \xrightarrow{\;p\;} D^0$$

of sets and functions, with D^0 being a singleton, together with, for any $n \in \omega$ and $x \in D^n$ a specified *interval* structure, that is, a (nonempty) linear order with a bottom and a top element, on the set $p^{-1}(x)$, subject to the following condition: for any $n \in \omega$, and for the functions

$$\begin{tikzcd} D^{n+1} & D^n \arrow[l, shift right, "t^n"'] \arrow[l, shift left, "b^n"] & D^{n-1} \arrow[l, shift right, "t^{n-1}"'] \arrow[l, shift left, "b^{n-1}"] \end{tikzcd}$$

assigning to x the top and bottom elements in $p^{-1}(x)$, for x in D^{n-1} or D^n, we have that the set

$$\mathrm{Equ}(t^n, b^n) = \{x \in D^n : t^n(x) = b^n(x)\}$$

equals the set $\mathrm{Im}(t^{n-1}) \cup \mathrm{Im}(b^{n-1})$ (here, D^{-1} is understood to be the empty set).

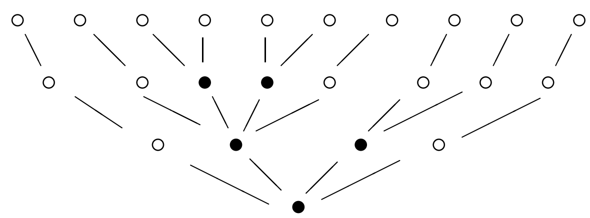
Thus, in a disk, in dimension 0, we have exactly one element. In each positive dimension $n = 1, 2, \ldots$, we have a *bundle* of intervals over the previous level, i.e., a disjoint union of intervals, each interval being mapped by the projection to a single element on the previous level. An element x in dimension $n > 0$ is *singular*, that is, an endpoint of an interval in the bundle, if and only if the fiber $p^{-1}(x)$, an interval in the bundle in dimension $n+1$, is a singleton.

The *inner* nodes of a disk are the non-singular nodes. Note that the unique node in D^0 is inner. The planar tree $\iota(D)$ of the inner nodes of a disk D completely determines the disk: any isomorphism $\iota(D) \xrightarrow{\cong} \iota(D')$ can be uniquely extended to an isomorphism $D \xrightarrow{\cong} D'$. (In $\iota(D)$ the nodes over a given node have a specified linear order on them: hence the adjective "planar".)

A disk is *finite* if it has finitely many inner nodes.

A *morphism* $f : D \to E$ maps D_n into E_n so that f is compatible with the projections, and for any x the induced map $f : p^{-1}(x) \to p^{-1}(f(x))$ preserves the interval structure: it is order- (that is, $\leq$-) preserving, and maps end-points to end-points. $\mathcal{D}$, *the category of finite disks*, is thereby defined.

Here is an example of the first 3 bundles or 4 levels of a disk:

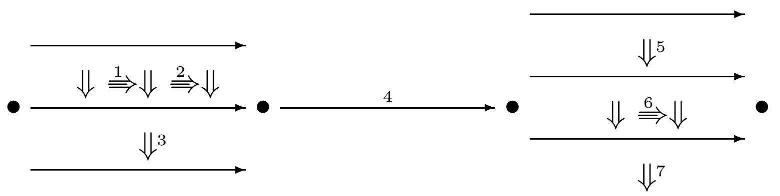


showing inner nodes in black ($\bullet$), singular ones in white ($\circ$).

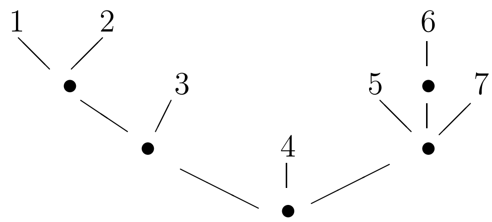
The main theorem of the paper is that $\mathcal{S}^{op}$ and $\mathcal{D}$ are equivalent categories.

The category $\mathcal{I}$ of finite strict intervals (non-empty finite linear orders with a first and a last element, which are distinguished in the structure and preserved by the morphisms) and the category Δ of finite non-empty linear orders are dual to one another. The Stone-type adjunction based on a schizophrenic object establishing this duality is described in detail in section 2.7 (except that we deal with Δ^+ instead of Δ and the corresponding part of $\mathcal{T}$ instead of $\mathcal{T}$). This is the duality Joyal describes in section 1.1 of [J]. Also, the same duality appears in [SGL], p. 455. Our treatment of this duality serves a purely expository purpose. We display the ingredients of the schizophrenic object in detail to make the generalization to the ω-dimensional case easier to follow.

In the remaining parts of the paper we show that the higher-dimensional analogs of these notions are related to each other in a similar manner. On the way to this result we study the structure of $\mathcal{D}$, the category of finite disks, and that of $\mathcal{S}$, the category of simple ω-categories. Among other things, we define an internal disk $\mathbf{D}$ in $\mathcal{S}$ and an internal ω-category $\mathbf{C}$ in $\mathcal{D}$. In a sense, $\mathbf{D}$ is a transposition of $\mathbf{C}$ and this structure is a schizophrenic object for those categories, defining via hom functors a Stone-type adjunction which establishes an equivalence of the categories $\mathcal{S}$ and $\mathcal{D}$, in a way similar to the duality presented in section 2.7 for $\mathcal{I}$ and Δ. Although the full proof of this fact is long, the correspondence between a particular simple ω-graph G and the internal tree $\iota(D)$ of the disk D dual to the simple ω-category generated by G is easy to understand in particular cases. For example, the simple ω-graph and the (planar) tree drawn below correspond in this sense to each other.
An ω-graph:



and the corresponding tree:



In the ω-graph above, we numbered those cells that are neither domains nor codomains of other cells. The nodes of the tree correspond to those cells in the ω-graph that are not codomains of any other cell. The leaves correspond the cells which are neither domains nor codomains of any other cell in the ω-graph. To indicate the correspondence, we marked the leaves and corresponding cells with the same numbers.

The correspondence indicated above between planar trees and simple ω-graphs is due to M. Batanin [B]. The correspondence described here is the same as his mapping from T to T^*, for T a planar tree, and T^* the corresponding specific globular set given in [B].

We use extensively the technical notion of a *ud-vector*, ('up-and-down-vector') a vector $\vec{n} = n_0, n_1, \ldots, n_{2k}$ of natural numbers in which elements with odd subscripts are smaller than the neighboring elements with even ones. The ud-vectors completely characterize both finite disks and simple ω-categories up to isomorphism, constituting a simple and useful invariant for objects of both categories. Therefore, we are able to define and prove many things concerning these categories by induction on ud-vectors. A finite disk with a given invariant corresponds via duality to a simple ω-category with the same invariant.

We note that there are other ways, cosidered in the literature, to use finite sequences of natural numbers to describe the same combinatorial objects, cf. [L], [Ma].

To form an idea how the ud-vector associated with a tree is formed, consider the example of a tree given above. Now, the ud-vector is $3, 2, 3, 1, 2, 0, 1, 0, 2, 1, 3, 1, 2$. This is a sequence of (non-negative) integers; we start numbering the positions with 0. In the seven even positions, we find the heights of the seven leaves, in the order from left to right. In each odd position, we find the height where the leaves corresponding to the neighboring even positions 'meet'.

Let us comment on the connections of our work to other people's work.

It goes without saying that one of the starting points for the present paper is A.Joyal's preprint [J], produced in September 1997. In addition, Joyal conjectured the exact statement of our main result, the equivalence of $\mathcal{S}^{op}$ and $\mathcal{D}$, with the small difference that in his version of $\mathcal{S}$, the ingredient 'composable ω-graph' is replaced by 'globular cardinal' (which, as we said, we proved to be equivalent concepts). We learned about this fact from an e-mail message by Joyal on June 23, 1999, when our work had been completed, and a version of the present paper had been written, and was ready for electronic dissemination. Upon receiving a description of our result, in the e-mail message mentioned above, Joyal wrote, among others: 'I am happy you have proved this duality. I had suspected it shortly after writing my notes 'disk, ...' [which are [J]], but had no proof. I like your description of composable ω-graph'.

Recently we realized that the paper [BS] (that appeared in the year 2000, but which had been put on the internet already in November 1997) contains, in essence, a statement of Joyal's conjecture. As we had been unaware of [BS], this source did not influence our work. As we indicated at the beginning of this Introduction, the idea of the category $\mathcal{S}$ came to the first author in the Spring of 1998 independently of considerations involving disks.

As we mentioned above, A. Joyal called his category Θ the category of Batanin cells, indicating connections of his work to Michael Batanin's work. In [B], Batanin introduced , and used extensively, the planar trees mentioned above. His construction of the ω-graph T^* out of the tree T is the same as what we described above as the correspondence between simple ω-graphs and trees; in particular, simple ω-graphs are the same as the ones of the form T^*, for T a planar tree. In Proposition 4.2 in [B], Batanin gives a construction of the free ω-category generated by an arbitrary ω-graph, one that uses trees and, ultimately, simple ω-graphs; we will reprove his result in this paper (in sections 4.1 and 6.4).

At the end of the paper [BS], we find a statement concerning Θ, Joyal's category mentioned above, the formal dual of the category of finite disks. When one compares Joyal's paper [J] with the description of Θ given in [BS], the coincidence of the two descriptions turns out to be nothing but the main theorem of our paper! We have recently learned from a private communication by Professors Street and Batanin that the description of Θ given in [BS] resulted from conversations of theirs and Andre Joyal's in which Joyal described his conjecture, "identifying" the original description of Θ and the one in [BS], which is essentially the same as that of $\mathcal{S}$.

In [BS], there is no proof of our theorem, neither is Joyal's original description of Θ presented. Therefore the reader who does not know Joyal's paper gets the misleading impression that the description of [BS] is merely a reformulation of Joyal's definition. Joyal's purely combinatorial definition of Θ (already given above in this introduction) is very different from that of the category of simple ω-categories; the equivalence of the two categories is far from obvious as Joyal's statement '[I] had no proof' quoted above also indicates. Let us point out that we got the idea of the duality theorem of this paper in August 1998, and its proof shortly thereafter.

Since A. Joyal's preprint [J] is unpublished, we find it appropriate to point out that, despite the appearence of the word 'duality' in the title, the paper does not contain an indication of the possibility of the statement of our duality theorem. In fact, [J] makes no reference to (strict) ω-categories at all.

A theorem related to our main result is Theorem 1.13 of [Be]. Note, however, that the expression $\Theta(S, T)$ used in the statement of the theorem is defined in a way that is not directly related to hom-sets in Joyal's category Θ.

Our paper is organized as follows. In chapter 2 we introduce the notions and some notation concerning the main notions used in the paper: disks, simple ω-categories, and ud-vectors. In the last section, 2.7, we present the well known duality for finite linear orders and finite intervals in a way that can be generalized to the case of simple ω-categories and of finite disks. This presentation is more involved than it could be, but we think that doing this exercise will help the reader to understand the main result of the paper.

In chapter 3 we investigate the category of finite disks. In section 3.1, we introduce some notation and state some basic facts concerning disks. In section 3.11, we study certain special pullbacks of disks, and for this purpose we use some simple results discussed in section 3.6, concerning similar limits in some related categories of posets. We obtain a presentation of any finite disk as a multi-pullback of very simple disks, and we associate with every disk a ud-vector which describes it up to isomorphism. In section 3.21, we define three special kinds of morphisms in $\mathcal{D}$ and we show that every morphism can be, essentially uniquely, presented as a composition of such morphisms. In section 3.25, we define the internal ω-category $\mathbf{C}$ in $\mathcal{D}$ and we show that homming into it defines a contravariant functor from $\mathcal{D}$ to $\mathcal{S}$.

In chapter 4 the simple ω-graphs and simple ω-categories are investigated. In section 4.1, some notation concerning simple ω-graphs is introduced and the construction of a free ω-category on an ω-graph is presented. The construction is based on simple ω-graphs.

We verify that the construction is correct by relating it to a more general one presented in Appendix 6.8. In section 4.7, we prove that simple ω-graphs are exactly those that are composable. In section 4.9 , we introduce some notation for simple ω-categories, prove some of their properties, define an internal disk $\mathbf{D}$ in $\mathcal{S}$. We show that homming into it defines a contravariant functor from $\mathcal{S}$ to $\mathcal{D}$.

In chapter 5 we state and prove the main result of the paper. We show that the contravariant functors mentioned above form a Stone adjunction between $\mathcal{S}$ and $\mathcal{D}$, which is an equivalence of categories. In section at the end of the section 5.5, we indicate the correspondence of objects and some morphisms in categories $\mathcal{D}$ and $\mathcal{S}$ via the established duality.

The final section of this chapter contains some applications of our work. Among other things we define a nerve functor for ω-categories, i.e. a full and faithful functor

$$\mathcal{N}_\omega : \omega\mathrm{Cat} \longrightarrow \mathrm{Set}^{\mathcal{D}}$$

In this way, we identify ω-categories as special pullbacks preserving cellular sets, i.e. a special kind of θ-categories.

The chapter 6 contains four appendices. We spell out the full (elementary) definitions of an internal disk and an internal ω-category. In Appendix 6.3 we prove some facts concerning internal ω-categories. Among other things we prove some general form of the associativity law for ω-categories. In Appendix 6.8 we give a construction of a free internal ω-category over an internal ω-graph and we prove that the free ω-category functor preserves pullbacks. This construction in based on ud-vectors.

For the convenience of the reader, all the notation introduced in the paper is collected in chapter 7.

In the whole paper, ω denotes the set of (von Neuman) natural numbers, i.e. if $n \in \omega$ then $n = \{0, \ldots, n-1\}$, and ω^+ the set of positive natural numbers. Set is the category of (small) sets.

Acknowledgements

We thank M.Batanin and R.Street for the valuable informations that they provided for us concerning the historical background to our work.

The first autor thanks the participants of the Montreal Category Seminar who in the Winter and Spring 1999, listened to several talks by him on the subject of the present paper.

The diagrams for this paper were prepared with a help of *catmac* of Michael Barr.

2. Preliminaries

2.1. The category $\mathcal{D}$ of finite disks. The category of finite disks $\mathcal{D}$ was introduced by A. Joyal in [J]. In this section, we repeat this definition using the original terminology.

In order to introduce the category of finite disks we need to introduce the category of finite trees.

A *bundle of linear orders* over the set B is a linear order in Set$/B$, i.e. a map $p : E \longrightarrow B$ with each fiber linearly ordered. A *(planar) tree* T is a sequence of bundles of linear orders

$$\cdots \xrightarrow{p^{s+1}} T^{s+1} \xrightarrow{p^s} T^s \quad \cdots \quad T^1 \xrightarrow{p^0} T^0 \cong 1$$

We often omit the superscript s of projection p^s, when it does not lead to confusion.

A *morphism of trees* $f : T \longrightarrow T'$ is a set of functions $\{f^s : T^s \longrightarrow T'^s\}_{s\in\omega}$ preserving projections and order in fibers. A tree is *finite* if all T_n's are finite and almost all are empty. Let $\mathcal{T}ree$ and $\mathcal{T}$ denotes the categories of trees and finite trees, respectively.

If $n > s$, by $p^{(s)} : T^n \longrightarrow T^s$ we denote the composition of $n-s$ projections. By convention, if $n = s$, then $p^{(s)}(x) = x$.

We introduce notation for some finite trees. For $n \in \omega$, θ_n is a tree such that, for $s \in \omega$

$$\theta_n^s \;=\; \begin{cases} \{s\} & \text{if } s \leq n \\ \emptyset & \text{if } s > n \end{cases}$$

and the projections are the obvious ones. Thus, for example, θ_3 can be drawn as

$$\begin{array}{c} \mathbf{3} \\ | \\ \mathbf{2} \\ | \\ \mathbf{1} \\ | \\ \mathbf{0} \end{array}$$

A *leaf* x of a tree T is a node of T, such that $p^{-1}(x) = \emptyset$. Clearly, any tree morphism $f : T \longrightarrow T'$ in uniquely determined by the function f restricted to the leaves of T.

A *bundle of intervals over set* B is an interval in Set$/B$, i.e. a diagram of sets and functions

$$\begin{tikzcd} E \arrow[r, "p"] & B \arrow[l, shift right=2, "b"'] \arrow[l, shift left=2, "t"] \end{tikzcd}$$

with each fiber $p^{-1}(x)$, for $x \in B$ being an *interval*, i.e. a linear order with endpoints $b(x)$ and $t(x)$. The equalizer of b and t, a subset of B, is the *singular set* of the bundle. A *disk* D is a sequence of bundles of intervals

$$\cdots \xrightarrow{p^{s+1}} D^{s+1} \xrightarrow{p^s} D^s \quad \cdots \quad D^1 \xrightarrow{p^0} D^0 \cong 1$$

such that the singular set $eq(b,t)$ of $p : D^{n+1} \longrightarrow D^n$ is equal to $b(D^{n-1}) \cup t(D^{n-1})$. (Here, by convention, $D^{-1} = \emptyset$, and the functions $b, t : D^{-1} \longrightarrow D^0$ are thereby defined.) We call this property the *disk condition*.

As a consequence of the definition, we have $bb = tb$ and $bt = tt$. We define the *boundary* $\partial(D^n)$ to be $b(D^{n-1}) \cup t(D^{n-1})$ and the *interior* $\imath(D^n)$ to be $D^n \setminus \partial(D^n)$. By the previous convention $\partial(D^0) = \emptyset$. The nodes in $\imath(D^n)$ are called *inner* and the nodes in $\partial(D^n)$ are called *outer*. Because of the 'disk condition', the projections p send inner nodes to inner nodes and hence, restricting the projections to the interiors, we obtain the *internal tree* $\imath(D)$ of the disk D

$$\cdots \xrightarrow{\ p\ } \imath(D^{s+1}) \xrightarrow{\ p\ } \imath(D^s) \quad \cdots \quad \imath(D^1) \xrightarrow{\ p\ } \imath(D^0) \cong 1$$

We say that the disk D is *finite* if $\imath(D)$ is.

A *morphism of disks* $f : D \longrightarrow E$ is a set of functions $\{f^s : D^s \longrightarrow E^s\}_{s\in\omega}$ preserving projections, order and endpoints in fibers. Let $\mathcal{D}k$ and $\mathcal{D}$ denotes the categories of disks and finite disks, respectively. From now on, by a disk we mean a finite disk, unless explicitly stated otherwise.

Similarly as for trees, if $n \geq s$, by $p^{(s)} : D^n \longrightarrow D^s$ we denote the composition of $n-s$ projections.

There is an obvious forgetful functor $|-|$ from disks to trees, forgetting the endpoints, which has a left adjoint $\overline{(-)}$:

$$\begin{tikzcd} \mathcal{T}ree \arrow[r, shift right, "\overline{(-)}"'] & \mathcal{D}k \arrow[l, shift right, "|-|"'] \end{tikzcd}$$

For a tree T, $\overline{T}$ is the unique disk D, such that $\imath(D)$ is isomorphic to T. In fact, $\mathcal{D}k$ is both Kleisli and Eilenberg-Moore category for the monad induced by this adjunction. The above adjunction restricts to finite disks and finite trees:

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift right, "\overline{(-)}"'] & \mathcal{D} \arrow[l, shift right, "|-|"'] \end{tikzcd}$$

It follows that, in order to define a disk morphism $f : D \longrightarrow E$, it is enough to define a tree morphism $f' : \imath(D) \longrightarrow E$. In the Appendix 6.1, we give an internal version of the notion of a disk in an arbitrary category, so that, disks (not necessarily finite) defined above become internal disks in Set.

2.2. The category $\mathcal{S}$ of simple ω-categories . An ω*-graph* G in a category C has for each $n \in \omega$ an object G_n of n-cells in C and operations

$$\begin{tikzcd} G_{n+1} \arrow[r, shift left, "d_n"] \arrow[r, shift right, "c_n"'] & G_n \end{tikzcd}$$

of *domain* and *codomain*. We usually omit the subscripts of the morphisms d_n and c_n. Furthermore, in the diagram

$$\begin{tikzcd} \cdots\ G_{n+1} \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_n \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_{n-1}\ \cdots \end{tikzcd}$$

we have $d \circ d = d \circ c$ and $c \circ d = c \circ c$. A morphism between ω-graphs G and G' is a family of arrows $\{\varphi_n : G_n \longrightarrow G'_n\}_{n\in\omega}$ in C commuting with operations of domain and codomain. By ωGr we denote *the category of* ω*-graphs* in Set.

A *simple* ω*-graph* is an ω-graph G in Set, such that

1. G is non-empty and finite, i.e. each G_n in finite $G_0 \neq \emptyset$ and almost all are empty; the height of G is $\mathrm{ht}(G) = \max\{n : G_n \neq \emptyset\}$;

2. for $n \in \omega$, G_n is partially ordered; for any $x, y \in G_n$ the subset of G_{n+1}
$$G_{n+1}(x, y) = \{u \in G_{n+1} \ :\ d(u) = x \quad \text{and} \quad c(u) = y\}$$
is linearly ordered by $\geq$; as well as G_0; let $\rhd$ denote the immediate predecessor relation: $u \rhd v$ means that $u, v \in G_0$ and v is an immediate predecessor of u or $u, v \in G_{n+1}(x, y)$ for some $x, y \in G_n$ such that $x \rhd y$, and moreover v is a predecessor of u in that order;

3. for $n \in \omega$, if $x, y \in G_n$ then
$$x \rhd y \quad \text{iff} \quad G_{n+1}(x, y) \neq \emptyset$$

Let $s\omega$Gr denote the full subcategory of ωGr, whose objects are simple ω-graphs.

The full definition of an ω-category in a category $\mathcal{C}$ is given in the Appendix 6.2. Below, we briefly sketch the definition.

An ω-category $\mathbf{A}$ is an ω-graph together with operations of identity, for $n, l \in \omega$, $n \leq l$,

$$\iota^A_{(n,l)} = \iota^A_{(l)} : A_n \longrightarrow A_l$$

and compositions

$$m^A_{n_0,n_1,n_2} : A_{n_0,n_1,n_2} \longrightarrow A_{\max(n_0,n_2)}$$

where for a 3-tuple $\langle n_0, n_1, n_2\rangle$ such that $n_1 < n_0, n_2$, the diagram

$$\begin{tikzcd}
A_{n_0,n_1,n_2} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & A_{n_2} \arrow[d, "d^A"] \\
A_{n_0} \arrow[r, "c^A"'] & A_{n_1}
\end{tikzcd}$$

is a pullback. This data is subject to conditions concerning domains and codomains of identities and compositions, neutrality of identities, associativity of compositions, and middle exchange law. The morphisms of ω-categories are defined as the morphisms of the underlying graphs preserving additionally compositions and identities. ωCat denotes *the category of ω-categories* in Set.

The forgetful functor

$$\mathcal{U} : \omega\mathrm{Cat} \longrightarrow \omega\mathrm{Gr}$$

has a left adjoint,

$$[-] : \omega\mathrm{Gr} \longrightarrow \omega\mathrm{Cat}$$

associating to a graph G, the free ω-category $[G]$ generated by G. A specific construction of this functor using simple ω-graphs is given in section 4.1.

If G' is a sub-ω-graph of G, then $[G']$ is a sub-ω-category of $[G]$. Moreover the ω-category $[G]$ determines uniquely (up to an isomorphism) the ω-graph G, as the ω-graph of those cells in $[G]$ that are not compositions of two other non-identity cells in $[G]$. We have that if an ω-functor $\varphi : [G] \longrightarrow [G']$ is an isomorphism then there is a unique isomorphism of ω-graphs $\psi : G \longrightarrow G'$ such that $[\psi]$ is φ. All this can be easily deduced from the description of the functor $[-] : \omega\mathrm{Gr} \longrightarrow \omega\mathrm{Cat}$ given in the section 4.1.

Let G be an ω-graph. A cell e in $[G]$, is said to be *maximal* if it is not an identity cell and it is not contained in $[G']$, for any proper sub-ω-graph G' of G. An ω-category S is simple iff for some graph G, it is isomorphic to the category $[G]$ containing exactly one maximal cell. The unique maximal cell in $[G]$ and S, if exists, will be denoted by mac_G and mac_S, respectively. An ω-graph is *composable* if $[G]$ is a simple category. The name composable comes from the intuition that if a graph G is composable then it has well define composition in $[G]$ which is the maximal cell mac_G.

Note, that a 1-category (i.e. ω-category without non-identity cells of dimension bigger then 1) is simple iff it is generated by a composable string of arrows, e.g. the category $[G_1]$, generated by the graph G_1:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "g"'] & \bullet \arrow[d, "h"] \\
\bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}$$

has no maximal arrow, and in the category $[G_2]$, generated by the graph G_2:

$$\begin{tikzcd} \bullet \arrow[loop above, "f"] \end{tikzcd}$$

all non-identity arrows are maximal.

Later (Theorem 4.8) we shall prove that simple ω-categories are exactly those which are free ω-categories generated by a simple ω-graphs, i.e. that a ω-graph is composable iff it is simple.

We shall define a functor

$$Tr : s\omega\mathrm{Gr} \longrightarrow \mathcal{T}$$

Let G be a simple ω-graph. Let $\max(G_n)$ be the set of maximal elements in G_n and $\max(G_n, x, y)$ the maximal element in $G_n(x, y)$, provided $x, y \in G_{n-1}$ and $x \rhd y$. We put

$$Tr(G)^n \ = \ \max(G_n)$$

Thus $Tr(G)^0$ contains only $\max(G_0)$, the maximal element of G_0 and for $n > 0$ and $x \in G_n$ we have that $x \in Tr(G)^n$ iff $x = \max(G_n, d(x), c(x))$.

The projection $p_n : Tr(G)^{n+1} \longrightarrow Tr(G)^n$ is defined as follows. For $x \in G_1$, $p^1(x) = \max(G_0)$. For $n > 1$ and $x \in G_n$,

$$p^n(x) \ = \max(G_{n-1}, d_{(n-2)}(x), c_{(n-2)}(x))$$

The order in the fibers of $Tr(G)$ is given as follows. If $u, v \in Tr(G)^1$ then $u \geq v$ iff $d(u) \geq d(v)$ in G_0, and if $u, v \in Tr(G)^n$ for $n > 1$, then $u \geq v$ iff $d(u) \geq d(v)$ in $G_{n-1}(d_{(n-2)}(x), c_{(n-2)}(x))$.

Let $f : G \longrightarrow G'$ be a morphism of simple ω-graphs. Then for $x = \max(G_0) \in Tr(G)^0$ we put $Tr(f)^0(x) \ = \ \max(G'_0)$ and for $x \in Tr(G)^n$ for $n > 0$, we put $Tr(f)^n(x) = \max(G'_n, d(f_n(x)), c(f_n(x)))$.

The functor Tr is neither full nor faithful, but it is conservative (i.e. reflects isomorphisms) and essentially surjective. We shall sketch why the last property holds. Let T be a tree. We shall define a simple ω-graph G such that $Tr(G)$ is isomorphic to T. To this end, we define a predecessor function

$$\mathrm{pre}_T : \bigcup_{s=1}^{\infty} T^s \longrightarrow \bigcup_{s=0}^{\infty} T^s$$

such that, for $x \in T^n$,

$$\mathrm{pre}_T(x) \quad = \quad \begin{cases} \text{next element in the fiber} & \text{if exists} \\ p^{n-1}(x) & \text{otherwise} \end{cases}$$

for $n \geq 1$. We put

$$G_n \ = \ T^n + T^{n+1}$$

for $x \in T^n \hookrightarrow G_n$

$$d_n(x) = \mathrm{pre}_T(x) \in G_{n-1}$$

$$c_n(x) = x \in T^n \hookrightarrow G_{n-1}$$

for $x \in T^{n+1} \hookrightarrow G_n$

$$d_n(x) = \mathrm{pre}_T(p^n(x)) \in G_{n-1}$$

$$c_n(x) = p^n(x) \in T^n \hookrightarrow G_{n-1}$$

Now, one can check that $Tr(G)$ is isomorphic to T.

Putting together the functors that we have mentioned so far, we get the following diagram of categories and functors

$$\begin{tikzcd}
\mathcal{D} \arrow[d, shift left, "|-|"] & \mathcal{S} \arrow[r] & \omega\mathrm{Cat} \arrow[d, shift left, "\mathcal{U}"] \\
\mathcal{T} \arrow[u, shift left, "\overline{(-)}"] & s\omega\mathrm{Gr} \arrow[l, "Tr"] \arrow[u, "{[-]}"] \arrow[r] & \omega\mathrm{Gr} \arrow[u, shift left, "{[-]}"]
\end{tikzcd}$$

where horizontal arrow going right are inclusions.

2.3. The ud-vectors. In this section we introduce ud-vectors, some vectors of natural numbers. They characterize up to an isomorphism both disks and simple ω-categories, being much simpler then either of them.

The reason, the notion of a ud-vector is rather technical as opposed the other two is that there is no reasonable easy notion of a morphism of ud-vectors. However, it is very convenient to describe domains and codomains of disk morphisms and ω-functors between simple ω-categories using ud-vectors. Some (important) pullbacks in $\mathcal{D}$ can be described in terms of operation of amalgamation of ud-vectors. For $l \in \omega$, we introduce an l-size of ud-vectors. This will allow us to show easily many properties of disks and simple categories, by induction on l-size.

By an *up-and-down vector*, *ud-vector* for short, $\vec{u}$, we mean a sequence of natural numbers $\vec{u} = \langle u_0, \ldots, u_{2k} \rangle$ with $k \in \omega$, such that $u_{2i+1} < u_{2i}, u_{2i+2}$, for $i \in k$. By the *length* $\mathrm{lh}(\vec{u})$ of a ud-vector $\vec{u}$, we mean the number of even-numbered elements in it. (To help the intuition of the reader, let us note that, with T the tree corresponding to $\vec{u}$ in the sense indicated in the Introduction, we have that $\mathrm{lh}(\vec{u})$ is the number of the leaves in T. This, and the similar parenthetical remarks that follow, are not needed for the technical development.) Thus, in the above case, $\mathrm{lh}(\vec{u}) = k+1$. The *height* $\mathrm{ht}(\vec{u})$ of a ud-vector $\vec{u}$, we mean the maximum number in $\vec{u}$, i.e. $\max(\vec{u})$. ($\mathrm{ht}(\vec{u})$ is the maximal dimension of a cell occuring in G, the simple graph corresponding to $\vec{u}$.) If we write $\vec{u} = \vec{u'}, z, \vec{u''}$ then we mean that the ud-vector $\vec{u}$ is a concatenation of ud-vector $\vec{u'}$ followed by a single term z, followed by ud-vector $\vec{u''}$.

Let $l, k \in \omega$, $\vec{u}$ a ud-vector of length $k+1$. We say that $\vec{u}$ is *l-primitive* iff $\min(\vec{u}) \geq l$ and $\max(\vec{u}) > l$. The *l-size* of a ud-vector $\vec{n}$ we define as follows

$$\mathrm{size}_{(l)}(\vec{u}) \quad = \quad \begin{cases} 1 & \text{if } \vec{u} = u_0 \leq l \\ 1 & \text{if } \vec{u} \text{ is } l\text{-primitive} \\ \mathrm{size}_{(l)}(\vec{u'}) + size_{(l)}(\vec{u''}) & \text{if } \vec{u} = \vec{u'}, z, \vec{u''},\ z = \min(\vec{u}) < l \end{cases}$$

We shall prove many statements involving ud-vectors by induction on l-size. (With G as above $\mathrm{size}_{(l)}(\vec{u})$ is the number of equivalence classes of the relation 'a parallel to b' for l-cells a, b, where the l-cells a, b are said to be parallel if either $l = 0$, or $d(a) = d(b)$ and $c(a) = c(b)$.)

The ud-vector $\mathrm{tr}_{(l)}(\vec{u})$, the *l-truncation of* $\vec{u}$ is defined, by induction on l-size of $\vec{u}$, as follows

$$\mathrm{tr}_{(l)}(\vec{u}) \quad = \quad \begin{cases} u_0 & \text{if } \vec{u} = u_0 \leq l \\ l & \text{if } \vec{u} \text{ is } l\text{-primitive} \\ \mathrm{tr}_{(l)}(\vec{u'}), z, \mathrm{tr}_{(l)}(\vec{u''}) & \text{if } \vec{u} = \vec{u'}, z, \vec{u''} \text{ and } z = \min(\vec{u}) < l \end{cases}$$

(With G as above, $\mathrm{tr}_{(l)}(\vec{u})$ is the ud-vector associated with the 'l-truncation' G' of G, where G' is obtained from G by deleting every cell of dimension greater then l, and replacing each parallelism class of l-cells by just one l-cell.)

For $l \in \omega$ and ud-vectors $\vec{u}$, $\vec{v}$, such that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$, i.e. $\vec{u}$ and $\vec{v}$ are *l-compatible*, we define recursively a ud-vector $[\vec{u}, l, \vec{v}]$, an *l-amalgam of ud-vectors* $\vec{u}$ and $\vec{v}$, by induction on l-size of $\vec{w}$, as follows

$$[\vec{u}, l, \vec{v}] \quad = \quad \begin{cases} \vec{u} & \text{if } \vec{v} = v_0 \leq l, \\ \vec{v} & \text{if } \vec{u} = u_0 \leq l, \\ \vec{u}, l, \vec{v} & \text{if both } \vec{u} \text{ and } \vec{v} \text{ are } l\text{-primitive}, \\ [\vec{u'}, l, \vec{v'}], z, [\vec{u''}, l, \vec{v''}] & \text{if } \vec{u} = \vec{u'}, z, \vec{u''},\ \vec{v} = \vec{v'}, z, \vec{v''}, \\ & \quad \mathrm{tr}_{(l)}(\vec{u'}) = \mathrm{tr}_{(l)}(\vec{v'}), \\ & \quad \mathrm{tr}_{(l)}(\vec{u''}) = \mathrm{tr}_{(l)}(\vec{v''}) \\ & \quad \text{and } z = \min(\vec{u}) < l \end{cases}$$

If we write $[\vec{u}, l, \vec{v}]$, we always presuppose that $\vec{u}$ and $\vec{v}$ are l-compatible. (The l-amalgam $[\vec{u}, l, \vec{v}]$ can be related to simple graphs as folows. Let G, H and I be the simple graphs corresponding to $\vec{u}$, $\vec{v}$ and $\vec{w} = \mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v})$, respectively. Consider the map $c : I \longrightarrow G$, $d : I \longrightarrow H$ defined as follows. On cells of dimensions less that l, c and d are 'inclusions'. For a in I of dimension l, $c(a)$ is the *last* element in the parallelism class of elements of G corresponding to a, $d(a)$ is the *last* element of the corresponding class in H. Finally, let J be the pushout (in the category of ω-graphs) of the maps c and d. J is the simple graph corresponding to $[\vec{u}, l, \vec{v}]$.)

The (n_1, n_3)-*amalgam* $[\vec{u}, n_1, \vec{v}, n_3, \vec{w}]$ of three ud-vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, such that $\vec{u}$ and $\vec{v}$ are n_1-compatible and $\vec{v}$ and $\vec{w}$ are n_3-compatible is defined as follows:

$$[\vec{u}, n_1, \vec{v}, n_3, \vec{w}] \quad = \quad \begin{cases} [[\vec{u}, n_1, \vec{v}], n_3, \vec{w}] & \text{if } n_1 \geq n_3 \\ [\vec{u}, n_1, [\vec{v}, n_3, \vec{w}]] & \text{if } n_1 < n_3 \end{cases}$$

We list the following easy relations between the notions introduced above.

2.4. LEMMA. *Let $l, n_1, n_3 \in \omega$, $\vec{u}$, $\vec{v}$, $\vec{w}$ ud-vectors, such that $\vec{u}$ and $\vec{v}$ are n_1-complatible and $\vec{v}$ and $\vec{w}$ are n_3-complatible. Then*

1. *if* $\mathrm{ht}(\vec{u}) \leq l$ *then* $\mathrm{tr}_{(l)}(\vec{u}) = \vec{u}$;
2. *if* $l \leq n_1$ *then* $\mathrm{tr}_{(l)}(\mathrm{tr}_{(n_1)}(\vec{u})) = \mathrm{tr}_{(l)}(\vec{u})$;
3. $\mathrm{ht}(\mathrm{tr}_{(l)}(\vec{u})) = \min(l, \mathrm{ht}(\vec{u}))$;
4. $\mathrm{size}_{(l)}(\vec{u}) = \mathrm{size}_{(l)}(\mathrm{tr}_{(l)}(\vec{u})) = \mathrm{lh}(\mathrm{tr}_{(l)}(\vec{u}))$;
5. $\mathrm{ht}([\vec{u}, l, \vec{v}]) = \max(\mathrm{ht}(\vec{u}), \mathrm{ht}(\vec{v}))$;
6. *if* $l \leq n_1$ *then* $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}([\vec{u}, n_1, \vec{v}]) = \mathrm{tr}_{(l)}(\vec{v})$;
7. *if* $n_1 < l$ *then* $\mathrm{tr}_{(l)}([\vec{u}, n_1, \vec{v}]) = [\mathrm{tr}_{(l)}(\vec{u}), n_1, \mathrm{tr}_{(l)}(\vec{v})]$;
8. *if* $n_1 = n_3$ *then* $[[\vec{u}, n_1, \vec{v}], n_3, \vec{w}] = [\vec{u}, n_1, [\vec{v}, n_3, \vec{w}]]$;
9. *if* $n_1 < n_3$ *then* $[\vec{u}, n_1, \vec{v}, n_3, \vec{w}] = [[\vec{u}, n_1, \vec{v}], n_3, [\mathrm{tr}_{(n_3)}(\vec{u}), n_1, \vec{w}]]$;
10. *if* $n_1 > n_3$ *then*
$$[\vec{u}, n_1, \vec{v}, n_3, \vec{w}] = [[\vec{u}, n_3, \mathrm{tr}_{(n_1)}(\vec{w})], n_1, [\vec{u}, n_3, \vec{w}]];$$
11. $[\mathrm{tr}_{(n_1)}(\vec{u}), n_1, \vec{u}] = \vec{u} = [\vec{u}, n_1, \mathrm{tr}_{(n_1)}(\vec{u})]$;
12. *if* $n_1 < l$ *then*
$$\mathrm{tr}_{(l)}([\vec{u}, n_1, \vec{v}]) = \mathrm{tr}_{(l)}([\mathrm{tr}_{(l)}(\vec{u}), n_1, \vec{v}]) = \mathrm{tr}_{(l)}([\vec{u}, n_1, \mathrm{tr}_{(l)}(\vec{v})]);$$
13. *if* $n_1 < l$ *then*
$$[\vec{u}, n_1, \vec{v}] = [[\vec{u}, n_1, \mathrm{tr}_{(l)}(\vec{v})], l, [\mathrm{tr}_{(l)}(\vec{u}), n_1, \vec{v}]] =$$
$$[[\mathrm{tr}_{(l)}(\vec{u}), n_1, \vec{v}], l, [\vec{u}, n_1, \mathrm{tr}_{(l)}(\vec{v})]].$$

PROOF. Exercise. ∎

The free ω-category on the terminal ω-graph can be conveniently described in terms of ud-vectors and the operations introduced above. We denote this ω-category by UD.

Intuitively, UD is constructed by formally composing ud-vectors of length 1 (the generating cells of UD) in all possible ways. The cells in UD are diagrams of (generating) cells with some prescribed compatibility (the domains of some cells match the codomains of some other cells, in such a way that they 'compose' altogether to a single cell). In ud-vector $\vec{u}$ the i-th generating cell u_{2i} matches $i+1$-st generating cell u_{2i+2} at level u_{2i+1}.

The set of n-cells UD_n consists of the ud-vectors of height at most n. The domain and the codomain operations:

$$d_{(l)} = d^{\mathrm{UD}}_{(l)}, c_{(l)} = c^{\mathrm{UD}}_{(l)} : \mathrm{UD}_n \longrightarrow \mathrm{UD}_l$$

for $l \leq n$, are given by the l-truncation, i.e. for a $\vec{u} \in \mathrm{UD}_n$, we have

$$d_{(l)}(\vec{u}) = c_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{u})$$

The operations are well defined by Lemma 2.4 3.

The identity operations:

$$\iota_{(n)} = \iota^{\mathrm{UD}}_{(n)} : \mathrm{UD}_l \longrightarrow \mathrm{UD}_n$$

are inclusions, for $l \leq n$.

The compositions in UD are given by the operations of amalgamation. The set $\mathrm{UD}_{n_0,n_1,n_2}$ is the set of n_1-compatible pairs of ud-vectors $(\vec{u}, \vec{v})$, such that $\mathrm{ht}(\vec{u}) \leq n_0$ and $\mathrm{ht}(\vec{v}) \leq n_2$. The composition

$$m_{n_0,n_1,n_2} : \mathrm{UD}_{n_0,n_1,n_2} \longrightarrow \mathrm{UD}_{\max(n_0,n_2)}$$

is given, for the pair of ud-vectors $(\vec{u}, \vec{v}) \in \mathrm{UD}_{n_0,n_1,n_2}$ by

$$m_{n_0,n_1,n_2}(\vec{u}, \vec{v}) = [\vec{u}, n_1, \vec{v}]$$

The composition is well defined by Lemma 2.4 5.

2.5. Proposition. UD *defined above is the free ω-category over the terminal ω-graph* **1**.

Proof. First we shall verify that UD is an ω-category and then we shall show that it is free over **1**. To prove this we are going to use Lemma 2.4.

By Lemma 2.4 2, we have,

$$d \circ d = d \circ c = c \circ d = c \circ c$$

so UD is an ω-graph. We shall indicate, why conditions (vi)-(xi) in 6.2 of the definition of an ω-category are satisfied.

(vi) .1: let $(\vec{u},\vec{v}) \in \mathrm{UD}_{n_0,n_1,n_2}$, $l \leq n_1$. Then, using Lemma 2.4 6, we get

$$\begin{aligned} d_{(l)} \circ \pi_0(\vec{u},\vec{v}) &= d_{(l)}(\vec{v}) \\ &= \mathrm{tr}_{(l)}(\vec{v}) = \mathrm{tr}_{(l)}([\vec{u},n_1,\vec{v}]) \\ &= d_{(l)}([\vec{u},n_1,\vec{v}]) = d_{(l)} \circ m(\vec{u},\vec{v}) \end{aligned}$$

(vi) .2: use Lemma 2.4 6.

(vi) .3 and (vi).4 are similar.

(vii) : use Lemma 2.4 1.

(viii) : use Lemma 2.4 1, 8, 9, 10.

(ix) : says that the compositions, of inclusions is an inclusion.

(x) : use Lemma 2.4 11.

(xi) : use Lemma 2.4 12, 13.

This shows that UD is indeed an ω-category.

Now we shall show that UD free on $\mathbf{1}$. We define

$$\eta : \mathbf{1} \longrightarrow \mathrm{UD}$$

so that $\eta_n(*) = n \in \mathrm{UD}_n$, for $n \in \omega$. Let $F : \mathbf{1} \longrightarrow A$ be an ω-graph morphism into an ω-category A. We define an ω-graph functor $\overline{F} : \mathrm{UD} \longrightarrow A$, as follows:

$$\overline{F}_n(\vec{u}) \quad = \quad \begin{cases} \iota_{(n)}(F(\vec{u_0})) & \text{if } \vec{u} = u_0 \\ m^A_{n,l,n}(\overline{F}(\vec{u'}),\overline{F}(\vec{u''})) & \text{if } \vec{u} = \vec{u'},l,\vec{u''}, \text{ and } = \min(\vec{u}) = l < \min(\vec{u'}) \end{cases}$$

It is easy to see that, more generaly, the equality which results by droping the condition $l < \min(\vec{u'})$ is also true.

The verification that $\overline{F} \circ \eta = F$ is trivial. Since every cell in UD is obtained by applaying the ι and the m operations repeatedly starting with ud-vectors of length 1, if follows that $\overline{F}$ must be unique, if it exists. It remains to show that $\overline{F}$ is indeed an ω-functor. $\overline{F}$ *preserves domains.* Let $l < n$ and $\vec{u} \in \mathrm{UD}_n$. We need to show that $d_{(l)}(\overline{F}_n(\vec{u})) = \overline{F}_l((d_{(l)}(\vec{u}))$.

The argument is by induction on l-size of $\vec{u}$.

If $\mathrm{size}_{(l)}(\vec{u}) = 1$, we argue by induction on lenght of $\vec{u}$. If $\mathrm{lh}(\vec{u}) = 1$, i.e. $\vec{u} = u_0$, we have for $l \leq u_0$:

$$\overline{F}_l(d_{(l)}(\vec{u})) = \overline{F}_l(d_{(l)}(u_0)) = F_l(*)) =$$

$$= F_l(d_{(l)}(*)) = d_{(l)}(F_l(*)) = d_{(l)} \circ \iota_{(n)}(F_{u_0}(*)) =$$

$$= d_{(l)}(\overline{F}(u_0)) = d_{(l)}(\overline{F}(\vec{u}))$$

and for $l > u_0$:

$$\overline{F}_l(d_{(l)}(\vec{u})) = \overline{F}_l(u_0) =$$

$$= \iota_{(l)}(F_{u_0}(*)) = d_{(l)} \circ \iota_{(n)}(F_{u_0}(*)) =$$

$$= d_{(l)}(\overline{F}_n(u_0)) = d_{(l)}(\overline{F}(\vec{u}))$$

If $\mathrm{lh}(\vec{u}) > 1$ then $\vec{u} = \vec{u'}, k, \vec{u''}$, where $n > k = \min(\vec{u}) \geq l$. Using the inductive hypothesis, we get

$$\overline{F}_l(d_{(l)}(\vec{u})) = \overline{F}_l(l) = \overline{F}_l(d_{(l)}(*)) = d_{(l)}(F_{u_0}(*) =$$

$$= d_{(l)}(\overline{F}(\vec{u'}) = d_{(l)}(m_{n,k,n}(\overline{F}(\vec{u'}), \overline{F}(\vec{u''})) = d_{(l)}(\overline{F}(\vec{u}))$$

Now assume that $\mathrm{size}_{(l)}(\vec{u}) > 1$, $\vec{u} = \vec{u'}, k, \vec{u''}$, and $k = \min(\vec{u}) < l$. Then, using the inductive assumption and the axioms of ω-categories, we have:

$$\overline{F}_l(d_{(l)}(\vec{u})) = \overline{F}_l(\mathrm{tr}_{(l)}(\vec{u})) =$$

$$= \overline{F}_l(\mathrm{tr}_{(l)}(\vec{u'}), k, \mathrm{tr}_{(l)}(\vec{u''})) = m_{l,k,l}(\overline{F}_l(\mathrm{tr}_{(l)}(\vec{u'})), \overline{F}_l(\mathrm{tr}_{(l)}(\vec{u''}))) =$$

$$= m_{l,k,l}(\overline{F}_l(d_{(l)}(\vec{u'})), \overline{F}_l(d_{(l)}(\vec{u''}))) = m_{l,k,l}(d_{(l)}(\overline{F}_n(\vec{u'})), d_{(l)}(\overline{F}_n(\vec{u''}))) =$$

$$d_{(l)}(m_{n,k,n}(\overline{F}_n(\vec{u'}), d_{(l)}(\overline{F}_n(\vec{u''}))) = d_{(l)}(\overline{F}_n(\vec{u}))$$

For the codomains, the argument is the same.

$\overline{F}$ *preserves identities.* Let $n < l$ and $\vec{u} \in \mathrm{UD}_n$. We need to show that $\iota_{(l)}(\overline{F}_n(\vec{u})) = \overline{F}_l((\iota_{(l)}(\vec{u}))$.

The argument is by induction on lenght of $\vec{u}$. If $\vec{u} = u_0$ then

$$\iota_{(l)}(\overline{F}_n(\vec{u})) = \iota_{(l)}(\iota_{(n)}(\overline{F}_{u_0}(*))) =$$

$$= \iota_{(l)}(\overline{F}_{u_0}(*)) = \overline{F}_l(*) = \overline{F}_l(\iota_{(l)}(*))$$

If $\vec{u} = \vec{u'}, k, \vec{u''}$ where $k = \min(\vec{u})$, we have

$$\iota_{(l)}(\overline{F}_n(\vec{u})) = \iota_{(l)}(m_{l,k,l}(\overline{F}_n(\vec{u'}), \overline{F}_n(\vec{u''}))) =$$

$$= m_{l,k,l}(\iota_{(l)}(\overline{F}_n(\vec{u'})), \iota_{(l)}(\overline{F}_n(\vec{u''}))) = m_{l,k,l}(\overline{F}_l(\iota_{(l)}(\vec{u'})), \overline{F}_l(\iota_{(l)}(\vec{u''}))) =$$

$$= m_{l,k,l}(\overline{F}_l(\vec{u'}), \overline{F}_l(\vec{u''})) = \overline{F}_l(\vec{u}) = \overline{F}_l(\iota_{(l)}(\vec{u}))$$

$\overline{F}$ *preserves compositions.* Let $(\vec{u}, \vec{v}) \in \mathrm{UD}_{n_0,n_1,n_2}$, $n = \max(n_0, n_2)$. We need to show that $m_{n_0,n_1,n_2}(\overline{F}_{n_0}(\vec{u}), \overline{F}_{n_2}(\vec{v})) = \overline{F}_n(m_{n_0,n_1,n_2}(\vec{u}, \vec{v}))$.

The argument is by induction on n_1-size of $\vec{w} = \mathrm{tr}_{n_1}(\vec{u})$.

If $\vec{u} = u_0 \leq n_1$ then $\vec{u} = \mathrm{tr}_{(n_1)}(\vec{v}) = \iota_{(n_0)}(d_{(n_1)}(\vec{v})$, $n = n_1$, and we have

$$m_{n_1}(\overline{F}_{n_0}(\vec{u}), \overline{F}_{n_2}(\vec{v})) = m_{n_1}(\overline{F}_{n_0}(\iota_{(n_0)}(d_{(n_1)}(\vec{v})), \overline{F}_{n_2}(\vec{v})) =$$

$$= m_{n_1}(\iota_{(n_0)}(d_{(n_1)}(\overline{F}_{n_2}(\vec{v})), \overline{F}_{n_2}(\vec{v})) = \overline{F}_n(\vec{v}) =$$

$$\overline{F}_n([\mathrm{tr}_{(n_1)}(\vec{v}), \vec{v}]) = \overline{F}_n(m_{n_1}(\vec{u}, \vec{v}))$$

The case $\vec{v} = v_0 \leq n_1$ is similar.

If both $\vec{u}$ and $\vec{v}$ are n_1-primitive then

$$m_{n_1}(\overline{F}_{n_0}(\vec{u}), \overline{F}_{n_2}(\vec{v})) == \overline{F}_n(\vec{u}, n_1, \vec{v}) =$$

$$= \overline{F}_n([\vec{u}, n_1, \vec{v}]) = \overline{F}_n(m_{n_1}(\vec{u}, \vec{v}))$$

Now, let $\vec{u} = \vec{u'}, l, \vec{u''}$, $\vec{v} = \vec{v'}, l, \vec{v''}$, $l = \min(\vec{u}) = \min(\vec{v}) < n_1$, $\mathrm{tr}_{n_1}(\vec{u'}) = \mathrm{tr}_{n_1}(\vec{v'})$, and $\mathrm{tr}_{n_1}(\vec{u''}) = \mathrm{tr}_{n_1}(\vec{v''})$. Then, using inductive hypothesis, axioms (vi), (viii), and (xi) of the definiton of ω-category, and the fact that

$$d_{(n_1)}(\overline{F}_{n_2}(\vec{v'})) = c_{(n_1)}(\overline{F}_{n_0}(\vec{u'})), \qquad c_{(n_1)}(\overline{F}_{n_0}(\vec{u''})) = d_{(n_1)}(\overline{F}_{n_2}(\vec{v''})) \tag{1}$$

we have (we drop indices in $\overline{F}$)

$$\overline{F}(m_{n_1}(\vec{u}, \vec{v})) = \overline{F}([\vec{u}, n_1, \vec{v}]) =$$

(assumption on $\vec{u}$ and $\vec{v}$)

$$= \overline{F}([[\vec{u'}, l, \vec{u''}], n_1, [\vec{v'}, l, \vec{v''}]]) =$$

(definition of n_1-amalgam)

$$= \overline{F}([\vec{u'}, n_1, \vec{v'}], l, [\vec{u''}, n_1, \vec{v''}]]) =$$

(definition of $\overline{F}$)

$$= m_l(\overline{F}([\vec{u'}, n_1, \vec{v'}]), l, \overline{F}([\vec{u''}, n_1, \vec{v''}])) =$$

(inductive hypothesis)

$$= m_l(m_{n_1}(\overline{F}(\vec{u'}), \overline{F}(\vec{v'})), m_{n_1}(\overline{F}(\vec{u''}), \overline{F}(\vec{v''}))) =$$

(axiom viii.4)

$$= m_{n_1}(m_l(\overline{F}(\vec{u'}), d_{(n_1)}(m_{n_1}(\overline{F}(\vec{u''}), \overline{F}(\vec{v''})))),$$
$$m_l(\overline{F}(\vec{v'}), m_{n_1}(\overline{F}(\vec{u''}), \overline{F}(\vec{v''})))) =$$

(axiom viii.2)

$$= m_{n_1}(m_l(\overline{F}(\vec{u'}), d_{(n_1)}(m_{n_1}(\overline{F}(\vec{u''}), \overline{F}(\vec{v''})))),$$
$$m_{n_1}(m_l(\overline{F}(\vec{v'}), \overline{F}(\vec{u''})), m_l(c_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{v''})))) =$$

(axiom vi.1)

$$= m_{n_1}(m_l(\overline{F}(\vec{u'}), d_{(n_1)}(\overline{F}(\vec{u''}))),$$
$$m_{n_1}(m_l(\overline{F}(\vec{v'}), \overline{F}(\vec{u''})), m_l(c_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{v''})))) =$$

(axiom ix, middle exchange law)

$$= m_{n_1}(m_l(\overline{F}(\vec{u'}), d_{(n_1)}(\overline{F}(\vec{u''}))),$$
$$m_{n_1}(m_{n_1}(m_l(d_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{u''})), m_l(\overline{F}(\vec{v'}), c_{(n_1)}(\overline{F}(\vec{u''})))),$$
$$m_l(c_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{v''})))) =$$

(axiom viii.1)

$$= m_{n_1}(m_{n_1}(m_l(\overline{F}(\vec{u'}), d_{(n_1)}(\overline{F}(\vec{u''}))), m_l(d_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{u''}))),$$
$$m_{n_1}(m_l(\overline{F}(\vec{v'}), c_{(n_1)}(\overline{F}(\vec{u''}))), m_l(c_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{v''})))) =$$

(equations (1))

$$= m_{n_1}(m_{n_1}(m_l(\overline{F}(\vec{u'}), d_{(n_1)}(\overline{F}(\vec{u''}))), m_l(c_{(n_1)}(\overline{F}(\vec{u'})), \overline{F}(\vec{u''}))),$$
$$m_{n_1}(m_l(\overline{F}(\vec{v'}), d_{(n_1)}(\overline{F}(\vec{v''}))), m_l(c_{(n_1)}(\overline{F}(\vec{v'})), \overline{F}(\vec{v''})))) =$$

(axiom xi, middle exchange law)

$$= m_{n_1}(m_l(\overline{F}(\vec{u'}), \overline{F}(\vec{u''})), m_l(\overline{F}(\vec{v'}), \overline{F}(\vec{v''}))) =$$

(definition of $\overline{F}$)

$$= m_{n_1}(\overline{F}(\vec{u}), \overline{F}(\vec{v}))$$

This ends the proof of the proposition. ∎

2.6. Some notation concerning ω-graphs and ω-categories. In this section we introduce some notation concerning ω-graphs and ω-categories. For any ω-graph G and ud-vector $\vec{u}$, we define the pullbacks $G_{\vec{u}}$ of $\mathrm{lh}(\vec{u})$-tuples of compatible cells and between such objects we define the 'multi'-versions of the operations of domain $d^G_{\vec{u};l}$, codomain $c^G_{\vec{u};l}$, and some projections. If additionally, G is an ω-category we define the 'multi'-versions of the operations of composition $m_{\vec{u}}$. These 'multi'-operations satisfy some generalizations of the laws defining ω-categories. The explicit statements and proofs of these laws are in Appendix 6.3.

Since we will use these definitions not only when the ambient category is Set but also in the category $\mathcal{D}$ which is not even finitely complete, we assume that we have a fixed ω-graph G and ω-category A in a category $\mathcal{C}$ with all (finite) limits that are explicitely taken.

We assume that in $\mathcal{C}$, for any ud-vector $\vec{u}$, we can form the following limit

$$\begin{tikzcd}
 & & & G_{\vec{u}} \arrow[dlll, "\pi^G_0"'] \arrow[dl, "\pi^G_2"'] \arrow[dr, "\pi^G_{2k-2}"] \arrow[drrr, "\pi^G_{2k}"] & & & \\
G_{u_0} \arrow[dr, "c"'] & & G_{u_2} \arrow[dl, "d"] & & G_{u_{2k-2}} \arrow[dr, "c"'] & & G_{u_{2k}} \arrow[dl, "d"] \\
 & G_{u_1} & & \cdots & & G_{u_{2k-1}} &
\end{tikzcd}$$

If $0 \leq i \leq j < \mathrm{lh}(\vec{u})$ we denote by

$$\pi^G_{i..j} : G_{\vec{u}} \longrightarrow G_{u_{2i},\ldots,u_{2j}}$$

the obvious projection.

Let $l \in \omega$ and $\vec{u}$ be a ud-vector, $\mathrm{lh}(\vec{u}) = k+1$. The morphisms of *multi-domain* and *multi-codomain* in an ω-graph G

$$d^G_{\vec{u};l}, c^G_{\vec{u};l} : G_{\vec{u}} \longrightarrow G_{\mathrm{tr}_{(l)}(\vec{u})}$$

are defined recursively, as follows

$$d^G_{\vec{u};l} \;=\; \begin{cases} 1_{G_{u_0}} & \text{if } \vec{u} = u_0 \leq l; \\ d^G_{(l)} \circ \pi_0 & \text{if } \vec{u} \text{ is } l\text{-primitive}; \\ d^G_{\vec{u'};l} \times d^G_{\vec{u''};l} & \text{if } \vec{u} = \vec{u'}, w, \vec{u''} \text{ and } w = \min(\vec{u}) < l. \\ & \ \min(\vec{u'}) > l. \end{cases}$$

and

$$c^G_{\vec{u};l} \;=\; \begin{cases} 1_{G_{u_0}} & \text{if } \vec{u} = u_0 \leq l; \\ c^G_{(l)} \circ \pi_k & \text{if } \vec{u} \text{ is } l\text{-primitive}; \\ c^G_{\vec{u'};l} \times c^G_{\vec{u''};l} & \text{if } \vec{u} = \vec{u'}, w, \vec{u''} \text{ and } w = \min(\vec{u}) < l. \\ & \ \min(\vec{u'}) > l. \end{cases}$$

It is convenient to define morphisms $d^G_{(l)}, c^G_{(l)} : G_n \longrightarrow G_{\min(l,n)}$ for any $l, n \in \omega$, by putting $d^G_{(l)} = c^G_{(l)} = id_{G_n}$, if $l \geq n$.

For $\vec{u}, \vec{v} \in \mathrm{UD}_{n_0,n_1,n_2}$ the projection morphisms in $\mathcal{C}$

$$\pi^G_{0;\vec{u}} : G_{[\vec{u},n_1,\vec{v}]} \longrightarrow G_{\vec{u}} \qquad\qquad \pi^G_{1;\vec{v}} : G_{[\vec{u},n_1,\vec{v}]} \longrightarrow G_{\vec{v}}$$

are defined as follows

$$\pi^G_{0;\vec{u}} = \begin{cases} d^G_{\vec{v};u_0} & \text{if } \vec{u} = u_0 \leq n_1; \\ \\ \pi^G_{0..k} & \text{if } \vec{u} \text{ is } n_1\text{-primitive, and } \mathrm{lh}(\vec{u}) = k+1; \\ \\ \pi^G_{0;\vec{u'}} \times \pi^G_{0;\vec{u''}} & \text{if } \vec{u} = \vec{u'}, z, \vec{u''},\ \vec{v} = \vec{v'}, z, \vec{v''}, \\ & \ \mathrm{tr}_{(n_1)}(\vec{u'}) = \mathrm{tr}_{(n_1)}(\vec{v'}),\ \mathrm{tr}_{(n_1)}(\vec{u''}) = \mathrm{tr}_{(n_1)}(\vec{v''}), \\ & \ \text{and } z = \min(\vec{u}) < n_1. \end{cases}$$

and

$$\pi^G_{1;\vec{v}} = \begin{cases} c^G_{\vec{u};v_0} & \text{if } \vec{v} = v_0 \leq n_1; \\ \\ \pi^G_{k+1..k'} & \text{if } \vec{v} \text{ is } n_1\text{-primitive, } \mathrm{lh}(\vec{u}) = k+1, \\ & \ \text{and } \mathrm{lh}(\vec{u}, n_1, \vec{v}) = k'+1; \\ \pi^G_{1;\vec{v'}} \times \pi^G_{1;\vec{v''}} & \text{if } \vec{u} = \vec{u'}, z, \vec{u''},\ \vec{v} = \vec{v'}, z, \vec{v''}, \\ & \ \mathrm{tr}_{(n_1)}(\vec{u'}) = \mathrm{tr}_{(n_1)}(\vec{v'}),\ \mathrm{tr}_{(n_1)}(\vec{u''}) = \mathrm{tr}_{(n_1)}(\vec{v''}), \\ & \ \text{and } z = \min(\vec{u}) < n_1. \end{cases}$$

In Lemma 6.5 we show that, with the above definitions, the square

$$\begin{tikzcd} G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "\pi^G_{1;\vec{v}}"] \arrow[d, "\pi^G_{0;\vec{u}}"'] & G_{\vec{v}} \arrow[d, "d^G_{\vec{v};n_1}"] \\ G_{\vec{u}} \arrow[r, "c^G_{\vec{u};n_1}"'] & G_{\vec{w}} \end{tikzcd}$$

is a pullback.

Let $\vec{u}$ a ud-vector, $\mathrm{lh}(\vec{u}) = k+1 > 2$. We put

$$u = \max\{u_{2i+1} : i \in k\} \quad \text{and} \quad j = \min\{i \in k : u_{2i+1} = u\}$$

and then $\vec{u} = \vec{u'}, u_{2j-1}, u_{2j}, u, u_{2j+2}, u_{2j+3}, \vec{u''}$, for some ud-vectors $\vec{u'}$ and $\vec{u''}$. The morphisms

$$A_{\vec{u}} \xrightarrow{\ 1_{A_{\vec{u'}}} \times m^A_{u_{2j},u,u_{2j+2}} \times 1_{A_{\vec{u''}}}\ } A_{\vec{u'},u_{2j-1},\max(u_{2j},u_{2j+2}),u_{2j+3},\vec{u''}} \tag{2}$$

is the *one-step ($\vec{u}$-compatible) composition morphism*, and the morphisms

$$A_{\vec{u}} \xrightarrow{\quad m^A_{\vec{n}} \quad} A_{\mathrm{ht}(\vec{u})} \tag{3}$$

which is a composition of one-step composition morphisms is *the canonical composition morphism.* By convention, if $\vec{u} = u_0$ then $m^A_{\vec{u}} = id_{A_{u_0}}$.

2.7. The small duality for $\mathcal{S}_1$ and $\mathcal{D}^1$. For $n \in \omega^+$, by $\mathcal{S}_n$ we denote the category of simple n-categories i.e. the simple ω-categories truncated to the first n levels, and by $\mathcal{D}^n$ we denote the category of finite n-disks, i.e. disks truncated to the first n levels. In this section we sketch the well know duality between $\mathcal{S}_1$ and $\mathcal{D}^1$ in the way we will prove duality between $\mathcal{S}$ and $\mathcal{D}$. In fact, for any positive n, the duality between $\mathcal{S}_n$ and $\mathcal{D}^n$ holds and it is a restriction of the duality between $\mathcal{S}$ and $\mathcal{D}$.

$\mathcal{S}_1$ is equivalent to the category of finite non-empty linear orders and monotone maps and $\mathcal{D}^1$ is equivalent to the category of finite linear orders with (necessarily different) endpoints and monotone functions preserving endpoints. Thus, it is well know that they are dually equivalent. An easy explanation of this fact can be found in [SGL]. Below, we will exhibit a more involved but, as we believe, more instructive explanation of this fact, that can be generalized to the higher dimensional categories, of which it is a special case.

Before we prove the duality theorem we need to develop some notation. Both elements on level 1 in a disk in $\mathcal{D}^1$ and objects in categories in $\mathcal{S}_1$ comes with an order. However it is convenient to draw these orders in reverse directions, in disk as *increasing* orders, and in categories as *decreasing* orders. At the end of this section we will return to this point.

The 1-disks. An object of $\mathcal{D}^1$, i.e. a 1-disk in Set, is a finite linear order with different endpoints, i.e. it is isomorphic to one of the form

$$\begin{tikzcd} \{\langle k,l\rangle : 0 \leq k \leq l \leq n\} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \{k : 0 \leq k \leq n\} \arrow[r, "p"] & \{\star\} \arrow[l, shift right, "b"'] \arrow[l, shift left, "t"] \end{tikzcd}$$

for some $n \in \omega^+$, where $b(\star) = 0$ and $t(\star) = n$. Such a 1-disk will be denoted by $\lfloor n \rfloor$.

We introduce a notation for maps into $\lfloor 1 \rfloor$ in $\mathcal{D}^1$. For $l \in n$ we have disk maps

$$\overline{l; l+1} : \lfloor n \rfloor \longrightarrow \lfloor 1 \rfloor$$

defined by

$$\overline{l; l+1}(i) \;=\; \begin{cases} 0 & \text{if} \quad i \leq l \\ 1 & \text{otherwise} \end{cases}$$

We write $\overline{l;}$ or $\overline{;l+1}$ for $\overline{l; l+1}$, as well. These are all the maps from $\lfloor n \rfloor$ to $\lfloor 1 \rfloor$.

A map $g : \lfloor n' \rfloor \longrightarrow \lfloor n \rfloor$ induces a dual map by composition

$$\mathcal{D}^1(g, \lfloor 1 \rfloor) : \mathcal{D}^1(\lfloor n \rfloor, \lfloor 1 \rfloor) \;\longrightarrow\; \mathcal{D}^1(\lfloor n' \rfloor, \lfloor 1 \rfloor)$$

$$\mathcal{D}^1(g, \lfloor 1 \rfloor)(\overline{l;}) \;=\; \overline{l;} \circ g$$

for $l \in n$.

The category $\mathcal{D}^1$ does not have pullbacks of arbitrary pairs of morphisms with common codomain. But it has some pullbacks. In fact, the morphisms $f : \lfloor n \rfloor \longrightarrow \lfloor l \rfloor$ and $f' : \lfloor n' \rfloor \longrightarrow \lfloor l \rfloor$ have a pullback in $\mathcal{D}^1$ iff for any $x \in \lfloor l \rfloor$ either $f^{-1}(x)$ or $f'^{-1}(x)$ has at most one element. If this is the case, the pullback is computed as in the category of posets. Let $\mathbf{d}, \mathbf{c} : \lfloor 2 \rfloor \longrightarrow \lfloor 1 \rfloor$ be morphism in $\mathcal{D}^1$, such that $\mathbf{d} = \overline{0;1}$ and $\mathbf{c} = \overline{1;2}$. Then $\mathbf{d}$ and $\mathbf{c}$ satisfy the above condition and we have a pullback

$$\begin{tikzcd}
\lfloor 3 \rfloor \arrow[r, "\boldsymbol{\pi}_1"] \arrow[d, "\boldsymbol{\pi}_0"'] & \lfloor 2 \rfloor \arrow[d, "\mathbf{d}"] \\
\lfloor 2 \rfloor \arrow[r, "\mathbf{c}"'] & \lfloor 1 \rfloor
\end{tikzcd}$$

Similarly, we have a pullback

$$\begin{tikzcd}
\lfloor 4 \rfloor \arrow[r, "\boldsymbol{\pi}_{23}"] \arrow[d, "\boldsymbol{\pi}_{12}"'] & \lfloor 3 \rfloor \arrow[d, "\boldsymbol{\pi}_1"] \\
\lfloor 3 \rfloor \arrow[r, "\boldsymbol{\pi}_2"'] & \lfloor 2 \rfloor
\end{tikzcd}$$

Let

$$\mathbf{m} : \lfloor 3 \rfloor \longrightarrow \lfloor 2 \rfloor \qquad \boldsymbol{\iota} : \lfloor 1 \rfloor \longrightarrow \lfloor 2 \rfloor$$

be morphisms in $\mathcal{D}^1$ such that $\mathbf{m}(1) = \mathbf{m}(2) = 1$. In this way, we have defined an internal category $\mathbf{C}_1$ in $\mathcal{D}^1$ given by the diagram

$$\begin{tikzcd}
\lfloor 1 \rfloor \arrow[r, "\boldsymbol{\iota}" description] & \lfloor 2 \rfloor \arrow[l, shift right=3, "\mathbf{d}"'] \arrow[l, shift left=3, "\mathbf{c}"] & \lfloor 3 \rfloor \arrow[l, shift right=3, "\boldsymbol{\pi}_1"'] \arrow[l, "\mathbf{m}" description] \arrow[l, shift left=3, "\boldsymbol{\pi}_2"] & \lfloor 4 \rfloor \arrow[l, shift right=4, "\boldsymbol{\pi}_{12}"'] \arrow[l, shift right=1, "1 \times \mathbf{m}"'] \arrow[l, shift left=1, "\mathbf{m} \times 1"] \arrow[l, shift left=4, "\boldsymbol{\pi}_{23}"]
\end{tikzcd}$$

The simple 1-categories. The objects of $\mathcal{S}_1$ are free categories generated by finite (possibly empty) strings of arrows. Let for $n \in \omega$, $\lceil n \rceil$ denote the free category generated by the string

$$n \longrightarrow (n-1) \longrightarrow \ldots \longrightarrow 1 \longrightarrow 0$$

Any object in $\mathcal{S}_1$ is isomorphic to $\lceil n \rceil$ for some $n \in \omega$.

Similarly, we introduce a notation for functors into $\lceil 1 \rceil$ in $\mathcal{S}_1$. Since in $\mathcal{S}_1$ the functors are uniquely determined by their values on objects, we will define only their object functions. We define functors

$$0\!\downarrow,\ (k+1\!\downarrow\! k),\ \downarrow\! n \;:\; \lceil n \rceil \longrightarrow \lceil 1 \rceil$$

with $k \in n$, so that for $i \in \{0, \dots, n\}$

$$\begin{aligned} (0\!\downarrow)(i) &= 1 \\ (\downarrow\! n)(i) &= 0 \\ (k+1\!\downarrow\! k)(i) &= \begin{cases} 1 & \text{if } i > k \\ 0 & \text{if } k \geq i \end{cases} \end{aligned}$$

We abbreviate $(k+1\!\downarrow\! k)$ to either $k+1\!\downarrow$ or $\downarrow\! k$.

A functor $\varphi : \lceil n' \rceil \longrightarrow \lceil n \rceil$ induces a dual map by composition

$$\begin{aligned} \mathcal{S}_1(\varphi, \lceil 2 \rceil) : \mathcal{S}_1(\lceil n \rceil, \lceil 2 \rceil) &\longrightarrow \mathcal{S}_1(\lceil n' \rceil, \lceil 2 \rceil) \\ \mathcal{S}_1(\varphi, \lceil 2 \rceil)(h) &= h \circ \varphi \end{aligned}$$

In $\mathcal{S}_1$ we have an internal 1-disk, denoted by $\mathbf{D}_1$

$$\begin{tikzcd} \lceil 2 \rceil \arrow[r, shift left=2, "\boldsymbol{\rho}^1"] \arrow[r, shift right=2, "\boldsymbol{\rho}^2"'] & \lceil 1 \rceil \arrow[r, "\mathbf{p}"] & \lceil 0 \rceil \arrow[l, shift right=3, "\mathbf{b}"'] \arrow[l, shift left=3, "\mathbf{t}"] \end{tikzcd}$$

with $\boldsymbol{\rho}^1 = 2\!\downarrow\!1$, $\boldsymbol{\rho}^2 = 1\!\downarrow\!0$, $\mathbf{b} = \downarrow\! n$, and $\mathbf{t} = 0\!\downarrow$.

The schizophrenic object. The category $\mathbf{C}_1$ in $\mathcal{D}^1$ and the disk $\mathbf{D}^1$ in $\mathcal{S}_1$ form together a schizophrenic object. By this, we mean, that we can form the following diagram in Set

$$
\begin{array}{ccccc}
\star \; \substack{\xrightarrow{b} \\ \xleftarrow{p} \\ \xrightarrow[t]{}} & \begin{matrix} X \\ Y \end{matrix} & \substack{\xleftarrow{\rho'^0} \\ \xleftarrow[\rho'^1]{}} & & \begin{matrix} X & Y \\ & Z \end{matrix} \\
\vdots & c \uparrow \; \iota \downarrow \; \uparrow d & & & c \uparrow \; \iota \downarrow \; \uparrow d \\
\star \; \substack{\xrightarrow{b} \\ \xleftarrow{p} \\ \xrightarrow[t]{}} & \begin{matrix} X \\ f \\ Y \end{matrix} & \substack{\xleftarrow{\rho^0} \\ \xleftarrow[\rho^1]{}} & & \begin{matrix} X & f & Y \\ & gf & g \\ & & Z \end{matrix} \\
\vdots & \pi_2 \uparrow \; m \uparrow \; \uparrow \pi_1 & & & \pi_2 \uparrow \; m \uparrow \; \uparrow \pi_1 \\
\star \; \substack{\xrightarrow{b} \\ \xleftarrow{p} \\ \xrightarrow[t]{}} & \begin{matrix} (X,X) \\ (X,f) \\ (f,Y) \\ (Y,Y) \end{matrix} & \substack{\xleftarrow{\rho^{00}} \\ \xleftarrow[\rho^{11}]{}} & & \begin{matrix} (X,X) & (X,f) & (f,Y) & (Y,Y) \\ & (X,gf) & (f,g) & (Y,g) \\ & & (gf,Z) & (g,Z) \\ & & & (Z,Z \end{matrix} \\
\vdots & m \times 1 \uparrow \; \uparrow 1 \times m & & & m \times 1 \uparrow \; \uparrow 1 \times m \\
\star \; \substack{\xrightarrow{b} \\ \xleftarrow{p} \\ \xrightarrow[t]{}} & \begin{matrix} (X,X,X) \\ (X,X,f) \\ (X,f,Y) \\ (f,Y,Y) \\ (Y,Y,Y) \end{matrix} & \substack{\xleftarrow{\rho^{000}} \\ \xleftarrow[\rho^{111}]{}} & & \begin{matrix} (X,X,X) & (X,X,f) & (X,f,Y) & (f,Y,Y) & (Y,Y,Y) \\ & (X,X,gf) & (X,f,g) & (f,Y,g) & (Y,Y,g) \\ & & (X,gf,Z) & (f,g,Z) & (Y,g,Z) \\ & & & (gf,Z,Z) & (g,Z,Z) \\ & & & & (Z,Z,Z) \end{matrix}
\end{array}
$$

In this diagram the rows are disks isomorphic to the internal category $\mathbf{C}_1$ in $\mathcal{D}$, and the columns are categories isomorphic to the internal disk $\mathbf{D}^1$ in $\mathcal{S}$. Before we will see this we shall describe the data in the above diagram, in details.

The sets in this diagram are given by lists or arrays of its elements, so that the projections were easy to understand, e.g. in the second row of the right column array

$$
\begin{matrix} X & f & Y \\ & gf & g \\ & & Z \end{matrix}
$$

represents the the set $\{id_X, f, id_Y, gf, g, id_Z\}$.

As we said, the columns are diagrams of free categories on simple 1-graphs, on one object, on one arrow $X \xrightarrow{f} Y$ and, on a string of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, respectively. Thus they are equivalent to $\lceil 0 \rceil$, $\lceil 1 \rceil$, and $\lceil 2 \rceil$. In the first row there are objects, in the second there are morphisms, in the third there are composable pairs of morphism, and in the fourth there are composable triples of morphism. The object X, Y, Z in 2nd, 3rd and 4th rows stand for identities on them, i.e. for 1_X, 1_Y, 1_Z, respectively.

The rows are disks isomorphic to $\lfloor 1 \rfloor$, $\lfloor 2 \rfloor$,$\lfloor 3 \rfloor$, and $\lfloor 4 \rfloor$, respectively. In the first column there are one-element sets, i.e. level 0 parts of the disks, in the second column there are universes of disks, and in the third row there are linear orders on these universes.

The morphism ρ^i and ρ'^i, for $i = 0, 1$ are 1st and 2nd projections, e.g. $\rho'^0(X) = \rho'^1(Y) = X$, $\rho'^1(Y) = \rho'^1(Z) = Y$, $\rho^0(g) = \rho^0(gf) = f$, $\rho^1(g) = \rho^1(Z) = Y$, etc. The morphisms ρ^{ii} and ρ^{iii}, for $i = 0, 1$, denote $\rho^i \times \rho^i$ and $\rho^i \times \rho^i \times \rho^i$, respectively. The morphisms p are uniquely determined, b and t picks the least and the largest elements in the order, respectively, e.g. in the forth row $b(\star) = (X, X, X)$. The way the morphism in the columns are defined is easy to guess: d, c, ι, and m are morphisms of domain, codomain, identity and composition, respectively.

Note that respective morphisms in rows and columns commute, e.g.

$$m \circ \rho^{ii} = \rho^i \circ m \qquad\qquad d \circ \rho^i = \rho'^i \circ d$$

and so on. This is equivalent to saying, that if we pick the corresponding morphisms in three columns, then they form a disk morphism between disks in the rows, e.g. m's form a disk morphism $\mathbf{m}$ form the disk in third row to the disk in the second row. The same thing remains true if we exchange the role of the rows and columns. This is the essence of being a schizophrenic object.

The duality. We claim

2.8. Theorem. *The contravariant functors of homming into $\mathbf{C}_1$ and into $\mathbf{D}^1$ give rise to a Stone adjunction*

$$\begin{tikzcd} \mathcal{D}^1 \arrow[rr, shift left, "\mathcal{D}^1(-,\mathbf{C}_1)"] & & \mathcal{S}_1 \arrow[ll, shift left, "\mathcal{S}_1(-,\mathbf{D}^1)"] \end{tikzcd}$$

which is an equivalence of categories. Thus the categories $\mathcal{S}_1$ and $\mathcal{D}^1$ are dually equivalent.

Proof. To fix the notation, in the following we consider the category $\mathbf{C}_1$ in $\mathcal{D}^1$ as being formed of the rows in the diagram describing the schizophrenic object, e.g. the disk universe of object of objects $\lfloor 1 \rfloor$ will be $\{X, Y\}$ rather than $\{0, 1\}$. Similarly, we consider the disk $\mathbf{D}^1$ in $\mathcal{S}_1$ as being formed from columns in the same diagram, e.g. $\lfloor 2 \rfloor$ in $\mathbf{D}^1$ stands for the second row. This give us certain convenient identifications, e.g. the universe of the disk $\lfloor 2 \rfloor$ is equal to the set of maps of the category $\lceil 1 \rceil$.

For $n \in \omega^+$, the objects of $\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1)$ form the set

$$\mathcal{D}^1(\lfloor n \rfloor, \lfloor 1 \rfloor) = \{\overline{0;1}, \overline{1;2}, \ldots, \overline{n-1;n}\}$$

and the category $\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1)$ isomorphic to $\lceil n-1 \rceil$.

For $n \in \omega$, the universe of the disk $\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1)$ is the set

$$\mathcal{S}_1(\lceil n \rceil, \lceil 1 \rceil) = \{\downarrow n, n{\downarrow}n-1, \ldots, 1{\downarrow}0, 0{\downarrow}\}$$

and with the increasing order so that $\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1)$ is isomorphic to $\lfloor n+1 \rfloor$. Thus the functors have the appropriate codomains.

The statement of the theorem says that we have two natural isomorphisms η and ε, satisfying the triangular equalities.

We shall define these natural transformations. The component of η at $\lfloor n \rfloor$ in $\mathcal{D}^1$ is a disk map

$$\eta_{\lfloor n \rfloor} : \lfloor n \rfloor \longrightarrow \mathcal{S}_1(\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1), \mathbf{D}^1)$$

such that on universes it is a function

$$\eta_{\lfloor n \rfloor} : \{0, \ldots, n\} \longrightarrow \mathcal{S}_1(\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1), \lceil 1 \rceil)$$

so that, for $i \in \{0, \ldots, n\}$,

$$\eta_{\lfloor n \rfloor}(i) : \mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1) \longrightarrow \lceil 1 \rceil$$

is a functor with the object function

$$\eta_{\lfloor n \rfloor}(i)_0 : \mathcal{D}^1(\lfloor n \rfloor, \lfloor 1 \rfloor) \longrightarrow \{X, Y\}$$

given by

$$h \longmapsto h(i)$$

Identifying $\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1)$ with $\lceil n-1 \rceil$ we have, for $i \in \{0, \ldots, n\}$

$$\eta_{\lfloor n \rfloor}(i) : \lceil n-1 \rceil \longrightarrow \lceil 1 \rceil \quad = \quad \begin{cases} i{\downarrow} & \text{if } i < n \\ \downarrow n-1 & \text{if } i = n \end{cases}$$

So $\eta_{\lfloor n \rfloor}$ is iso.

The component of ε at $\lceil n \rceil$ in $\mathcal{S}_1$ is a functor

$$\varepsilon_{\lceil n \rceil} : \lceil n \rceil \longrightarrow \mathcal{D}^1(\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1), \mathbf{C}_1)$$

with object function

$$\varepsilon_{\lceil n \rceil, 0} : \{0, \ldots, n\} \longrightarrow \mathcal{D}^1(\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1), \lfloor 1 \rfloor)$$

such that, for $i \in \{0, \ldots, n\}$,

$$\varepsilon_{\lceil n \rceil, 0}(i) : \mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1) \longrightarrow \lfloor 1 \rfloor$$

is a disk map, given by the function on universes

$$\varepsilon_{\lceil n \rceil, 0}(i) : \mathcal{S}_1(\lceil n \rceil, \lceil 1 \rceil) \longrightarrow \{X, Y\}$$

by

$$h \longmapsto h(i)$$

Identifying $\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1)$ with $\lfloor n+1 \rfloor$ we have, for $i \in \{0, \ldots, n\}$, that

$$\varepsilon_{\lceil n \rceil, 0}(i) = \overline{i; i+1} : \lfloor n+1 \rfloor \longrightarrow \lfloor 1 \rfloor$$

So $\varepsilon_{\lceil n \rceil}$ is iso.

We leave to the reader the verification that $\varepsilon_{\lceil n \rceil}$ and $\eta_{\lfloor n \rfloor}$ are well defined, η and ε are natural, and that the following triangles

$$\begin{tikzcd}
 & \mathcal{S}_1(\mathcal{D}^1(\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1), \mathbf{C}_1), \mathbf{D}^1) \arrow[dr, "\mathcal{S}_1(\varepsilon_{\lceil n \rceil}, \mathbf{D}^1)"] & \\
\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1) \arrow[ur, "\eta_{\mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1)}"] \arrow[rr, "\mathrm{Id}"'] & & \mathcal{S}_1(\lceil n \rceil, \mathbf{D}^1)
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{D}^1(\mathcal{S}_1(\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1), \mathbf{D}^1), \mathbf{C}_1) \arrow[dr, "\mathcal{D}^1(\eta_{\lfloor n \rfloor}, \mathbf{C}_1)"] & \\
\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1) \arrow[ur, "\varepsilon_{\mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1)}"] \arrow[rr, "\mathrm{Id}"'] & & \mathcal{D}^1(\lfloor n \rfloor, \mathbf{C}_1)
\end{tikzcd}$$

commute. This is end the proof. ■

We will make a comment on the orders in 1-disks and simple 1-categories or rather finite intervals (strict with endpoints) and finite non-empty linear orders. In [J] there is a very suggestive picture explaining this duality:

$$| * | * | * | * |$$

showing that the interval

$$| \quad | \quad | \quad | \quad |$$

is the set of Dedekind cuts of the order

$$* \quad * \quad * \quad *$$

The trick is that the orders on $*$'s and $|$'s are going in the different directions i.e. to have increasing order on cuts

$$| \leq | \leq | \leq | \leq |$$

we need to have decreasing order on stars

$$* \geq * \geq * \geq *$$

3. Disks

3.1. Some notation concerning disks. We introduce some notation concerning disks. Let x, y be nodes in a disk D. We call x a *left endpoint* (*right endpoint, bi-endpoint*) iff it is the least (largest, both) element in the linear order of the fiber over $p(x)$. A node x is a *leaf* iff the fiber $p^{-1}(x)$ contain exactly two elements. Note that, x is a leaf iff it is the image of a leaf of the tree $\imath(D)$ under the unit morphism $\eta : \imath(D) \longrightarrow D$. Since our disks are finite, over every inner node there is a leaf. We write $x \lhd y$ to mean that both x and y are nodes in the same fiber and y is the immediate successor of x. Since $p^{(0)}(x) = p^{(0)}(y)$ we can define the number

$$\mu^D_{x,y} = \max\{l : p^{(l)}(x) = p^{(l)}(y)\}$$

We usually omit the superscript D. Note that, if $x \in D^l$, $y \in D^k$ and $\mu_{x,y} < \min\{l, k\}$ then the node $p^{(\mu_{x,y})}(x) = p^{(\mu_{x,y})}(y)$ must be inner (there are at least two elements in the fiber over it), moreover the nodes $p^{(\mu_{x,y}+1)}(x)$, $p^{(\mu_{x,y}+1)}(y)$ are comparable. This last observation allows us to introduce a natural linear order on the whole set D^l extending the existing orders on fibers of $p : D^l \longrightarrow D^{l-1}$. Namely, if $x, y \in D^l$ are different nodes then we put

$$x < y \quad \text{iff} \quad p^{(\mu_{x,y}+1)}(x) < p^{(\mu_{x,y}+1)}(y)$$

But this order is *not* preserved by the disk morphisms, in general.

If x, y are different leaves of D then we always have $\mu_{x,y} < \min\{l, k\}$. Thus we can also introduce a linear order on leaves by the above formula. We denote this order by $\preceq$ and by $\ll$ we denote the immediate successor relation given by this order.

If x, y are elements of a poset, we write $x \perp y$ if x and y are comparable and $x \not\perp y$ if they are not.

The category $\mathcal{BL}$ is a full subcategory of Poset containing finite sums of finite linear orders, i.e. a poset X is in $\mathcal{BL}$ iff $\perp$ is an equivalence relation on X (i.e. $\perp$ is transitive on X). $\mathcal{BI}$ is a subcategory of $\mathcal{BL}$ containing the same objects, and morphisms in $\mathcal{BI}$ preserve additionally minimal and maximal elements.

Thus the category $\mathcal{BL}$ is equivalent to the category of finite bundles of finite linear orders and the category $\mathcal{BI}$ is equivalent to the category of finite bundles of finite intervals.

Note that the inclusion functor $\mathcal{BI} \longrightarrow \mathcal{BL}$ has a right adjoint $\overline{(-)} : \mathcal{BL} \longrightarrow \mathcal{BI}$ which adds to a non-empty object X of $\mathcal{BL}$ new minimal and maximal elements at the ends of each maximal linear order in X; $\overline{(\emptyset)}$ is the one-element (non-strict) interval.

The following easy lemma provides an equivalent elementary description of disks.

3.2. Lemma. *The sequence of functions*

$$
\begin{tikzcd}
\cdots \arrow[r, "p"] & D^{s+1} \arrow[l, shift right=2, "b"'] \arrow[l, shift left=2, "t"] \arrow[r, "p"] & D^{s} \arrow[l, shift right=2, "b"'] \arrow[l, shift left=2, "t"] & \cdots & D_1 \arrow[r, "p"] & D^0 \cong 1 \arrow[l, shift right=2, "b"'] \arrow[l, shift left=2, "t"]
\end{tikzcd}
$$

between objects of $\mathcal{BI}$ is a disk iff for $s \in \omega$,

1. *for $x \in D^s$, $bp(x) \leq x \leq tp(x)$;*
2. *p maps D^{s+1} onto D^s, so that, for $x, y \in D^{s+1}$, $p(x) = p(y)$ iff $x \perp y$;*
3. *for $x \in D^s$, $b(x) = t(x)$ iff $x = bp(x)$ or $x = tp(x)$.*

Proof. The conditions 1. and 2. say that $p : D^{s+1} \longrightarrow D^s$ is a bundle of intervals and the condition 3. expresses the disk condition. ■

For $n \in \omega$, we give a description of the disk γ_n which is the image of θ_n under the functor $\overline{(-)}$. The *universe* of γ_n is given by

$$\gamma_n^l \quad = \quad \begin{cases} 2l+1 & \text{if } 0 \leq l \leq n \\ 2n+2 & \text{if } n < l \end{cases}$$

the *projections* $p : \gamma_n^{l+1} \longrightarrow \gamma_n^l$ are defined as follows: for $0 \leq l < n$

$$p(i) \quad = \quad \begin{cases} i & \text{if } 0 \leq i \leq l \\ i-1 & \text{if } i = l+1 \\ i-2 & \text{if } l+1 < i \leq 2l \end{cases}$$

for $l = n$

$$p(i) \quad = \quad \begin{cases} i & \text{if } 0 \leq i \leq n \\ i-1 & \text{if } n < i \leq 2n+1 \end{cases}$$

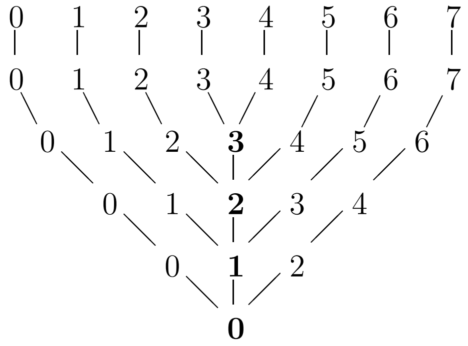
and for $l > n$

$$p(i) = i$$

The *orders in fibers* of γ_n are defined as follows: if $i, j \in \gamma_n^l$, then

$$i < j \ \text{ iff } \begin{cases} l \leq n \text{ and } l-1 \leq i < j \leq l+1 \\ \textit{or} \\ l = n+1 \text{ and } i = n \text{ and } j = n+1 \end{cases}$$

In order to make this simple but tedious definition easier to grasp we present below the first six levels of γ_3. The inner nodes are marked bold.



Thus in γ_3 at level $l = 0, 1, 2, 3$ there is exactly one inner node l, and $l - 1 < l < l + 1$, and there are no inner nodes at higher levels. Moreover, at level 4 we have $3 < 4$.

3.3. LEMMA. *Let $l, n_1, n_2 \in \omega$, $n_1 < n_2, l$. Then*

1. $\gamma^l_{n_2} \;=\; \gamma^l_{n_1} + \gamma^{l-n_1-1}_{n_2-n_1-1}$

2. $\min\{x \in \gamma^l_{n_2} : p^{(n_1)}(x) = n_1\} \;=\; n_1$

3. $\max\{x \in \gamma^l_{n_2} : p^{(n_1)}(x) = n_1\} \;=\; n_1 + 1 + \gamma^{l-n_1-1}_{n_2-n_1-1} \;=\; \gamma^l_{n_2} - n_1 - 1$

PROOF. Exercise. ■

We introduce some notions and notation concerning disks γ_n. Let $l, n \in \omega$ and $x \in \gamma^l_n$. We say that x is in the *left side* of γ_n iff either $l \leq n$ and $x < l$ or $n < l$ and $x \leq n$. We say that x is in the *right side* of γ_n iff either $l \leq n$ and $x > l$ or $n < l$ and $x > n$. We say that x is in the *far left side* of γ_n iff x is in the left side of γ_n and it is a right endpoint. We say that x is in the *far right side* of γ_n iff x is in the right side of γ_n and it is a left endpoint. By $ls(\gamma_n)$, $fls(\gamma_n)$, $rs(\gamma_n)$, $frs(\gamma_n)$, we denote left, far left, right, far right sides of γ_n, respectively. Note that a node in the left (right) side is left (right) endpoints and a node is far left or far right side iff it is a bi-endpoint.

For later reference, we list below the numeric conditions for a node of to be in the above parts of the disk γ_n.

3.4. LEMMA. *Let $l, n \in \omega$ and $x \in \gamma_n$. Then*

1. $x \in ls(\gamma_n)$ *iff* ($l \leq n$ *and* $x < l$) *or* ($l > n$ *and* $x \leq n$)

2. $x \in fls(\gamma_n)$ *iff* ($l \leq n + 1$ *and* $x < l - 1$) *or* ($l > n + 1$ *and* $x \leq n$)

3. $x \in rs(\gamma_n)$ *iff* ($l \leq n$ *and* $x > l$) *or* ($l > n$ *and* $x > n$)

4. $x \in frs(\gamma_n)$ *iff* ($l \leq n$ *and* $x > l + 1$) *or* ($l = n + 1$ *and* $x > n + 1$) *or* ($n + 1 < l$ *and* $x \geq n + 1$). ■

Before defining some specific morphisms in $\mathcal{D}$, we shall make some general observations concerning them.

3.5. Lemma.

1. *Any disk morphism $f : D \longrightarrow E$ is uniquely determined by its values on leaves of D.*

2. *Let $n \in \omega$ and D be a disk. Then the disk morphism $f : \gamma_n \longrightarrow D$ is uniquely determined by the value $f^n(n) \in D^n$. Moreover, for any $x \in D^n$ there is a unique morphism $f : \gamma_n \longrightarrow D$, such that $f^n(n) = x$. Thus, we have a bijection*

$$\mathcal{D}(\gamma_n, D) \longrightarrow D^n$$

which is natural in D.

3. *Let $n \in \omega$ and D be a disk. Then the disk morphism $f : D \longrightarrow \gamma_n$ is uniquely determined by the value $\{f^l\}_{l \leq n+1}$.*

Proof.

Ad 1 We already noted that any disk morphisms $f : D \longrightarrow E$ is uniquely determined by its values on inner nodes of D. But since over each inner node there is a leaf and disk morphisms preserves projections, once we fix values of a morphisms on leaves the other values on inner nodes are uniquely determined.

Ad 2 The node $n \in \gamma_n^n$ is the unique leaf of γ_n. Thus the value $f^n(n) \in D^n$ indeed uniquely determines the morphism f. On the other hand, for any $x \in D^n$ we can define a tree morphism $f' : \theta_n \longrightarrow D$ by the formula $f'^l(l) = p^{(l)}(x)$ for $0 \leq l \leq n$ and f' extends uniquely to a disk morphism $f : \gamma_n \longrightarrow D$, such that $f^n(n) = x$.

Ad 3 This follows from the fact that at levels l greater than $n+1$ all the intervals contains one element. So, for $x \in D^l$, we have $f^l(x) = f^{n+1}(p^{(n+1)}(x))$.

■

Let $n_0, n_1, n_2 \in \omega$ and $n_1 < n_0, n_2$. We define the codomain morphism

$$\mathbf{c}_{(n_1)} = \mathbf{c} : \gamma_{n_0} \longrightarrow \gamma_{n_1}$$

by the condition $\mathbf{c}^{n_0}(n_0) = n_1$, and the domain morphism

$$\mathbf{d}_{(n_1)} = \mathbf{d} : \gamma_{n_2} \longrightarrow \gamma_{n_1}$$

by the condition $\mathbf{d}^{n_2}(n_2) = n_1 + 1$. Since these morphisms play very important role in this paper we shall give their full descriptions. For $x \in \gamma^l_{n_0}$ we have

$$\mathbf{c}^l(x) \quad = \quad \begin{cases} x & \text{if } (0 \leq l \leq n_1) \text{ or } (x < n_1) \\ n_1 & \text{if } n_1 < l \text{ and } n_1 \leq x < \gamma^l_{n_0} - n_1 - 1 \\ n_1 + 1 & \text{if } n_1 < l \text{ and } x = \gamma^l_{n_0} - n_1 - 1 \\ x - \gamma^l_{n_0} + 2n_1 + 2 & \text{if } n_1 < l \text{ and } \gamma^l_{n_0} - n_1 - 1 < x < \gamma^l_{n_0} \end{cases}$$

and for $y \in \gamma^l_{n_2}$ we have

$$\mathbf{d}^l(y) \;=\; \begin{cases} y & \text{if } (0 \le l \le n_1) \text{ or } (n_1 < l \text{ and } y < n_1) \\ n_1 & \text{if } n_1 < l \text{ and } y = n_1 \\ n_1 + 1 & \text{if } n_1 < l \text{ and } n_1 < y \le \gamma^l_{n_2} - n_1 - 1 \\ y - \gamma^l_{n_2} + 2n_1 + 2 & \text{if } n_1 < l \text{ and } \gamma^l_{n_2} - n_1 - 1 < y < \gamma^l_{n_2} \end{cases}$$

3.6. The limits in $\mathcal{BI}$. We want to study the limits in $\mathcal{D}$. In particular, we want to show that all the disk are the multi-pullbacks constructed using objects γ_n and morphisms $\mathbf{c}$ and $\mathbf{d}$. Before we do this, we shall characterize, in this section, the multi-pullbacks in $\mathcal{BI}$.

For $k \in \omega$, by a *zigzag of morphisms* of length $k+1$ in a category we mean a diagram

$$\begin{tikzcd}[column sep=small] C_0 \arrow[dr, "f_1"'] & & C_2 \arrow[dl, "f_2"] \arrow[dr, "f_3"'] & & C_4 \arrow[dl, "f_4"] & \cdots & C_{2k-2} \arrow[dr, "f_{2k-1}"'] & & C_{2k} \arrow[dl, "f_{2k}"] \\ & C_1 & & C_3 & & & & C_{2k-1} & \end{tikzcd} \tag{4}$$

The limit of such a zigzag, if exists, is called *multi-pullback*.

Given a zigzag (4) in Poset. By a *compatible tuple* for this zigzag, we mean, a $k{+}1$-tuple $\langle x_0, \ldots, x_k\rangle$ such, that $x_i \in C_i$ for $i \in k+1$ and $f_{2i+1}(x_i) = f_{2i+2}(x_{i+1})$ for $i \in k$.

Thus, the limit of (4) in Poset is the set of compatible tuples of the zigzag (4) with pointwise order.

The limits in the category of linear orders, it they exists, are computed in the category of partial orders Poset. This is mainly due to the fact that, one- and two-element linear orders are objects of this category. The same is true for the category of (non-strict) intervals $\mathcal{I}$, since if all the maps in a diagram of linear orders preserves endpoints, then so do all the projections. Thus in $\mathcal{I}$ the limit of a diagram exists if its limit in Poset is a linear order. For similar reasons the limits in both $\mathcal{BL}$ and $\mathcal{BI}$, if exists, are computed in Poset.

We say that a zigzag (4) in $\mathcal{I}$ is *thin* iff for any compatible tuples $\vec{x}$, $\vec{y}$ if there is $0 \le i_0 \le k$ such, that $x_{i_0} < y_{i_0}$ and $f_{2i_0+1}(x_{i_0}) = f_{2i_0+1}(y_{i_0})$ than $x_i = y_i$, for all $i_0 < i \le k$.

We have

3.7. Lemma. *The limit of a zigzag of morphisms (4) in $\mathcal{I}$ exists in $\mathcal{I}$ iff (4) is thin.* ∎

We say that the zigzag (4) in Poset is *thin* iff for any compatible tuples $\vec{x}$, $\vec{y}$ such that, $x_i \perp y_i$, for $i \in k+1$, $x_{i_0} < y_{i_0}$ and $f_{2i_0+1}(x_{i_0}) = f_{2i_0+1}(y_{i_0})$ for some $i_0 \in k$, we have $x_i = y_i$, for all $i > i_0$. We state some equivalent conditions for a zigzag in $\mathcal{BI}$ to be thin.

3.8. Lemma. *Let (4) be a zigzag of morphisms in* Poset. *Then, the following are equivalent*

1. *the zigzag (4) is thin;*

2. *for any compatible tuples $\vec{x}$, $\vec{y}$ such that, $x_i \perp y_i$, for $i \in k+1$, $x_{i_0} < y_{i_0}$ and $f_{2i_0}(x_{i_0}) = f_{2i_0}(y_{i_0})$, for some $0 < i_0 \leq k$, we have $x_i = y_i$, for all $i < i_0$;*

3. *there are no compatible tuples $\vec{x}$, $\vec{y}$ such that, $x_i \perp y_i$, for $i \in k+1$, $x_{i_0} < y_{i_0}$ and $x_{i_1} > y_{i_1}$, for some $i_0, i_1 \in k+1$.*

Proof. We shall prove that 1. is equivalent to 3. The condition 2. is 'symmetric' to 1. and its equivalence with 3. can be proved similarly.

Suppose 3. does not hold. Let $\vec{x}$, $\vec{y}$ be compatible tuples for (4) such, that $x_i \perp y_i$ for all $i \in k+1$, and $x_{i_0} < y_{i_0}$, $x_{i_1} > y_{i_1}$ for some $i_0, i_1 \in k+1$. Without loss of generality, we can assume that $i_0 < i_1$. Let

$$i_2 \;=\; \max\{i < i_1 : x_i < y_i\}$$

Then $x_{i_2} < y_{i_2}$, $x_{i_2+1} \geq y_{i_2+1}$, $f_{2i_2+1}(x_{i_2}) \leq f_{2i_2+1}(y_{i_2})$ and

$$f_{2i_2+1}(x_{i_2}) \;=\; f_{2i_2+2}(x_{i_2+1}) \;\geq\; f_{2i_2+2}(y_{i_2+1}) \;=\; f_{2i_2+1}(y_{i_2})$$

So, we have $f_{2i_2+1}(x_{i_2}) = f_{2i_2+1}(y_{i_2})$, $i_1 > i_2$ and $x_{i_1} \neq y_{i_1}$, i.e. the zigzag (4) is not thin.

To prove the converse, suppose that 1. does not hold. Let $\vec{x}$, $\vec{y}$ be compatible tuples for (4) such, that $x_i \perp y_i$ for all $i \in k+1$, and for some $0 \leq i_0 < i_1 \leq k$, $x_{i_0} < y_{i_0}$, $f_{2i_0+1}(x_{i_0}) = f_{2i_0+1}(y_{i_0})$, $x_{i_1} \neq y_{i_1}$. If $x_{i_1} > y_{i_1}$, then this contradicts 3. immediately. If $x_{i_1} < y_{i_1}$ then the tuples

$$\vec{x'} \;=\; \langle x_0, \ldots, x_{i_0}, y_{i_0+1}, \ldots, y_k\rangle \qquad \vec{y'} \;= < y_0, \ldots, y_{i_0}, x_{i_0+1}, \ldots, x_k >$$

are compatible and contradicts 3. ■

3.9. Lemma. *Let (4) be a zigzag of morphisms in* Poset*, $\vec{x}$, $\vec{y}$ compatible tuples such that, $x_i \perp y_i$, for $i \in k+1$. Then the tuple $\vec{z} = \langle z_i\rangle_{i\in k+1}$, where $z_i = \min\{x_i, y_i\}$ is compatible and $\vec{z} \leq \vec{x}, \vec{y}$.*

Proof. We shall prove, for $i \in k$, that

$$f_{2i+1}(z_i) = f_{2i+2}(z_{i+1}) \tag{5}$$

Fix $i \in k$. If z_i and z_{i+1} comes from the same tuple $\vec{x}$ or $\vec{y}$, then (5) holds, since both tuples are compatible.

Suppose that $z_i = x_i < y_i$ and $z_{i+1} = y_{i+1} < x_{i+1}$. The other case is similar. We have

$$f_{2i+1}(x_i) \;=\; f_{2i+2}(x_{i+1}) \;\geq\; f_{2i+2}(y_{i+1})$$

and

$$f_{2i+1}(x_i) \;\leq\; f_{2i+1}(y_i) \;=\; f_{2i+2}(y_{i+1})$$

i.e. (5) holds. ■

We have

3.10. Proposition. *The zigzag (4) in $\mathcal{BI}$ has a limit iff it is thin. Moreover, if this is the case the limit is computed in* Poset.

Proof. Assume that (4) is thin. Let $\vec{x}$, $\vec{y}$, $\vec{z}$ be compatible tuples, such that $\vec{x} \perp \vec{y} \perp \vec{z}$. We shall show that $\vec{x} \perp \vec{z}$.

Since the relation $\perp$ is transitive in C_{2i} we have $\vec{x_i} \perp \vec{z_i}$, for $i \in k+1$. If $\vec{x} = \vec{z}$ then clearly $\vec{x} \perp \vec{z}$. So, assume that $\vec{x} \neq \vec{z}$. Let

$$i_0 \;=\; \min\{i : x_i \neq z_i\}$$

With out loss of generality, we can assume that $x_{i_0} < z_{i_0}$. Let

$$i_1 \;=\; \max\{i : x_i \leq z_i\}$$

If $i_1 < k$ then we have $x_{i_0} < z_{i_0}$ and $x_{i_1+1} > z_{i_1+1}$ and the zigzag (4) is not thin, by Lemma 3.8. Thus $i_1 = k$ and $\vec{x} \leq \vec{z}$, so $\vec{x} \perp \vec{z}$, as required.

Thus, we have shown that the limit of (4) in Poset is in $\mathcal{BI}$. Hence this zigzag has a limit in $\mathcal{BI}$.

To prove the converse, assume that the zigzag (4) is not thin and let (L, π) be a limiting cone of (4) in Poset. Then, by Lemma 3.8, we have compatible tuples $\vec{x}$, $\vec{y}$ such that $x_i \perp y_i$, for $i \in k+1$, $x_{i_0} < y_{i_0}$ and $x_{i_1} > y_{i_1}$, for some $i_0, i_1 \in k+1$. So, in particular $\vec{x} \not\perp \vec{y}$. By Lemma 5 the tuple $\vec{z} = \langle z_i \rangle_{i \in k+1}$, where $z_i = \min\{x_i, y_i\}$, for $i \in k+1$ is compatible and $\vec{z} < \vec{x}, \vec{y}$. So $\perp$ is not transitive on L, L is not in $\mathcal{BI}$, and hence the zigzag (4) does not have a limit in $\mathcal{BI}$. ∎

3.11. The limits in $\mathcal{D}$. In this section we study the limits in the category $\mathcal{D}$. We show that each disk has a canonical presentation as a multi-pullback of disks of form γ_n. Such a presentation describes the correspondence between the disks and the ud-vectors, the numerical invariants for disks, i.e. ud-vectors determine the disks up to isomorphisms.

Note that, for $l \in \omega$, we have functors

$$(-)^l : \mathcal{D} \longrightarrow \mathcal{BI}$$

sending disk D to its l-th level D^l (not forgetting the order).

3.12. Proposition. *The limits in $\mathcal{D}$, if exists, are computed pointwise in* Poset*, i.e. for a cone in $\mathcal{D}$ to be a limiting cone it is necessary and sufficient that, for each $l \in \omega$, its image under every functor $(-)^l$ be a limiting cone in* Poset.

Proof. Fix $l \in \omega$.

Recall from Lemma 3.5 that, for any disk D, we have a bijective correspondence between nodes $x \in D^l$ and disk morphisms $f_x : \gamma_l \longrightarrow D$, such that $f^l_x(l) = x$. Let $\gamma = \overline{\theta}$ be a disk, such that θ is a tree with one node at levels smaller than l, two nodes $l \leq l+1$, at level l, and no other nodes. It easy to see that, we have a bijective correspondence between pairs of nodes $x, y \in D^l$, such that $x \leq y$ and morphisms $f_{x,y} : \gamma \longrightarrow D$, such that $f^l_{x,y}(l) = x$ and $f^l_{x,y}(l+1) = y$.

Let (D, π), be a limit in $\mathcal{D}$ of the functor $F : J \longrightarrow \mathcal{D}$. Then, for $l \in \omega$, we have a bijective correspondences

$$\begin{array}{c} x \in D^l \\ \hline f_x : \gamma_l \longrightarrow D \\ \hline \pi \circ f_x : \gamma_l \longrightarrow F \\ \hline \pi^l \circ f^l_x\lceil_{\{l\}} : \{l\} \longrightarrow F^l \end{array}$$

and

$$\begin{array}{c} x, y \in D^l, \ \ x \leq y \\ \hline f_{x,y} : \gamma \longrightarrow D \\ \hline \pi \circ f_{x,y} : \gamma \longrightarrow F \\ \hline \pi^l \circ f^l_{x,y}\lceil_{\{l,l+1\}} : \{l \leq l+1\} \longrightarrow F^l \\ \hline \pi^l \circ f^l_{x,y}\lceil_{\{l\}} : \{l\} \longrightarrow F^l \ \leq \ \pi^l \circ f^l_{x,y}\lceil_{\{l+1\}} : \{l+1\} \longrightarrow F^l \end{array}$$

where the second bijection, in both cases, follows from the fact that (D, π) is a limit of F. These correspondences show that, the universe of D^l is in bijective correspondence, via π^l, with with cones from one-element set to F^l and moreover the order of elements in D^l corresponds to the pointwise order of cones from one-element set to F^l, i.e. (D^l, π^l) is a limit of F^l in Poset.

To finish the proof, note that, the morphisms p, b, t in disk D are morphisms into limits (D^l, π^l) from cones induced by morphisms p, b, t in disks $F(j)$, for $j \in Ob(J)$. Thus they are unique. ∎

We call a zigzag of morphisms in $\mathcal{D}$ *thin* iff its image under every functor $(-)^l$, for $l \in \omega$, is thin in $\mathcal{BI}$. We have

3.13. Lemma. *The limit of a zigzag of morphisms in* $\mathcal{D}$

$$\begin{tikzcd} D_0 \arrow[dr, "f_1"'] & & D_2 \arrow[dl, "f_2"] \arrow[dr, "f_3"'] & & D_4 \arrow[dl, "f_4"] & \cdots & D_{2k-2} \arrow[dr, "f_{2k-1}"'] & & D_{2k} \arrow[dl, "f_{2k}"] \\ & D_1 & & D_3 & & & & D_{2k-1} & \end{tikzcd} \quad (6)$$

exists iff the zigzag (6) is thin.

Proof. By Propositions 3.12, 3.10, the condition is clearly necessary.

Assume that, the zigzag (6) is thin. We shall construct a limit (D, π) of (6) in $\mathcal{D}$.

Since (6) is thin, for $l \in \omega$, we have a limit (D^l, π^l) in $\mathcal{BI}$ of the zigzag $(6)^l$, the image of (6) under the functor $(-)^l$. Thus $x = \langle x_i \rangle_{i \in k+1} \in D^l$ iff $f_{2i+1}(x_i) = f_{2i+2}(x_{i+1})$, for $i \in k$.

Let $p : D^{l+1} \longrightarrow D^l$ be the unique morphism from the cone $(D^{l+1}, p \circ \pi^{l+1})$ into the limiting cone (D^l, π^l), so that $p \circ \pi^{l+1} = \pi^l \circ p$. Similarly, $b, t : D^l \longrightarrow D^{l+1}$, are the unique morphisms, into the limiting cone (D^{l+1}, π^{l+1}), such that

$$b \circ \pi^l = \pi^{l+1} \circ b \quad \text{and} \quad b \circ \pi^l = \pi^{l+1} \circ b$$

This ends the construction of (D, π).

Note that π preserves order, and commutes with p, b, t by construction. Now, by Lemma 3.12, to show that (D, π) is a limit of (6) in $\mathcal{D}$, it is enough to prove that D is a disk. To this end, we shall verify that it satisfy the conditions from Lemma 3.2.

Fix $l \in \omega$. Clearly, D^l is an object of $\mathcal{BI}$.

If $x \in D^{l+1}$, then

$$b(p(x)) \ \leq \ x \leq \ t(p(x)) \quad \text{iff} \quad b(p(x_i)) \ \leq \ x_i \leq \ t(p(x_i)) \quad \text{for } i \in k+1$$

and the latter condition holds, since D_{2i} is a disk, for $i \in k+1$.

Let $x, y \in D^{l+1}$. If $p(x) = p(y)$ then $b(p(x)) \leq x, y$, and since D^{l+1} is in $\mathcal{BI}$, we have $x \perp y$. On the other hand, if $x \perp y$, then $x_i \perp y_i$, for $i \in k+1$, and then

$$p(x) \ = \ \langle p(x_i) \rangle_{i \in k+1} \ = \ \langle p(y_i) \rangle_{i \in k+1} \ = \ p(y)$$

Thus, it remains to show that, for $x \in D^{l+1}$

$$b(x) \ = \ t(x) \quad \text{iff} \quad x \ = \ b(p(x)) \ \text{ or } \ x \ = \ t(p(x)) \tag{7}$$

If $x = b(p(x))$, then $x_i = b(p(x_i))$, and hence $b(x_i) = t(x_i)$, for $i \in k+1$. Thus $b(x) = t(x)$.

If $x = t(p(x))$ the argument is similar. This proves one side of (7).

For the converse, we assume that $b(x) = t(x)$. Then $y = b(p(x)) \leq x \leq t(p(x)) = z$. We need to show, that one of those inequalities is, in fact, an equality.

Assume contrary, that $y < x < z$. Note that, for $i \in k+1$, since $b(x_i) = t(x_i)$, the node x_i is outer and then either $x_i = y_i$ or $x_i = z_i$. There are $i_1, i_2 \in k+1$, such that

$$x_{i_1} \ = \ y_{i_1} \ < \ z_{i_1} \quad \text{and} \quad y_{i_2} \ < \ z_{i_2} \ = \ x_{i_2}$$

Suppose that $i_1 < i_2$. The case $i_2 < i_1$ can be treated similarly. Let

$$i_0 \ = \ \max\{i < i_2 : x_i \ = \ y_i \ < \ z_i\}$$

Then $i_0 < i_2$ and

$$x_{i_0} \ = \ y_{i_0} \ < \ z_{i_0} \quad \text{and} \quad y_{i_0+1} \ \leq \ z_{i_0+1} \ = \ x_{i_0+1}$$

Thus, we have

$$f_{2i_0+1}(y_{i_0}) \ = \ f_{2i_0+1}(x_{i_0}) \ = \ f_{2i_0+2}(x_{i_0+1}) \ = \ f_{2i_0+2}(z_{i_0+1}) \ = \ f_{2i_0+1}(z_{i_0})$$

$i_2 > i_0$, $z_{i_0} \neq y_{i_0}$ and $y_i \perp z_i$, for $i \in k+1$. But this contradicts the fact, that 6^l is thin. Thus $x = b(p(x))$ or $x = t(p(x))$, and (7) holds. ∎

Let n_0, n_1, n_2, $\mathbf{c}$, $\mathbf{d}$ be as above, $l \in \omega$, $x \in \gamma^l_{n_0}$ and $y \in \gamma^l_{n_2}$. If $\mathbf{c}^l(x) = \mathbf{d}^l(y)$, we say that x and y are *connected*, and more specifically, *x is connected to the right to y* and *y is connected to the left to x*. In order to study limits of some diagrams involving $\mathbf{c}$ and $\mathbf{d}$, we need to analyze properties of connected nodes. When two elements are connected they may be from different sides of the disks and if one of the elements is inner the other may be not. However, the connected elements inherit some properties one from another. Since the following lemma contains many conditions we add some slogans to help understand and remember them.

3.14. Lemma. *Let $l, n_0, n_1, n_2 \in \omega$, $n_1 < n_0, n_2$, $x \in \gamma^l_{n_0}$ and $y \in \gamma^l_{n_2}$, and $\mathbf{c}^l(x) = \mathbf{d}^l(y)$. Then*

1. *if x is in the (far) left side of γ_{n_0} then if y is in the (far) left side of γ_{n_2} and y is the unique node of $\gamma^l_{n_2}$ with the property $\mathbf{c}^l(x) = \mathbf{d}^l(y)$*

   (the slogan: elements from the (far) left side are connected to the right to elements on the (far) left side and they are connected in a unique way);

2. *if y is in the (far) right side of γ_{n_2} then if x is in the (far) right side of γ_{n_0} and x is the unique node of $\gamma^l_{n_0}$ with the property $\mathbf{c}^l(x) = \mathbf{d}^l(y)$*

   (the slogan: elements from the (far) right side are connected to the left to elements on the (far) right side and they are connected in a unique way);

3. *if y is in the left side of γ_{n_2} and x is in the right side of γ_{n_0} then either x is in the far right side of γ_{n_0} or y is in the far left side of γ_{n_2}*

   (the slogan: if an element in the left side is connected to the left to an element on an element on the right side one of the elements must be in the far side);

4. *$l \leq n_1$ then x is inner iff y is inner;*

5. *if x is inner and $n_1 < l$ then $y = n_1$, i.e. y is in the left side;*

6. *if y is inner and $n_1 < l$ then $x = \gamma^l_{n_0} - n_1 - 1$, i.e. x is in the right side;*

   (the slogan the last three conditions: the inner nodes are connected to unique elements; moreover they are connected either to inner nodes or to the left to elements in the right side or to the right to elements in the left side).

Proof. Recall the numeric conditions from Lemma 3.4 for elements to be in special parts of the disk γ_n. For the whole proof we assume that $l, n_0, n_1, n_2 \in \omega$, $n_1 < n_0, n_2$, and $x \in \gamma^l_{n_0}$ $y \in \gamma^l_{n_2}$, and $\mathbf{c}^l(x) = \mathbf{d}^l(y)$.

Note that, if $l \leq n_1$ then, if one of the nodes x, y is in either side or far side or is inner then so is the other node. Thus for $l \leq n_1$, all the statements of the lemma are obvious. Therefore, we assume further that $l > n_1$.

Ad 1. Let $x \in fls(\gamma_{n_0})$. If $x < n_1$ then

$$y \ = \ x \ < \ n_1 \ \leq \ \min\{l-1, n_2-1\}$$

and hence $y \in fls(\gamma_{n_2})$. If $x \geq n_1$ then $l > n_1 + 1$ and then

$$\min\{l, n_2\} \; > \; n_1 \; = \; \mathbf{c}^l(x) \; = \; \mathbf{d}^l(y) \; = y$$

and $y \in fls(\gamma_{n_2})$, as well.

Now assume that, $x \in (ls(\gamma_{n_0}) \setminus fls(\gamma_{n_0}))$. Then $l \leq n_0 + 1$, $x = l - 1$, and we have again

$$\min\{l, n_2\} \; > \; n_1 \; = \; \mathbf{c}^l(x) \; = \; \mathbf{d}^l(y) \; = y$$

i.e. $y \in ls(\gamma_{n_2})$. Note that in all the above cases the node y connected to x is unique.

Ad 2. Let $y \in frs(\gamma_{n_2})$. If $y > \gamma^l_{n_2} - n_1 + 1$ then $\mathbf{d}^l(y) > n_1 + 1$. It follows that, $c^l(x) > n_1 + 1$, and we get

$$x \; = \; \mathbf{c}^l(x) + \gamma^l_{n_0} - 2n_1 - 2 \; > \; \gamma^l_{n_0} - n_1 - 1 > \min\{l, n_0\}$$

This means that $x \in frs(\gamma_{n_0})$. If $y \leq \gamma^l_{n_2} - n_1 - 1$ then $l > n_1 + 1$, $n_1 + 1 = \mathbf{d}^l(y) = \mathbf{c}^l(x)$ and

$$x \; = \; \gamma^l_{n_0} - n_1 - 1 > \min\{l, n_0 + 1\}$$

Hence $x \in frs(\gamma_{n_0})$.

Now assume that, $y \in (rs(\gamma_{n_2}) \setminus frs(\gamma_{n_2}))$. Then, either $l \leq n_2$ and $y = l + 1$ or $l = n_2 + 1$ and $y = n_2 + 2$. In either case, $\mathbf{c}^l(x) = \mathbf{d}^l(y) = n_1 + 1$. So $x = \gamma^l_{n_0} - n_1 - 1$. If $l > n_1 + 1$, then we already know from the above argument that $x \in frs(\gamma_{n_0}) \subseteq rs(\gamma_{n_0})$. If $l = n_1 + 1$ then $x = n_1 + 2$ and $x \in frs(\gamma_{n_0})$. As before, it is easy to see that in all the cases the node x connected to y is unique.

Ad 3. Let $x \in frs(\gamma_{n_0})$ and $y \in fls(\gamma_{n_2})$. Then, either $\mathbf{c}^l(x) = n_1$ or $\mathbf{c}^l(x) = n_1 + 1$.

In the former case, $y = n_1$ and it is a bi-endpoint, unless $l = n_1 + 1$. But then $x \in \{n_1, n_1 + 1\}$ i.e. $x \notin rs(\gamma_{n_0})$.

In the latter case, $x = \gamma^l_{n_0} - n_1 - 1$ and it is a bi-endpoint, unless $l = n_1 + 1$. But then $y \in \{n_1 + 1, n_1 + 2\}$ i.e. $y \notin ls(\gamma_{n_2})$.

The remaining cases are left for the reader as an exercise. ■

3.15. Lemma. *Let $\vec{u}$ be a ud-vector of length $k+1$. Then the zigzag*

$$\begin{tikzcd}[column sep=small]
\gamma_{u_0} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_2} \arrow[dl, "\mathbf{d}"] \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_4} \arrow[dl, "\mathbf{d}"] & \cdots & \gamma_{u_{2k-2}} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_{2k}} \arrow[dl, "\mathbf{d}"] \\
& \gamma_{u_1} & & \gamma_{u_3} & & & & \gamma_{u_{2k-1}} &
\end{tikzcd} \tag{8}$$

is thin.

Proof. Fix $l \in \omega$, and let $\vec{x}$, $\vec{y}$ be compatible tuples in the zigzag $(8)^l$, the image of (8) under the functor $(-)^l$, such that for some $i_0 \in k+1$, $x_{i_0} < y_{i_0}$ and $\mathbf{c}(x_{i_0}) = \mathbf{c}(y_{i_0})$. By Lemma 3.14, $l > u_{2i_0+1}$ and the node $x_{i_0+1} = y_{i_0+1} = u_{2i_0+1}$ must be in the left side of $\gamma_{u_{2i_0+2}}$, and then for all $i \geq i_0$ we have $x_i = y_i$. Hence $(8)^l$ is thin. ■

Thus, by Proposition 3.12 and Lemmas 3.13, 3.15, for any ud-vector $\vec{u}$, the zigzag of form (8) has a limit in $\mathcal{D}$. We shall describe a specific limit $(\gamma_{\vec{u}}, \boldsymbol{\pi})$ of the diagram (8). We have a limiting cone in $\mathcal{D}$

$$\begin{tikzcd}
 & & & \gamma_{\vec{u}} \arrow[dlll, "\boldsymbol{\pi}_0"'] \arrow[dl, "\boldsymbol{\pi}_2"'] \arrow[dr, "\boldsymbol{\pi}_{2k-2}"] \arrow[drrr, "\boldsymbol{\pi}_{2k}"] & & & \\
\gamma_{u_0} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_2} \arrow[dl, "\mathbf{d}"] & & \gamma_{u_{2k-2}} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_{2k}} \arrow[dl, "\mathbf{d}"] \\
 & \gamma_{u_1} & & \cdots & & \gamma_{u_{2k-1}} &
\end{tikzcd} \tag{9}$$

where for $l \in \omega$, $\gamma^l_{\vec{u}}$ is a set of compatible tuples for the zigzag $(8)^l$, i.e. $x_i \in \gamma^l_{u_{2i}}$, for $i \in k+1$, and $\mathbf{c}^l(x_i) = \mathbf{d}^l(x_{i+1})$, for $i \in k$. The order on $\gamma^l_{\vec{u}}$ is defined pointwise, as well, as the operations p, b, t on $\gamma_{\vec{u}}$, e.g. $p(\vec{x}) = \langle p(x_i)\rangle_{i \in k+1}$. The projections $\boldsymbol{\pi}_{2i} : \gamma_{\vec{u}} \longrightarrow \gamma_{u_{2i}}$, for $i \in k+1$, are the usual projections, i.e. $\boldsymbol{\pi}^l_{2i}(\vec{x}) = x_i$, for $\vec{x} \in \gamma^l_{\vec{u}}$.

The following lemma gives more informations about $\gamma_{\vec{u}}$.

3.16. Lemma. *All the unexplained notation refers to (9). Let $l \in \omega$, $\vec{x} \in \gamma^l_{\vec{u}}$. Then*

1. $\mid \gamma^l_{\vec{u}} \mid = \sum_{i=0}^{2k} (-1)^i \gamma^l_{u_i}$;
2. *there are $0 \leq i_0 \leq i_1 \leq k+1$, such that*
$$x_i \in rs(\gamma_{2u_i}) \quad \textit{for} \quad i < i_0$$
$$x_i \in ls(\gamma_{2u_i}) \quad \textit{for} \quad i \geq i_1$$
$$x_i \textit{ is inner } (x_i = l) \quad \textit{for} \quad i_0 \leq i < i_1;$$
3. *$\vec{x}$ is inner iff $x_{i_0} \in \gamma^l_{u_{2i_0}}$ is inner, for some $i_0 \in k+1$;*
4. *there are $k+1$ leaves: $\vec{x}^0, \ldots, \vec{x}^k$ in $\gamma_{\vec{u}}$, such that $\vec{x}^j \in \gamma^{u_{2j}}_{\vec{u}}$, for $j \in k+1$ and $\vec{x}^j \ll \vec{x}^{j+1}$, for $j \in k$;*
5. *$\vec{x}$ is a j-th leaf iff $x_j = u_{2j}$, moreover $x^j_j \in \gamma_{u_{2j}}$ is the only inner node in $\vec{x}^j$;*
6. *for $j \in k$, $\mu_{\vec{x}^j, \vec{x}^{j+1}} = u_{2j+1}$.*

Proof. 1. is left as an exercise.

2. follows from Lemma 3.14, since elements in the right sides are connected to the left to elements in the right side, elements in the left sides are connected to the right to elements in the left side, and inner nodes are either connected to inner nodes or to the left to elements in the right side, and to the right to elements in the left side.

Fix $l \in \omega$ and $\vec{x} \in \gamma^l_{\vec{u}}$. If, for some $i_0 \in k+1$, $x_{i_0} \in \gamma^l_{u_{2i_0}}$ is inner then $b(x_{i_0}) \neq t(x_{i_0})$ and hence $b(\vec{x}) \neq t(\vec{x})$. So, $\vec{x}$ is inner, as well. If x_i is outer then $b(x_i) = t(x_i)$. So, if all x_i's are outer then $\vec{x}$ is outer, as well. This shows 3.

If $x_{i_0} \in \gamma^l_{u_{2i_0}}$ is a leaf, then $x_{i_0} = l = u_{2i_0}$ and $x_{i_0-1} \in rs(\gamma_{2u_0-2})$, $x_{i_0+1} \in ls(\gamma_{2u_0+2})$. Hence, by Lemma 3.14, for $i \neq i_0$, x_i is an outer node. It follows, that $b(\vec{x})$ and $t(\vec{x})$ are the only compatible tuples over $\vec{x}$, i.e. $\vec{x}$ is a leaf.

On the other hand, if $\vec{x} \in \gamma^l_{\vec{u}}$ is a leaf, then $b(\vec{x}) \neq t(\vec{x})$ are the only compatible tuples over $\vec{x}$. Then, there is $i_0 \in k+1$, such that $b(x_{i_0}) < t(x_{i_0})$.

In case x_{i_0} is not a leaf, then we have an inner node $y \in \gamma^{l+1}_{u_{2i_0}}$, such that $b(x_{i_0}) < y < t(x_{i_0})$.

In case, either x_{i_0-1} or x_{i_0+1} is inner, by Lemma 3.14, $l < u_{2i_0+1} < u_{2i_0}$ and we have, as in previous case, an inner node $y \in \gamma^{l+1}_{u_{2i_0}}$, such that $b(x_{i_0}) < y < t(x_{i_0})$.

Having y as in either case, we show that $\vec{x}$ is not inner. As, y is inner, there is a unique tuple $\vec{y} \in \gamma^l_{\vec{u}}$, such that $y_{i_0} = y$. Clearly, $b(\vec{x}) \neq \vec{y} \neq t(\vec{x})$. Since x_{i_0} is inner, $\vec{x}$ is the unique compatible tuple in $\gamma^l_{\vec{u}}$ with x_{i_0} on i_0-th place and $p(\vec{y})$ has $x_{i_0} = p(y)$ on i-th place, as well. Therefore $p(\vec{y}) = \vec{x}$, i.e. there are at least three elements over $\vec{x}$ and $\vec{x}$ is not a leaf.

Thus, we have shown that $\vec{x} \in \gamma^l_{\vec{u}}$ is a leaf iff there is a unique i_0, such that x_{i_0} is an inner node and moreover this x_{i_0} is a leaf in $\gamma^l_{u_{2i_0}}$. Then, we must have $x_{i_0} = l = u_{2i_0}$.

For $j \in k+1$, we define compatible tuples

$$\vec{x}^j = \langle x^j_i \rangle_{i \in k+1} \in \gamma^{u_{2j}}_{\vec{u}}$$

by the condition $x^j_j = u_{2j}$. Since $n_{2j} \in \gamma^{u_{2j}}_{u_{2j}}$ is the unique leaf of $\gamma_{u_{2j}}$, $\vec{x}^j$'s are well defined by the above condition. Then, by the above considerations, they are all and only leaves of $\gamma_{\vec{u}}$.

Since x^{j+1}_{j+1} is inner, for $l \leq u_{2j}, u_{2j+2}$, we have

$$p^{(l)}(\vec{x}^j) = p^{(l)}(\vec{x}^{j+1}) \quad \text{iff} \quad p^{(l)}(x^j_{j+1}) = p^{(l)}(x^{j+1}_{j+1})$$

We can easily check, that

$$x^j_{j+1} = u_{2j+1} \in \gamma^{u_{2j}}_{u_{2j+2}}$$

and, for $l \geq u_{2j+1}$

$$p^{(l)}(x^j_{j+1}) = p^{(l)}(u_{2j+1}) = u_{2j+1}$$

$$p^{(l)}(x^{j+1}_{j+1}) = p^{(l)}(u_{2j+2}) = l$$

Thus, we have

$$p^{(u_{2j+1})}(x^j_{j+1}) = p^{(u_{2j+1})}(x^{j+1}_{j+1})$$

$$p^{(u_{2j+1}+1)}(x^j_{j+1}) = u_{2j+1} < u_{2j+1} + 1 = p^{(u_{2j+1}+1)}(x^{j+1}_{j+1})$$

This shows, that $\mu_{\vec{x}^j,\vec{x}^{j+1}} = u_{2j+1}$ and that $\vec{x}^j \preceq \vec{x}^{j+1}$, for $j \in k$. So, we also have $\vec{x}^j \ll \vec{x}^{j+1}$, for $j \in k$. ∎

Using the above Lemma we can prove the following Proposition.

3.17. PROPOSITION. *Let $\vec{u}$, $\vec{v}$ be ud-vectors, such that the disks $\gamma_{\vec{u}}$ and $\gamma_{\vec{v}}$ are isomorphic. Then $\vec{u} = \vec{v}$.*

PROOF. Let $\vec{u}$, $\vec{v}$ be ud-vectors, and let $f : \gamma_{\vec{u}} \longrightarrow \gamma_{\vec{v}}$ be a disk isomorphism. The leaves, as well, as order on leaves are defined in terms of operations p, b, t, and order in disks. Thus, any disk isomorphism must send leaves to leaves bijectively. Hence, $\gamma_{\vec{u}}$ and $\gamma_{\vec{v}}$ have the same number of leaves, say $k+1$, i.e. the ud-vectors $\vec{u}$ and $\vec{v}$ have equal length. Let $\vec{x}^0, \ldots, \vec{x}^k$ be the leaves of $\gamma_{\vec{u}}$ and $\vec{y}^0, \ldots, \vec{y}^k$ be the leaves of $\gamma_{\vec{v}}$, listed in order. Then $f(\vec{x}^j) = \vec{y}^j$, for $j \in k+1$. Thus $u_{2j} = v_{2j}$, for $j \in k+1$. For $l \leq u_{2j}, u_{2j+2}$, we also have

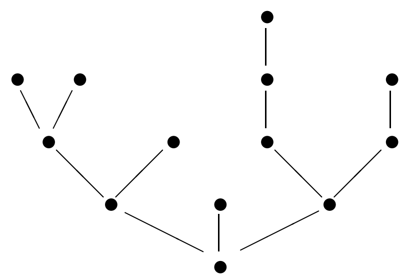
$$p^{(l)}(\vec{x}^j) = p^{(l)}(\vec{x}^{j+1}) \qquad p^{(l)}(\vec{y}^j) = p^{(l)}(\vec{y}^{j+1})$$

Thus

$$u_{2j+1} = \mu_{\vec{x}^j, \vec{x}^{j+1}} = \mu_{\vec{y}^j, \vec{y}^{j+1}} = v_{2j+1}$$

i.e. $\vec{u} = \vec{v}$. ■

We have shown that the inner nodes of the disk $\gamma_{\vec{u}}$, form a tree $\theta_{\vec{u}}$, which have $\mathrm{lh}(\vec{u})$ leaves. The j-th leaf is at the level u_{2j} and it 'meets' $j+1$-st leaf at the level u_{2j+1}. This describes $\theta_{\vec{u}}$ and hence $\gamma_{\vec{u}}$ uniquely. For example the tree $\theta_{3,2,3,1,2,0,1,0,4,1,3}$ looks like that:



We shall show, that any disk is isomorphic to a disk $\gamma_{\vec{u}}$, for some ud-vector $\vec{u}$. To this end we introduces some 'projection' morphisms from any disk into disks of form γ_n, and gives some connections between them.

3.18. PROPOSITION. *Let D be a disk, $n_0, n_1, n_2, n \in \omega$. Then*

1. *Let $x = x_n \in D^n$ be a leaf, and $x_l = p^{(l)}(x_n)$ for $0 \leq l < n$. Then, there is a unique disk morphism*

$$\overline{x} : D \longrightarrow \gamma_n$$

such that, for $0 \leq l \leq n$ and $y \in D^l$, we have

$$\overline{x}^l(y) = l \quad \textit{iff} \quad y = x_l \tag{10}$$

2. *Let $x \in D^{n_0}$, $y \in D^{n_2}$ be leaves, such that $x \ll y$, and $n_1 = \mu_{x,y}$. Then the diagram*

$$\begin{tikzcd} & D \arrow[dl, "\overline{x}"'] \arrow[dr, "\overline{y}"] & \\ \gamma_{n_0} \arrow[dr, "\mathbf{c}"'] & & \gamma_{n_2} \arrow[dl, "\mathbf{d}"] \\ & \gamma_{n_1} & \end{tikzcd} \tag{11}$$

commutes.

3. *Let $x \in D^{n_0}$, $y \in D^{n_2}$ and $n_1 = \mu_{x,y}$, such that $p^{n_1+1}(x) \lhd p^{n_1+1}(y)$. Then there is a unique morphism*
$$\overline{x;y} : D \longrightarrow \gamma_{n_1}$$
such that, for $0 \leq l \leq n_1$ and $z \in D^l$
$$\overline{x;y}^l(z) = l \qquad \textit{iff} \qquad z = p^{(l)}(x)\ (= p^{(l)}(y)) \tag{12}$$
and
$$\overline{x;y}^{n_0}(x) = n_1 \qquad \textit{and} \qquad \overline{x;y}^{n_2}(y) = n_1 + 1 \tag{13}$$

Proof. Ad 1. Let D, n, x, x_l be as in the statement. We put for $l \leq n$ and $y \in D^l$

$$\overline{x}^l(y) \quad = \quad \begin{cases} \mu_{x,y} & \text{if } p^{(\mu_{x,y}+1)}(y) < x_{\mu_{x,y}+1} \\ l & \text{if } y = x_l \\ 2l - \mu_{x,y} & \text{if } p^{(\mu_{x,y}+1)}(y) > x_{\mu_{x,y}+1} \end{cases}$$

and for $y \in D^{n+1}$

$$\overline{x}^{n+1}(y) \quad = \quad \begin{cases} \mu_{x,y} & \text{if } p^{(\mu_{x,y}+1)}(y) < x_{\mu_{x,y}+1} \\ n & \text{if } y = b(x) \\ n+1 & \text{if } y = t(x) \\ 2n+1-\mu_{x,y} & \text{if } p^{(\mu_{x,y}+1)}(y) > x_{\mu_{x,y}+1} \end{cases}$$

Since any node $y \in D^l$ is either equal to x_l or $p^{(\mu_{x,y}+1)}(y)$ is comparable but different than $p^{(\mu_{x,y}+1)}(x)$, it follows, that for $l \leq n$, the above formulas defines functions $\overline{x}^l : D^l \longrightarrow \gamma_n^l$. As x is a leaf, if $y \in D^l$ and $p(y) = x$ then $y \in \{b(x), t(x)\}$. Moreover, if $y \in D^l$ and $p(y) \neq x$ then $x_{\mu_{x,y}+1}$ and $p^{(\mu_{x,y}+1)}(y)$ are comparable but different. Hence $\overline{x}^{n+1} : D^{n+1} \longrightarrow \gamma_n^{n+1}$ is a well defined function, as well.

We need to verify that the functions $\{\overline{x}^l\}_{l \in n+2}$ preserves projections, order in fibers, and endpoints. Then, by Lemma 3.5, we will get a unique morphism $\overline{x} : D \longrightarrow \gamma_n$ extending the above set of functions.

First, we shall show that the orders are preserved.

Note that, if $l \leq n$, $y, z \in \gamma_n^l$ and $y < z$, then $p(y) = p(z)$ and either $\mu_{x,y} = \mu_{x,z} < l-1$ or $p(y) = x_{l-1} \in D^{l-1}$. In the former case, we simply have that $\overline{x}^l(y) = \overline{x}^l(z)$, by first or third close of the definition of $\overline{x}^l$. In the latter case, $y, z \in p^{-1}(x_{l-1})$. If $u \in p^{-1}(x_{l-1})$ then $u = x_{l-1}$ or $\mu_{x,u} = l-1$. Hence

$$\overline{x}^l(y) \;=\; \begin{cases} l-1 & \text{if } u < x_l \\ l & \text{if } u = x_l \\ l+1 & \text{if } u > x_l \end{cases}$$

Since, $y < z$, and $l-1 < l < l+1$ in order of γ_n^l, we have $\overline{x}^l(y) \leq \overline{x}^l(z)$.

If $y, z \in D^{n+1}$ and $y < z$, then either $y = b(x)$ and $z = t(x)$ and hence

$$\overline{x}^{n+1}(y) \;=\; n \;<\; \overline{x}^{n+1}(z)$$

or $\mu_{x,y} = \mu_{x,z} < n$ and then $\overline{x}^{n+1}(y) = \overline{x}^{n+1}(z)$. Thus, for $l \leq n+1$, the functions $\overline{x}^{n+1}$ preserves the orders.

To see, that projections are preserved, note that, if $l \leq n+1$ and $y \in D^l$, then either $y \in p^{-1}(l-1)$ or $\mu_{x,y} < l-1$. In both cases, we have that

$$p \circ \overline{x}^l(y) \;=\; \overline{x}^l(p(y))$$

i.e. projections are preserved.

For the preservation of endpoints, note that, if $l \leq n+1$ and $y \not\in p^{-1}(x_{l-1})$, then $\overline{x}^l(y)$ is the unique element in the fiber over $p \circ \overline{x}^l(y)$, i.e. it is a bi-endpoint. If $y \in p^{-1}(x_{l-1})$, then

$$\overline{x}^l(y) \;=\; \begin{cases} l-1 & \text{if } l \leq n \text{ and } y < x_l \\ n & \text{if } y = b(x_{l-1}) \\ n+1 & \text{if } y = t(x_{l-1}) \\ l+1 & \text{if } l \leq n \text{ and } y > x_l \end{cases}$$

Since all the elements are endpoints of th appropriate kind, the endpoints in $p^{-1}(x)$ are also preserved by $\overline{x}^l$, for $l \leq n+1$.

It remains to show that, the morphism $\overline{x}$ is a unique disk morphism satisfying (10). Suppose contrary, that $\pi : D \longrightarrow \gamma_n$ is a different morphism satisfying (10). Let

$$l = \min\{l' : \pi^{l'}(y) \neq \overline{x}_x^{l'}(y) \;\text{ for some }\; y \in D^{l'}\}$$

and fix $y \in D^l$, such that $\pi^l(y) \neq \overline{x}^l(y)$.

Since l is minimal we have

$$p \circ \pi^l(y) \;=\; \pi^{l-1}(p(y)) \;=\; \overline{x}^{l-1}(p(y)) \;=\; p \circ \overline{x}^l(y)$$

Thus $p(y) = l-1$, as $l-1$ is the only node in γ_n having more than one element in the fiber over it. So, $\overline{x}^l(y), \pi^l(y) \in \{l-1, l, l+1\}$. Then one of the following

$$x_l < y \qquad y = x_l \qquad y < x_l$$

holds. If $x_l < y$, then by (10) for $\overline{x}$ we have

$$\overline{x}^l(y) \neq \overline{x}^l(x_l) = l \quad \text{and} \quad \overline{x}^l(x_l) \leq \overline{x}^l(y)$$

Thus $\overline{x}^l(y) = l+1$. Since (10) holds for π, as well, we also have, that $\overline{x}^l(y) = l+1$, contrary to the choice of y.

The case $y < x_l$ is similar.

If $y = x_l$, then by (10) for both $\overline{x}$ and π, we have

$$\pi^l(y) = \pi^l(x_l) = \overline{x}^l(x_l) = \overline{x}^l(y)$$

again contrary to the choice of y. This contradiction shows, that $\overline{x}$ satisfying (10) is unique. Ad 2. Let $x_0 \in D^{n_0}$, $x_2 \in D^{n_2}$ be leaves, such that $x \ll y$, and $n_1 = \mu_{x_0,x_2}$. By Lemma 3.5, it is enough to verify that, for $l \leq n_1+1$ and $y \in D^l$

$$\mathbf{c}^l \circ \overline{x_0}^l(y) = \mathbf{d}^l \circ \overline{x_2}^l(y) \tag{14}$$

If $l \leq n_1$, then $\mu_{x_0,y} = \mu_{x_2,y}$. Moreover, $p^{(l)}(x_0) = p^{(l)}(x_2)$. Thus $\overline{x_0}^l(y) = \overline{x_2}^l(y)$. Since, for $l \leq n_1$, $\mathbf{c}^l = \mathbf{d}^l$, as well, (14) holds for $l \leq n_1$.

If $l = n_1+1$ and $y \notin \{p^{(n_1+1)}(x_0), p^{(n_1+1)}(x_2)\}$, then $\mu_{x_0,y} = \mu_{x_2,y}$ and, since $p^{(n_1+1)}(x_0) \lhd p^{(n_1+1)}(x_2)$, we have

$$y < p^{(n_1+1)}(x_0) \quad \text{iff} \quad y < p^{(n_1+1)}(x_2)$$

Thus $\overline{x_0}^{n_1+1}(y) = \overline{x_2}^{n_1+1}(y) \neq n_1+1$. But, for $n_1+1 \neq x \in \gamma^{n_1+1}_{n_0} = \gamma^{n_1+1}_{n_2}$, we have $\mathbf{c}^{n_1+1}(x) = \mathbf{d}^{n_1+1}(x)$, i.e. (14) holds for $y \notin \{p^{(n_1+1)}(x_0), p^{(n_1+1)}(x_2)\}$.

Since $p^{(n_1+1)}(x_0) \lhd p^{(n_1+1)}(x_2)$, we have

$$\begin{array}{rclcrcl}
\overline{x_0}^{n_1+1}(p^{(n_1+1)}(x_0)) & = & n_1+1 & \qquad & \overline{x_0}^{n_1+1}(p^{(n_1+1)}(x_2)) & = & n_1+2 \\
\overline{x_2}^{n_1+1}(p^{(n_1+1)}(x_0)) & = & n_1 & & \overline{x_2}^{n_1+1}(p^{(n_1+1)}(x_2)) & = & n_1+1 \\
\mathbf{c}^{n_1+1}(n_1+1) & = & n_1 & & \mathbf{c}^{n_1+2}(n_1+2) & = & n_1+1 \\
\mathbf{d}^{n_1+1}(n_1) & = & n_1 & & \mathbf{d}^{n_1+1}(n_1+1) & = & n_1+1
\end{array}$$

i.e. (14) holds for $y \in \{p^{(n_1+1)}(x_0), p^{(n_1+1)}(x_2)\}$. Thus (14) holds for $l = n_1+1$, as well. This proves that the diagram (11) commutes.

Ad 3. Let $x \in D^{n_0}$, $y \in D^{n_2}$ and $n_1 = \mu_{x,y}$, such that $x' = p^{n_1+1}(x) \lhd p^{n_1+1}(y) = y'$. We shall give the definition of $\overline{x;y}$ and leave to the reader the verification that, they satisfy the conditions (12) and (13).

By x'' and y'' we denote the largest leaf over x' and the least leaf over y', respectively, if they exists, i.e. if x' and y' are inner, respectively. We put

$$\overline{x;y} : D \longrightarrow \gamma_{n_1} = \begin{cases} \mathbf{c}_{(n_1)} \circ \overline{x''} & \text{if } x' \text{ is inner} \\ \mathbf{d}_{(n_1)} \circ \overline{y''} & \text{if } y' \text{ is inner} \\ \overline{p^{\mu_{x,y}}(x)} & \text{otherwise} \end{cases}$$

■

For further reference we give an explicit formula for $\overline{x;y}^l : D^l \longrightarrow \gamma_n^l$, where $x, y \in D^{n+1}$ and $x \lhd y$. For $z \in D^l$ we have

$$\overline{x;y}^l(z) \quad = \quad \begin{cases} n & \text{if } l > n \text{ and } p^{(n+1)}(z) = x \\ n+1 & \text{if } l > n \text{ and } p^{(n+1)}(z) = y \\ l & \text{if } l \leq n \text{ and } p^{(l)}(x) = z \\ \mu_{x,z} & \text{if } \mu_{x,z} < \min(l, n+1) \\ & \;\text{and } p^{(\mu_{x,z}+1)}(z) < p^{(\mu_{x,z}+1)}(x) \\ \gamma_n^l - \mu_{x,z} - 1 & \text{if } \mu_{x,z} < \min(l, n+1) \\ & \;\text{and } p^{(\mu_{x,z}+1)}(x) < p^{(\mu_{x,z}+1)}(z) \end{cases} \tag{15}$$

3.19. Theorem. *Let D be a disk, and let $\{x_i\}_{i \in k+1}$ be the set of leaves of D such that $x_i \in D^{u_{2i}}$ for $i \in k+1$, $x_i \ll x_{i+1}$ and $u_{2i+1} = \mu_{x_i, x_{i+1}}$ for $i \in k$. Then the diagram*

$$\begin{tikzcd} & & & D \arrow[dlll, "\overline{x_0}"'] \arrow[dl, "\overline{x_2}"'] \arrow[dr, "\overline{x_{2k-2}}"] \arrow[drrr, "\overline{x_{2k}}"] & & & \\ \gamma_{u_0} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_2} \arrow[dl, "\mathbf{d}"] & \cdots & \gamma_{u_{2k-2}} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_{2k}} \arrow[dl, "\mathbf{d}"] \\ & \gamma_{u_1} & & & & \gamma_{u_{2k-1}} & \end{tikzcd} \tag{16}$$

is a limiting cone in $\mathcal{D}$.

Proof. By Lemma 3.12, we need to verify that for $l \in \omega$, the image $(16)^l$ of cone (16) under the functor $(-)^l$ is a limiting cone in Poset, i.e. we need to verify, that $(16)^l$ is a limit in Set and that for $l \in \omega$ and $y, z \in D^l$, we have

$$y \leq z \quad \text{iff} \quad \overline{x_i}^l(y) \leq \overline{x_i}^l(z) \quad \text{for} \quad i \in k+1$$

First, we shall show that the diagram $(16)^l$ is a limit in Set. We have Claim 1. If $y, z \in D^l$ and $y \lhd z$, then there is $i_0 \in k+1$ such that $\overline{x_{i_0}}^l(y) \ < \ \overline{x_{i_0}}^l(z)$.

If y is inner, then for some $i_0 \in k+1$ there is a leaf x_{i_0} over y, and we have

$$\overline{x_{i_0}}^l(y) = l < l+1 = \overline{x_{i_0}}^l(z).$$

If z is inner, then for some $i_0 \in k+1$ there is a leaf x_{i_0} over z, and we have

$$\overline{x_{i_0}}^l(y) = l-1 < l = \overline{x_{i_0}}^l(z).$$

If both y and z are outer, then $p(y)$ is a leaf, i.e. for some $i_0 \in k+1$, $p(y) = x_{i_0}$. Then

$$\overline{x_{i_0}}^l(y) = l-1 < l =\leq \overline{x_{i_0}}^l(z).$$

This ends the proof of Claim 1.

Now, let $y, z \in D^l$ be arbitrary different nodes, $l' = \mu_{y,z} + 1$, $y' = p^{(l')}(y)$, $z' = p^{(l')}(z)$. Then the nodes $y', z' \in D^{l'}$ are comparable and different, say $y' < z'$. Thus, we have a node $y'' \in D^{l'}$, such that $y' \leq y'' \lhd z'$. By Claim 1, there is $i_0 \in k+1$, such that $\overline{x_{i_0}}^{l'}(y'') < \overline{x_{i_0}}^{l'}(z')$, and then

$$\overline{x_{i_0}}^{l}(y) \;=\; \overline{x_{i_0}}^{l'}(y') \;\leq\; \overline{x_{i_0}}^{l}(y'') \;<\; \overline{x_{i_0}}^{l'}(z') \;<\; \overline{x_{i_0}}^{l'}(z)$$

Hence $\overline{x_{i_0}}^{l}(y) \;<\; \overline{x_{i_0}}^{l}(z)$.

We shall show by induction on $l \in \omega$

Claim 2 For each compatible tuple $\{y_i\}_{i \in k+1}$ at level l, there is $y \in D^l$ such that

$$\overline{x_i}^{l}(y) \;=\; y_i \quad \text{for} \quad i \in k+1$$

For $l = 0$, Claim 2 is obvious, since at level 0 all disks have exactly one element.

Now, suppose that Claim 2 holds for $l-1$. Let $\{y_i\}_{i \in k+1}$ be a compatible tuple at level l, i_0 and i_1 as in Lemma 3.16.2 for that tuple. Since **c** and **d** commutes with projection the tuple $\{z_i\}_{i \in k+1}$ at level $l-1$, such that

$$z_i \;=\; p(y_i) \;\in\; \gamma^{l-1}_{n2i} \quad \text{for} \quad i \in k+1$$

is compatible at the level $l-1$. By induction hypothesis there is $z \in D^{l-1}$, such that

$$\overline{x_i}^{l-1}(z) \;=\; z_i \quad \text{for} \quad i \in k+1$$

We shall consider four cases

1. $i_0 = k+1$;
2. $i_1 = 0$;
3. $i_0 < i_1$;
4. $0 < i_0 = i_1 \leq k$.

Case 1. If $i_0 = k+1$ then all y_i's are right endpoints. Then, since $\overline{x_i}^{l}$'s preserves endpoints, the node $y = t(z)$ satisfy the Claim 2. *Case 2.* If $i_1 = 0$ then all y_i's are left endpoints, and similarly as before, $y = b(z)$ satisfy the Claim 2. *Case 3.* If $i_0 < i_1$ then $y_{i_0} = l$ is an inner node in γ_{2i_0}. We put $y = p^{(l)}(x_{i_0})$. Then

$$\overline{x_{i_0}}^{l}(y) = l = y_{i_0}$$

By Lemma 3.14, $\{y_i\}_{i \in k+1}$ is a unique compatible tuple $\{u_i\}_{i \in k+1}$ at level l, such that $u_{i_0} = y_{i_0}$. But, $\{\overline{x_i}^{l}(y)\}_{i \in k+1}$ is a compatible tuple with the same property, i.e. y is as in the Claim 2. *Case 4.* If $0 < i_0 = i_1 \leq k$ then y_{i_0} is in the right side of γ_{2i_0} and is connected to the right to y_{i_1} which is in the left side of γ_{2i_1}. Thus by Lemma 3.14 either y_{i_0} or y_{i_1} is in the far side.

If y_{i_0} is in far side then, again by Lemma 3.14, all y_i for $i \leq i_0$ are in the far right side. Thus y_i is a left endpoint for all $i \in k+1$. Thus as in Case 2, $y = b(z)$ satisfy the Claim 2.

If y_{i_1} is in far side then, for similar reasons as previously, all y_i for $i \leq i_0$ are in the far left side. Thus y_i is a right endpoint for all $i \in k+1$. Thus as in Case 1, $y = t(z)$ satisfy the Claim 2.

This shows that for $l \in \omega$, $(16)^l$ is a limiting cone in Set.

In order to show that $(16)^l$ is a limiting cone in Poset we need to verify, that if $y, z \in D^l$ and

$$\overline{x_i}^l(y) \;\leq\; \overline{x_i}^l(z) \;\text{ in }\; \gamma_{n_{2i}} \;\text{ for }\; i \in k+1 \;\text{ then }\; y \leq z \;\; D^l \tag{17}$$

Since, for $i \in k+1$

$$\overline{x_i}^{l-1}(p(y)) \;=\; p \circ \overline{x_i}^l(y) \;=\; p \circ \overline{x_i}^l(z) \;=\; \overline{x_i}^{l-1}(p(z))$$

it follows, form previous considerations, that $p(y) = p(z)$. Therefore y and z are comparable. If $y > z$ then, again from previous considerations, there is $i_0 \in k+1$, such that $\overline{x_{i_0}}^l(y) \neq \overline{x_{i_0}}^l(z)$. But $\overline{x_{i_0}}^l(y) \;\leq\; \overline{x_{i_0}}^l(z)$, so $\overline{x_{i_0}}^l(y) \;<\; \overline{x_{i_0}}^l(z)$, and we get a contradiction with the fact that $\overline{x_{i_0}}^l$ preserves order. Thus, we have $y \leq z$, i.e. (17) holds, and $(16)^l$ is a limiting cone in Poset, as was to be shown. ∎

Thus we have

3.20. Corollary. *For every disk D, there is a unique ud-vector $\vec{u}$ such that D is isomorphic to $\gamma_{\vec{u}}$.*

Proof. By Lemmas 3.13, 3.15, for any ud-vector $\vec{u}$, we have a disk $\gamma_{\vec{u}}$. By Theorem 3.19, for any disk D, there is a ud-vector $\vec{u}$ such, that D is isomorphic to $\gamma_{\vec{u}}$, and by Proposition 3.17 such a ud-vector $\vec{u}$ is unique. ∎

3.21. Factorizations in $\mathcal{D}$. In this section we shall study morphisms of disks.

It is easy to see that the epi in $\mathcal{D}$ are onto and mono are one-to-one and that these classes of morphisms form a factorization system in $\mathcal{D}$. However in $\mathcal{D}$ there is another factorization system.

Let $f : D \longrightarrow E$ be a disk morphism. We say that f is *inner* iff f sends inner nodes to inner nodes, i.e. $f(\imath(D)) \subseteq \imath(E)$. We say that f is *outer* iff f is epi and for any inner nodes x, y in D, if $f(x) = f(y)$ then $f(x)$ is an outer node.

The inner morphisms are exactly those that are induced by the maps of underlying trees. In fact, the category of finite disks with the inner maps as morphisms is equivalent to the category of finite trees.

If there is an inner morphism from $\gamma_{\vec{u}}$ to γ_l then it is unique, and it is the case iff $\mathrm{ht}(\vec{u}) \leq l$. We denote by

$$\mathbf{m}_{\vec{u}} : \gamma_{\vec{u}} \longrightarrow \gamma_{\mathrm{ht}(\vec{u})} \qquad\qquad \boldsymbol{\iota}_{(l)} : \gamma_n \longrightarrow \gamma_l$$

the unique inner morphisms between these disks, where $n \leq l$. Note that this notaton agree with the one in section 2.6 i.e. if we define for $n_1 < n_0, n_2$ the morphisms

$$\mathbf{m}_{n_0,n_1,n_2} : \gamma_{n_0,n_1,n_2} \longrightarrow \gamma_{\max(n_0,n_2)}$$

as the unique inner morphisms, then since the composition of inner morphisms is inner, $\mathbf{m}_{\vec{u}}$ is indeed the canonical internal composition morphism. This will be important in the next section.

We have

3.22. LEMMA.

1. *Outer morphisms are split epimorphisms with a unique splitting;*
2. *Let $\vec{u}$ be a ud-vector. Then, a disk morphism $f : D \longrightarrow \gamma_{\vec{u}}$ is outer iff $\boldsymbol{\pi}_i \circ f$ is outer for $i \in \mathrm{lh}(\vec{u})$;*
3. *For any ud-vector $\vec{u}$ the projection $\boldsymbol{\pi}_i : \gamma_{\vec{u}} \longrightarrow \gamma_{u_{2i}}$ is outer, for $i \in \mathrm{lh}(\vec{u})$;*
4. *The following are equivalent*
   (a) *A disk morphism $f : D \longrightarrow \gamma_n$ is outer;*
   (b) *there are $x, y \in D^{n+1}$, such that $x \lhd y$ and $f = \overline{x; y}$;*
   (c) *for $l \leq n$ there is a unique $z \in \imath(D)^l$ such that $f^l(z) = l \in \gamma^l_n$.*

PROOF. Ad 1. If $f : D \longrightarrow E$ is an outer morphism of disks, then the inverse image of each inner node of E contains one element. Thus $f^{-1}\lceil\imath(E) : \imath(E) \longrightarrow D$ is a tree morphisms. Therefore, it extends to a disk morphism $\overline{f} : E \longrightarrow D$. Since $f \circ (f^{-1}\lceil\imath(E)$ is identity on inner nodes $f \circ \overline{f} = id_E$. Since there are no choices involved in construction of $\overline{f}$ it is clear that $\overline{f}$ with the above property is unique.

Ad 2. Let $\vec{u}$ be a ud-vector and $f : D \longrightarrow \gamma_{\vec{u}}$ a disk morphism. For any $i \in \mathrm{lh}(\vec{u})$, $\boldsymbol{\pi}_i : \gamma_{\vec{u}} \longrightarrow \gamma_{u_{2i}}$ is outer. Hence if f is outer so is $\boldsymbol{\pi}_i \circ f$ for $i \in \mathrm{lh}(\vec{u})$.

Suppose now, that $\boldsymbol{\pi}_i \circ f$ for $i \in \mathrm{lh}(\vec{u})$. Let x, y be inner nodes in D such that $f(x) = f(y)$ and $f(x)$ is inner. Then, by Proposition 3.16.3, for some $i_0 \in \mathrm{lh}(\vec{u})$, $\boldsymbol{\pi}_{i_0} \circ f(x)$ is inner. Since $\boldsymbol{\pi}_{i_0} \circ f$ is outer, we have that $x = y$.

Ad 3. Follows from 2.

Ad 4. $(b) \Rightarrow (a)$. From the description in Proposition 3.18 easily follows, that $\overline{x; y}$ is outer.

$(a) \Rightarrow (c)$. If $f : D \longrightarrow \gamma_n$ is outer then, it is epi and there is a a node $z \in D^n$ such that $f^n(z) = n \in \gamma^n_n$. Since $n \in \gamma^n_n$ is inner z must be inner. And since f is outer z must be unique.

$(c) \Rightarrow (b)$. Having such z as above, in the fiber $p^{-1}(z) = \{z_1, \ldots, z_k\} \subseteq D^{n+1}$, there are nodes z_i and z_{i+1} such that $f(b(z)) = f(z_i)$ and $f(b(z)) = f(z_{i+1})$. Putting $x = z_i$ and $y = z_{i+1}$, we have $f = \overline{x; y}$. ∎

3.23. PROPOSITION. *Outer and inner morphisms of disks form a factorization system.*

Proof. For a disk morphism $f : D \longrightarrow E$ we define an equivalence relation $\sim$ on D so that for $x, y \in D$,

$$x \sim y \text{ iff } x \perp y \text{ and } f(x) = f(y) \text{ and } f(x) \text{ is outer}$$

On $D/_{\sim}$ we have an obvious disk structure inherited from D and the obvious morphisms

$$\begin{tikzcd} D \arrow[rr, "f"] \arrow[dr, "\overset{\circ}{f}"'] & & E \\ & D/_{\sim} \arrow[ur, "\overset{\bullet}{f}"'] & \end{tikzcd}$$

form the outer-inner factorization of f. If in the commuting diagram in $\mathcal{D}$

$$\begin{tikzcd} {} \arrow[r, "f_2"] \arrow[d, "f_1"'] & {} \arrow[d, "g_1"] \\ {} \arrow[r, "g_2"'] & {} \end{tikzcd}$$

f_1 is outer and g_1 is inner then $h = f_2 \circ \overline{f_1}$ is the unique lifting. It exists, by Lemma 3.22, since f_1 is outer, and the commutations of the triangles easily follows given the fact that g_1 is inner. ■

From the proposition we get

3.24. Corollary. *For any $n \in \omega$ and any disk morphism $f : D \longrightarrow \gamma_n$ there is a ud-vector $\vec{u}$, $\mathrm{ht}(\vec{u}) = u$, such that f factorizes as follows*

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\overset{\circ}{f}"'] & \gamma_n \\ \gamma_{\vec{u}} \arrow[r, "\mathbf{m}_{\vec{u}}"'] & \gamma_u \arrow[u, "\boldsymbol{\iota}_{(n)}"'] \end{tikzcd}$$

with $\overset{\circ}{f}$ outer part of f, and the morphisms $\boldsymbol{\iota}_{(n)}$, $\mathbf{m}_{\vec{u}}$ defined above.

Proof. In order to get this factorization of a disk morphism f we need to take the factorization $(\overset{\circ}{f}, \overset{\bullet}{f})$ from Proposition 3.23 and furthermore take epi-mono factorization of its inner part $\overset{\bullet}{f}$. ■

We call the factorization of a disk morphism f described in the above Corollary the *canonical factorization* of f.

3.25. An internal ω-category $\mathbf{C}$ in $\mathcal{D}$. In this section we define an internal ω-category $\mathbf{C}$ in $\mathcal{D}$, and we show that by homming into $\mathbf{C}$ we get a functor into the category of simple ω-categories $\mathcal{S}$.

For $n \in \omega$, γ_n is the object of n-cells of $\mathbf{C}$, for $n \geq l$

$$\mathbf{d}_{(l)}, \mathbf{c}_{(l)} : \gamma_n \longrightarrow \gamma_l$$

are the domain and codomain morphisms in $\mathbf{C}$, for $n \leq l$

$$\boldsymbol{\iota}_{(l)} : \gamma_n \longrightarrow \gamma_l$$

is the identity morphism, and for any ud-vector $\vec{n}$ of length 2

$$\mathbf{m}_{\vec{n}} : \gamma_{\vec{n}} \longrightarrow \gamma_{\mathrm{ht}(\vec{n})}$$

is the composition morphism in $\mathbf{C}$.

We have

3.26. Proposition. $\mathbf{C}$ *defined above is an internal* ω*-category in* $\mathcal{D}$*.*

Proof. We need to check, that the data specified above verify the conditions (vi)-(xi) of the definition of an ω-category given in Appendix 6.2. Thus, we need to check that many diagrams commutes. We shall only verify some chosen ones, and comment on the other leaving to the reader to calculate the rest.

Note that by Lemma 3.5 it is enough to verify the equality of morphisms at leaves only. Let for a ud-vector $\vec{u}$, $\lambda^i_{\vec{u}} \in \gamma^{u_{2i}}_{\vec{u}}$ be the i-th leaf of $\gamma_{\vec{u}}$, for $i \in \mathrm{lh}(\vec{u})$. Usually we drop subscript $\vec{u}$ in $\lambda^i_{\vec{u}}$.

The condition (vi).1 holds, i.e. for $l \leq n_1 < n_0, n_2$ the square

$$\begin{tikzcd} \gamma_{n_0,n_1,n_2} \arrow[r, "\mathbf{m}"] \arrow[d, "\pi_0"'] & \gamma_{\max(n_0,n_2)} \arrow[d, "\mathbf{d}_{(l)}"] \\ \gamma_{n_0} \arrow[r, "\mathbf{d}_{(l)}"'] & \gamma_l \end{tikzcd}$$

commutes, as the following calculations show

$$\mathbf{d}_{(l)} \circ \mathbf{m}(\lambda^0) = \mathbf{d}_{(l)}(n_0) = l + 1 = \mathbf{d}_{(l)}(n_0) = \mathbf{d}_{(l)} \circ \pi_0(\lambda^0)$$

$$\mathbf{d}_{(l)} \circ \mathbf{m}(\lambda^1) = \mathbf{d}_{(l)}(n_2) = l + 1 = d_{(l)}(\gamma^{n_2}_{\max(n_0,n_2)} - n_1 - 1) = d_{(l)} \circ \pi_0(\lambda^0)$$

The condition (vii) since, for $l \geq n$, we have

$$\mathbf{d}_{(l)} \circ \boldsymbol{\iota}_{(l)}(\lambda^0) = \mathbf{c}_{(l)}(n) = n = \mathbf{d}_{(n)} \circ \boldsymbol{\iota}_{(l)}(\lambda^0)$$

The conditions (vii)-(x) holds, since all the morphisms involved are inner and inner the morphism into a disk of form γ_n from a given disk is unique, if exists.

The condition (xi) holds as well, i.e. for $n_1 < l < n_0, n_2$, the diagram

$$\begin{tikzcd}
\gamma_{n_0,n_1,n_2} \arrow[rr, "\langle \mathbf{m}_{l,n_1,n_2}(\mathbf{d}_{(l)} \times 1), \mathbf{m}_{n_0,n_1,l}(1 \times \mathbf{c}_{(l)})\rangle"] \arrow[d, "\langle \mathbf{m}_{n_0,n_1,l}(1 \times \mathbf{d}_{(l)}), \mathbf{m}_{l,n_1,n_2}(\mathbf{c}_{(l)} \times 1)\rangle"'] \arrow[drr, "\mathbf{m}"] & & \gamma_{n_2,l,n_0} \arrow[d, "\mathbf{m}_{n_2,l,n_0}"] \\
\gamma_{n_0,l,n_2} \arrow[rr, "\mathbf{m}_{n_0,l,n_2}"'] & & \gamma_{\max(n_0,n_2)}
\end{tikzcd}$$

commutes. The morphism $\langle \mathbf{m}_{l,n_1,n_2}(\mathbf{d}_{(l)} \times 1), \mathbf{m}_{n_0,n_1,l}(1 \times \mathbf{c}_{(l)})\rangle$ glue the two branches up to the level l and switch the remaining parts, whereas the morphism $\langle \mathbf{m}_{n_0,n_1,l}(1 \times \mathbf{d}_{(l)}), \mathbf{m}_{l,n_1,n_2}(\mathbf{c}_{(l)} \times 1)\rangle$ just glue together the branches up to level l without switching them. Thus both morphisms are inner and composed with $\mathbf{m}_{n_2,l,n_0}$ and $\mathbf{m}_{n_0,l,n_2}$, respectively must be equal one to another. ■

From the proposition we get immediately

3.27. Corollary. *For any disk D, $\mathcal{D}(D, \mathbf{C})$ is an ω-category in* Set.

By an *l-D-cut* (or *l-cut*, *D-cut*) we mean an outer morphism $D \longrightarrow \gamma_l$.

Since both $\mathbf{d}$ and $\mathbf{c}$ are outer morphisms, on D-cuts we can define operations of domain and codomain simply by composing them with $\mathbf{d}$ and $\mathbf{c}$, respectively.

Let $\mathrm{Cut}(D)_n$ be the set of n-D-cuts. We have a sequence of sets and functions, denoted by $\mathrm{Cut}(D)$

$$\cdots \underset{\mathbf{c}}{\overset{\mathbf{d}}{\rightrightarrows}} \mathrm{Cut}(D)_n \underset{\mathbf{c}}{\overset{\mathbf{d}}{\rightrightarrows}} \cdots \ \mathrm{Cut}(D)_1 \underset{\mathbf{c}}{\overset{\mathbf{d}}{\rightrightarrows}} \mathrm{Cut}(D)_0$$

Note that if $f : D \longrightarrow \gamma_n$ is outer then there is a unique inner node $z \in D^n$, such that $f^n(z) = n \in \gamma_n^n$. Thus $b(z) \neq t(z)$ and there is a unique pair $x, y \in p^{-1}(z) \subseteq D^{n+1}$, such that $x \lhd y$, $f^{n+1}(x) = b(z)$ and $f^{n+1}(y) = t(z)$. By Proposition 3.18 and Lemma 3.5, we have $f = \overline{x;y}$. Clearly, for any pair $x, y \in D^{n+1}$, such that $x \lhd y$, $\overline{x;y}$ is an n-D-cut. Moreover, both first and the second component determines the pair uniquely. We write $\overline{x;}$ ($\overline{;x}$) for the outer morphisms $D \longrightarrow \gamma_n$ determined by the pair whose first (second) component is x. In this notation, we have

$$\mathbf{d} \circ \overline{x;y} = \overline{;p(x)} \qquad\qquad \mathbf{c} \circ \overline{x;y} = \overline{p(x);}$$

We define an order on n-D-cuts by putting for $f = \overline{x;x'}$ and $g = \overline{y;y'}$ such that $\mathbf{d} \circ f = \mathbf{d} \circ g$ and $\mathbf{c} \circ f = \mathbf{c} \circ g$ (i.e. $p(x) = p(y)$) that

$$f \geq g \quad \text{iff} \quad x \leq y \text{ in } D^{n+1}$$

3.28. PROPOSITION. Cut(D) *is a simple ω-graph and the inclusion*

$$\eta_D : \mathrm{Cut}(D) \longrightarrow \mathcal{D}(D, \mathbf{C})$$

is an ω-graph morphism.

PROOF. Since $\mathbf{d} \circ \mathbf{d} = \mathbf{d} \circ \mathbf{c}$ and $\mathbf{c} \circ \mathbf{d} = \mathbf{c} \circ \mathbf{c}$, Cut$(D)$ is indeed an ω-graph. Moreover, the domain and codomain operations in $\mathcal{D}(D, \mathbf{C})$ are defined by composition with $\mathbf{d}$ and $\mathbf{c}$, as well. Thus η_D is an inclusion of ω-graphs. It remains to show that Cut(D) is simple.

Clearly, for $n \in \omega$, Cut$(D)_n$ is finite.

Since fibers of p are linearly ordered, the above condition defines a linear order on those n-D-cuts whose domains and codomains coincide. Moreover, for f, g as above, we have

$$f \rhd g \quad \text{iff} \quad x' = y$$

But, then y must be an inner node, and $b(y) \neq t(y)$. So $\overline{b(y);} : D \longrightarrow \gamma_{n+1}$ is an $n+1$-D-cut for which

$$\mathbf{d} \circ \overline{b(y);} = f \quad \text{and} \quad \mathbf{c} \circ \overline{b(y);} = g$$

We have as well that

$$\mathbf{d} \circ f = \overline{;p(x)} \rhd \overline{p(x);} = \mathbf{c} \circ f$$

Thus Cut(D) is a simple ω-graph. ∎

From the proof we get a Corollary.

3.29. COROLLARY. *If $n \in \omega$ and f, g are n-D-cuts then $f \rhd g$ iff there is $x \in D^{n+1}$ such that $f = \overline{;x}$ and $g = \overline{x;}$.* ∎

3.30. PROPOSITION. *For any disk D, the inclusion*

$$\eta_D : \mathrm{Cut}(D) \longrightarrow \mathcal{D}(D, \mathbf{C})$$

is the universal morphism from graph Cut(D) *to the forgetful functor* ωCat $\longrightarrow$ ωGr.

PROOF. Let D be a disk, $\mathbf{A}$ an ω-category, and $F : \mathrm{Cut}(D) \longrightarrow \mathbf{A}$. We need to show that there is a unique ω-functor $G : \mathcal{D}(D, \mathbf{C}) \longrightarrow \mathbf{A}$ such that the triangle

$$\begin{tikzcd} \mathrm{Cut}(D) \arrow[r, "\eta_D"] \arrow[dr, "F"'] & \mathcal{D}(D, \mathbf{C}) \arrow[d, "G"] \\ & \mathbf{A} \end{tikzcd} \tag{18}$$

commutes. First we extend F to all outer morphisms. For any ud-vector $\vec{u}$, we shall define a function

$$\vec{F} : \mathcal{D}_{\mathrm{outer}}(D, \gamma_{\vec{u}}) \longrightarrow A_{\vec{u}}$$

by the formula

$$\vec{F}(e) \;=\; \langle F(\pi_i^A \circ e)\rangle_{i\in \mathrm{lh}(\vec{u})} \in A_{\vec{u}}$$

Since F preserves domains and codomains, $\vec{F}$ is well defined. Let $f : D \longrightarrow \gamma_n$ be a disk morphism, i.e. n-cell in $\mathcal{D}(D,\mathcal{C})$. By Corollary 3.24, we have a factorization

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\overset{\circ}{f}"'] & \gamma_n \\ \gamma_{\vec{u}} \arrow[r, "\mathbf{m}_{\vec{u}}"'] & \gamma_u \arrow[u, "\boldsymbol{\iota}_{(n)}"'] \end{tikzcd} \tag{19}$$

with $\overset{\circ}{f}$ outer. We put

$$G(f) \;=\; \boldsymbol{\iota}_{(n)}^A \circ m_{\vec{u}}^A(\vec{F}(\overset{\circ}{f})) \in A_n$$

First note that (19) shows that any n-cell is $\mathcal{D}(D,\mathbf{C})$ is a result of application of composition and identity operations to a compatible tuple of D-cuts. Thus G with the required properties, if exists is, unique. Secondly, for a D-cut $f : D \longrightarrow \gamma_n$ we have

$$G(\eta_D(f)) = G(f) = \boldsymbol{\iota}_{(n)}^A \circ \mathbf{m}_n^A(F(f)) = F(f)$$

i.e. (18) commutes. It remains to show that G defined above preserve domains, codomains, identities and compositions.

Before we show that G preserves compositions we shall prove *Claim.* For any $l \in \omega$, ud-vector $\vec{u}$ and outer morphism $e : D \longrightarrow \gamma_{\vec{u}}$ we have

$$\mathbf{d}_{(l)}^A \circ \vec{F}(e) \;=\; \vec{F}(\mathbf{d}_{\vec{u};l} \circ e) \tag{20}$$

$$\mathbf{c}_{(l)}^A \circ \vec{F}(e) \;=\; \vec{F}(\mathbf{c}_{\vec{u};l} \circ e). \tag{21}$$

We prove the Claim by induction on l-size of $\vec{u}$.

If $\vec{u} = u_0 < l$ then e is an u_0-cut, $\vec{F}(e) = F(e)$, $\mathbf{d}^{(l)} = 1_{\gamma_{u_0}}$ and $d_{(l)}^A = id_{A_{u_0}}$. Hence (20) holds in this case.

If $\vec{u}$ is l-primitive, then we have

$$\mathbf{d}_{\vec{u};l}^A(\vec{F}(e)) = \mathbf{d}_{(l)}^A \circ \pi_0^A(\vec{F}(e)) = \mathbf{d}_{(l)}^A(F(\pi_0 \circ e)) =$$

$$= F(\mathbf{d}_{(l)} \circ \pi_0 \circ e) = \vec{F}(\mathbf{d}_{\vec{u};l} \circ e) = \vec{F}(\mathbf{d}_{\vec{u};l} \circ e)$$

If $\vec{u} = \vec{u'}, z, \vec{u''}$, $\mathrm{lh}(\vec{u}) = k+1$, $\mathrm{lh}(\vec{u'}) = k'+1$, and $z = \min(\vec{u}) < l$, then using the inductive assumption on ud-vectors $\vec{u'}$ and $\vec{u''}$ of smaller l-size we have

$$\begin{aligned}
\mathbf{d}^{A,(l)}_{\vec{u}}(\vec{F}(e)) &= \mathbf{d}^{A}_{\vec{u'};l} \circ \pi^{A}_{0..k'} \circ (\vec{F}(e)) \times \mathbf{d}^{A}_{\vec{u''};l} \circ \pi^{A}_{k'+1..k} \circ (\vec{F}(e)) \\
&= \mathbf{d}^{A}_{\vec{u'};l}(\vec{F}(\pi_{0..k'} \circ e)) \times \mathbf{d}^{A}_{\vec{u''};l}(\vec{F}(\pi_{k'+1..k} \circ e)) \\
&= \vec{F}(\mathbf{d}_{\vec{u'};l} \circ \pi_{0..k'} \circ e) \times \vec{F}(\mathbf{d}_{\vec{u''};l} \circ \pi_{k'+1..k} \circ e) \\
&\stackrel{*}{=} \vec{F}(\langle \mathbf{d}_{\vec{u'};l} \circ \pi_{0..k'} \circ e, \mathbf{d}_{\vec{u''};l} \circ \pi_{k'+1..k} \circ e \rangle) \\
&= \vec{F}((\mathbf{d}_{\vec{u'};l} \times \mathbf{d}^{(l)}_{\vec{u''}}) \circ e) \\
&= \vec{F}(\mathbf{d}_{\vec{u};l} \circ e)
\end{aligned}$$

where $\stackrel{*}{=}$ follows from the fact that $z = \min(\vec{u})$ and hence

$$\mathbf{c}_{(z)} \circ \mathbf{d}_{\vec{u'};l} \circ \pi_{0..k'} \circ e = \mathbf{d}_{(z)} \circ \mathbf{d}_{\vec{u''};l} \circ \pi_{k'+1..k} \circ e$$

The proof of (21) is similar. This ends the proof of the Claim. ∎

We can show now, that G preserves domains and codomains. Let $l, n \in \omega$, and $f : D \longrightarrow \gamma_n$ be a disk map. By Lemma 6.4 we have the following commutative diagram

$$\begin{tikzcd}
D \arrow[d, "\mathring{f}"'] \arrow[drr, "f"] & & \\
\gamma_{\vec{u}} \arrow[r, "\mathbf{m}_{\vec{u}}"] \arrow[d, "\mathbf{d}_{\vec{u};l}"'] & \gamma_u \arrow[r, "\iota^{(n)}"] \arrow[d, "\mathbf{d}_{(l)}"] & \gamma_n \arrow[d, "\mathbf{d}_{(l)}"] \\
\gamma_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[r, "\mathbf{m}_{\mathrm{tr}_{(l)}(\vec{u})}"'] & \gamma_{\min(l,u)} \arrow[r, "\boldsymbol{\iota}_{(l)}"'] & \gamma_l
\end{tikzcd}$$

with $\mathring{f}$ outer. Then using the Claim we obtain

$$\begin{aligned}
G(\mathbf{d}_{(l)} \circ f) &= \boldsymbol{\iota}^{A}_{(l)} \circ \mathbf{m}^{A}_{\mathrm{tr}_{(l)}(\vec{u})}(\vec{F}(\mathbf{d}_{\vec{u};l} \circ \mathring{f}) \\
&= \boldsymbol{\iota}^{A}_{(l)} \circ \mathbf{m}^{A}_{\mathrm{tr}_{(l)}(\vec{u})} \circ \mathbf{d}^{A}_{\vec{u};l}(\vec{F}(\mathring{f}) \\
&= \boldsymbol{\iota}^{A}_{(l)} \circ \mathbf{d}^{A}_{(l)} \circ \mathbf{m}^{A}_{\vec{u}}(\vec{F}(\mathring{f}) \\
&= \mathbf{d}^{A}_{(l)} \circ \iota^{A}_{(n)} \circ \mathbf{m}^{A}_{\vec{u}}(\vec{F}(\mathring{f}) \\
&= \mathbf{d}^{A}_{(l)}(G(f))
\end{aligned}$$

Thus G preserves domains. The preservation of codomains can be proved similarly.

To see that G preserves identities, let $l, n \in \omega$, $l > n$, and $f : D \longrightarrow \gamma_n$ be a disk map. Then, we can form a commuting diagram

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, "\overset{\circ}{f}"'] & \gamma_n \arrow[r, "\iota_n^{(l)}"] & \gamma_l \\
\gamma_{\vec{u}} \arrow[r, "\mathbf{m}_{\vec{u}}"'] & \gamma_u \arrow[u, "\boldsymbol{\iota}_{(n)}"] \arrow[ur, "\boldsymbol{\iota}_{(l)}"'] &
\end{tikzcd}$$

with $\overset{\circ}{f}$ outer. Thus, we have

$$G(\boldsymbol{\iota}_{(l)} \circ f) = G(\boldsymbol{\iota}_{(l)} \circ \mathbf{m}_{\vec{u}} \circ \overset{\circ}{f}) = \boldsymbol{\iota}^A_{(l)} \circ \mathbf{m}^A_{\vec{u}}(G(\overset{\circ}{f}))$$

$$= \boldsymbol{\iota}^A_{(l)} \circ \iota^A_{(n)} \circ \mathbf{m}^A_{\vec{u}}(G(\overset{\circ}{f})) = \boldsymbol{\iota}^A_{(l)}(G(f))$$

i.e. G preserves identities.

It remains to show that G preserves compositions. Let $u', v', l \in \omega$, $l < u', v'$ and $f : D \longrightarrow \gamma_{u'}$, $f : D \longrightarrow \gamma_{v'}$ be disk morphisms, such that $\mathbf{c}_{(l)} \circ f = \mathbf{d}_{(l)} \circ g$. We have the following commutative diagrams displaying the canonical factorizations of f and $\mathbf{c}_{(l)} \circ f$,

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, "\overset{\circ}{f}"'] & \gamma_{u'} \arrow[r, "\mathbf{c}_{(l)}"] & \gamma_l \\
\gamma_{\vec{u}} \arrow[r, "\mathbf{m}_{\vec{u}}"'] \arrow[dr, "\mathbf{c}_{\vec{u};l}"'] & \gamma_u \arrow[u, "\boldsymbol{\iota}_{(u')}"] \arrow[r, "\mathbf{c}_{(l)}"'] & \gamma_{\min(u,l)} \arrow[u, "\boldsymbol{\iota}_{(l)}"'] \\
& \gamma_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[ur, "\mathbf{m}_{(\mathrm{tr}_{(l)}(\vec{u})}"'] &
\end{tikzcd}$$

and the canonical factorizations of g and $\mathbf{d}_{(l)} \circ g$

$$\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, "\overset{\circ}{g}"'] & \gamma_{v'} \arrow[r, "\mathbf{d}_{(l)}"] & \gamma_l \\
\gamma_{\vec{v}} \arrow[r, "\mathbf{m}_{\vec{v}}"'] \arrow[dr, "\mathbf{d}_{\vec{v};l}"'] & \gamma_v \arrow[u, "\boldsymbol{\iota}_{(v')}"] \arrow[r, "\mathbf{d}_{(l)}"'] & \gamma_{\min(v,l)} \arrow[u, "\boldsymbol{\iota}_{(l)}"'] \\
& \gamma_{\mathrm{tr}_{(l)}(\vec{v})} \arrow[ur, "\mathbf{m}_{(\mathrm{tr}_{(l)}(\vec{v})}"'] &
\end{tikzcd}$$

Since $\mathbf{c}_{(l)} \circ f = \mathbf{d}_{(l)} \circ g$ and the canonical factorizations are unique, we have that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v})$, $\min(v, l) = \min(u, l)$, and $\mathbf{c}_{\vec{u};l} \circ \mathring{f} = \mathbf{d}_{\vec{v};l} \circ \mathring{g}$. Thus we can form a commuting diagram

$$
\begin{tikzcd}
& D \arrow[d, "e"] \arrow[ddl, "\mathring{f}"'] \arrow[ddrr, "\mathring{g}"] & & \\
& \gamma_{[u,l,v]} \arrow[dl, "\pi_{0;\vec{u}}"'] \arrow[drr, "\pi_{1;\vec{v}}"] \arrow[ddd, "\mathbf{m}_{\vec{u}} \times \mathbf{m}_{\vec{v}}"] & & \\
\gamma_{\vec{u}} \arrow[drr, "\mathbf{c}_{\vec{u};l}"'] \arrow[ddd, "\mathbf{m}_{\vec{u}}"'] & & & \gamma_{\vec{v}} \arrow[dl, "\mathbf{d}_{\vec{v};l}"] \arrow[ddd, "\mathbf{m}_{\vec{v}}"] \\
& & \gamma_{\vec{w}} \arrow[ddd, "\mathbf{m}_{\vec{w}}"] & \\
& \gamma_{[u,w,v]} \arrow[dl, "\pi_0"'] \arrow[drr, "\pi_1"] \arrow[ddd, "\boldsymbol{\iota}_{(u')} \times \boldsymbol{\iota}_{(v')}"] & & \\
\gamma_u \arrow[drr, "\mathbf{c}_{(w)}"'] \arrow[ddd, "\boldsymbol{\iota}_{(u')}"'] & & & \gamma_v \arrow[dl, "\mathbf{d}_{(w)}"] \arrow[ddd, "\boldsymbol{\iota}_{(v')}"] \\
& & \gamma_w \arrow[ddd, "\boldsymbol{\iota}_{(l)}"] & \\
& \gamma_{u,l,v} \arrow[dl, "\pi_0"'] \arrow[drr, "\pi_1"] & & \\
\gamma_u \arrow[drr, "\mathbf{c}^{(l)}"'] & & & \gamma_v \arrow[dl, "\mathbf{d}^{(l)}"] \\
& & \gamma_l &
\end{tikzcd}
$$

in which the three horizontal squares are pullbacks, and e is the induced map into a pullback. Moreover, by Lemma 6.7, for $z = \min(u, v)$ and $z' = \min(u', v')$ we have a commuting diagram

$$
\begin{tikzcd}
\gamma_{[\vec{u},l,\vec{v}]} \arrow[d, "\mathbf{m}_{\vec{u}} \times \mathbf{m}_{\vec{v}}"'] \arrow[dr, "\mathbf{m}_{[\vec{u},l,\vec{v}]}"] & \\
\gamma_{[u,l,v]} \arrow[r, "\mathbf{m}_{[u,l,v]}"] \arrow[d, "\boldsymbol{\iota}_{(u')} \times \boldsymbol{\iota}_{(v')}"'] & \gamma_z \arrow[d, "\boldsymbol{\iota}_{(z')}"] \\
\gamma_{u',l,v'} \arrow[r, "\mathbf{m}_{u',l,v'}"'] & \gamma_{z'}
\end{tikzcd}
$$

Thus we have

$$G(f) = \boldsymbol{\iota}^A_{(u')} \circ \mathbf{m}_{\vec{u}}(\vec{F}(\mathring{f}))$$

and

$$G(g) = \boldsymbol{\iota}^A_{(v')} \circ \mathbf{m}_{\vec{v}}(\vec{F}(\mathring{g}))$$

Now using Lemma 6.7 we have

$$
\begin{aligned}
G(\mathbf{m}_{u',l,v'})(f,g)) &= \boldsymbol{\iota}^A_{(z')} \circ \mathbf{m}^A_{[\vec{u},l,\vec{v}]}(\vec{F}(e)) \\
&= \boldsymbol{\iota}^A_{(z')} \circ \mathbf{m}^A_{[u,l,v]} \circ (\mathbf{m}^A_{\vec{u}} \times \mathbf{m}^A_{\vec{v}})(\vec{F}(e)) \\
&= \mathbf{m}^A_{u',l,v'} \circ (\boldsymbol{\iota}^A_{(u')} \times \boldsymbol{\iota}^A_{(v')}) \circ (\mathbf{m}^A_{\vec{u}} \times \mathbf{m}^A_{\vec{v}})(\langle \vec{F}(\pi_{0;\vec{u}} \circ e), \vec{F}(\pi_{1;\vec{v}} \circ e)\rangle) \\
&= \mathbf{m}^A_{u',l,v'} \circ \langle \iota^A_{(u')} \circ \mathbf{m}^A_{\vec{u}}(\vec{F}(\mathring{f}), \boldsymbol{\iota}^A_{(v')} \circ \mathbf{m}^A_{\vec{v}}(\vec{F}(\mathring{g}))\rangle \\
&= \mathbf{m}^A_{u',l,v'}(G(f), G(g))
\end{aligned}
$$

i.e. the compositions are preserved as well. ∎

We get

3.31. Corollary. *The essential image of the hom-functor* $\mathcal{D}(-, \mathbf{C})$ *into* ωCat *is contained in the category of simple* ω*-categories* $\mathcal{S}$*, and hence we have a contravariant hom-functor*

$$\mathcal{D}(-, \mathbf{C}) : \mathcal{D}^{op} \longrightarrow \mathcal{S}$$

Proof. By Proposition 4.8 an ω-category is simple iff it is a free ω-category on a simple ω-graph. By Proposition 3.28 Cut(D) is a simple ω-graph and by Proposition 3.30 $\mathcal{D}(D, \mathbf{C})$ is a free ω-category on Cut(D), for any disk D. Thus $\mathcal{D}(D, \mathbf{C})$ is a simple ω-category. ∎

4. Simple ω-categories

4.1. Simple ω-graphs and free ω-categories. This section begins the investigation of simple ω-categories. We shall introduce some notation and prove some basic facts concerning simple ω-graphs. In Appendix 6.8, we give an internal construction of a free ω-category over an ω-graph, in an ambient category satisfying some mild exactness properties. The construction uses essentially ud-vectors. When the ambient category is Set the same construction can be described more conveniently, using simple ω-graphs. We shall present the construction at the end of the section.

We shall define some specific ω-graphs $\alpha^{\vec{u}}$, for any ud-vector $\vec{u}$. For $\vec{u} = n$, we put

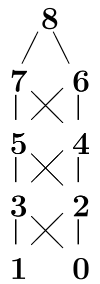
$$\alpha^n_l \quad = \quad \begin{cases} \emptyset & \text{if } l > n \\ \{2n\} & \text{if } l = n \\ \{2l+1,\, 2l\} & \text{if } 0 \leq l < n \end{cases}$$

$$d,\, c : \alpha^n_l \longrightarrow \alpha^n_{l-1}$$

$$d(x) = 2l - 1 \qquad\qquad c(x) = 2l - 2$$

for $x \in \alpha^n_l$, and $1 \leq l \leq n$.

For example α^4 can be pictured as follows:



For any two cells e, e' in a simple ω-graph G we define a number

$$\nu^G_{e,e'} = \max\{l : d_{(l)}(e) \perp d_{(l)}(e')\}$$

Since G_0 is linearly ordered $\nu^G_{e,e'}$ is well defined. We have

4.2. Lemma. *Let $f : G \longrightarrow H$ be a morphisms of simple ω-graphs. Then f is one-to-one and the image of f is a sub-ω-graph of H. In particular, if $m \in \omega$ and $e, e', e'' \in G'_m$ such that $e > e' > e''$ and e, e'' are in the image of f, so is e'.*

PROOF. First note that if a function between finite linear orders preserves the successor relation then it reflects the order and is one-to-one.

Let $f : G \longrightarrow H$ be an ω-graph morphism between simple ω-graphs. f maps G_0 to H_0 and, for $n \in \omega$, $x, y \in G_n$ f maps $G_{n+1}(x, y)$ to $H_{n+1}(f(x), f(y))$. In particular, if $G_{n+1}(x, y)$ is not empty, so is $H_{n+1}(f(x), f(y))$. Since G and H are simple f preserves $\rhd$. Thus, by the remark above, it also reflects it and is one-to-one when restricted to these sets. But if $x, y \in G_n$ and $x \not\perp y$ then $l = \nu_{x,y} < n$. Hence $d_{(l)}(x) \neq d_{(l)}(y)$ and $d_{(l)}(x) \perp d_{(l)}(y)$. Thus by the above $d_{(l)}(f(x)) \neq d_{(l)}(f(y))$, and then $f(x) \neq f(y)$. Therefore f is one-to-one. ∎

For $l < n$ we define, the *bottom* and the *top* ω-graph morphisms

$$\mathbf{b}^n = \mathbf{b}\ , \mathbf{t}^n = \mathbf{t} : \alpha^l \longrightarrow \alpha^n$$

by the conditions $\mathbf{b}_n(2l) = 2l$ and $\mathbf{t}_n(2l) = 2l + 1$.

Let $\vec{u}$ be a ud-vector. The set of nodes in the simple ω-graph $\alpha^{\vec{u}}$ is the set of pairs $\langle i, x\rangle$ such that $i \in \mathrm{lh}(\vec{u})$ and $0 \leq x \leq 2u_{2i}$ divided by the equivalence relation $\sim$ generated by the relation $\sim_0$ such that

$$\langle i, x\rangle \sim_0 \langle i', x'\rangle \ \text{ iff } \begin{cases} i = i' \text{ and } x = x' \\ \text{or} \\ i + 1 = i' \text{ and } x = x' \text{ and } x < 2u_{2i+1} \\ \text{or} \\ i + 1 = i' \text{ and } x = 2u_{2i+1} \text{ and } x' = 2u_{2i+1} + 1 \end{cases}$$

We write $[i, x]$ for the equivalence class of the pair $\langle i, x\rangle$. We put, for $l \in \omega$,

$$\alpha^{\vec{u}}_l = \{[i, x] \in \alpha^{\vec{u}} : \lfloor x/2\rfloor = l\}$$

and domain and codomain functions

$$d^{\alpha^{\vec{u}}}, c^{\alpha^{\vec{u}}} : \alpha^{\vec{u}}_{l+1} \longrightarrow \alpha^{\vec{u}}_l$$

are given by

$$d^{\alpha^{\vec{u}}}([i, x]) = [i, d(x)] \qquad\qquad c^{\alpha^{\vec{u}}}([i, x]) = [i, c(x)]$$

for $[i, x] \in \alpha^{\vec{u}}_{l+1}$.

The embedding morphisms

$$\boldsymbol{\kappa}^i : \alpha^{u_{2i}} \longrightarrow \alpha^{\vec{u}}$$

are defined by the condition $\boldsymbol{\kappa}^i(2u_{2i}) = [i, 2u_{2i}]$, for $i \in \mathrm{lh}(\vec{u})$.

We have

4.3. Proposition. *The diagram*

$$
\begin{tikzcd}
 & & & \alpha^{\vec{u}} & & & \\
\alpha^{u_0} \arrow[urrr, "\boldsymbol{\kappa}^0"] & & \alpha^{u_2} \arrow[ur, "\boldsymbol{\kappa}^2"] & & \alpha^{u_{2k-2}} \arrow[ul, "\boldsymbol{\kappa}^{2k-2}"'] & & \alpha^{u_{2k}} \arrow[ulll, "\boldsymbol{\kappa}^{2k}"'] \\
 & \alpha^{u_1} \arrow[ul, "\mathbf{b}"] \arrow[ur, "\mathbf{t}"'] & & \cdots & & \alpha^{u_{2k-1}} \arrow[ul, "\mathbf{b}"] \arrow[ur, "\mathbf{t}"'] &
\end{tikzcd}
\tag{22}
$$

in $s\omega\mathrm{Gr}$ *is a colimiting cone in* $\omega\mathrm{Gr}$.

Proof. $\alpha^{\vec{u}}$ described above is clearly a colimit of (22) in $\omega\mathrm{Gr}$. The verification that $\alpha^{\vec{u}}$ is a simple ω-graph is left for the reader. ∎

For $l \in \omega$, and ud-vector $\vec{u}$ the *multi-bottom* and the *multi-top* ω-graph morphisms

$$\mathbf{b}^{\vec{u};l}, \mathbf{t}^{\vec{u};l} : \alpha^{\mathrm{tr}_{(l)}(\vec{u})} \longrightarrow \alpha^{\vec{u}}$$

are defined as follows

$$\mathbf{b}^{\vec{u};l} \quad = \quad \begin{cases} 1_{\alpha_{u_0}} & \text{if } \vec{u} = u_0 \leq l; \\ \kappa^{\mathrm{lh}(\vec{u})-1} \circ \mathbf{b}^{(u_0)} & \text{if } \vec{u} \text{ is } l\text{-primitive;} \\ \mathbf{b}^{\vec{u'};l} + \mathbf{b}^{\vec{u''};l} & \text{if } \vec{u} = \vec{u'}, w, \vec{u''} \text{ and } w = \min(\vec{u}) < l. \end{cases}$$

$$\mathbf{t}^{\vec{u};l} \quad = \quad \begin{cases} 1_{\alpha_{u_0}} & \text{if } \vec{u} = u_0 \leq l; \\ \kappa^{0} \circ \mathbf{t}^{(u_0)} & \text{if } \vec{u} \text{ is } l\text{-primitive;} \\ \mathbf{t}^{\vec{u'};l} + \mathbf{t}^{\vec{u''};l} & \text{if } \vec{u} = \vec{u'}, w, \vec{u''} \text{ and } w = \min(\vec{u}) < l. \end{cases}$$

For $l \in \omega$ and ud-vectors $\vec{u}$ and $\vec{v}$ such that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$ the embedding morphisms in $s\omega\mathrm{Gr}$

$$\boldsymbol{\kappa}^{0;\vec{u}} : \alpha^{\vec{u}} \longrightarrow \alpha^{[\vec{u},l,\vec{v}]} \qquad\qquad \boldsymbol{\kappa}^{1;\vec{v}} : \alpha^{\vec{v}} \longrightarrow \alpha^{[\vec{u},l,\vec{v}]}$$

are as follows.

$$\boldsymbol{\kappa}^{0;\vec{u}} \quad = \quad \begin{cases} \mathbf{t}^{\vec{v};u_0} & \text{if } \vec{u} = u_0 \leq n_1; \\ \boldsymbol{\kappa}^{0..k} & \text{if } \vec{u} \text{ is } n_1\text{-primitive, and } \mathrm{lh}(\vec{u}) = k+1; \\ \boldsymbol{\kappa}^{0;\vec{u'}} + \boldsymbol{\kappa}^{0;\vec{u''}} & \text{if } \vec{u} = \vec{u'}, z, \vec{u''},\ \vec{v} = \vec{v'}, z, \vec{v''}, \\ & \ \mathrm{tr}_{(n_1)}(\vec{u'}) = \mathrm{tr}_{(n_1)}(\vec{v'}),\ \mathrm{tr}_{(n_1)}(\vec{u''}) = \mathrm{tr}_{(n_1)}(\vec{v''}), \\ & \ \text{and } z = \min(\vec{u}) < n_1. \end{cases}$$

and

$$\boldsymbol{\kappa}^{1;\vec{v}} \quad = \quad \begin{cases} \mathbf{b}^{\vec{u};u_0} & \text{if } \vec{u} = u_0 \leq n_1; \\ \boldsymbol{\kappa}^{k+1..k'} & \text{if } \vec{v} \text{ is } n_1\text{-primitive, } \mathrm{lh}(\vec{u}) = k+1, \\ & \ \text{and } \mathrm{lh}(\vec{u}, n_1, \vec{v}) = k'+1; \\ \boldsymbol{\kappa}^{1;\vec{v'}} + \boldsymbol{\kappa}^{1;\vec{v''}} & \text{if } \vec{u} = \vec{u'}, z, \vec{u''},\ \vec{v} = \vec{v'}, z, \vec{v''}, \\ & \ \mathrm{tr}_{(n_1)}(\vec{u'}) = \mathrm{tr}_{(n_1)}(\vec{v'}),\ \mathrm{tr}_{(n_1)}(\vec{u''}) = \mathrm{tr}_{(n_1)}(\vec{v''}), \\ & \ \text{and } z = \min(\vec{u}) < n_1. \end{cases}$$

We have

4.4. PROPOSITION. *For $l \in \omega$ and ud-vectors $\vec{u}$ and $\vec{v}$ such that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$ the diagram in $s\omega$Gr*

$$\begin{tikzcd}
\alpha^{[\vec{u},l,\vec{v}]} & \alpha^{\vec{v}} \arrow[l, "\boldsymbol{\kappa}^{1;\vec{v}}"'] \\
\alpha^{\vec{u}} \arrow[u, "\boldsymbol{\kappa}^{0;\vec{u}}"] & \alpha^{\vec{w}} \arrow[l, "\mathbf{b}^{\vec{u};l}"] \arrow[u, "\mathbf{t}^{\vec{v};l}"']
\end{tikzcd} \tag{23}$$

is a pushout in ωGr.

PROOF. Let $\vec{u}$, $\vec{v}$, and l be as in the proposition. We prove the Proposition by induction on l-size of $\mathrm{tr}_{(l)}(\vec{u})$.

If $\vec{u} = u_0 \leq l$ then (23) is

$$\begin{tikzcd}
\alpha^{\vec{v}} & \alpha^{\vec{v}} \arrow[l, "1"'] \\
\alpha^{u_0} \arrow[u, "\mathbf{t}^{\vec{u};l}"] & \alpha^{u_0} \arrow[l, "1"] \arrow[u, "\mathbf{t}^{\vec{u};l}"']
\end{tikzcd}$$

so it is a pushout.

The case $\vec{v} = v_0 \leq l$ is similar.

If $\vec{u}$, $\vec{v}$ are l-primitive and then $[\vec{u}, l, \vec{v}] = \vec{u}, l, \vec{v}$ and (23) is

$$\begin{tikzcd}
\alpha^{\vec{u},l,\vec{v}} & \alpha^{\vec{v}} \arrow[l, "\boldsymbol{\kappa}^{k+1..k'}"'] \\
\alpha^{\vec{u}} \arrow[u, "\boldsymbol{\kappa}^{0..k}"] & \alpha^{l} \arrow[l, "\mathbf{b}^{\vec{u};l}"] \arrow[u, "\mathbf{t}^{\vec{v};l}"']
\end{tikzcd}$$

where $k = \mathrm{lh}(\vec{u})$ and $k' = ln(\vec{u}, l, \vec{v}) + 1$. So by Proposition 4.4 it is again pushout.

Finally, if $z = \min(\vec{w})$ $\vec{u} = \vec{u'}, z, \vec{u''}$, $\vec{v} = \vec{v'}, z, \vec{v''}$ $\mathrm{tr}_{(l)}(\vec{u'}) = \mathrm{tr}_{(l)}(\vec{v'}) = \vec{w'}$, $\mathrm{tr}_{(l)}(\vec{u''}) = \mathrm{tr}_{(l)}(\vec{v''}) = \vec{w''}$ then $[\vec{u}, l, \vec{v}] = [\vec{u'}, l, \vec{v'}], z, [\vec{u''}, l, \vec{v''}]$ and (23) is the following square

$$\begin{tikzcd}
\alpha^{[\vec{u'},l,\vec{v'}],w,[\vec{u''},l,\vec{v''}]} & \alpha^{\vec{v'},w,\vec{v''}} \arrow[l, "\boldsymbol{\kappa}^{1;\vec{v'}} + \boldsymbol{\kappa}^{1;\vec{v''}}"'] \\
\alpha^{\vec{u'},w,\vec{u''}} \arrow[u, "\boldsymbol{\kappa}^{0;\vec{u'}} + \boldsymbol{\kappa}^{0;\vec{u''}}"] & \alpha^{\vec{u'},w,\vec{u''}} \arrow[l, "\mathbf{b}^{\vec{u'};l} + \mathbf{b}^{\vec{u''};l}"] \arrow[u, "\mathbf{t}^{\vec{v'};l} + \mathbf{t}^{\vec{v''};l}"']
\end{tikzcd}$$

which arises as a pushout over α^z of two squares, which are pushouts by inductive assumption for vectors $\vec{w'}$ and $\vec{w''}$ with smaller l-size then $\vec{w'}, z, \vec{w''}$. Thus it is again a pushout. ∎

4.5. Proposition.

1. *For any simple ω-graph G, there is a ud-vector $\vec{u}$ such that G is isomorphic to $\alpha^{\vec{u}}$.*
2. *Let ud-vectors $\vec{u}$, $\vec{v}$ be ud-vectors. If $f : \alpha^{\vec{u}} \longrightarrow \alpha^{\vec{v}}$ is an isomorphism then $\vec{u} = \vec{v}$ and $f = 1_{\alpha^{\vec{u}}}$.*

Proof. 2. follows easily from 1. For 1. we give only a sketch of the proof.

Let G be a simple ω-graph. By T_G we denote the set of cells in G which are neither domains nor codomains of other cells in G. We define a linear order on T_G, so that for $a, b \in T_G$

$$a \leq b \quad \text{iff} \quad d_{(\nu_{a,b})}(a) \geq d_{(\nu_{a,b})}(b)$$

Let $T_G = \{a_0, \ldots, a_k\}$ such that $a_i \leq a_{i+1}$ for $0 \leq i \leq k$. Let u_{2i} be such a number that $a_i \in G_{u_{2i}}$, for $0 \leq i \leq k$, and $u_{2i+1} = \nu_{a_i,a_{i+1}}$ for $0 \leq i \leq k-1$. Then, we can prove, by induction on the cardinality of T_G that G is isomorphic to $\alpha^{\vec{u}}$, where $\vec{u} = u_0, \ldots, u_{2k}$. ∎

Let G be an arbitrary ω-graph. In Appendix 6.8 the construction of the free internal ω-category $[G]$ over an internal ω-graph G is given, when the ambient category has finite limits and disjoint and universal coproducts. In Set, this construction can be described more conveniently using simple ω-graphs.

To distinguish (temporarily) between these two constructions by $[[G]]$ we will denote the ω-category defined in terms of simple ω-graphs.

The set of n-cells is

$$[[G]]_n = \{f : \alpha^{\vec{u}} \to G : \mathrm{ht}(\vec{u}) \leq n\}$$

The set of n_1-compatible pairs of n_0- and n_2-cells is

$$\begin{aligned}[[G]]_{n_0,n_1,n_2} = \{\langle f, g\rangle :\ & f : \alpha^{\vec{u}} \to G, \quad g : \alpha^{\vec{v}} \to G, \\ & \mathrm{ht}(\vec{u}) \leq n_0, \quad \mathrm{ht}(\vec{v}) \leq n_2, \quad \text{and} \quad f \circ \mathbf{b}^{\vec{u};n_1} = g \circ \mathbf{t}^{\vec{v};n_1}\}\end{aligned}$$

For $l \leq n$, the domain and the codomain operations

$$d_l^{[[G]]}, c_l^{[[G]]} : [[G]]_n \longrightarrow [[G]]_l$$

for $f : \alpha^{\vec{u}} \to G \in [[G]]_n$, are defined by composition

$$d_l^{[[G]]}(f) = f \circ \mathbf{t}^{\vec{u};l} : \alpha^{\mathrm{tr}_{(l)}(\vec{u})} \to G,$$

$$c_l^{[[G]]}(f) = f \circ \mathbf{b}^{\vec{u};l} : \alpha^{\mathrm{tr}_{(l)}(\vec{u})} \to G.$$

For $n \leq l$, the identity operations

$$\iota_l^{[[G]]} : [[G]]_n \longrightarrow [[G]]_l$$

are inclusions. For $n_1 < n_0, n_2$, the compositions

$$m_{n_0,n_1,n_2}^{[[G]]} : [[G]]_{n_0,n_1,n_2} \longrightarrow [[G]]_{\max(n_0,n_2)}$$

are defined, via pushouts in $s\omega$Gr. By Proposition 4.4, for any pair $\langle f, g\rangle \in [[G]]_{n_0,n_1,n_2}$ there is a unique morphism $[f, g] : \alpha^{[\vec{u},n_1,\vec{v}]} \longrightarrow G$ making the diagram

$$\begin{tikzcd}
G & & \\
& \alpha^{[\vec{u},n_1,\vec{v}]} \arrow[ul, "{[f,g]}"'] & \alpha^{\vec{v}} \arrow[l, "\boldsymbol{\kappa}^{1;\vec{v}}"'] \arrow[ull, "g"'] \\
& \alpha^{\vec{u}} \arrow[u, "\boldsymbol{\kappa}^{0;\vec{u}}"] \arrow[uul, "f"] & \alpha^{\vec{w}} \arrow[l, "\mathbf{t}^{\vec{u};n_1}"] \arrow[u, "\mathbf{b}^{\vec{v};n_1}"']
\end{tikzcd}$$

commutes, where $\vec{w} = \mathrm{tr}_{(n_1)}(\vec{u})(= \mathrm{tr}_{(n_1)}(\vec{v}))$. We put

$$m^{[[G]]}_{n_0,n_1,n_2}(\langle f, g\rangle) = [f, g]$$

We define

$$\eta_G : G \longrightarrow [[G]]$$

such that, for $x \in G_n$, $\eta_G(x) : \alpha^n \longrightarrow G$ so that $\eta_G(x)(2n) = x$.

Thus by Theorem 6.9 we obtain

4.6. Theorem. *For any ω-graph G, $[[G]]$ as described above is an ω-category. Moreover, the association $G \longmapsto [[G]]$ is functorial in G and the functor*

$$[[-]] : \omega\mathrm{Gr} \longrightarrow \omega\mathrm{Cat}$$

is a left adjoint to the forgetful functor $\omega\mathrm{Cat} \longrightarrow \omega\mathrm{Gr}$, with η as the unit of adjunction, i.e. $[[G]]$ is the free ω-category on the ω-graph G.

Remark. This is essentially a special case of Proposition 4.2 in [B].

Proof. We shall show that $[[G]]$ defined above is isomorphic to $[G]$ defined in Appendix 6.8.

To this end, we define a functor

$$\xi^G : s\omega\mathrm{Gr}^{op} \longrightarrow \mathrm{Set}$$

such that, for a ud-vector $\vec{u}$, we put

$$\xi^G(\alpha^{\vec{u}}) = \omega\mathrm{Gr}(\alpha^{\vec{u}}, G).$$

On morphisms ξ^G acts by composition. Since colimits in ωGr are computed pointwise, ξ^G sends pointwise colimits in $s\omega$Gr to limits in Set. In particular ξ^G sends the diagrams (22) and (23) in $s\mathcal{G}$ to the limiting diagrams in Set.

For $n \in \omega$, we have a bijection

$$\varphi_n : \xi^G(\alpha^n) \longrightarrow G_n$$

such that $\varphi_n(f) = f(2n)$, for $f \in \xi^G(\alpha^n)$. Moreover, it is easy to see that the diagram

$$\begin{tikzcd}
\xi^G(\alpha^n) \arrow[rr, "\varphi_n"] \arrow[d, shift right=4, "\xi^G(\mathbf{t})"'] \arrow[d, shift left=4, "\xi^G(\mathbf{b})"] & & G_n \arrow[d, shift right=4, "d^G_{(l)}"'] \arrow[d, shift left=4, "c^G_{(l)}"] \\
\xi^G(\alpha^l) \arrow[rr, "\varphi_l"'] & & G_l
\end{tikzcd}$$

commutes serially, i.e. ξ^G 'sends' α^n to G_n, $\mathbf{t}$ to d^G, and $\mathbf{b}$ to c^G. It follows, that for any ud-vector $\vec{u}$, the colimit $\alpha^{\vec{u}}$ with the coprojections $\boldsymbol{\kappa}^j : \alpha^{u_{2i}} \longrightarrow \alpha^{\vec{u}}$ is sent by ξ^G to $G_{\vec{u}}$ with the projections $\pi_j : G^{\vec{u}} \longrightarrow G^{u_{2i}}$. We denote by $\varphi_{\vec{u}} : \xi^G(\alpha^{\vec{u}}) \longrightarrow G_{\vec{u}}$ the induced morphisms commuting with the projections. Thus we have $\varphi_{\vec{u}}(f : \alpha^{\vec{u}} \to G) = \langle f \circ \kappa^i(2u_{2i})\rangle_{i\in \mathrm{lh}(\vec{u})}$. Then, for any pair of vectors $\vec{u}$, $\vec{v} \in \mathrm{UD}_{n_0,n_1,n_2}$, ξ^G sends the pushout $\alpha^{[\vec{u},n_1,\vec{v}]}$ with the coprojections $\boldsymbol{\kappa}^{0;\vec{u}}$, $\boldsymbol{\kappa}^{1;\vec{v}}$ to the pullback $G_{[\vec{u},n_1,\vec{v}]}$ with the projections $\pi_{0;\vec{u}}$, $\pi_{1;\vec{v}}$.

For $l \in \omega$, and a ud-vector $\vec{u}$ the definition of $d^G_{\vec{u};l}$ is dual to the definition of $\mathbf{t}^{\vec{u};l}$, one is using d^G's, projections, and pullbacks and the other one is using $\mathbf{t}$'s, coprojections, and pushouts. Therefore ξ^G is sending $\mathbf{t}^{\vec{u};l}$ to $d^G_{\vec{u};l}$, and for the similar reason ξ^G is sending $\mathbf{b}^{\vec{u};l}$ to $c^G_{\vec{u};l}$, i.e. the diagram

$$\begin{tikzcd}
\xi^G(\alpha^{\vec{u}}) \arrow[rr, "\Phi_{\vec{u}}"] \arrow[d, shift right=4, "\xi^G(\mathbf{t}^{\mathrm{tr}_{(l)}(\vec{u})})"'] \arrow[d, shift left=4, "\xi^G(\mathbf{b}^{\mathrm{tr}_{(l)}(\vec{u})})"] & & G_{\vec{u}} \arrow[d, shift right=4, "d^G_{\vec{u};l}"'] \arrow[d, shift left=4, "c^G_{\vec{u};l}"] \\
\xi^G(\alpha^{\mathrm{tr}_{(l)}(\vec{u})}) \arrow[rr, "\Phi_{\mathrm{tr}_{(l)}(\vec{u})}"'] & & G_{\mathrm{tr}_{(l)}(\vec{u})}
\end{tikzcd} \tag{24}$$

commutes serially.

Since, for $n \in \omega$,

$$[[G]]_n = \coprod_{\vec{u}\in \mathrm{UD}_n} \xi^G(\alpha^{\vec{u}}) \tag{25}$$

we have a bijection

$$\Phi_n = \coprod_{\vec{u}\in \mathrm{UD}_n} \varphi_{\vec{u}} : [[G]]_n \longrightarrow [G]_n.$$

It remain to show that Φ commutes with the ω-category operations.

The preservations of the domains and the codomains follows from (25) and the diagram 24. Preservation of identities is trivial. Finally, to show that the compositions are preserved consider the following diagram

$$\begin{tikzcd}
\xi^G(\alpha^{[\vec{u},n_1,\vec{v}]}) \arrow[rrr, "\varphi_{[\vec{u},n_1,\vec{v}]}"] \arrow[dr, "\zeta_{\vec{u};\vec{v}}"] \arrow[ddr] & & & G_{[\vec{u},n_1,\vec{v}]} \arrow[dl, "\kappa_{\vec{u},\vec{v}}"'] \arrow[ddl, "\kappa_{[\vec{u},n_1,\vec{v}]}"] \\
& {[[G]]_{n_0,n_1,n_2}} \arrow[r, "\Phi_{n_0}\times\Phi_{n_2}"] \arrow[d, "m"] & {[G]_{n_0,n_1,n_2}} \arrow[d, "m"] & \\
& {[[G]]_n} \arrow[r, "\Phi_n"'] & {[G]_n} &
\end{tikzcd}$$

where $n = \max(n_0, n_2)$, $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n,n_1,n_2}$, $\zeta_{\vec{u},\vec{v}}$ is the coprojection morphism, i.e. the morphism of composing with coprojections $\boldsymbol{\kappa}^{0;\vec{u}}$ and $\boldsymbol{\kappa}^{1;\vec{v}}$, and the unnamed morphism is an inclusion. The commutation of the left triangle follows from the universal property of the pushout, the right triangle commutes by definition of $m^G_{n_0,n_1,n_2}$, and the outer square commutes by definition of Φ_n. The upper square commutes, since for $\langle f : \alpha^{\vec{u}} \to G, g : \alpha^{\vec{v}} \to G \rangle \in [[G]]_{n_0,n_1,n_2}$, we have

$$\Phi_{n_0} \times \Phi_{n_2}(f, g) = \varphi_{[\vec{u},n_1,\vec{v}]}([f, g])$$

Hence the inner square commutes as well, i.e. Φ preserves compositions and it is an isomorphism of ω-categories $[[G]]$ and $[G]$, as required. ∎

In the following, we shall write $[G]$, rather then $[[G]]$.

4.7. Composable ω-graphs. In this section we make the connection between simple ω-graphs and simple ω-categories. The latter are defined as free ω-categories with a unique maximal cell, i.e. free on composable ω-graphs. We have

4.8. Proposition. *An ω-graph is composable iff it is simple.*

Proof. Let G be an ω-graph. A cell $a : \alpha^{\vec{u}} \longrightarrow G$ in $[G]_n$ is a non-identity cell iff $\mathrm{ht}(\vec{u}) = n$. We have

Claim.

a is a maximal in $[G]$ iff a is surjective.

Consider the surjective-injective factorization of a in $\omega\mathrm{Gr}$

$$\begin{tikzcd}
\alpha^{\vec{u}} \arrow[rr, "a"] \arrow[dr, "e"'] & & G \\
& G' \arrow[ur, "i"'] &
\end{tikzcd}$$

Then a is in the image of $[i] : [G'] \longrightarrow [G]$. If a is maximal, then, by definition, i an iso and a is a surjection. On the other hand, if a is a surjection then so is i, and since it is an injection it mast be an iso. Hence a is maximal. This proves the Claim.

Thus if G is a simple ω-graph then there is a unique surjection $a : \alpha^{\vec{u}} \longrightarrow G$ in $\omega\mathrm{Gr}$, for some ud-vector $\vec{u}$, which is in fact an iso, as any morphism of simple ω-graphs is one-to-one. Hence $[G]$ is a simple ω-category, with the unique maximal cell $a \in G_{\mathrm{ht}(\vec{u})}$.

Now assume that $[G]$ is a simple ω-category. We need to show that G is a simple ω-graph. Let $m \in [G]_n$ be the unique maximal arrow in $[G]$. By the Claim m is surjective. To finish the proof it is enough to show that m is injective, as well. Suppose not, let k be the minimal such that $m_k : \alpha^{\vec{u}}_k \longrightarrow G_k$ is not one-to-one, and let $x, y \in \alpha^{\vec{u}}_k$, such that $m_k(x) = m_k(y)$. By minimality of k, we have $d(x) = d(y) = x'$ and $c(x) = c(y) = y'$. We can divide the set of those cells o in $\alpha^{\vec{u}}$ for which $d_{(k-1)}(o) = x'$ and $c_{(k-1)}(o) = x'$ into three classes which we can picture schematically as follows

$$\cdots\ x' \quad \begin{array}{c} \longrightarrow \\ \mathbf{BLUE} \\ \overset{x}{\longrightarrow} \\ \mathbf{RED} \\ \overset{y}{\longrightarrow} \\ \mathbf{GREEN} \\ \longrightarrow \end{array} \quad y'\ \cdots$$

where the **BLUE** class contains those cells o in $\alpha^{\vec{u}}$ for which $c_(k)(o) > x$, the **RED** class contains those cells o for which $x \geq d_(k)(o)$ and $c_(k)(o) \geq y$, and the **GREEN** class contains those cells o for which $y > d_(k)(o)$. Clearly x and y are in the **RED** class. We shall construct another simple ω-graph H by dubbling the **RED** class in $\alpha^{\vec{u}}$. The relevant part of the ω-graph H we can draw schematically as follows

$$\begin{array}{c} \longrightarrow \\ \mathbf{BLUE} \\ \overset{x}{\longrightarrow} \\ \mathbf{RED} \\ \cdots x' \overset{y=\bar{x}}{\longrightarrow} y' \cdots \\ \mathbf{PINK} \\ \overset{\bar{y}}{\longrightarrow} \\ \mathbf{GREEN} \\ \longrightarrow \end{array}$$

where the **PINK** class is the second copy of the **RED** class. A cell in the **PINK** class corresponding to a cell in the **RED** class we distinguish by putting a bar over it, i.e. $\bar{z}$ is the **PINK** version of the **RED** cell z. The cell y is identified with $\bar{x}$. The domains and codomains in H remains as they were in $\alpha^{\vec{u}}$ except that for those cells o in the **GREEN** class for which we had $d_{(k)}(o) = y$ now we put $d_{(k)}(o) = \bar{y}$ and for the cell $\bar{o}$ in the **PINK** class we put for domains

$$d_{(l)}(\bar{o}) \quad = \quad \begin{cases} \bar{o}' & \text{if } l > k \text{ and } d_{(l)}(o) = o' \\ o' & \text{if } l \leq k \text{ and } d_{(l)}(o) = o' \end{cases}$$

and similarly for codomains

$$c_{(l)}(\bar{o}) \quad = \quad \begin{cases} \bar{o}' & \text{if } l > k \text{ and } c_{(l)}(o) = o' \\ o' & \text{if } l \leq k \text{ and } c_{(l)}(o) = o' \end{cases}$$

We have a ω-graph morphism

$$m' : H \longrightarrow G$$

by $m'(o) = m(o)$ for o which are not in **PINK** and $m'(\bar{o}) = m(o)$ for o in **RED** class. Since $m(x) = m(y)$ the morphism m' is well defined. By Proposition 4.5, we can assume that $H = \alpha^{\vec{v}}$ for some ud-vector $\vec{v}$. Since H in not equal to $\alpha^{\vec{u}}$, $m \neq m'$. But m' is surjective (since m was), so by the Claim, it is a maximal cell in $[G]$ as well, i.e. G is not composable contrary to the supposition. ∎

4.9. An internal disk **D** in $\mathcal{S}$. We introduce below some notation for simple ω-categories and prove some basic facts about them.

For any ud-vector $\vec{s}$, $\delta^{\vec{s}}$ is, by definition, the free ω-category on $\alpha^{\vec{s}}$. However, for some simple ω-categories we need a more precise description. Below, we give a specific presentation for the simple categories δ^s and $\delta^{s+1,s,s+1}$, for $s \in \omega$.

It is maybe much simpler to look at the picture of δ^4 given below to understand what δ^s is, however before giving the picture we put the general definition of δ^s.

The n-cells of δ^s are

$$\delta^s_n = \begin{cases} 2n+2 & \text{if } n < s \\ 2s+1 & \text{if } n \geq s \end{cases}$$

and the domain, codomain, and identity functions

$$d_n, c_n : \delta^s_{n+1} \longrightarrow \delta^s_n \qquad \iota_{n+1} : \delta^s_n \longrightarrow \delta^s_{n+1}$$

are, for $n < s-1$

$$d_n(x) = \begin{cases} x & \text{if } x \leq n \\ x-2 & \text{if } x > n \end{cases}$$

$$c_n(x) = \begin{cases} x & \text{if } x < n \\ x-2 & \text{if } x \geq n \end{cases}$$

$$\iota_{n+1}(x) = \begin{cases} x & \text{if } x \leq n \\ x+2 & \text{if } x > n \end{cases}$$

for $n = s-1$

$$d_{s-1}(x) = \begin{cases} x & \text{if } x \leq s \\ x-1 & \text{if } x > s \end{cases}$$

$$c_{s-1}(x) = \begin{cases} x & \text{if } x < s \\ x-1 & \text{if } x \geq s \end{cases}$$

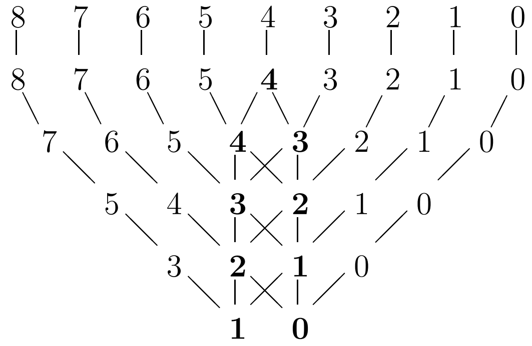
$$\iota_s(x) = \begin{cases} x & \text{if } x < s \\ x+1 & \text{if } x \geq s \end{cases}$$

and for $n \geq s$

$$d_n = c_n = \iota_n = id$$

We don't give an explicit formula for compositions m_{n_0,n_1,n_2} since any compositions are compositions with identities.

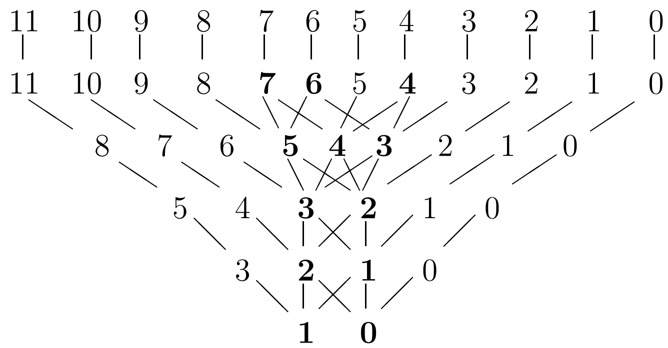
The first six levels of δ^4 can be pictured as follows:



For $s \in \omega^+$, the n-cells of $\delta^{s,s-1,s}$ are

$$\delta_n^{s,s-1,s} \quad = \quad \begin{cases} 2n+2 & \text{if } n < s-1 \\ 2s+1 & \text{if } n = s-1 \\ 2s+4 & \text{if } n \geq s \end{cases}$$

and the definitions of domain, codomain, identity and compositions in $\delta^{s,s-1,s}$ are left for the reader. For example $\delta^{4,3,4}$ can be pictured as follows (no-identity arrows are marked bold):



For $s < s'$, and ud-vector $\vec{s}$ and $i \in \mathrm{lh}(\vec{s})$ the ω-functors

$$\mathbf{b}, \mathbf{t} : \delta^s \longrightarrow \delta^{s'} \qquad\qquad \mathbf{b}^{\vec{s};l}, \mathbf{t}^{\vec{s};l} : \delta^{\mathrm{tr}_{(l)}(\vec{s})} \longrightarrow \delta^{\vec{s}}$$

$$\boldsymbol{\kappa}^i : \delta^{s_{2i}} \longrightarrow \delta^{\vec{s}}$$

and for $l \in \omega$ and ud-vectors $\vec{u}$ and $\vec{v}$ such that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$ the ω-functors and

$$\boldsymbol{\kappa}^{0;\vec{u}} : \delta^{\vec{u}} \longrightarrow \delta^{[\vec{u},l,\vec{v}]} \qquad\qquad \boldsymbol{\kappa}^{1;\vec{v}} : \delta^{\vec{v}} \longrightarrow \delta^{[\vec{u},l,\vec{v}]}$$

are defined to be the images of the ω-graph morphisms with the same name in $s\omega\mathrm{Gr}$ under the free ω-category functor $[-]$. Thus, as an immediate corollary from Propositions 4.3 and 4.4 we obtain

4.10. COROLLARY.

1. *For any ud-vector $\vec{s}$, the diagram*

$$\begin{tikzcd}
 & & & \delta^{\vec{s}} & & & \\
\delta^{s_0} \arrow[urrr, "\boldsymbol{\kappa}^0"] & & \delta^{s_2} \arrow[ur, "\boldsymbol{\kappa}^2"] & \cdots & \delta^{s_{2k-2}} \arrow[ul, "\boldsymbol{\kappa}^{2k-2}"'] & & \delta^{s_{2k}} \arrow[ulll, "\boldsymbol{\kappa}^{2k}"'] \\
 & \delta^{s_1} \arrow[ul, "\mathbf{t}"] \arrow[ur, "\mathbf{b}"'] & & & & \delta^{s_{2k-1}} \arrow[ul, "\mathbf{t}"] \arrow[ur, "\mathbf{b}"'] &
\end{tikzcd} \tag{26}$$

is a colimiting cone in $\mathcal{S}$.

2. *For $l \in \omega$ and ud-vectors $\vec{u}$ and $\vec{v}$ such that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$ the diagram*

$$\begin{tikzcd}
\delta^{[\vec{u},l,\vec{v}]} & \delta^{\vec{v}} \arrow[l, "\boldsymbol{\kappa}^{1;\vec{v}}"'] \\
\delta^{\vec{u}} \arrow[u, "\boldsymbol{\kappa}^{0;\vec{u}}"] & \delta^{\vec{w}} \arrow[l, "\mathbf{t}^{\vec{u};l}"] \arrow[u, "\mathbf{b}^{\vec{v};l}"']
\end{tikzcd} \tag{27}$$

is a pushout in $\mathcal{S}$. ■

Let G be a simple ω-graph and $s \in \omega$. We shall introduce some notation for the morphisms

$$[G] \longrightarrow \delta^s \quad \text{and} \quad [G] \longrightarrow \delta^{s+1,s,s+1}$$

There is a unique ω-functor $[G] \longrightarrow \delta^0$, which we denote by $\downarrow$. For $s \in \omega$, $e, e' \in G_s$ such that, $e \rhd e'$ we define an ω-functor

$$e{\downarrow}e' : [G] \longrightarrow \delta^{s+1}$$

so that its composition with the ω-graph morphisms $\eta_G : G \longrightarrow [G]$, is given for $n \geq 0$ and $x \in G_n$ by

$$(e{\downarrow}e' \circ \eta_G)_n(x) \quad = \quad \begin{cases} s+1 & \text{if } n > s \text{ and } d_{(s)}(e) = x \\ l & \text{if } n \geq l \text{ and } c_{(l)}(e') \geq d_{(l)}(x) \\ \delta^{s+1}_n - l - 1 & \text{if } n \geq l \text{ and } c_{(l)}(x) \geq d_{(l)}(e) \end{cases}$$

The ω-functor $e{\downarrow}e'$ sends e and e' to the only two non-identity s-cells and the remaining cells in $[G]$ to the only cell in δ^{s+1} with the suitable domain and codomain. It is the only ω-functor from $[G]$ to δ^{s+1} distinguishing e and e'. If $x \in G_n$ then we can define

$$l = \max\{l' : \ l' \leq s, n \ \text{ and } \ d_{(l')}(e) \perp d_{(l')}(x)\}$$

Then, either

$$l = s \text{ and } d_{(l)}(x) = e \text{ and } c_{(l)}(x) = e'$$

or

$$c_{(l)}(e') \geq d_{(l)}(x) \text{ and } d_{(l')}(e') = d_{(l')}(x) \text{ for} l' < l$$

or

$$c_{(l)}(x) \geq d_{(l)}(e) \text{ and } c_{(l')}(e) = c_{(l')}(x) \text{ for} l' < l$$

Thus the formula for $e{\downarrow}e' \circ \eta_G$ defines the only graph morphism such that $e{\downarrow}e' \circ \eta_G(e) = s+1$ and $e{\downarrow}e' \circ \eta_G(e') = s$. We subsume the properties of $e{\downarrow}e'$ in the following lemma.

4.11. Lemma. *Let $s \in \omega$ and G be a simple ω-graph.*

1. *Let $e, e' \in G_s$, such that $e \rhd e'$. Then $e{\downarrow}e' : [G] \longrightarrow \delta^{s+1}$ is a well defined ω-functor and it is the unique ω-functor from $[G]$ to δ^{s+1} such that*

$$(e{\downarrow}e')_s(e) = s+1 \qquad\qquad (e{\downarrow}e')_s(e') = s$$

2. *The ω-functor $f : [G] \longrightarrow \delta^{s+1}$ is onto iff there is $e \in G_{s+1}$ such that $f = (d(e){\downarrow}c(e))$.*

Proof. For 1. we argued before the statement of the lemma.

The ω-functor $f : [G] \longrightarrow \delta^{s+1}$ is onto iff there is an $s+1$-cell $e \in G_{s+1}$ such that $f_{s+1}(e) = s+1$. But then $f = (d(e){\downarrow}c(e))$. Thus 2. holds as well. ■

If e is minimal in $G_s(d(e), c(e))$ then, we define

$$e{\downarrow} = \mathbf{b} \circ (d(e){\downarrow}c(e)) : [G] \longrightarrow \delta^{s+1}$$

and if e is maximal in $G_s(d(e), c(e))$ then, we define

$${\downarrow}e = \mathbf{t} \circ (d(e){\downarrow}c(e)) : [G] \longrightarrow \delta^{s+1}$$

Since in the morphism $(e{\downarrow}e')$ we have $e \rhd e'$, both e and e' determine the morphism $(e{\downarrow}e')$ uniquely. Hence we can write $e{\downarrow} = e{\downarrow}e' = {\downarrow}e'$, for short. Note that in this way, for any s-cell e the morphisms ${\downarrow}e$ and $e{\downarrow}$ are defined, no matter e has successor and predecessor or not.

Using this notation we can define the some ω-functors. Let $s \in \omega$. We have

$$\mathbf{p}^s : \delta^{s+1} \longrightarrow \delta^s$$

such that $\mathbf{p}^s = (s{\downarrow}s-1)$, i.e. $\mathbf{p}^s$ 'glue together the only non-identity s-cells $s+1$ and s. Moreover, we define

$$\boldsymbol{\rho}^{0;s+1}, \boldsymbol{\rho}^{1;s+1} : \delta^{s+1,s,s+1} \longrightarrow \delta^{s+1}e$$

such that $\boldsymbol{\rho}^{0;s+1} = (s+2{\downarrow}s+1)$ and $\boldsymbol{\rho}^{1;s+1} = (s+1{\downarrow}s)$ with $s+2, s+1, s \in \delta^{s+1,s,s+1}_s$, and

$$\boldsymbol{\rho}^{s+1} : \delta^{s+1} \longrightarrow \delta^{s+1,s,s+1}$$

such that $\boldsymbol{\rho}^{s+1}_{s+1}(s+1) = s+3$, where $s+3 \in \delta^{s+1,s,s+1}_{s+1}$ is the maximal cell in $\delta^{s+1,s,s+1}$.

We shall prove that any $s \in \omega$, the diagram

$$\begin{tikzcd} \delta^{s+1,s,s+1} \arrow[r, shift left=2, "\boldsymbol{\rho}^0"] \arrow[r, shift right=2, "\boldsymbol{\rho}^1"'] & \delta^{s+1} \arrow[r, "\mathbf{p}"] & \delta^{s} \arrow[l, shift right=3, "\mathbf{b}"'] \arrow[l, shift left=3, "\mathbf{t}"] \end{tikzcd} \tag{28}$$

is an internal bundle of intervals in $\mathcal{S}$. To this end, we introduce below and in the following lemma more notation and state some properties of the ω-functors that we have defined so far.

For ω-functors $z_0, z_1 : [G] \longrightarrow \delta^{s+1}$, we write $z_0 \leq z_1$ iff there is an ω-functors $f : [G] \longrightarrow \delta^{s+1,s,s+1}$ such that $\boldsymbol{\rho}^i \circ f = z - i$, for $i = 0, 1$.

4.12. Lemma. *Let $s \in \omega$ and $[G]$ be a simple ω-category, $e, e' \in G_s$. Then*

1. *We have*
$$(\downarrow e)_s(e) = s \qquad (e\downarrow)_s(e) = s+1$$
Moreover, if e is maximal (minimal) in $G_(d(e), c(e))$ then the equality $\downarrow e_s(e) = s$ ($e\downarrow_s(e) = s+1$) determines the ω-functor $\downarrow e$ ($e\downarrow$) uniquely.

2. *If $s = 0$ then*
$$\mathbf{p} \circ (\downarrow e) = \mathbf{p} \circ (e\downarrow) = \downarrow;$$

3. *If $s > 0$ then*
$$\mathbf{p} \circ (\downarrow e) = \mathbf{p} \circ (e\downarrow) = (d(e)\downarrow c(e));$$

4. *If $e \rhd e'$ then*
$$\mathbf{b} \circ (e\downarrow e') = (\min(G_s(e, e')\downarrow) \qquad \mathbf{t} \circ (e\downarrow e') = (\downarrow\max(G_s(e, e'));$$

5. *$\boldsymbol{\rho}^i \circ \boldsymbol{\rho} = id$ for $i = 0, 1$;*

6. *If $e \geq e'$ then there is a unique morphism*
$$\downarrow e..e'\downarrow : [G] \longrightarrow \delta^{s+1,s,s+1}$$
such that for $e'' \in G_s$
$$\downarrow e..e'\downarrow(e'') = s+1 \quad \text{iff} \quad e \geq e'' \geq e'$$
If $e = e'$ we write $\downarrow e\downarrow$ for $\downarrow e..e\downarrow$;

7. *If $e \geq e'$ then*
$$\boldsymbol{\rho}^0 \circ \downarrow e..e'\downarrow = \downarrow e \qquad \boldsymbol{\rho}^1 \circ \downarrow e..e'\downarrow = e'\downarrow$$

8. *if $z_0, z_1 : [G] \longrightarrow \delta^{s+1}$ are two different ω-functors and $z_0 \leq z_1$ then there are $e_0, e_1 \in G_s$ such that $e_0 \geq e_1$ and $z_0 = \downarrow e_0$ and $z_1 = e_1\downarrow$;*

9. *Let $z : [G] \longrightarrow \delta^{s+1}$ be an ω-functor. If $z_s(e) = s$ ($z_s(e) = s+1$) then there is $e_0 \in G_s$ ($e_1 \in G_s$) such that $e_0 \geq e$ ($e \geq e_1$) and $z = \downarrow e_0$ ($z = e_1\downarrow$);*

PROOF. Exercise. ∎

Now we can prove

4.13. LEMMA. *For any $s \in \omega$ the diagram (28) is a bundle of intervals $\mathcal{S}$.*

PROOF. We shall verify the conditions 1. - 4. in Appendix 6.1.

To see 1. we calculate

$$\mathbf{p}_s \circ \mathbf{b}_s(s) = \mathbf{p}_s(s+1) = s = \mathbf{p}_s(s) = \mathbf{p}_s \circ \mathbf{t}_s(s)$$

Hence we have $\mathbf{p} \circ \mathbf{b} = 1_{\delta^s} = \mathbf{p} \circ \mathbf{t}$.

2. is easy .

Ad 3. let $z_0, z_1, z_0 : [G] \longrightarrow \delta^{s+1}$ be ω-functors with G simple.

Then (a) holds since, by Lemma 4.12 $\boldsymbol{\rho}^i \circ \boldsymbol{\rho} \circ z_0 = z_0$, for $i = 0, 1$.

To show (b), we suppose contrary, that $z_0 \leq z_1$ and $z_1 \leq z_0$ but $z_0 \neq z_1$. Then by Lemma 4.12 there are cells $e_0, e_1, e_0', e_1' \in G_s$ such that $e_0 \geq e_1$, $e_0' \geq e_1'$ and $z_0 = {\downarrow} e_0 = e_0' {\downarrow}$, $z_1 = e_1 {\downarrow} = {\downarrow} e_0'$. Thus we get

$$e_0' \rhd e_0 \geq e_1 \rhd e_1' \geq e_0'$$

which is a contradiction.

Ad (c). Suppose that $z_0 \leq z_1$ and $z_1 \leq z_2$. We shall show that $z_0 \leq z_2$. We assume that $z_0 \neq z_1$, $z_1 \neq z_2$ since otherwise the thesis is obvious. Thus, by Lemma 4.12, there are e_0, e_1, e_1', e_2, such that $e_0 \geq e_1$, $e_1' \geq e_2$ and

$$\boldsymbol{\rho}^0 \circ {\downarrow} e_0..e_1 {\downarrow} = z_0 \qquad \boldsymbol{\rho}^1 \circ {\downarrow} e_0..e_1 {\downarrow} = z_1$$

$$\boldsymbol{\rho}^0 \circ {\downarrow} e_1'..e_2 {\downarrow} = z_1 \qquad \boldsymbol{\rho}^1 \circ {\downarrow} e_1'..e_2 {\downarrow} = z_2$$

Hence, in particular $e_1 \rhd e_1'$. But then $e_0 \geq e_2$ and

$$\boldsymbol{\rho}^0 \circ {\downarrow} e_0..e_2 {\downarrow} = z_0 \qquad \boldsymbol{\rho}^1 \circ {\downarrow} e_0..e_2 {\downarrow} = z_2$$

i.e. $z_0 \leq z_2$, as required.

To show (d), we assume that $z_0 \neq z_1$. If $\mathbf{p} \circ z_0 = \mathbf{p} \circ z_1$ then there is $e \in G_s$ such that either $z_0(e) = s$ and $z_1(e) = s+1$ or $z_0(e) = s+1$ and $z_1(e) = s$. We suppose that the former condition holds. The latter can be treated similarly. There are s-cells $e_0, e_1 \in G_s$ such that $e_0 \geq e \geq e_1$ and $z - 0 = {\downarrow} e_0$ and $z_1 = e_1 {\downarrow}$. So, we have

$$\boldsymbol{\rho}^0 \circ {\downarrow} e_0..e_1 {\downarrow} = z_0 \qquad \boldsymbol{\rho}^1 \circ {\downarrow} e_0..e_1 {\downarrow} = z_1$$

i.e. $z_0 \leq z_1$, as required. On the other hand, if $z_0 \leq z_1$ then, there are $e_0, e_1 \in G_s$, such that $e_0 \geq e_1$, $z_0 = {\downarrow} e_0$ and $z_1 = e_1 {\downarrow}$. But then

$$\mathbf{p} \circ z_0 = \mathbf{p}_0 {\downarrow} e_0 = d(e_0) {\downarrow} c(e_0) = d(e_1) {\downarrow} c(e_1) = \mathbf{p} \circ e_1 = \mathbf{p} \circ z_1$$

as required.

Ad 4. Note that for morphisms $\boldsymbol{\kappa}^0, \boldsymbol{\kappa}^1 : \delta^{s+1} \longrightarrow \delta^{s+1,s,s+1}$ we have $\boldsymbol{\kappa}^0 = \downarrow s \downarrow$, $\boldsymbol{\kappa}^1 = \downarrow s+1 \downarrow$. Therefore, we have

$$\boldsymbol{\rho}^0 \circ \boldsymbol{\kappa}^0 = s\downarrow = \mathbf{b} \circ \mathbf{p} \qquad \boldsymbol{\rho}^1 \circ \boldsymbol{\kappa}^0 = \downarrow s = 1$$

and

$$\boldsymbol{\rho}^0 \circ \boldsymbol{\kappa}^1 = s+1\downarrow = \mathbf{t} \circ \mathbf{p} \qquad \boldsymbol{\rho}^1 \circ \boldsymbol{\kappa}^0 = s+1\downarrow = 1$$

i.e. $\mathbf{b} \circ \mathbf{p} \leq 1 \leq \mathbf{t} \circ \mathbf{p}$, as required. ■

Thus for any $m \in \omega$ and simple ω-category $[G]$ there is a partial order $\leq$ on the set $\mathcal{S}([G], \delta^{s+1})$. Moreover, since (28) is a bundle of intervals each element has at most one immediate successor. We denote by $\lhd$ the immediate successor relation induced by this partial order. We have

4.14. Lemma. *For any $s \in \omega$ and ω-functors $z_0, z_1 : [G] \longrightarrow \delta^{s+1}$ we have $z_0 \lhd z_1$ iff there is $e \in G_s$ such that $z_0 = \downarrow e$ and $z_1 = e\downarrow$.*

Proof. We can assume that $z_0 \neq z_1$. By Lemma 4.12, $z_0 < z_1$ iff there are $e_0, e_1 \in G_s$ such that $e_0 \geq e_1$, $z_0 = \downarrow e_0 = \boldsymbol{\rho}^0 \circ \downarrow e_0..e_1 \downarrow$, and $z_1 = e_1\downarrow = \boldsymbol{\rho}^1 \circ \downarrow e_0..e_1\downarrow$. Thus $z_0 \lhd z_1$ iff $e_0 = e_1$, as required. ■

4.15. Lemma. *Let $s > 0$ and $[G]$ be a simple ω-category. Then any ω-functor $f : [G] \longrightarrow \delta^s$ is either onto and there are $e, e' \in G_{s-1}$ such that, $e \rhd e'$ and $f = (e\downarrow e')$ or f admits one of the following factorizations*

$$\begin{tikzcd} {[G]} \arrow[rr, "f"] \arrow[dr, "\downarrow"'] & & \delta^s \\ & \delta^0 \arrow[ur, "\mathbf{b}"'] & \end{tikzcd} \qquad \begin{tikzcd} {[G]} \arrow[rr, "f"] \arrow[dr, "\downarrow"'] & & \delta^s \\ & \delta^0 \arrow[ur, "\mathbf{t}"'] & \end{tikzcd} \tag{29}$$

or there are $0 < s' < s$ and $e, e' \in G_{s'-1}$ such that, $e \rhd e'$ and f admits one of the following factorizations

$$\begin{tikzcd} {[G]} \arrow[rr, "f"] \arrow[dr, "e\downarrow e'"'] & & \delta^s \\ & \delta^{s'} \arrow[ur, "\mathbf{b}"'] & \end{tikzcd} \qquad \begin{tikzcd} {[G]} \arrow[rr, "f"] \arrow[dr, "e\downarrow e'"'] & & \delta^s \\ & \delta^{s'} \arrow[ur, "\mathbf{t}"'] & \end{tikzcd} \tag{30}$$

Proof. Let G be a simple ω-graph and $f : [G] \longrightarrow \delta^s$ be an ω-graph morphism. If f is onto then the thesis follows from Lemma 4.11. Suppose that f is not onto and let s' be the the maximal number so that the ω-functor

$$f' : [G] \xrightarrow{f} \delta^s \xrightarrow{\mathbf{p}} \delta^{s'}$$

is onto. If $s' = 0$ then $f' = \downarrow$ and if $s > 0$ then, by Lemma 4.11 there is $e \in G'_s$ such that $f' = d(e)\downarrow c(e)$. Thus in either case there is $e \in G_{s'}$ such that $f'_{s'}(e) = s'$. Since $\mathbf{p}(s') = \{s'+1, s'\}$, we have that either $f_{s'}(e) = s'+1$ or $f_{s'}(e) = s'$. Then from definitions of $\mathbf{p}$, $\mathbf{b}$, and $\mathbf{t}$ follows that in the former case $f = \mathbf{t} \circ d(e)\downarrow c(e)$ and in the latter case $f = \mathbf{b} \circ d(e)\downarrow c(e)$, as required. ■

Thus, we have defined all the objects and morphisms of the following diagram $\mathbf{D}$

$$\begin{tikzcd}
 & \delta^{s+1,s,s+1} \arrow[d, shift right, "\boldsymbol{\rho}^{0;s+1}"'] \arrow[d, shift left, "\boldsymbol{\rho}^{1;s+1}"] & \delta^{s,s-1,s} \arrow[d, shift right, "\boldsymbol{\rho}^{0;s}"'] \arrow[d, shift left, "\boldsymbol{\rho}^{1;s}"] & \delta^{s-1,s-2,s-1} \arrow[d, shift right, "\boldsymbol{\rho}^{0;s-1}"'] \arrow[d, shift left, "\boldsymbol{\rho}^{1;s-1}"] & & \delta^{1,0,1} \arrow[d, shift right, "\boldsymbol{\rho}^{0;1}"'] \arrow[d, shift left, "\boldsymbol{\rho}^{1;1}"] & \\
\cdots & \delta^{s+1} \arrow[r, "\mathbf{p}^{s}"] & \delta^{s} \arrow[l, shift right=3, "\mathbf{b}^{s+1}"'] \arrow[l, shift left=3, "\mathbf{t}^{s+1}"] \arrow[r, "\mathbf{p}^{s-1}"] & \delta^{s-1} \arrow[l, shift right=3, "\mathbf{b}^{s}"'] \arrow[l, shift left=3, "\mathbf{t}^{s}"] & \cdots & \delta^{1} \arrow[r, "\mathbf{p}^{1}"] & \delta^0 \cong 1 \arrow[l, shift right=3, "\mathbf{b}^{1}"'] \arrow[l, shift left=3, "\mathbf{t}^{1}"]
\end{tikzcd}$$

in $\mathcal{S}$.

Now, we are ready to define a contravariant functor from $\mathcal{S}$ to $\mathcal{D}$. We have

4.16. Proposition. *The diagram* $\mathbf{D}$ *defined above is an internal disk in* $\mathcal{S}$.

Proof. We need to verify that the elementary axioms of an internal disk given in Appendix 6.1 are satisfied. The first for axioms holds by Lemma 4.13. Thus, it remains to verify the 'disk condition'. As we remarked earlier δ^0 is a terminal object in $\mathcal{S}$. From the proof of Theorem 4.6 follows that, for $s \in \omega$, both $\mathbf{b}^s$ and $\mathbf{t}^s$ equalizes $\mathbf{b}^{s+1}$ and $\mathbf{t}^{s+1}$. Moreover, if an ω-functor $f : [G] \longrightarrow \delta^{s+1}$, equalizes $\mathbf{b}^{s+1}$ and $\mathbf{t}^{s+1}$ then f is not onto. Therefore by Lemma 4.15, there is $f' : [G] \longrightarrow \delta^{s+1}$, such that either $f = \mathbf{b} \circ f'$ or $f = \mathbf{t} \circ f'$, i.e. 'disk condition' holds. ∎

From the proposition we get

4.17. Corollary. *The internal disk* $\mathbf{D}$ *in* $\mathcal{S}$ *induces a contravariant hom-functor*

$$\mathcal{S}(-, \mathbf{D}) : \mathcal{S}^{op} \longrightarrow \mathcal{D}.$$

Proof. Proposition 4.16, is essentially saying that hom functors send the diagram $\mathbf{D}$ in $\mathcal{S}$ into a disk in Set. To see that it is finite, it is enough to notice that, there is finitely many morphism from any ω-category S into δ^s, for $s \in \omega$. Moreover, by Lemma 4.15, for any S the numbers of such morphisms is bound by 2 times the cardinality of the simple ω-graph G, such that $[G] \simeq S$. ∎

5. The duality

5.1. Stone adjunction. In this section we show that the the contravariant hom-functors

$$\begin{tikzcd}
\mathcal{D} \arrow[r, shift left=2, "\mathcal{D}(-, \mathbf{C})"] & \mathcal{S} \arrow[l, shift left=2, "\mathcal{S}(-, \mathbf{D})"]
\end{tikzcd}$$

give rise to a Stone adjunction based on a schizophrenic object. Then, in the next section we shall show that it is in fact a duality.

First we make some easy observation, showing the connections between γ's and δ's, and between internal operations and and external morphisms on those. This explains, in what sense, the collections of δ^s, for $s \in \omega$ and the collection of γ_n, for $n \in \omega$, are equal. In fact, it is the essence of being the schizophrenic object for the adjunctions between $\mathcal{D}$ and $\mathcal{S}$.

5.2. Lemma.

1. *For any $n, s \in \omega$ we have*
$$\gamma_n^s = \delta_n^s;$$

2. *for any $s \in \omega$ and $n \in \omega^+$, the diagrams in* Set
$$\begin{tikzcd}[column sep=huge] \gamma^s_{n,n-1,n} \arrow[r, shift left=4, "\boldsymbol{\pi}^s_{0,n}"] \arrow[r, "\mathbf{m}^s_{n,n-1,n}"] \arrow[r, shift right=4, "\boldsymbol{\pi}^s_{1,n}"'] & \gamma^s_n \arrow[r, shift left=4, "\mathbf{d}^s_{n-1}"] \arrow[r, shift right=4, "\mathbf{c}^s_{n-1}"'] & \gamma^s_{n-1} \arrow[l, "\boldsymbol{\iota}^s_n"'] \end{tikzcd}$$
and
$$\begin{tikzcd}[column sep=huge] \delta^s_n \times_{\delta^s_{n-1}} \delta^s_n \arrow[r, shift left=4, "\pi^s_{0,n}"] \arrow[r, "m^s_{n,n-1,n}"] \arrow[r, shift right=4, "\pi^s_{1,n}"'] & \delta^s_n \arrow[r, shift left=4, "d^s_{n-1}"] \arrow[r, shift right=4, "c^s_{n-1}"'] & \delta^s_{n-1} \arrow[l, "\iota^s_n"'] \end{tikzcd}$$
are equal;

3. *for any $n \in \omega$ and $s \in \omega^+$, the diagrams in* Set
$$\begin{tikzcd}[column sep=huge] \delta^{s,s-1,s}_n \arrow[r, shift left=3, "\boldsymbol{\rho}^{0;s}_n"] \arrow[r, shift right=3, "\boldsymbol{\rho}^{1;s}_n"'] & \delta^s_n \arrow[r, "\mathbf{p}^{s-1}_n"] & \delta^{s-1}_n \arrow[l, shift right=4, "\mathbf{b}^s_n"'] \arrow[l, shift left=4, "\mathbf{t}^s_n"] \end{tikzcd}$$
and
$$\begin{tikzcd}[column sep=huge] \leq^s_n \arrow[r, shift left=3, "\rho^{0;s}_n"] \arrow[r, shift right=3, "\rho^{1;s}_n"'] & \gamma^s_n \arrow[r, "p^{s-1}_n"] & \gamma^{s-1}_n \arrow[l, shift right=4, "b^s_n"'] \arrow[l, shift left=4, "t^s_n"] \end{tikzcd}$$
are equal, where $\leq^s_n$ is the order on γ^s_n in the disk γ_n. ∎

This observation allow to define the following morphisms.

Let $s \in \omega$, D be a disk and $x \in D^s$. Then we can define an *evaluation ω-functor*

$$\mathrm{ev}_x : \mathcal{D}(D, \mathbf{C}) \longrightarrow \delta^s$$

such that, for $n \in \omega$,

$$\mathrm{ev}_{x,n} : \mathcal{D}(D, \gamma_n) \longrightarrow \delta^s_n$$

is a function

$$f : D \to \gamma_n \;\longmapsto f^s(x) \in \gamma^s_n = \delta^s_n$$

Let $n \in \omega$, S be a simple ω-category and $e \in S_n$. Then we can similarly define an *evaluation disk map*

$$\mathrm{ev}_e : \mathcal{S}(S, \mathbf{D}) \longrightarrow \gamma_n$$

such that, for $m \in \omega$,

$$\mathrm{ev}^s_e : \mathcal{S}(S, \delta^s) \longrightarrow \gamma^s_n$$

is a function

$$g : S \to \delta^s \;\longmapsto g_n(e) \in \delta^s_n = \gamma^s_n$$

Having these evaluation maps we can define the following two natural transformations:

$$\eta : 1_{\mathcal{D}} \longrightarrow \mathcal{S}(\mathcal{D}(-, \mathbf{C}), \mathbf{D})$$

such that, for a disk D,

$$\eta_D : D \longrightarrow \mathcal{S}(\mathcal{D}(D, \mathbf{C}), \mathbf{D})$$

is a disk map such that, for $s \in \omega$ and $x \in D^m$,

$$\eta^s_D(x) = \mathrm{ev}_x$$

$$\varepsilon : 1_{\mathcal{S}} \longrightarrow \mathcal{D}(\mathcal{S}(-, \mathbf{D}), \mathbf{C})$$

such that, for a simple ω-category S,

$$\varepsilon_S : S \longrightarrow \mathcal{D}(\mathcal{S}(S, \mathbf{D}), \mathbf{C})$$

is an ω-functor such that, for $n \in \omega$ and $e \in S_n$,

$$\varepsilon_{S,n}(e) = \mathrm{ev}_e$$

We have

5.3. LEMMA. *η and ε are well defined natural transformations.*

PROOF. We need to verify that

1. evaluation disk maps are indeed disk maps;
2. evaluation ω-functors are indeed ω-functors;
3. for any disk D, η_D is disk map;
4. for any simple ω-category S, ε_S is ω-functor;
5. η is a natural transformation;
6. ε is a natural transformation.

In the following, we shall use Lemma 5.2 many times.

Ad 1. Let D be a disk, $s \in \omega$, $x \in D^s$. We shall show that

$$\mathrm{ev}_x : \mathcal{D}(D, \mathbf{C}) \longrightarrow \delta^s$$

is an ω-functor. Let $n \in \omega$, $f : D \longrightarrow \gamma_n$. We have

$$\mathrm{ev}_{x,n-1}(\mathbf{d} \circ f) = \mathbf{d}^s_{n-1} \circ f^s(x) = d^s_{n-1}(f^s(x)) = d^s_{n-1}(\mathrm{ev}_{x,n}(f))$$

i.e. ev_x preserves domains. Preservation of codomains can be shown similarly. Moreover, we have

$$\mathrm{ev}_{x,n+1}(\boldsymbol{\iota} \circ f) = \boldsymbol{\iota}^s_{n+1} \circ d^s(x) = \iota^s_{n+1}(f^s(x)) = \iota^s_{n+1}(\mathrm{ev}_{x,n}(f))$$

i.e. ev_x preserves identities. To show that ev_x preserves compositions, let $f, g : D \longrightarrow \gamma_n$ be disk maps, such that $\mathbf{c} \circ f = \mathbf{d} \circ g$. Then, we have

$$\mathrm{ev}_{x,n}(\mathbf{m}_{n,n-1,n} \circ \langle f, g\rangle) = \mathbf{m}^s_{n,n-1,n}(f^s(x), g^s(x)) =$$

$$= m^s_{n,n-1,n}(f^s(x), g^s(x)) = m^s_{n,n-1,n}(\mathrm{ev}_{x,n}(f), \mathrm{ev}_{x,n}(g))$$

i.e. ev_x preserves compositions as well and it is an ω-functor.

Ad 2. Let S be a simple ω-category, $n \in \omega$, and $e \in S_n$. We shall show that

$$\mathrm{ev}_e : \mathcal{S}(S, \mathbf{D}) \longrightarrow \gamma_n$$

is a disk map. Let $s \in \omega^+$, $\varphi : S \longrightarrow \delta^s$ be an ω-functor. Then

$$\mathrm{ev}^{s-1}_e(\mathbf{p}^{s-1} \circ \varphi) = (\mathbf{p}^{s-1} \circ \varphi)_n(e) = p^{s-1}_n(\varphi_n(e)) = p^{s-1}_n(\mathrm{ev}^s_e(\varphi))$$

i.e. ev_e preserves projections. Moreover, for $s \in \omega$, and φ as above, we have

$$\mathrm{ev}^{s+1}_e(\mathbf{b}^{s+1} \circ \varphi) = (\mathbf{b}^{s+1} \circ \varphi)_n(e) = b^{s+1}_n(\varphi_n(e)) = b^{s+1}_n(\mathrm{ev}^s_e(\varphi))$$

i.e. ev_e preserves left end-points. For right end-points the calculations are similar. To see that ev_e preserves the order let $s \in \omega^+$ and $\varphi^0, \varphi^1 : S \longrightarrow \delta^s$ be ω-functors, such that $\varphi^0 \leq \varphi^1$, i.e. there is $\varphi : S \longrightarrow \delta^{s,s-1,s}$ and $\varphi^i = \boldsymbol{\rho}^i \circ \varphi$, for $i = 0, 1$. Then, we have

$$\mathrm{ev}_e(\varphi^0) = \varphi^0_n(e) = \boldsymbol{\rho}^0_n \circ \varphi_n(e) = \rho^0_n \circ \varphi_n(e)$$

and

$$\mathrm{ev}_e(\varphi^1) = \varphi^1_n(e) = \boldsymbol{\rho}^1_n \circ \varphi_n(e) = \rho^1_n \circ \varphi_n(e)$$

i.e. $\mathrm{ev}_e(\varphi^0) \leq \mathrm{ev}_e(\varphi^1)$ in γ^s_n and ev_e preserves the order, as well.

Ad 3. Let D be a disk. We shall show that

$$\eta_D : D \longrightarrow \mathcal{S}(\mathcal{D}(D, \mathbf{C}), \mathbf{D})$$

is a disk map. For $s \in \omega^+$

$$\eta^s_D : D \longrightarrow \mathcal{S}(\mathcal{D}(D, \mathbf{C}), \delta^s)$$

Then, for and $x \in D^s$, $n \in \omega$ and $f : D \longrightarrow \gamma_n$ we have

$$\eta^{s-1}_D(p(x))_n(f) = \mathrm{ev}_{p(x),n}(f) = f^{s-1}(p(x)) = p^{s-1}_n(f^s(x)) =$$

$$= \mathbf{p}^{s-1}_n(\mathrm{ev}_{x,n}(f)) = \mathbf{p}^{s-1}_n \circ \mathrm{ev}_{x,n}(f) = \mathbf{p}^{s-1}_n \circ \eta^s_D(x)_n(f)$$

i.e. $\eta^{s-1}_D(p(x)) = \mathbf{p}^{s-1}_n \circ \eta^s_D(x)$ and η_D preserves the projections. To see that η_D preserves the endpoints, we take $s \in \omega$ and n and f as above. We have

$$\eta^{s+1}_D(b(x))_n(f) = \mathrm{ev}_{b(x),n}(f) = f^{s+1}(b(x)) = b^{s+1}_n(f^s(x)) =$$

$$= \mathbf{b}^{s+1}_n(\mathrm{ev}_{x,n}(f)) = \mathbf{b}^{s+1}_n \circ \mathrm{ev}_{x,n}(f) = \mathbf{p}^{s+1}_n \circ \eta^s_D(x)_n(f)$$

i.e. $\eta^{s+1}_D(b(x)) = \mathbf{b}^{s+1}_n \circ \eta^s_D(x)$ and η_D preserves the left endpoints. For the right endpoints the argument is similar. Finally, let $x_0, x_1 \in D^s$, $x_0 \leq x_1$, and $f : D \longrightarrow \gamma_n$. Then

$$\eta^s_D(x_0)_n(f) = f^s(x_0) = f^s(x_1) = \eta^s_D(x_1)_n(f)$$

i.e. $\eta^s_D(x_0) \leq \eta^s_D(x_1)$, and η_D preserves the orders, so it is a disk map.

Ad 4. Let S be a simple ω-category. We shall show that

$$\varepsilon_S : S \longrightarrow \mathcal{D}(\mathcal{S}(S, \mathbf{D}), \mathbf{C})$$

is an ω-functor. For $n \in \omega$,

$$\varepsilon_{S,n} : S_n \longrightarrow \mathcal{D}(\mathcal{S}(S, \mathbf{D}), \gamma_n)$$

Fix $s \in \omega^+$, $e \in S_n$, and $\varphi : S \longrightarrow \delta^s$. Then, we have

$$\varepsilon_{S,n-1}(d(e))^s(\varphi) = \varphi_{n-1}(d(e)) = d^s_{n-1}(\varphi_s(e)) =$$

$$= \mathbf{d}^s_{n-1}(\varphi_n(e)) = \mathbf{d}^s_{n-1}(\varepsilon_{S,n-1}(e)^s)(\varphi)) = (\mathbf{d}_{n-1}(\varepsilon_{S,n-1}(e))^s(\varphi))$$

i.e. ε_S preserves the domains. The argument for preservation of codomains is similar. To show that ε_S preserves the identities we take $n \in \omega$ and s, e, φ as above. We have

$$\varepsilon_{S,n+1}(\iota(e))^s(\varphi) = \varphi_{n+1}(\iota(e)) = \iota^s_{n+1}(\varphi_n(e)) =$$

$$= \boldsymbol{\iota}^s_{n+1}(\varepsilon_{S,n}(e))^s(\varphi)) = (\boldsymbol{\iota}^s_{n+1} \circ \varepsilon_{S,n}(e))^s(\varphi))$$

i.e. $\varepsilon_{S,n+1} \circ \iota = \boldsymbol{\iota}^s_{n+1} \circ \varepsilon_{S,n}$ and ε_S preserves identities. For preservation of compositions, let $e, e' \in S_n$, such that $c(e) = d(e')$. Then

$$\varepsilon_{S,n}(m(e,e'))^s(\varphi) = \varphi_n(m(e,e')) = m^s_{n,n-1,n}(\varphi_n(e), \varphi_n(e')) =$$

$$= \mathbf{m}^s_{n,n-1,n}(\varphi_n(e), \varphi_n(e')) = \mathbf{m}^s_{n,n-1,n}(\varepsilon_{S,n}(e)^s(\varphi), \varepsilon_{S,n}(e')^s(\varphi)) =$$

$$= (\mathbf{m}^s_{n,n-1,n} \circ \langle \varepsilon_{S,n}(e)^s(\varphi), \varepsilon_{S,n}(e')\rangle)^s(\varphi)$$

i.e. $\varepsilon_{S,n}(m(e,e')) = \mathbf{m}^s_{n,n-1,n} \circ \langle \varepsilon_{S,n}(e)^s(\varphi), \varepsilon_{S,n}(e')\rangle$ and the compositions are preserved as well. Hence ε_S is an ω-functor indeed, as required.

Ad 5. Let $f : D \longrightarrow E$ be a disk map. We need to show that the square

$$\begin{tikzcd} S \arrow[r, "\eta_D"] \arrow[d, "f"'] & \mathcal{S}(\mathcal{D}(D,\mathbf{C}),\mathbf{D}) \arrow[d, "\mathcal{S}(\mathcal{D}(f,\mathbf{C}),\mathbf{D})"] \\ E \arrow[r, "\eta_E"'] & \mathcal{S}(\mathcal{D}(E,\mathbf{C}),\mathbf{D}) \end{tikzcd}$$

commutes. Let $n, s \in \omega$, $x \in D^s$ and $g : E \longrightarrow \gamma_n$ a disk map. Then, we have

$$(\mathcal{S}(\mathcal{D}(f,\mathbf{C}),\mathbf{D}) \circ \eta_D)(x)_n(g) = (\mathcal{S}(\mathcal{D}(f,\mathbf{C}),\mathbf{D})(\eta^s_D(x)))_n(g) =$$

$$(\eta^s_D(x) \circ \mathcal{D}(f,\mathbf{C}))_n(g) = \mathrm{ev}x, n \circ \mathcal{D}(f,\mathbf{C})_n(g) = \mathrm{ev}_{x,n}(g \circ f) = (g \circ f)^s(x) =$$

$$= (g^s(f^s(x)) = \mathrm{ev}_{f(x),n}(g) = \eta_E(f(x))_n(g) = ((\eta_E \circ f)(x))_n(g)$$

i.e. the above square commutes.

Ad 6. Let $\varphi : E \longrightarrow T$ be an ω-functor. We need to show that the square

$$\begin{tikzcd} S \arrow[r, "\varepsilon_S"] \arrow[d, "\varphi"'] & \mathcal{D}(\mathcal{S}(S,\mathbf{D}),\mathbf{C}) \arrow[d, "\mathcal{D}(\mathcal{S}(\varphi,\mathbf{D}),\mathbf{C})"] \\ T \arrow[r, "\varepsilon_T"'] & \mathcal{D}(\mathcal{S}(T,\mathbf{D}),\mathbf{C}) \end{tikzcd}$$

commutes. Let $n, s \in \omega$, $e \in S_n$ and $\psi : T \longrightarrow \delta^s$ an ω-functor. Then, we have

$$(\mathcal{D}(\mathcal{S}(\varphi,\mathbf{D}),\mathbf{C}) \circ \varepsilon_S)_n(e))^s(\psi) = (\mathcal{D}(\mathcal{S}(\varphi,\mathbf{D}),\mathbf{C})_n(\varepsilon_{S,n}(e))^s(\psi) =$$

$$= \mathrm{ev}^s_e \circ \mathcal{S}(\varphi,\mathbf{D})^s(\psi) = \mathrm{ev}^s_e(\psi \circ \varphi) = \psi_n \circ \varphi_n(e) =$$
$$= \mathrm{ev}^s_{\varphi_n(e)}(\psi) = \varepsilon_{T,n}(\varphi_n(e))^s(\psi) = (\varepsilon_T \circ \varphi)_n(e)^s(\psi)$$

i.e. the above square commutes.

This ends the proof of the Lemma. ∎

5.4. PROPOSITION. *The natural transformations η and ε are the unit and counit of the Stone adjunction*

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left, "\mathcal{D}(-,\mathbf{C})"] & \mathcal{S} \arrow[l, shift left, "\mathcal{S}(-,\mathbf{D})"] \end{tikzcd}$$

PROOF. We need to verify that the following triangles

$$\begin{tikzcd} & \mathcal{D}(\mathcal{S}(\mathcal{D}(-,\mathbf{D}),\mathbf{C}),\mathbf{D}) \arrow[dr, "\mathcal{D}(\eta,\mathbf{C})"] & \\ \mathcal{D}(-,\mathbf{C}) \arrow[ur, "\varepsilon_{\mathcal{D}(-,\mathbf{C})}"] \arrow[rr, "1_{\mathcal{D}(-,\mathbf{C})}"'] & & \mathcal{D}(-,\mathbf{C}) \end{tikzcd} \tag{31}$$

and

$$\begin{tikzcd} & \mathcal{S}(\mathcal{D}(\mathcal{S}(-,\mathbf{D}),\mathbf{C}),\mathbf{D}) \arrow[dr, "\mathcal{S}(\varepsilon,\mathbf{D})"] & \\ \mathcal{S}(-,\mathbf{D}) \arrow[ur, "\eta_{\mathcal{S}(-,\mathbf{D})}"] \arrow[rr, "1_{\mathcal{S}(-,\mathbf{D})}"'] & & \mathcal{S}(-,\mathbf{D}) \end{tikzcd} \tag{32}$$

commute.

For commutation of (31), let D be a disk, $n, s \in \omega$, $f : D \longrightarrow \gamma_n$ a disk map, and $x \in D^s$. Then

$$((\mathcal{D}(\eta_D, \mathbf{C}) \circ \varepsilon_{\mathcal{D}(D,\mathbf{C})})_n(f))^s(x) =$$

$$= (\mathcal{D}(\eta_D, \mathbf{C})_n(\mathrm{ev}_f))^s(x) = (\mathrm{ev}_f \circ \eta_D)^s(x) =$$

$$= \mathrm{ev}_f^s(\mathrm{ev}_x) = (\mathrm{ev}_x)_n(f) = f^s(x)$$

i.e. the triangle (31) commutes.

For commutation of (32), let S be a simple ω-category, $n, s \in \omega$, $\varphi : S \longrightarrow \delta^s$ an ω-functor, and $e \in S_n$. Then

$$((\mathcal{S}(\varepsilon_S, \mathbf{D}) \circ \eta_{\mathcal{S}(S,\mathbf{D})})^s(\varphi))_n(e) =$$

$$= (\mathcal{S}(\varepsilon_S, \mathbf{D})^s(\mathrm{ev}_\varphi))_n(e) = \mathrm{ev}_{\varphi,n} \circ \varepsilon_{D,n}(e) =$$

$$\mathrm{ev}_{\varphi,n}(\mathrm{ev}_e) = (\mathrm{ev}_e)^s(\varphi) = \varphi(e)$$

i.e. the triangle (32) commutes. ■

5.5. The duality theorem. This section is devoted to the proof of the main theorem. We show that the Stone adjunction that we established in the previous section is an equivalence of categories. At the end, we list some specific corresponding objects and morphisms in both categories.

5.6. Lemma. *Let $s \in \omega$, D be a disk and $z \in D^s$. Then the ω functors ev_z and $\overline{;z}{\downarrow}\overline{z;}$, from $\mathcal{D}(D, \mathbf{C})$ to δ^s, are equal.*

Proof. Fix $s \in \omega$ and $z \in D^s$. Then, we have disk morphisms $\overline{;z}$, $\overline{z;} : D \longrightarrow \gamma_{s-1}$, such that $\overline{;z} \rhd \overline{z;}$ and an ω-functors

$$\overline{;z}{\downarrow}\overline{z;}\,,\ \mathrm{ev}_z : \mathcal{D}(D, \mathbf{C}) \longrightarrow \delta^s$$

We have

$$\overline{;z}^s(z) = s \qquad\qquad \overline{z;}^s(z) = s + 1$$

Then

$$\overline{;z}{\downarrow}\overline{z;}_{s-1}(\overline{;z}) = s + 1 = \overline{;z}^s(z) = (\mathrm{ev}_z)_{s-1}(\overline{;z})$$

and

$$\overline{;z}{\downarrow}\overline{z;}_{s-1}(\overline{z;}) = s = \overline{z;}^s(z) = (\mathrm{ev}_z)_{s-1}(\overline{z;})$$

Hence, by Lemma4.11 we get that $\overline{;z}{\downarrow}\overline{z;} = \mathrm{ev}_z$, as required. ■

We have

5.7. PROPOSITION. *For every disk D, the disk map*

$$\eta_D : D \longrightarrow \mathcal{S}(\mathcal{D}(D, \mathbf{C}), \mathbf{D})$$

is an isomorphism.

PROOF. Let D be a disk. First we shall show that η_D is mono. Fix $n \in \omega$ and $x, y \in D^n$ such that $x \neq y$. Then $l = \mu_{x,y} < n$ and the nodes $p^{(l+1)}(x)$ and $p^{(l+1)}(y)$ are comparable but different. Suppose $p^{(l+1)}(x) < p^{(l+1)}(y)$. Thus, with the disk morphism $\overline{x; y} : D \longrightarrow \gamma_l$ in $\mathcal{D}(D, \gamma_l)$, we have

$$\eta^n_D(x)_l(\overline{x; y}) = \mathrm{ev}_{x,l}(\overline{x; y}) = \overline{x; y}_n(x) = l \neq l + 1 =$$

$$= \overline{x; y}_n(y) = \mathrm{ev}_{y,l}(\overline{x; y}) = \eta^n_D(y)_l(\overline{x; y})$$

so, $\eta^n_D(x) \neq \eta^n_D(y)$, i.e. η_D mono, as required.

It remains to show that η_D is epi. Let $s \in \omega$, and $f : \mathcal{D}(D, \mathbf{C}) \longrightarrow \delta^s$ be an ω-functor.

If $s = 0$ then $f = \downarrow$ and $\eta^0_D(0) = \downarrow = f$.

If $s > 0$ and there are $e, e' \in \mathrm{Cut}(D)_{s-1}$ such that $e \rhd e'$ and $f = e{\downarrow}e'$. By, Corollary 3.29, there is $z \in D^m$ such that $e = \overline{;z}$ and $e' = \overline{z;}$. Thus $e{\downarrow}e' = \overline{;z}{\downarrow}\overline{z;}$. So, by Lemma 5.6,

$$\eta^s_D(z) = \mathrm{ev}_z = \overline{;z}{\downarrow}\overline{z;} = e{\downarrow}e' = f$$

Now, if $f : \mathcal{D}(D, \mathbf{C}) \longrightarrow \delta^s$ is an arbitrary ω-functor, by Lemma 4.15, it can be factorized as $\mathbf{b}$ or $\mathbf{t}$ followed by an ω-functor considered above, i.e. in the image of η_D. But η_D is a disk morphism so it preserves endpoints i.e. operations b and t. Hence, any morphism $f : \mathcal{D}(D, \mathbf{C}) \longrightarrow \delta^s$ is in the image of η_D, so η_D is epi, as required. ■

5.8. LEMMA. *Let $n \in \omega$, G a simple ω-graph, and $e \in G_n$.*

1. *Then the disk map*
$$\mathrm{ev}_e : \mathcal{S}([G], \mathbf{D}) \longrightarrow \gamma_n$$
is outer;

2. *the disk maps ev_e and $\overline{{\downarrow}e; e{\downarrow}}$ are equal.*

PROOF. Ad 1. This is an immediate corollary from Lemmas 3.22.4(c) and 4.11.

Ad 2. First note that by 1. and Lemma 3.22 both morphisms ev_e and $\overline{{\downarrow}e; e{\downarrow}}$ are outer. Moreover, we have

$$(\overline{{\downarrow}e; e{\downarrow}})^{n+1}({\downarrow}e) = n = ({\downarrow}e)_n(e) = (\mathrm{ev}_e)^{n+1}({\downarrow}e)$$

and

$$(\overline{{\downarrow}e; e{\downarrow}})^{n+1}(e{\downarrow}) = n + 1 = (e{\downarrow})n(e) = (\mathrm{ev}_e)^{n+1}(e{\downarrow})$$

By, Proposition 3.18.3, $\overline{{\downarrow}e; e{\downarrow}}$ is the unique morphism with this property so $\mathrm{ev}_e = \overline{{\downarrow}e; e{\downarrow}}$, as required. ■

5.9. Proposition. *For every simple ω-category S the ω-functor*

$$\varepsilon_S : S \longrightarrow \mathcal{D}(\mathcal{S}(S, \mathbf{D}), \mathbf{C})$$

is an equivalence of ω-categories.

Proof. By Proposition 4.8, we can assume that $S = [G]$ for some simple ω-graph G. Thus we will show that for any $n \in \omega$ the function

$$\varepsilon_{[G],n} : [G]_n \longrightarrow \mathcal{D}(\mathcal{S}([G], \mathbf{D}), \gamma_n).$$

is a bijection.

First we shall show that $\varepsilon_{[G]}$ is mono. Let $m \in \omega$, $\sigma, \sigma' \in [G]_m$ and $\sigma \neq \sigma'$. Then, there is $s' \leq s$ and $e \in G_{s'}$ such that either $e \sqsubseteq \sigma$ and $e \not\sqsubseteq \sigma'$ or $e \not\sqsubseteq \sigma$ and $e \sqsubseteq \sigma'$. We can assume that s' is minimal such that $e \sqsubseteq \sigma$ and $e \not\sqsubseteq \sigma'$. By minimality of s', we have that $d(e)$, $c(e) \sqsubseteq \sigma'$. Then, there is $e' \in G_{s'}$ such that $d(e) = d(e')$ and $c(e) = c(e')$. Then, either $e > e'$ or $e' > e$. Since both cases are similar, we consider the case $e > e'$. We have

$$(e{\downarrow})_s(\sigma) \;=\; \begin{cases} s'+1 & \text{if there is } e'' \in G_{s'+1} \\ & \quad \text{such that } d^{s'}(e'') = e \text{ and } e' \sqsubseteq \sigma \\ \delta_s^{s'+1} - s' - 1 & \text{otherwise} \end{cases}$$

Thus $(e{\downarrow})_s(\sigma) > s'$. By Lemma 4.2, if $e'' \in G_{s'}$, $e'' \sqsubseteq \sigma'$ and $e'' \perp e$ then $e > e''$. Therefore $(e{\downarrow})_s(\sigma') = s'$ and

$$(\mathrm{ev}_\sigma)^{s'+1}(e{\downarrow}) = (e{\downarrow})_s(\sigma) > s' = (e{\downarrow})_s(\sigma') = (\mathrm{ev}_{\sigma'})^{s'+1}(e{\downarrow})$$

i.e. $(\mathrm{ev}_\sigma) \neq (\mathrm{ev}_{\sigma'})$ and $\varepsilon_{[G]}$ is mono, as required.

Next, we show that $\varepsilon_{[G]}$ is onto. Let $f : \mathcal{S}([G], \mathbf{D}) \longrightarrow \gamma_n$ be a disk map. In the following diagram

$$\begin{tikzcd}
 & \mathcal{S}(S, \mathbf{D}) \arrow[r, "f"] \arrow[dl, "\overline{x;y}"'] \arrow[d, "\mathring{f}"] & \gamma_n \\
\gamma_{u_{2i}} & \gamma_{\vec{u}} \arrow[l, "\pi_i"] \arrow[r, "\mathbf{m}_{\vec{u}}"'] & \gamma^u \arrow[u, "\iota_n"']
\end{tikzcd}$$

the square is the canonical factorization of f. Thus f is obtained by the operation of composition $\mathbf{m}_{\vec{u}}$ and identity ι_n from the compatible tuple $\mathring{f}$ of D-cuts, $\pi_i \circ \mathring{f}$, for $i \in \mathrm{lh}(\vec{u})$. Since $\varepsilon_{[G]}$ is an ω-functor it preserves compositions and identities it is enough to show that the tuple $\mathring{f}$ is in the image of $\varepsilon_{[G]}$. But $\varepsilon_{[G]}$ is mono so it both preserves and reflects domains and codomains operations. Thus it is enough to show that for $i \in \mathrm{lh}(\vec{u})$, $\pi_i \circ \mathring{f} : \mathcal{S}([G], \mathbf{D}) \longrightarrow \gamma_{u_{2i}}$ is in the image of $\varepsilon_{[G]}$.

Fix $i \in \mathrm{lh}(\vec{u})$. Since both $\overset{\circ}{f}$ and π_i are outer so is the composition. Hence there are ω-functors $x, y : [G] \longrightarrow \delta^{u_{2i}+1}$ such that $x \lhd y$ and $\overline{x; y} = \pi_i \circ \overset{\circ}{f}$. By Lemma 4.14 there is $e \in G_{u_{2i}}$ such that $x = \downarrow e$ and $y = e\downarrow$. Thus, by Proposition 5.8.2, we have

$$\pi_i \circ \overset{\circ}{f} = \overline{\downarrow e; e\downarrow} = \mathrm{ev}_e = \varepsilon_{[G],u_{2i}}(e)$$

i.e. $\pi_i \circ \overset{\circ}{f}$ is in the image of $\varepsilon_{[G]}$, and hence $\varepsilon_{[G]}$ is epi, as required. ■

5.10. Theorem. *The Stone adjunction*

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left, "\mathcal{D}(-,\mathbf{C})"] & \mathcal{S} \arrow[l, shift left, "\mathcal{S}(-,\mathbf{D})"] \end{tikzcd}$$

is an equivalence of categories.

Proof. The theorem follows from Propositions 5.4, 5.7, 5.9. ■

In the table below we show how objects and some morphisms in $\mathcal{S}$ and $\mathcal{D}$ correspond one another via the above duality. We need two notions concerning simple ω-categories. We say, that an ω-fumctor $g : S \longrightarrow T$ between simple ω-categories is an *immersion* iff it sends indecomposable[1] cells to indecomposable cells. Let $\mathrm{ht}(S) = u \geq v = \mathrm{ht}(T)$. We say that g is *essential* iff $g(\mathrm{mac}_S) = \iota_{(u)}(\mathrm{mac}_T)$.

The morphisms $\boldsymbol{\kappa}^i$, $\boldsymbol{\kappa}^{0;\vec{u}}$, $\boldsymbol{\kappa}^{1;\vec{v}}$, $\mathbf{b}$, $\mathbf{t}$, $\mathbf{b}^{\vec{u};l}$, $\mathbf{t}^{\vec{u};l}$ are typical immersions. The morphism $\mathbf{p}$ is an essential epi. By $\boldsymbol{\varrho}^{\vec{u}} : \delta^u \longrightarrow \delta^{\vec{u}}$, where $u = \mathrm{ht}(\vec{u})$, we denote the unique essential epimorphism sending the maximal cell $\mathrm{mac}_{\delta^u} = u \in \delta^u_u$ to the maximal cell $\mathrm{mac}_{\delta^{\vec{u}}}$ of $\delta^{\vec{u}}$. Thus, for $n \in \omega$, $\boldsymbol{\varrho}^{n+1,n,n+1} = \boldsymbol{\rho}^{n+1}$.

In the following, we denote by $(-)^*$ both dualising functors $\mathcal{D}(-,\mathbf{C})$ and $\mathcal{S}(-,\mathbf{D})$.

In the table $k, l, n \in \omega$ and $\vec{u}$, $\vec{v}$ are ud-vectors, $u = \mathrm{ht}(\vec{u})$, and if necessary $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{u})$.

[1]Recall that a cell a is indecomposable if it is not an identity of some other cell and if $a = a' \circ a'$, then either a' of a'' is an identity.

in $\mathcal{D}$	in $\mathcal{S}$
γ_n	δ^n
$\gamma_{\vec{u}}$	$\delta^{\vec{u}}$
$\boldsymbol{\pi}_i : \gamma_{\vec{u}} \longrightarrow \gamma_{u_{2i}}$	$\boldsymbol{\kappa}^i : \delta^{u_{2i}} \longrightarrow \delta^{\vec{u}}$
$\boldsymbol{\pi}_{0;\vec{u}} : \gamma_{[\vec{u},l,\vec{v}]} \longrightarrow \gamma_{\vec{u}}$	$\boldsymbol{\kappa}^{0;\vec{u}} : \delta^{\vec{u}} \longrightarrow \delta^{[\vec{u},l,\vec{v}]}$
$\boldsymbol{\pi}_{1;\vec{v}} : \gamma_{[\vec{u},l,\vec{v}]} \longrightarrow \gamma_{\vec{v}}$	$\boldsymbol{\kappa}^{1;\vec{v}} : \delta^{\vec{v}} \longrightarrow \delta^{[\vec{u},l,\vec{v}]}$
$\mathbf{d} : \gamma_n \longrightarrow \gamma_k$	$\mathbf{t} : \delta^k \longrightarrow \delta^n$
$\mathbf{c} : \gamma_n \longrightarrow \gamma_k$	$\mathbf{b} : \delta^k \longrightarrow \delta^n$
$\mathbf{d}_{\vec{u};l} : \gamma_{\vec{u}} \longrightarrow \gamma_{\mathrm{tr}_{(l)}(\vec{u})}$	$\mathbf{t}^{\vec{u};l} : \delta^{\mathrm{tr}_{(l)}(\vec{u})} \longrightarrow \delta^{\vec{u}}$
$\mathbf{c}_{\vec{u};l} : \gamma_{\vec{u}} \longrightarrow \gamma_{\mathrm{tr}_{(l)}(\vec{u})}$	$\mathbf{b}^{\vec{u};l} : \delta^{\mathrm{tr}_{(l)}(\vec{u})} \longrightarrow \delta^{\vec{u}}$
$\boldsymbol{\iota} : \gamma_k \longrightarrow \gamma_n$	$\mathbf{p} : \delta^n \longrightarrow \delta^k$
$\mathbf{m}_{n+1,n,n+1} : \gamma_{n+1,n,n+1} \longrightarrow \gamma_{n+1}$	$\boldsymbol{\varrho} : \delta^{n+1} \longrightarrow \delta^{n+1,n,n+1}$
$\mathbf{m}_{\vec{u}} : \gamma_{\vec{u}} \longrightarrow \gamma_u$	$\boldsymbol{\varrho}^{\vec{u}} : \delta^u \longrightarrow \delta^{\vec{u}}$
$f : D \to E$ is outer	$f^* : E^* \to D^*$ is an immersion
$f : D \to E$ is inner	$f^* : E^* \to D^*$ is essential

We shall make some comments about the above table. It is easy to see that

$$\mathrm{Cut}(\gamma_n)_k \quad = \quad \begin{cases} \emptyset & \text{if } k > n \\ 1_{\gamma_n} & \text{if } k = n \\ \{d_k \geq c_k\} & \text{if } 0 \leq k < n \end{cases}$$

i.e. $\mathrm{Cut}(\gamma_n)$ is isomorphic to α_n, by isomorphism sending 1_{γ_n} to $2n$. But the ω-categories $\mathcal{D}(\gamma_n, \mathbf{C})$ and δ^n are free over the ω-graphs $\mathrm{Cut}(\gamma_n)$ and α_n, respectively. Thus they are isomorphic via the unique isomorphisms sending 1_{γ_n} to n. Hence $(\gamma_n)^* \cong \delta^n$.

The morphism $\mathbf{t} : \alpha^n \longrightarrow \alpha^{n+1}$ is determined by the condition $\mathbf{b}_n(2n) = 2n + 1$ so it corresponds via the above isomorphism of the ω-graphs to the morphism $\mathrm{Cut}(\gamma_n) \longrightarrow \mathrm{Cut}(\gamma_{n+1})$ sending 1_{γ_n} to $\mathbf{d} : \gamma_{n+1} \longrightarrow \gamma_n$. Thus the square

$$\begin{tikzcd} \mathcal{D}(\gamma_n, \mathbf{C}) \arrow[r, "\cong"] \arrow[d, "..\circ \mathbf{d}"'] & \delta^n \arrow[d, "\mathbf{t}"] \\ \mathcal{D}(\gamma_{n+1}, \mathbf{C}) \arrow[r, "\cong"'] & \delta^{n+1} \end{tikzcd}$$

commute, i.e. $\mathbf{d}^* \cong \mathbf{t}$. Similarly we can prove that $\mathbf{c}^* \cong \mathbf{b}$. The object $\gamma_{\vec{u}}$ is a limit of a diagram in which the morphism are of form $\mathbf{c}$ and $\mathbf{d}$ and $\delta^{\vec{u}}$ is a colimit of the dual of that diagram, see pages 156 and 186. Thus $(\gamma_{\vec{u}})^* \cong \delta^{\vec{u}}$. Since the projections from the limit correspond to coprojections into the colimit, we also have $(\boldsymbol{\pi}_i)^* \cong \boldsymbol{\kappa}_i$.

The morphism $\mathcal{D}(\boldsymbol{\iota}, \mathbf{C}) : \mathcal{D}(\gamma_{n+1}, \mathbf{C}) \longrightarrow \mathcal{D}(\gamma_n, \mathbf{C})$ sends both $\mathbf{c}$, $\mathbf{d} : \gamma_{n+1} \longrightarrow \gamma_n$ to $1_{\gamma_n} = \mathbf{c} \circ \boldsymbol{\iota} = \mathbf{d} \circ \boldsymbol{\iota}$. So the diagram

$$\begin{tikzcd} \mathcal{D}(\gamma_{n+1},\mathbf{C}) \arrow[r, "\cong"] \arrow[d, "\mathcal{D}(\boldsymbol{\iota},\mathbf{C})"'] & \delta^{n+1} \arrow[d, "\mathbf{p}"] \\ \mathcal{D}(\gamma_n,\mathbf{C}) \arrow[r, "\cong"'] & \delta^n \end{tikzcd}$$

commutes, i.e. $\boldsymbol{\iota}^* \cong \mathbf{p}$.

The morphism $\mathbf{m}_{\vec{u}} : \gamma_{\vec{u}} \longrightarrow \gamma_u$, is the maximal arrow in $\mathcal{D}(\gamma_{\vec{u}},\mathbf{C})$, so the square

$$\begin{tikzcd} \mathcal{D}(\gamma_u,\mathbf{C}) \arrow[r, "\cong"] \arrow[d, "\mathcal{D}(\mathbf{m}_{\vec{u}},\mathbf{C})"'] & \delta^u \arrow[d, "\boldsymbol{\varrho}^{\vec{u}}"] \\ \mathcal{D}(\gamma_{\vec{u}},\mathbf{C}) \arrow[r, "\cong"'] & \delta^{\vec{u}} \end{tikzcd}$$

commutes as both morphism $\mathcal{D}(\mathbf{m}_{\vec{u}},\mathbf{C})$ and $\boldsymbol{\varrho}^{\vec{u}}$ sends the maximal arrow to the maximal arrows. Thus $(\mathbf{m}_{\vec{u}})^* \cong \boldsymbol{\rho}^{\vec{u}}$.

Let D be a disk. Then, an n-cell $a : D \longrightarrow \gamma_n$ in D^* is indecomposable iff a is an outer disk morphism. The maximal cell D^* is the unique inner disk morphism
$mac_{D^*} : D \longrightarrow \gamma_{\mathrm{ht}(D)}$.

Let $f : D \longrightarrow E$ be a disk morphism, and $\mathrm{ht}(D) = v \leq u = \mathrm{ht}(E)$. Recall that the inner morphism from D to γ_u is unique. Moreover, if $g \circ f$ is an inner morphism so is f. Consider the following diagram

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\mathrm{mac}_{D^*}"'] & E \arrow[d, "\mathrm{mac}_{E^*}"] \\ \gamma_v \arrow[r, "\iota_{(u)}"'] & \gamma_u \end{tikzcd} \tag{33}$$

We have:

$$f : D \to E \text{ is inner}$$

iff

$$\text{the square (33) is commutes}$$

iff

$$f^*(\mathrm{mac}_{E^*}) = \iota_{(u)}(\mathrm{mac}_{D^*})$$

iff

$$f^* : E^* \to D^* \text{ is essential}$$

Since outer morphisms do compose, f^* is an immersion whenever f is an outer morphism.

Finally, assume that $g : S \longrightarrow T$ is an immersion of simple ω-categories, $n > 0$, and $x, y : T \longrightarrow \delta^n$ are two nodes of the disk T^*, such that $x \circ g = y \circ g$ is an inner node of

S^*. Then $x \circ g$ has a splitting $z : \delta^n \longrightarrow S$, such that $x \circ g \circ z = id_{\delta^n} = y \circ g \circ z$, and moreover $z_n(n)$ is an indecomposable cell. Hence

$$x \circ g = d(z_n(n)) \downarrow c(z_n(n)) = y \circ g.$$

Since f preserves indecomposable cells, we have

$$x = g(d(z_n(n))) \downarrow g(c(z_n(n))) = y.$$

Thus g^* is an outer morphism.

5.11. Closing remarks. In this final section, we shall show that the notion of a disk plays a similar role for ω-categories to the role the notion of an interval plays for categories. In picture

$$\frac{\mathcal{D}}{\omega\text{-categories}} = \frac{\mathcal{I}}{\text{categories}}$$

The syntactic category for ω-categories

Since the theory of ω-categories is a finite limit theory it has an obvious syntactic category, the opposite of the category of the finitely presented ω-categories. However, there is a smaller category which is much easier to describe and contains all the necessary syntactic data. We shall describe below what we mean by that.

Let $\mathbf{S}_\omega$ be the sketch for the notion of ω-category, which is the extension of the one defined in Appendix 6.2 by objects $O_{\vec{u}}$ and cones stating that they are multipullbacks of O_{u_i}'s. Clearly, it is still true that, for any category $\mathcal{C}$ with finite limits, the category of internal ω-categories in $\mathcal{C}$, $\omega\mathrm{Cat}(\mathcal{C})$ is (equivalent to) the category of models of $\mathbf{S}_\omega$ in $\mathcal{C}$, $\mathrm{Mod}_{\mathcal{C}}(\mathbf{S}_\omega)$. Such an extension is natural in the sense that any 'reasonable' morphism definable in the theory of ω-categories has $O_{\vec{u}}$ as its domain, for some ud-vector $\vec{u}$. For example, $O_n \times O_k$ is not a 'reasonable' type since any definable function from $O_n \times O_k$ to O_l must be a projection followed by a definable function, i.e. there is no operation involving two (or more) cells without a specification on what level the codomain of the first cell match the domain of the second cell. $O_{\vec{u}}$ is a type of tuples of cells with a prescribed matching condition.

We can set a formal system to derive terms in contexts of form

$$\vec{x} \,:\, O_{\vec{u}} \,\vdash\, \vec{t} \,:\, O_{\vec{v}} \tag{34}$$

and to derive equations of terms in contexts of form

$$\vec{x} \,:\, O_{\vec{u}} \,\vdash\, \vec{t} = \vec{r} \,:\, O_{\vec{v}} \tag{35}$$

Intuitively (34) means that $\vec{x}$ is a sequence of variable, the variable x_i has type $O_{u_{2i}}$, the variables x_i and x_{i+1} match in the sense that $c_{(u_{2i+1})}(x_i) = d_{(u_{2i+1})}(x_{i+1})$. Moreover, $\vec{t}$ is a sequence of matching terms t_i of type $O_{v_{2i}}$ ($c_{(v_{2i+1})}(t_i) = d_{(v_{2i+1})}(t_{i+1})$) over $\mathbf{S}_\omega$, i.e. it is build from variables in $\vec{x}$ and the operations of domains, codomains, composition, and identity. (35) means that the theory of ω-categories proves that the terms $\vec{t}$ and $\vec{r}$ are equal.

Using this, we can define a syntactic category $\mathbf{T}_\omega$ with objects $O_{\vec{u}}$ for any ud-vector $\vec{u}$ and whose morphism from $O_{\vec{u}}$ to $O_{\vec{v}}$ are terms in context (34) divided by equations (35). The identity on $O_{\vec{u}}$ is

$$\vec{x} \ : \ O_{\vec{u}} \ \vdash \ \vec{x} \ : \ O_{\vec{u}}$$

and the composition is defined by substitution in the obvious way.

Each term (34) corresponds to a morphism in $\mathcal{S}$

$$\|\vec{x} \vdash \vec{t}\| : \delta^{\vec{v}} \longrightarrow \delta^{\vec{u}} \tag{36}$$

The definition of $\| - \|$ can be given along with term forming rules. If $\vec{v} = v_0$ then

$$\|\vec{x} \vdash t_0\| : \delta^{v_0} \longrightarrow \delta^{\vec{u}}$$

sends the only non-identity v_0-cell u_0 in $\delta^{v_0}_{v_0}$ to the value of the term t_0 in $\delta^{\vec{u}}$, where the variables in $\vec{x}$ are interpreted as the obvious cells generating $\delta^{\vec{u}}$. For vectors $\vec{v}$ of larger length we can use the fact that $\delta^{\vec{v}}$ is a colimit of δ^{v_i}'s.

Using the fact that all the ω-categories $\delta^{\vec{u}}$ are free, we obtain

5.12. Proposition. *For any pair of tuples of terms $\vec{t}$, $\vec{r}$, we have*

$$\vec{x} : O_{\vec{u}} \ \vdash \ \vec{t} = \vec{r} \ : \ O_{\vec{v}} \quad \text{iff} \quad \|\vec{x} \vdash \vec{t}\| \ = \ \|\vec{x} \vdash \vec{r}\| \ \ \Box$$

The above proposition allow us to define a functor

$$\| - \| : \mathbf{T}_\omega \longrightarrow \mathcal{S}^{op} \tag{37}$$

sending equivalence class of (34) to a morphism (36). It can be shown that the functor $\| - \|$ is an equivalence of categories. Since the diagram

$$\begin{tikzcd}
 & & & \gamma_{\vec{u}} \arrow[dlll, "\boldsymbol{\pi}_0"'] \arrow[dl, "\boldsymbol{\pi}_2"'] \arrow[dr, "\boldsymbol{\pi}_{2k-2}"] \arrow[drrr, "\boldsymbol{\pi}_{2k}"] & & & \\
\gamma_{u_0} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_2} \arrow[dl, "\mathbf{d}"] & \cdots & \gamma_{u_{2k-2}} \arrow[dr, "\mathbf{c}"'] & & \gamma_{u_{2k}} \arrow[dl, "\mathbf{d}"] \\
 & \gamma_{u_1} & & & & \gamma_{u_{2k-1}} &
\end{tikzcd} \tag{38}$$

is a limiting cone in $\mathcal{D}$ (we call it a *special pullback*), the corresponding diagram in $\mathbf{T}_\omega$

$$\begin{tikzcd}
 & & & O_{\vec{u}} \arrow[dlll, "{[\vec{x} \vdash x_0]}"'] \arrow[dl] \arrow[dr] \arrow[drrr, "{[\vec{x} \vdash x_k]}"] & & & \\
O_{n_0} \arrow[dr, "{[x_0 \vdash c(x_0)]}"'] & & O_{u_2} \arrow[dl, "{[x_1 \vdash d(x_1)]}"] & \cdots & O_{u_{2k-2}} \arrow[dr] & & O_{u_{2k}} \arrow[dl, "{[x_k \vdash d(x_k)]}"] \\
 & O_{u_1} & & & & O_{u_{2k-1}} &
\end{tikzcd} \tag{39}$$

is a limiting cone, as well. We have an obvious sketch morphism

$$|-|:\mathbf{S}_\omega \longrightarrow \mathbf{T}_\omega \tag{40}$$

which is identity on objects and sends

$$m_{n_0,n_1,n_2}: O_{n_0,n_1,n_2} \longrightarrow O_{\max(n_0,n_2)}$$

to

$$[x_0,x_1 \vdash m_{n_0,n_1,n_2}(x_0,x_1)] : O_{n_0,n_1,n_2} \longrightarrow O_{\max(n_0,n_2)}$$

etc. Composing (37) with (40) and the equivalence $\mathcal{S}(-,\mathbf{D})$, we get a sketch morphism, also denoted by $[-]$

$$[-]:\mathbf{S}_\omega \xrightarrow{\ |-|\ } \mathbf{T}_\omega \xrightarrow{\ \|-\|\ } \mathcal{S}^{op} \xrightarrow{\ \mathcal{S}(-,\mathbf{D})\ } \mathcal{D} \tag{41}$$

Recall that if S is a sketch and A is a category then a model of S in A is an association of objects and morphisms of S to objects and morphisms of A preserving domains, codomains, sending of identity specifications and commutative diagram specifications in S to identities and commutative diagrams, respectively, moreover sending the specified (co)cones to (co)limiting (co)cones. A *realized sketch* S is a sketch based on a category so that identity on S is a model of S in S.

Let $\mathcal{D}_{\rm spb}$ be the finite limit realized sketch based on the category $\mathcal{D}$, with cones being the special pullbacks (38) in $\mathcal{D}$. We have

5.13. Proposition.

1. *The morphism $[-]:\mathbf{S}_\omega \longrightarrow \mathcal{D}$ is a model of $\mathbf{S}_\omega$ in $\mathcal{D}$. Moreover, for any realized finite limit sketch S and a sketch morphism ψ there is a unique sketch morphism φ such that the diagram*

$$\begin{tikzcd} \mathbf{S}_\omega \arrow[r, "{[-]}"] \arrow[dr, "\psi"'] & \mathcal{D}_{\rm spb} \arrow[d, "\varphi"] \\ & S \end{tikzcd}$$

   commutes, i.e. $\mathcal{D}_{\rm spb}$ is the realized sketch generated by $\mathbf{S}_\omega$.

2. *For any category $\mathcal{C}$ with finite limits, the functor of composing with $[-]:\mathbf{S}_\omega \longrightarrow \mathcal{D}$*

$$\mathrm{Mod}_{\mathcal{C}}(\mathcal{D}_{\rm spb}) \longrightarrow \mathrm{Mod}_{\mathcal{C}}(\mathbf{S}_\omega)$$

   is equivalence of categories, i.e. the category of functors from $\mathcal{D}$ to $\mathcal{C}$ preserving special pullbacks is equivalent to the category of internal ω-categories in $\mathcal{C}$.

The nerve of ω-categories and geometric realization

The above result is to be compared with the well known fact that the category $\mathrm{Cat}(\mathcal{C})$ of internal categories in a category with finite limits $\mathcal{C}$, is equivalent to the category of functor from the category of intervals $\mathcal{I}$ to $\mathcal{C}$, preserving analogous special pullbacks. The essential inverse functor is the *nerve functor*

$$\mathcal{N} : \mathrm{Cat}(\mathcal{C}) \longrightarrow \mathrm{CAT}(\mathcal{I}, \mathcal{C})$$

associating to any small category its nerve. $\mathcal{N}$ is full, faithful and its essential image consists of special pullbacks preserving functors.

A similar ω-*nerve* functor can be defined for ω-categories

$$\mathcal{N}_\omega : \omega\mathrm{Cat}(\mathcal{C}) \longrightarrow \mathrm{CAT}(\mathcal{D}, \mathcal{C})$$

with similar properties, i.e. $\mathcal{N}_\omega$ is full, faithful and its essential image consists of special pullbacks preserving functors. For $\mathcal{C}$ being the category of sets Set, it can be defined by the formula

$$\mathcal{N}_\omega(A) = \omega\mathrm{Cat}((-, \mathbf{C}), A) : \mathcal{D} \longrightarrow \mathrm{Set}$$

for any small ω-category A.

The ω-nerve functor composed with the geometric realization functor

$$\mathcal{R} : \mathrm{Set}^{\mathcal{D}} \longrightarrow \mathrm{Top}$$

defined in [J] give rise to a geometric realization functor for arbitrary ω-categories

$$\mathcal{R} : \omega\mathrm{Cat} \longrightarrow \mathrm{Top}$$

The image of $\mathcal{N}_\omega$ can be identified in a different way, using the notion of a homogenous theories, cf. [Be]. In our terminology this can be explained as follows. The canonical topology J on the category of simple ω-graphs $s\omega\mathrm{Gr}$, consists of the jointly surjective families of morphisms. Moreover, we have two faithful, essentially surjective functors:

$$[-] : s\omega\mathrm{Gr} \longrightarrow \mathcal{S}$$

the free ω-category functor restricted to the simple ω-graphs, and

$$\mathcal{D}(|-|, \mathbf{C}) : \mathcal{T} \longrightarrow \mathcal{S}$$

associating to a given tree T the simple ω-category dual to the disk whose internal tree is T. The morphisms in the image of $[-]$ are immersions, and the morphisms in the image of $\mathcal{D}(|-|, \mathbf{C})$ are essential (homogenous in the terminology of [Be]). Since the immersions and the essential ω-functors form a factorization system on $\mathcal{S}$, $\mathcal{S}$ together with the above embedding functors $[-]$, $\mathcal{D}(|-|, \mathbf{C})$, and the topology J form a homogenous theory. The models of this homogenous theory, i.e. the contravariant functors on $\mathcal{S}$, which are J-sheaves when restricted to $s\omega\mathrm{Gr}$ via $[-]$, constitute the essential image of the ω-nerve functor

$$\mathcal{N}'_\omega : \omega\mathrm{Cat} \longrightarrow \mathrm{Set}^{\mathcal{S}^{op}}$$

given by

$$\mathcal{N}'_\omega(A) = \omega\mathrm{Cat}(-, A) : \mathcal{S}^{op} \longrightarrow \mathrm{Set}.$$

This is due to the fact, that for any presheaf $X : \mathcal{S}^{op} \longrightarrow \mathrm{Set}$, the functor

$$\mathcal{D} \xrightarrow{\mathcal{D}(-,\mathbf{C})} \mathcal{S}^{op} \xrightarrow{X} \mathrm{Set}$$

preserves the special pullbacks iff the functor

$$s\mathcal{G} \xrightarrow{[-]} \mathcal{S}^{op} \xrightarrow{X} \mathrm{Set}$$

is a J-sheaf.

ω-categories as θ-categories

$\mathcal{N}_\omega$ is also an embedding of strict ω-categories into θ-categories, i.e. the strict ω-categories are those θ-categories which preserves special pullbacks. To be more precise, recall from [J] that a *face* of disk $\gamma_{\vec{u}}$ is any disk epimorphism with domain $\gamma_{\vec{u}}$. The *dimension* of disk $\gamma_{\vec{u}}$ is $\sum_{i=0}^{\mathrm{lh}(\vec{u})-1}(-1)^i u_i$. A *$\theta$-category* is a *cellular set* X (i.e. an object in $\mathrm{Set}^{\mathcal{D}}$) such that any compatible filling of all but one inner face of $\gamma_{\vec{u}}$ in X can be extended to a compatible filling of all faces of $\gamma_{\vec{u}}$ in X, for any ud-vector $\vec{u}$. Note that this condition can be rephrased as saying that X is a *θ-category* iff it sends limits of inner horns in $\mathcal{D}$ to weak limits. The strict ω-categories satisfy the following much stronger 'horn-filling' condition: for any ud-vector $\vec{u}$ if outer faces of the shape $\boldsymbol{\pi}_i : \gamma_{\vec{u}} \longrightarrow \gamma_{u_{2i}}$ are filled in X in a compatible way then all other faces of $\gamma_{\vec{u}}$, including $\gamma_{\vec{u}}$ itself have a unique filling in X extending the given one. However it is not the case that the nerves of ω-categories are necesserily flat functors. For example, they don't preserve the following equaliser:

$$\begin{tikzcd} \gamma_n \arrow[r, "\boldsymbol{\iota}"] & \gamma_{n+1} \arrow[r, shift left, "\mathbf{d}"] \arrow[r, shift right, "\mathbf{c}"'] & \gamma_n \end{tikzcd}$$

in general.

The disk classifier

In [SGL] it is shown that the topos $\mathrm{Set}^{\mathcal{I}}$ classifies intervals. Using essentially the same method, adopted for the disk case it can be shown that the topos $\mathrm{Set}^{\mathcal{D}}$ classifies disks, cf. Theorem 1 of [J].

6. Appendices

6.1. The notion of an internal disk. In this Appendix, we present the concept of disk. Note that this concept is not (generalized) algebraic; we need a category with rich-enough internal logic in which existential quantifier and disjunction can be interpreted. We take a category C, and the only assumption we make on C is that it has a terminal object. Our concept of an internal disk 'in C' will have all its data located in the category C; however, the conditions imposed on the data will be evaluated in $\mathrm{Set}^{C^{op}}$, a topos, into which C

is embedded be the Yoneda functor. e also give a reformulation of the resulting notion that is purely elementary, without reference to $\mathrm{Set}^{C^{op}}$. We do not distinguish between an object in C and its image under Yoneda. A diagram

$$\begin{tikzcd} R \arrow[r, shift left=2, "r^0"] \arrow[r, shift right=2, "r^1"'] & X \arrow[r, "p"] & B \arrow[l, shift right=2, "b"'] \arrow[l, shift left=2, "t"] \end{tikzcd}$$

in C is a *bundle of intervals over* B if it is a linear order with (not necessarily different) endpoints b and t in the topos $\mathrm{Set}^{C^{op}}/B$, cf. [SGL] p. 455.

A *disk* D in a category C is a diagram

$$\begin{tikzcd} & R^{n+1} \arrow[d, shift right=2, "r^{0;n+1}"'] \arrow[d, shift left=2, "r^{1;n+1}"] & R^n \arrow[d, shift right=2, "r^{0;n}"'] \arrow[d, shift left=2, "r^{n;1}"] & R^{n-1} \arrow[d, shift right=2, "r^{0;n-1}"'] \arrow[d, shift left=2, "r^{1;n-1}"] & & R^1 \arrow[d, shift right=2, "r^{0;1}"'] \arrow[d, shift left=2, "r^{1;1}"] & \\ \cdots & D^{n+1} \arrow[r, "p^{n+1}"] & D^n \arrow[l, shift right=2, "b^{n+1}"'] \arrow[l, shift left=2, "t^{n+1}"] \arrow[r, "p^n"] & D^{n-1} \arrow[l, shift right=2, "b^n"'] \arrow[l, shift left=2, "t^n"] & \cdots & D^1 \arrow[r, "p^1"] & D^0 \cong 1 \arrow[l, shift right=2, "b^1"'] \arrow[l, shift left=2, "t^1"] \end{tikzcd}$$

in C, such that for $n > 0$ the diagram

$$\begin{tikzcd} R^n \arrow[r, shift left=2, "r^{0;n}"] \arrow[r, shift right=2, "r^{1;n}"'] & D^n \arrow[r, "p^n"] & D^{n-1} \arrow[l, shift right=2, "b^n"'] \arrow[l, shift left=2, "t^n"] \end{tikzcd}$$

is a bundle of intervals over D^{n-1} and moreover, in $\mathrm{Set}^{C^{op}}$, *the boundary of* D^n, i.e. the subobject $\partial(D^n) = im(b^n) \cup im(t^n)$ of D^n, is the singular set for the bundle p^{n+1}, i.e. the equalizer of b^{n+1} and t^{n+1}, for $n = 1, 2, \ldots$. By convention, we put $\partial(D^0) = \emptyset$. A *morphism of disks* in C is a family of morphisms $\{f_n : D^n \longrightarrow D'_n\}_{n\in\omega}$ preserving all the additional structure, i.e. projections, order in fibers and endpoints in the obvious internal sense.

The notion of a disk can be expressed in elementary terms as follows:

1. b^n and t^n are splittings of p^n, i.e. $p^n \circ b^n = p^n \circ t^n = 1_{D^{n-1}}$;

2. $r^{0;n}$, $r^{1;n}$ are jointly mono;

3. R^n is a linear order in fibers of p^n. Explicitely, for $z_0, z_1 : Z \longrightarrow D^n$, we write $z_0 \leq_{R^n} z_1$, whenever there is $z : Z \longrightarrow R^n$, such that $r^{i;n} \circ z = z_i$, for $i = 0, 1$. Then for any $z_0, z_1, z_2 : Z \longrightarrow D^n$ we have

   (a) $z_0 \leq_{R^n} z_0$

(b) if $z_0 \leq_{R^n} z_1$ and $z_1 \leq_{R^n} z_0$ then $z_0 = z_1$;

(c) if $z_0 \leq_{R^n} z_1$ and $z_1 \leq_{R^n} z_2$ then $z_0 \leq_{R^n} z_2$;

(d) if $p^n \circ z_0 = p^n \circ z_1$ iff either $z_0 \leq_{R^n} z_1$ or $z_1 \leq_{R^n} z_0$;

4. b and t are bottom and top endpoints in the fibers, i.e. $b^n \circ p^n \leq_{R^n} 1_{D^n} \leq_{R^n} t^n \circ p^n$;

5. disk condition: for $z : Z \longrightarrow D^n$ we have $b^{n+1} \circ z = t^{n+1} \circ z$ iff there is $z' : Z \longrightarrow D^{n-1}$ such that either $z = b^n \circ z'$ or $z = t^n \circ z'$.

6.2. The notion of an internal ω-category. It is well known that the concept of ω-category is equational over the category of ω-graphs. In this section, give the details of a definition of ω-category that exhibits its equational character over ω-graphs, and which works internally in any category. Although of course the idea of an internal equational description is common place, the details here are crucial for our purposes.

Our procedure here is slightly different from Appendix 6.1. We do not assume that the category C has all finite limits, but our definition implicitly assumes that certain finite limits exists in C. This will be seen e.g. in condition (ii) where the definition requires that we have *certain* pullbacks in C as part of the data for the internal ω-category. In the application, when the category C will be the category $\mathcal{D}$ of finite disks, we will have one particular internal ω-category for which all the required pullbacks do indeed exist in $\mathcal{D}$, despite the fact that $\mathcal{D}$ is not a finitely complete category.

Since the notion of ω-category is generalized algebraic, we do not need to refer to the topos $\mathrm{Set}^{C^{op}}$.

An ω-category $\mathbf{A}$ in C consists of data (i)-(v) subject to the conditions (vi)-(xi).

Data

(i) an ω-*graph* (or ω-*globular object*): i.e. the following diagram

$$\cdots \quad A_n \underset{c}{\overset{d}{\rightrightarrows}} A_{n-1} \quad \cdots \quad A_2 \underset{c}{\overset{d}{\rightrightarrows}} A_1 \underset{c}{\overset{d}{\rightrightarrows}} A_0$$

such that the *globularity conditions* hold

$$d \circ d = d \circ c \qquad c \circ d = c \circ c$$

Notation: If $l < n$, $d_{(l)} = d_{(n;l)} : A_n \longrightarrow A_l$ we denote a compositions of d's with the codomain A_l, and similarly by $c_{(l)} = c_{(n;l)} : A_n \longrightarrow A_l$ we denote a compositions of c's with the codomain A_l. By convention, if $l \geq n$, $d_{(l)}$ and $c_{(l)}$ denotes the identity on A_n. When both domain and codomain of $d_{(l)}$ and $c_{(l)}$ is clear from the context we also write d and c for $d_{(l)}$ and $c_{(l)}$, respectively.

(ii) for a 3-tuple $\langle n_0, n_1, n_2\rangle$ such that $n_1 < n_0, n_2$ a pullback

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & A_{n_2} \arrow[d, "d"] \\ A_{n_0} \arrow[r, "c"'] & A_{n_1} \end{tikzcd}$$

(iii) for a 5-tuple $\langle n_0, n_1, n_2, n_3, n_4 \rangle$ such that $n_1 < n_0, n_2$, $n_3 < n_2, n_4$ a (triple) pullback

$$\begin{tikzcd} A_{n_0} \arrow[d, "c"'] & A_{n_0,n_1,n_2,n_3,n_4} \arrow[l, "\pi_0"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A_{n_4} \arrow[d, "d"] \\ A_{n_1} & A_{n_2} \arrow[l, "d"] \arrow[r, "c"'] & A_{n_3} \end{tikzcd}$$

(iv) *composition morphisms*

$$m_{n_0,n_1,n_2} = m_{n_1} = m : A_{n_0,n_1,n_2} \longrightarrow A_{\max(n_0,n_2)}$$

(v) *identity morphisms*, for $n \leq l$

$$\iota_{(l)} = \iota_{(n;l)} : A_n \longrightarrow A_l$$

Conditions

(vi) *Domains and codomains of compositions*

We have the following commutative squares:

1. for $l \leq n_1$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "m"] \arrow[d, "\pi_0"'] & A_{\max(n_0,n_2)} \arrow[d, "d_{(l)}"] \\ A_{n_0} \arrow[r, "d_{(l)}"'] & A_l \end{tikzcd}$$

2. for $n_1 < l \leq \max(n_0, n_2)$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "m"] \arrow[d, "d_{(l)} \times d_{(l)}"'] & A_{\max(n_0,n_2)} \arrow[d, "d_{(l)}"] \\ A_{\min(n_0,l),n_1,\min(n_2,l)} \arrow[r, "m"'] & A_l \end{tikzcd}$$

and similarly for codomains,

3. for $l \leq n_1$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "m"] \arrow[d, "\pi_1"'] & A_{\max(n_0,n_2)} \arrow[d, "c_{(l)}"] \\ A_{n_0} \arrow[r, "c_{(l)}"'] & A_l \end{tikzcd}$$

4. for $n_1 < l \leq \max(n_0, n_2)$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "m"] \arrow[d, "c_{(l)} \times c_{(l)}"'] & A_{n_0} \arrow[d, "c_{(l)}"] \\ A_{\min(n_0,l),n_1,\min(n_2,l)} \arrow[r, "m"'] & A_l \end{tikzcd}$$

(vii) *Domains and codomains of identities*
for $n \leq l$

1.

$$\begin{tikzcd} A_n \arrow[r, "\iota_{(l)}"] \arrow[dr, "1"'] & A_l \arrow[d, "c_{(n)}"] \\ & A_n \end{tikzcd}$$

2.

$$\begin{tikzcd} A_n \arrow[r, "\iota_{(l)}"] \arrow[dr, "1"'] & A_l \arrow[d, "d_{(n)}"] \\ & A_n \end{tikzcd}$$

(viii) *Associativity of compositions*

1. for $n_1 = n_3$

$$\begin{tikzcd} A_{n_0,n_1,n_2,n_3,n_4} \arrow[r, "m \times 1"] \arrow[d, "1 \times m"'] & A_{\max(n_0,n_2),n_3,n_4} \arrow[d, "m"] \\ A_{n_0,n_1,\max(n_2,n_4)} \arrow[r, "m"'] & A_{\max(n_0,n_2,n_4)} \end{tikzcd}$$

2. for $n_1 < n_3 < n_0$

$$\begin{tikzcd}
A_{n_0,n_1,n_2,n_3,n_4} \arrow[r, "\langle m(\pi_0,\pi_1), m(c_{(n_3)}\pi_0,\pi_2)\rangle"] \arrow[d, "1\times m"'] & A_{\max(n_0,n_2),n_3,n_4} \arrow[d, "m"] \\
A_{n_0,n_1,\max(n_2,n_4)} \arrow[r, "m"'] & A_{\max(n_0,n_2,n_4)}
\end{tikzcd}$$

3. for $n_0 \leq n_3$

$$\begin{tikzcd}
A_{n_0,n_1,n_2,n_3,n_4} \arrow[r, "\langle m(\pi_0,\pi_1), m(\pi_0,\pi_2)\rangle"] \arrow[d, "1\times m"'] & A_{n_2,n_3,n_4} \arrow[d, "m"] \\
A_{n_0,n_1,\max(n_2,n_4)} \arrow[r, "m"'] & A_{\max(n_2,n_4)}
\end{tikzcd}$$

4. for $n_3 < n_1 < n_4$

$$\begin{tikzcd}
A_{n_0,n_1,n_2,n_3,n_4} \arrow[r, "m\times 1"] \arrow[d, "\langle m(\pi_0, d_{(n_1)}\pi_2), m(\pi_1,\pi_2)\rangle"'] & A_{\max(n_0,n_2),n_3,n_4} \arrow[d, "m"] \\
A_{n_0,n_1,\max(n_2,n_4)} \arrow[r, "m"'] & A_{\max(n_0,n_2,n_4)}
\end{tikzcd}$$

5. for $n_4 \leq n_1$

$$\begin{tikzcd}
A_{n_0,n_1,n_2,n_3,n_4} \arrow[r, "m\times 1"] \arrow[d, "\langle m(\pi_0,\pi_2), m(\pi_1,\pi_2)\rangle"'] & A_{\max(n_0,n_2),n_3,n_4} \arrow[d, "m"] \\
A_{n_0,n_1,n_2} \arrow[r, "m"'] & A_{\max(n_0,n_2,n_4)}
\end{tikzcd}$$

(ix) *Associativity of identities*

for $n_0 < n_1 < n_2$

$$\begin{tikzcd} A_{n_0} \arrow[r, "\iota_{(n_1)}"] \arrow[dr, "\iota_{(n_2)}"'] & A_{n_1} \arrow[d, "\iota_{(n_2)}"] \\ & A_{n_2} \end{tikzcd}$$

(x) *Compositions with identities*
On the left:

1. for $l \geq n_0$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "\iota_{(l)} \times 1"] \arrow[d, "m"'] & A_{l,n_1,n_2} \arrow[d, "m"] \\ A_{n_0} \arrow[r, "\iota"'] & A_{\max(l,n_2)} \end{tikzcd}$$

2.

$$\begin{tikzcd} A_{n_2} \arrow[r, "\langle \iota_{(n_0)} \circ d_{(n_1)}, 1 \rangle"] \arrow[dr, "\iota"'] & A_{n_0,n_1,n_2} \arrow[d, "m"] \\ & A_{\max(n_0,n_2)} \end{tikzcd}$$

On the right:

3. for $l \geq n_2$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[r, "1 \times \iota_{(l)}"] \arrow[d, "m"'] & A_{n_0,n_1,l} \arrow[d, "m"] \\ A_{n_2} \arrow[r, "\iota"'] & A_{\max(n_0,l)} \end{tikzcd}$$

4.

$$\begin{tikzcd} A_{n_0} \arrow[r, "{\langle 1, \iota_{(n_2)} \circ c_{(n_1)} \rangle}"] \arrow[dr, "\iota"'] & A_{n_0,n_1,n_2} \arrow[d, "m"] \\ & A_{\max(n_0,n_2)} \end{tikzcd}$$

(xi) *Middle Exchange Law*
for $n_1 < l < n_0, n_2$

$$\begin{tikzcd} A_{n_0,n_1,n_2} \arrow[rr, "{\langle m(d_{(l)} \times 1), m(1 \times c_{(l)}) \rangle}"] \arrow[d, "{\langle m(1 \times d_{(l)}), m(c_{(l)} \times 1) \rangle}"'] \arrow[drr, "m"] & & A_{n_2,l,n_0} \arrow[d, "m"] \\ A_{n_0,l,n_2} \arrow[rr, "m"'] & & A_{\max(n_0,n_2)} \end{tikzcd}$$

6.3. Some properties of ω-categories. In this Appendix we shall prove that the analogs of the axioms of ω-categories, expressed for the 'multi'-version of the operations of ω-categories ($d_{\vec{u};l}$, $c_{\vec{u};l}$, $m_{\vec{u}}$, etc.) defined in section 2.6, holds. For example, Lemma 6.7.1, gives some general form of the associativity law for ω-categories. We also prove, a more general form of the Middle Exchange Law. The definition of an ω-category is in Appendix 6.2.

In the whole section A denotes is an internal ω-category in a category $\mathcal{C}$. We assume that for any ud-vector $\vec{u}$ the 'multi'-pullback $A_{\vec{u}}$ exists.

We have

6.4. Lemma. *Let A be an ω-category in a category $\mathcal{C}$, $l \in \omega$, $\vec{u}$ a ud-vector, $l \leq u = \mathrm{ht}(\vec{u})$. Then the following squares*

$$\begin{tikzcd} A_{\vec{u}} \arrow[r, "d^A_{\vec{u};l}"] \arrow[d, "m^A_{\vec{u}}"'] & A_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[d, "m^A_{\mathrm{tr}_{(l)}(\vec{u})}"] \\ A_u \arrow[r, "d^A_{(l)}"'] & A_l \end{tikzcd} \tag{42}$$

and

$$\begin{tikzcd}
A_{\vec{u}} \arrow[r, "c^A_{\vec{u};l}"] \arrow[d, "m^A_{\vec{u}}"'] & A_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[d, "m^A_{\mathrm{tr}_{(l)}(\vec{u})}"] \\
A_u \arrow[r, "c^A_{(l)}"'] & A_l
\end{tikzcd} \tag{43}$$

commute.

Proof. We shall prove, by induction on l-size of $\vec{u}$ that the square (42) commutes. The commutation of the square (43) is similar and is left for the reader.

If $\vec{u} = u_0 \leq l$ then all morphisms in (42) are identities on A_{u_0}.

If $\vec{u}$ is l-primitive then $\mathrm{tr}_{(l)}(\vec{u}) = l$, $d^A_{\vec{n};l} = d^A_{(l)} \circ \pi_0$, and $m^A_{\mathrm{tr}_{(l)}(\vec{u})} = id_{A_l}$. Thus the commutativity of (42) reduces to the commutation of the following triangle

$$\begin{tikzcd}
A_{\vec{n}} \arrow[r, "d^A_{\vec{u};l}"] \arrow[d, "m^A_{\vec{u}}"'] & A_l \\
A_u \arrow[ur, "d^A_{(l)}"'] &
\end{tikzcd} \tag{44}$$

and the commutation of (44), we can show by induction on the length of the l-primitive vector $\vec{u}$.

If $\mathrm{lh}(\vec{u}) = 1$ then (44) clearly commutes. Assume that (44) commutes for vectors of length smaller then $\mathrm{lh}(\vec{u}) > 1$. Let $z = \min(\vec{u}) \geq l$, $\vec{u} = \vec{u'}, z, \vec{u''}$, $k' = \mathrm{lh}(\vec{u'})$, $u' = \mathrm{ht}(\vec{u'})$, and $u'' = \mathrm{ht}(\vec{u''})$. Then the triangles

$$\begin{tikzcd}
A_{\vec{u}} \arrow[r, "m^A_{\vec{u'}} \times m^A_{\vec{u''}}"] \arrow[dr, "d^A_{\vec{u'};l} \circ \pi_{0..k'}"'] & A_{u',z,u''} \arrow[r, "m^A_{u',z,u''}"] \arrow[d, "d^A_{(l)} \circ \pi^A_0"] & A_u \arrow[dl, "d^A_{(l)}"] \\
& A_l &
\end{tikzcd}$$

commute. The left one by inductive assumption on $\vec{u'}$ and the right one by the axioms (vi).1 of the definition of an ω-category. Moreover, since

$$d^A_{\vec{u'};l} \circ \pi_{0..k'} = d^A_{(l)} \circ \pi^A_0 \circ \pi^A_{0..k'} = d^A_{(l)} \circ \pi^A_0 = d^A_{\vec{u};l}$$

and

$$m^A_{\vec{u}} = m^A_{u',z,u''} \circ (m^A_{\vec{u'}} \times m^A_{\vec{u''}})$$

the outer triangle is (44). Thus (44) commutes for any ud-vector $\vec{u}$.

Finally, let $\mathrm{size}_l(\vec{u}) > 1$ and (42) commutes for ud-vectors of l-size smaller than $size_l(\vec{u})$. Let $\vec{u} = \vec{u'}, z, \vec{u''}$, where $z = \min(\vec{u}) < l$, $u = \mathrm{ht}(\vec{u})$, $u' = \mathrm{ht}(\vec{u'})$, $u'' = \mathrm{ht}(\vec{u''})$, $l' = \min(l, u')$, and $l'' = \min(l, u'')$. Then in the following diagram

$$\begin{tikzcd}
A_{\vec{u'},z,\vec{u''}} \arrow[r, "m^A_{\vec{u'}} \times m^A_{\vec{u''}}"] \arrow[d, "d^A_{\vec{u'};l} \times d^A_{\vec{u''};l}"'] & A_{u',z,u''} \arrow[r, "m^A_{u',z,u''}"] \arrow[d, "d^A_{(l)} \times d^A_{(l)}"'] & A_u \arrow[d, "d^A_{(l)}"] \\
A_{\mathrm{tr}_{(l)}(\vec{u'}),z,\mathrm{tr}_{(l)}(\vec{u''}))} \arrow[r, "m^A_{\mathrm{tr}_{(l)}(\vec{u'})} \times m^A_{\mathrm{tr}_{(l)}(\vec{u''})}"'] & A_{l',z,l''} \arrow[r, "m^A_{l',z,l''}"'] & A_l
\end{tikzcd}$$

the left square commutes by inductive assumption on ud-vectors $\vec{u'}$ and $\vec{u''}$, and and the right one commutes by condition (vi).2 of the definition of an ω-category. Thus 42 commutes. The commutation of 43 is similar. ■

We have

6.5. Lemma. *For any ω-graph A and $l \in \omega$ and ud-vectors $\vec{u}$, $\vec{v}$ such that $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$ the square*

$$\begin{tikzcd}
A_{[\vec{u},l,\vec{v}]} \arrow[r, "\pi^A_{1;\vec{v}}"] \arrow[d, "\pi^A_{0;\vec{u}}"'] & A_{\vec{v}} \arrow[d, "d^A_{\vec{v};l}"] \\
A_{\vec{u}} \arrow[r, "c^A_{\vec{u};l}"'] & A_{\vec{w}}
\end{tikzcd} \tag{45}$$

is a pullback.

Proof. We show that (45) is a pullback by induction on l-size$(\vec{u})$.

If $\vec{v} = v_0 \leq l$ then (45) is the following commutative square

$$\begin{tikzcd}
A_{\vec{u}} \arrow[r, "c^A_{\vec{u};l}"] \arrow[d, "id_{A_{\vec{u}}}"'] & A_{v_0} \arrow[d, "id_{A_{v_0}}"] \\
A_{\vec{u}} \arrow[r, "c^A_{\vec{u};l}"'] & A_{v_0}
\end{tikzcd}$$

The case $\vec{u} = u_0 \leq l$ is similar.

If both $\vec{u}$ and $\vec{v}$ are l-primitive then (45) is

$$\begin{tikzcd} A_{\vec{u},l,\vec{v}} \arrow[r, "\pi^A_{k_1+1..k_1+k_2+1}"] \arrow[d, "\pi^A_{0..k_1}"'] & A_{\vec{v}} \arrow[d, "d^A_{\vec{v};l}"] \\ A_{\vec{u}} \arrow[r, "c^A_{\vec{u};l}"'] & A_l \end{tikzcd}$$

where $k_1 = \mathrm{lh}(\vec{u}) - 1$, and $k_2 = \mathrm{lh}(\vec{v}) - 1$. Since $c^A_{\vec{u};l} = c^A_{(l)} \circ \pi^A_{k_1}$ and $d^A_{\vec{v};l} = d^A_{(l)} \circ \pi^A_0$, it is a pullback, as well.

Assume now that l-size$(\vec{u}) > 1$. Let $w = \min(\vec{u}) < l$, $\vec{u} = \vec{u'}, w, \vec{u''}$, $\vec{v} = \vec{v'}, w, \vec{v''}$, $\mathrm{tr}_{(l)}(\vec{u'}) = \mathrm{tr}_{(l)}(\vec{v'}) = \vec{w'}$, and $\mathrm{tr}_{(l)}(\vec{u''}) = \mathrm{tr}_{(l)}(\vec{v''}) = \vec{w''}$. Then (45) is

$$\begin{tikzcd} A_{[\vec{u'},l,\vec{v'}],w,[\vec{u''},l,\vec{v''}]} \arrow[r, "\pi^A_{1;\vec{v'}} \times \pi^A_{1;\vec{v''}}"] \arrow[d, "\pi^A_{0;\vec{u'}} \times \pi^A_{0;\vec{u''}}"'] & A_{\vec{v'},w,\vec{v''}} \arrow[d, "d^A_{\vec{v'};l} \times d^A_{\vec{u''};l}"] \\ A_{\vec{u'},w,\vec{u''}} \arrow[r, "c^A_{\vec{u'};l} \times c^A_{\vec{u''};l}"'] & A_{\vec{w'},w,\vec{w''}} \end{tikzcd}$$

and it is a pullback, since by inductive hypothesis on $\vec{u'}$ and $\vec{u''}$ it is a pullback of two pullback squares. This ends the proof. ∎

In Lemma 6.7 we shall prove among other things a kind of general associativity law. But for this, we shall need the following form of the Middle Exchange Law.

6.6. Lemma. *Let $u', u'', v', v'', l, z \in \omega$, $z < u', u'', v', v'', l$. Moreover*

$$\min(u', l) = \min(v', l) = l', \quad \min(u'', l) = \min(v'', l) = l''$$

$$u = \max(u', u''), \quad v = \max(v', v''),$$

$$n' = \max(u', v'), \quad n'' = \max(u'', v''), \quad n = \max(u, v).$$

Then the square

$$\begin{tikzcd} A_{[u',l,v'],z,[u'',l,v'']} \arrow[r, "\langle m(\pi_0,\pi_2), m(\pi_1,\pi_3)\rangle"] \arrow[d, "m_{[u',l,v']} \times m_{[u'',l,v'']}"'] & A_{[u,l,v]} \arrow[d, "m_{[u,l,v]}"] \\ A_{n',z,n''} \arrow[r, "m_{n',z,n''}"'] & A_n \end{tikzcd}$$

commutes.

PROOF. We need to show that the outer square in the following the diagram

$$
\begin{tikzcd}
 & & A_{[u',l,v'],z,[u'',l,v'']} \arrow[dll, "m\times m"'] \arrow[d] \arrow[drr] \arrow[drrr, "\langle m(\pi_0,\pi_2), m(\pi_1,\pi_3), \rangle"] \arrow[dl, phantom, "{[1]}"] \arrow[dr, phantom, "{[2]}"] \arrow[drr, phantom, "{[3]}"', shift right=2] & & & \\
A_{n',z,n''} \arrow[r] \arrow[drr, "m"'] & \bullet \arrow[dr] & \bullet \arrow[l] \arrow[r] \arrow[d, phantom, "{[5]}"] & \bullet \arrow[dl] & \bullet \arrow[l] \arrow[r] & A_{[u,l,v]} \arrow[dlll, "m"] \\
 & & A_n & & &
\end{tikzcd}
\tag{46}
$$

commutes. Below we explain what are the unexplained shapes in this diagram and why they commutes. The square [1] is

$$
\begin{tikzcd}
A_{[u',l,v'],z,[u'',l,v'']} \arrow[rr, "{\substack{< m(\pi_0, d_{(l)}\pi_2), m(\pi_1, d_{(l)}\pi_2), \\ m(c_{(l)}\pi_1, \pi_2), m(c_{(l)}\pi_1, \pi_3) >}}"] \arrow[d, "m\times m"'] & & A_{[[u',l,v'],l,[u'',l,v'']]} \arrow[d, "m\times m"] \\
A_{n',z,n''} \arrow[rr, "\langle m(\pi_0, d_{(l)}\pi_1), m(c_{(l)}\pi_0, \pi_1)\rangle"'] & & A_{[n',l,n'']}
\end{tikzcd}
$$

and it commutes by the axioms for domain and codomain of compositions (vi) and associativity (viii).3 and (viii).5.

The square [2] is

$$
\begin{tikzcd}
A_{[u',l,v'],z,[u'',l,v'']} \arrow[rr, "{\substack{< m(\pi_0, d_{(l)}\pi_2), m(d_{(l)}\pi_1, \pi_2), \\ m(\pi_1, c_{(l)}\pi_2), m(c_{(l)}\pi_1, \pi_3), >}}"] \arrow[d, "{\substack{< m(\pi_0, d_{(l)}\pi_2), m(\pi_1, d_{(l)}\pi_2), \\ m(c_{(l)}\pi_1, \pi_2), m(c_{(l)}\pi_1, \pi_3) >}}"] & & A_{[[u',l,u''],l,[v',l,v'']]} \arrow[d, "1\times m\times 1"] \\
A_{[[u',l,v'],l,[u'',l,v'']]} \arrow[rr, "1\times m\times 1"'] & & A_{[u',l,\max(v',u''),l,v'']}
\end{tikzcd}
$$

and it commutes by middle exchange law (xi).

The triangle [3] is

$$\begin{tikzcd}
A_{[u',l,v'],z,[u'',l,v'']} \arrow[d, "{< m(\pi_0, d_{(l)}\pi_2), m(d_{(l)}\pi_1, \pi_2), m(\pi_1, c_{(l)}\pi_2), m(c_{(l)}\pi_1, \pi_3) >}"'] \arrow[dr, "{\langle m(\pi_0,\pi_2), m(\pi_1,\pi_3)\rangle}"] & \\
A_{[[u',l,u''],l,[v',l,v'']]} \arrow[r, "m \times m"'] & A_{[u,l,v]}
\end{tikzcd}$$

and, again, it commutes by middle exchange law (xi).

The triangle [4] is

$$\begin{tikzcd}
A_{n',z,n''} \arrow[d, "{\langle m(\pi_0, d_{(l)}\pi_1), m(c_{(l)}\pi_0, \pi_1)\rangle}"'] \arrow[dr, "m"] & \\
A_{[n',l,n'']} \arrow[r, "m"'] & A_n
\end{tikzcd}$$

and, again, it commutes by middle exchange law (xi).

The square [5] is

$$\begin{tikzcd}
A_{[[u',l,v'],l,[u'',l,v'']]} \arrow[r, "1 \times m \times 1"] \arrow[d, "m \times m"'] & A_{[u',l,\max(v',u''),l,v'']} \arrow[d, "m"] \\
A_{[n',l,n'']} \arrow[r, "m"'] & A_n
\end{tikzcd}$$

and it commutes by associativity axiom (viii).1.

Finally, the square [6] is

$$\begin{tikzcd}
A_{[[u',l,u''],l,[v',l,v'']]} \arrow[r, "m \times m"] \arrow[d, "1 \times m \times 1"'] & A_{[u,l,v]} \arrow[d, "m"] \\
A_{[u',l,\max(v',u''),l,v'']} \arrow[r, "m"'] & A_n
\end{tikzcd}$$

and, again, it commutes by associativity axiom (viii).1.

Thus all the shapes including the outer square in the diagram (46) commute. ∎

6.7. LEMMA. *For any ω-category A.*

1. *for $l \in \omega$ and ud-vectors $\vec{u}$, $\vec{v}$ such that* $\mathrm{tr}_{(l)}(\vec{u}) = \mathrm{tr}_{(l)}(\vec{v}) = \vec{w}$ *the triangle*

$$\begin{tikzcd} A_{[\vec{u},l,\vec{v}]} \arrow[rr, "m^A_{[\vec{u},l,\vec{v}]}"] \arrow[dr, "m^A_{\vec{u}} \times m^A_{\vec{v}}"'] & & A_{\max(u,v)} \\ & A_{[u,l,v]} \arrow[ur, "m^A_{[u,l,v]}"'] & \end{tikzcd} \tag{47}$$

commutes;

2. *for any $u, v, u', v', z, z', l \in \omega$ such that $u \leq u'$, $v \leq v'$, $l < u', v'$, $z = \max(u,v)$ and $z' = \max(u',v')$, the square*

$$\begin{tikzcd} A_{[u,l,v]} \arrow[r, "m^A_{[u,l,v]}"] \arrow[d, "\iota^A_{(u')} \times \iota^A_{(v')}"'] & A_z \arrow[d, "\iota^A_{(z')}"] \\ A_{u',l,v'} \arrow[r, "m^A_{u',l,v'}"'] & A_{z'} \end{tikzcd} \tag{48}$$

commutes.

PROOF. Ad 1. We argue by induction on the l-size of $\vec{u}$.

If $\vec{v} = v_0$, then (47) is

$$\begin{tikzcd} A_{\vec{u}} \arrow[rr, "m^A_{\vec{u}}"] \arrow[dr, "m^A_{\vec{u}}"'] & & A_u \\ & A_u \arrow[ur, "1_{A_u}"'] & \end{tikzcd}$$

so it commutes. The case $\vec{u} = u_0$ is similar.

If $\vec{u}$ and $\vec{v}$ are l-primitive then (47) is the following triangle

$$\begin{tikzcd} A_{\vec{u},l,\vec{v}} \arrow[rr, "m^A_{\vec{u},l,\vec{v}}"] \arrow[dr, "m^A_{\vec{u}} \times m^A_{\vec{v}}"'] & & A_{\max(u,v)} \\ & A_{u,l,v} \arrow[ur, "m^A_{u,l,v}"'] & \end{tikzcd}$$

commutes, by definition of $m_{[\vec{u},l,\vec{v}]}$, since $l = \min([\vec{u}, l, \vec{v}])$.

Finally, we assume that $\mathrm{size}_l(\vec{u}) > 1$. Let

$$z = \min(\vec{u}), \quad \vec{u} = \vec{u'}, z, \vec{u''}, \quad \vec{v} = \vec{v'}, z, \vec{v''},$$

$$u = \mathrm{ht}(\vec{u}),\quad u' = \mathrm{ht}(\vec{u'}),\quad u'' = \mathrm{ht}(\vec{u''}),$$
$$v = \mathrm{ht}(\vec{v}),\quad v' = \mathrm{ht}(\vec{v'}),\quad v'' = \mathrm{ht}(\vec{v''}),$$
$$\vec{w'} = \mathrm{tr}_{(l)}(\vec{u'}) = \mathrm{tr}_{(l)}(\vec{v'}),\quad \vec{w''} = \mathrm{tr}_{(l)}(\vec{u''}) = \mathrm{tr}_{(l)}(\vec{v''}),$$
$$w' = \mathrm{ht}(\vec{w'}),\quad w'' = \mathrm{ht}(\vec{w''}),$$
$$l' = \min(l, u') = \min(l, v'),\quad l'' = \min(l, u'') = \min(l, v''),\quad \dot{l} = \min(l', l''),$$
$$\dot{\pi_0} = \pi_0 \circ \pi_{0;[u',l,v']},\quad \dot{\pi_1} = \pi_1 \circ \pi_{0;[u',l,v']},$$
$$\dot{\pi_2} = \pi_0 \circ \pi_{1;[u'',l,v'']},\quad \dot{\pi_3} = \pi_1 \circ \pi_{1;[u'',l,v'']}.$$

Then, the following diagram

$$\begin{tikzcd}
 & & A_{[\vec{u'},l,\vec{v'}],z,[\vec{u''},l,\vec{v''}]} \arrow[dll, "\pi_{0;\vec{u'}} \times \pi_{0;\vec{u''}}"'] \arrow[drr, "\pi_{1;\vec{v'}} \times \pi_{1;\vec{v''}}"] \arrow[ddd, "(m_{\vec{u'}} \times m_{\vec{v'}}) \times (m_{\vec{u''}} \times m_{\vec{v''}})"] & & \\
A_{\vec{u'},z,\vec{u''}} \arrow[drrr, "c_{\vec{u'};l} \times c_{\vec{u''};l}"'] \arrow[ddd, "m_{\vec{u'}} \times m_{\vec{u''}}"'] & & & & A_{\vec{v'},z,\vec{v''}} \arrow[dl, "d_{\vec{v'};l} \times d_{\vec{v''};l}"] \arrow[ddd, "m_{\vec{v'}} \times m_{\vec{v''}}"] \\
 & & & A_{\vec{w'},z,\vec{w''}} \arrow[ddd, "m_{\vec{w'}} \times m_{\vec{w''}}"] & \\
 & & A_{[u',l,v'],z,[u'',l,v'']} \arrow[dll, "\langle \dot{\pi_0}, \dot{\pi_2} \rangle"'] \arrow[drr, "\langle \dot{\pi_1}, \dot{\pi_3} \rangle"] \arrow[ddd, "\langle m(\pi_0,\pi_2), m(\pi_1,\pi_3) \rangle"] & & \\
A_{u',z,u''} \arrow[drrr, "c_{(l')} \times c_{(l'')}"'] \arrow[ddd, "m"'] & & & & A_{v',z,v''} \arrow[dl, "d_{(l')} \times d_{(l'')}"] \arrow[ddd, "m"] \\
 & & & A_{l',z,l''} \arrow[ddd, "m"] & \\
 & & A_{[u,l,v]} \arrow[dll, "\pi_0"'] \arrow[drr, "\pi_1"] & & \\
A_u \arrow[drrr, "c_{(l)}"'] & & & & A_v \arrow[dl, "d_{(l)}"] \\
 & & & A_{\dot{l}} &
\end{tikzcd}$$

commutes, in which, the three horizontal squares are pullbacks by Lemma 6.5, the four back squares commute formally, and four front squares commutes by Lemma 6.4. Moreover, by definition of $m^A_{\vec{u}}$, the composition of vertical morphisms on the left is equal to $m^A_{\vec{u}}$ and the composition of vertical morphisms on the right is equal to $m^A_{\vec{v}}$. This shows that the following triangle

$$\begin{tikzcd}
A_{[\vec{u'},l,\vec{v'}],z,[\vec{u''},l,\vec{v''}]} \arrow[rr, "m^A_{\vec{u}} \times m^A_{\vec{v}}"] \arrow[dr, "m^A_{\vec{u'}} \times m^A_{\vec{v'}} \times m^A_{\vec{u''}} \times m^A_{\vec{v''}}"'] & & A_{[u,l,v]} \\
& A_{[u',l,v'],z,[u'',l,v'']} \arrow[ur, "\langle m^A(\pi_0,\pi_2), m^A(\pi_1,\pi_3)\rangle"'] &
\end{tikzcd} \tag{49}$$

commutes. Then the triangle (47) commutes by Lemma 6.6.

Ad 2. To proof that (48) commutes we consider four cases according to whether the conditions

$$u \leq l \qquad\qquad v \leq l$$

hold true.

If $u \leq l$ and $v \leq l$, then the square (48) is

$$\begin{tikzcd}
A_z \arrow[r, "1_{A_z}"] \arrow[d, "\langle \iota_{(u')}, \iota_{(v')} \rangle"'] & A_z \arrow[d, "\iota_{(z')}"] \\
A_{u',l,v'} \arrow[r, "m"'] & A_{z'}
\end{tikzcd}$$

If $u' \leq v'$, it can be decomposed as a diagram

$$\begin{tikzcd}
A_z \arrow[r, "1_{A_z}"] \arrow[d, "\iota_{(u')}"'] & A_z \arrow[d, "\iota_{(v')}"] \\
A_{u'} \arrow[r, "\iota_{(v')}"'] \arrow[d, "\langle 1, \iota_{(v')} c_{(l)} \rangle"'] & A_{v'} \\
A_{u',l,v'} \arrow[ur, "m"'] &
\end{tikzcd}$$

in which the square commutes by associativity of identity (ix) and the triangle commutes by axiom (x).4, and if $u' > v'$, it can be decomposed as a diagram

$$\begin{tikzcd}
A_z \arrow[r, "1_{A_z}"] \arrow[d, "\iota_{(v')}"'] & A_z \arrow[d, "\iota_{(u')}"] \\
A_{v'} \arrow[r, "\iota_{(u')}"'] \arrow[d, "\langle \iota_{(v')} d_{(l)}, 1\rangle"'] & A_{u'} \\
A_{u',l,v'} \arrow[ur, "m"'] &
\end{tikzcd}$$

in which the square commutes by associativity of identity (ix) and the triangle commutes by axiom (x).2.

If $u > l$ and $v \leq l$, then the square (48) is the outer square in the diagram

$$\begin{tikzcd}
A_u \arrow[r, "1_{A_u}"] \arrow[d, "\langle 1, \iota_{(v)} c_{(l)}\rangle"'] & A_u \arrow[d, "1_{A_u}"] \\
A_{u,l,v} \arrow[r, "m"'] \arrow[d, "\iota_{(u')} \times 1"'] & A_u \arrow[d, "\iota_{(u')}"] \\
A_{u',l,v} \arrow[r, "m"'] \arrow[d, "1 \times \iota_{(v')}"'] & A_{u'} \arrow[d, "\iota_{(z')}"] \\
A_{u',l,v'} \arrow[r, "m"'] & A_{z'}
\end{tikzcd}$$

in which the top square commutes by (x).4, the middle square commutes by (x).1, and the bottom square commutes by (x).3.

The case $u \leq l$ and $v > l$ is similar to the previous one.

If $u > l$ and $v > l$, then the square (48) is the outer square in the diagram

$$\begin{tikzcd}
A_{u,l,v} \arrow[r, "m"] \arrow[d, "\iota_{(u)} \times 1"'] & A_z \arrow[d, "\iota_{(\max(u',v)}"] \\
A_{u',l,v} \arrow[r, "m"'] \arrow[d, "1 \times \iota_{(v')}"'] & A_{\max(u',v)} \arrow[d, "\iota_{(z')}"] \\
A_{u',l,v'} \arrow[r, "m"'] & A_{z'}
\end{tikzcd}$$

in which the top square commutes by (x).3 and bottom square commutes by (x).1.

This ends the proof of 2. ∎

6.8. Free internal ω-categories. Let us fix for the whole section a category $\mathcal{C}$ with pullbacks and stable disjoint countable coproducts, $\omega\mathrm{Gr}(\mathcal{C})$ is the category of ω-graphs in $\mathcal{C}$ and $\omega\mathrm{Cat}(\mathcal{C})$ is the category of ω-categories in $\mathcal{C}$. Then we have a forgetful functor

$$|-|:\omega\mathrm{Cat}(\mathcal{C})\longrightarrow\omega\mathrm{Gr}(\mathcal{C}).$$

In this section, we shall construct a left adjoint to $|-|$:

$$[-]:\omega\mathrm{Gr}(\mathcal{C})\longrightarrow\omega\mathrm{Cat}(\mathcal{C})$$

Let G be an ω-graph in $\mathcal{C}$. We shall construct an ω-category $[G]$, and ω-graph morphisms $\eta_G:G\longrightarrow[G]$, which is universal from G to $|-|$.

Since $\mathcal{C}$ has pullbacks, for any ud-vector $\vec{u}$, we can define, as in section 2.6, the multi-pullback $G_{\vec{u}}$, for $0\leq i\leq j<\mathrm{lh}(\vec{u})$, the projections

$$\pi^G_i:G_{\vec{u}}\longrightarrow G_{u_{2i}},\qquad \pi^G_{i..j}:G_{\vec{u}}\longrightarrow G_{u_{2i},\dots,u_{2j}},$$

for $l\in\omega$, the morphisms of multi-domain and the multi-codomain in an ω-graph G

$$d^G_{\vec{u};l},c^G_{\vec{u};l}:G_{\vec{u}}\longrightarrow G_{\mathrm{tr}_{(l)}(\vec{u})}$$

and for $\langle\vec{u},\vec{v}\rangle\in\mathrm{UD}_{n_0,n_1,n_2}$ the projection morphisms

$$\pi^G_{0;\vec{u}}:G_{[\vec{u},n_1,\vec{v}]}\longrightarrow G_{\vec{u}}\qquad\qquad \pi^G_{1;\vec{v}}:G_{[\vec{u},n_1,\vec{v}]}\longrightarrow G_{\vec{v}}.$$

Then, the object of n-cells of $[G]$ is the following coproduct

$$[G]_n=\coprod_{\vec{u}\in\mathrm{UD}_n}G_{\vec{u}}$$

with the embedding morphisms $\kappa_{\vec{u}}:G_{\vec{u}}\longrightarrow\coprod_{\vec{u}\in\mathrm{UD}_n}G_{\vec{u}}$, for any ud-vector $\vec{u}\in\mathrm{UD}_n$.

For $n\geq l$, the domain and the codomain morphisms

$$d^{[G]}_{(l)},c^{[G]}_{(l)};[G]_n\longrightarrow[G]_l$$

are so defined that, for any $\vec{u}\in\mathrm{UD}_n$, the squares

$$\begin{tikzcd}
{[G]_n} \arrow[r, "d^{[G]}_{(l)}"] & {[G]_l} \\
G_{\vec{u}} \arrow[u, "\kappa_{\vec{u}}"] \arrow[r, "d_{\vec{u};l}"'] & G_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[u, "\kappa_{\mathrm{tr}_{(l)}(\vec{u})}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{[G]_n} \arrow[r, "c^{[G]}_{(l)}"] & {[G]_l} \\
G_{\vec{u}} \arrow[u, "\kappa_{\vec{u}}"] \arrow[r, "c_{\vec{u};l}"'] & G_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[u, "\kappa_{\mathrm{tr}_{(l)}(\vec{u})}"']
\end{tikzcd}$$

commute. For $n \leq l$, the identity morphisms

$$\iota_{(l)}^{[G]} : [G]_n \longrightarrow [G]_l$$

are so defined that, for any $\vec{u} \in \mathrm{UD}_n$, the triangle

$$\begin{tikzcd}
{[G]_n} \arrow[r, "\iota_{(l)}^{[G]}"] & {[G]_l} \\
G_{\vec{u}} \arrow[u, "\kappa_{\vec{u}}"] \arrow[ur, "\kappa_{\vec{u}}"'] &
\end{tikzcd}$$

commutes.

In Lemma 6.5 we have shown that the square

$$\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "\pi_{1,\vec{v}}^G"] \arrow[d, "\pi_{0,\vec{u}}^G"'] & G_{\vec{v}} \arrow[d, "d_{\vec{v};n_1}^G"] \\
G_{\vec{u}} \arrow[r, "c_{\vec{u};n_1}^G"'] & G_{\vec{w}}
\end{tikzcd}$$

is a pullback. Since the countable coproducts are disjoint and universal in $\mathcal{C}$ they commutes with pullbacks and hence $[G]_{n_0,n_1,n_2}$ in the pullback

$$\begin{tikzcd}
{[G]_{n_0,n_1,n_2}} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & {[G]_{n_2}} \arrow[d, "d"] \\
{[G]_{n_0}} \arrow[r, "c"'] & {[G]_{n_1}}
\end{tikzcd}$$

is given by the coproduct

$$[G]_{n_0,n_1,n_2} = \coprod_{\langle \vec{u},\vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}} G_{[\vec{u},n_1,\vec{v}]}$$

with the embedding morphisms $\kappa_{\vec{u},\vec{v}} : G_{[\vec{u},n_1,\vec{v}]} \longrightarrow [G]_{n_0,n_1,n_2}$, for any $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$. The projections

$$\pi_0 : [G]_{n_0,n_1,n_2} \longrightarrow [G]_{n_0} \qquad \pi_1 : [G]_{n_0,n_1,n_2} \longrightarrow [G]_{n_2}$$

are so defined that, the squares

$$\begin{tikzcd}
{[G]_{n_0,n_1,n_2}} \arrow[r, "\pi_0"] & {[G]_{n_0}} \\
G_{[\vec{u},n_1,\vec{v}]} \arrow[u, "\kappa_{\vec{u},\vec{v}}"] \arrow[r, "\pi_{0,\vec{u}}"'] & G_{\vec{u}} \arrow[u, "\kappa_{\vec{u}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{[G]_{n_0,n_1,n_2}} \arrow[r, "\pi_1"] & {[G]_{n_2}} \\
G_{[\vec{u},n_1,\vec{v}]} \arrow[u, "\kappa_{\vec{u},\vec{v}}"] \arrow[r, "\pi_{1,\vec{v}}"'] & G_{\vec{v}} \arrow[u, "\kappa_{\vec{v}}"']
\end{tikzcd}$$

commute. Similarly, for a ud-vector n_0, n_1, n_2, n_3, n_4, the object of compatible triples of cells in $[G]$ is the coproduct

$$[G]_{n_0,n_1,n_2,n_3,n_4} = \coprod_{\langle\vec{u},\vec{v},\vec{w}\rangle\in \mathrm{UD}_{n_0,n_1,n_2,n_3,n_4}} G_{[\vec{u},n_1,\vec{v},n_3,\vec{w}]}$$

with the embedding morphisms $\kappa_{\vec{u},\vec{v},\vec{w}} : G_{[\vec{u},n_1,\vec{v},n_3,\vec{w}]} \longrightarrow [G]_{n_0,n_1,n_2,n_3,n_4}$, for any $\langle\vec{u},\vec{v},\vec{w}\rangle \in \mathrm{UD}_{n_0,n_1,n_2,n_3,n_4}$. We have a limiting diagram in $\mathcal{C}$

$$\begin{tikzcd}
{[G]_{n_0}} \arrow[d, "c"'] & {[G]_{n_0,n_1,n_2,n_3,n_4}} \arrow[l, "\pi_0"'] \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & {[G]_{n_4}} \arrow[d, "d"] \\
{[G]_{n_1}} & {[G]_{n_2}} \arrow[l, "d"] \arrow[r, "c"'] & {[G]_{n_3}}
\end{tikzcd}$$

The composition morphisms

$$m^{[G]}_{n_0,n_1,n_2} : [G]_{n_0,n_1,n_2} \longrightarrow [G]_{\max(n_0,n_2)}$$

are so defined that, for any $\langle\vec{u},\vec{v}\rangle \in \mathrm{UD}_{n_0,n_1,n_2}$, the triangle

$$\begin{tikzcd}
{[G]_{n_0,n_1,n_2}} \arrow[r, "m^{[G]}_{n_0,n_1,n_2}"] & {[G]_{\max(n_0,n_2)}} \\
G_{[\vec{u},n_1,\vec{v}]} \arrow[u, "\kappa_{\vec{u},\vec{v}}"] \arrow[ur, "\kappa_{[\vec{u},n_1,\vec{v}]}"'] &
\end{tikzcd}$$

commutes. This ends the construction of $[G]$. We have

6.9. THEOREM. *The functor* $|-| : \omega\mathrm{Cat}(\mathcal{C}) \longrightarrow \omega\mathrm{Gr}(\mathcal{C})$ *has a left adjoint*

$$[-] : \omega\mathrm{Gr}(\mathcal{C}) \longrightarrow \omega\mathrm{Cat}(\mathcal{C}).$$

Proof. We verify that $[G]$ is indeed an ω-category and that

$$\eta_G : G \longrightarrow [G]$$

is given, for $n \in \omega$, by

$$\eta_{G,n} = \kappa_n : G_n \longrightarrow [G]_n$$

is a universal morphism from G to $|-|$.

Note that since $[G]_n$, $[G]_{n_0,n_1,n_2}$, and $[G]_{n_0,n_1,n_2,n_3,n_4}$ are defined as coproducts the commutations of diagrams involving such objects reduces to commutations involving objects $G_{\vec{u}}$, $G_{\vec{u};\vec{v}}$, and $G_{\vec{u};\vec{v};\vec{w}}$, where $\vec{u} \in \mathrm{UD}_n$, $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$, and $\langle \vec{u}, \vec{v}, \vec{w} \rangle \in \mathrm{UD}_{n_0,n_1,n_2,n_3,n_4}$, respectively. After such reductions the commutations of the resulting diagrams follows from some equalities concerning ud-vectors and additionally, if these diagrams involve domains and codomains, some facts concerning muti-domains and multi-codomains of ω-graphs (from the previous section) are needed.

First we verify that $[G]$ is and ω-category in $\mathcal{C}$. To show that $d \circ d = d \circ c$, note that, for $n \in \omega$ and $\vec{u} \in \mathrm{UD}$, in the diagram

$$\begin{tikzcd}
{[G]_{n+2}} \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & {[G]_{n+1}} \arrow[r, "d"] & {[G]_n} \\
G_{\vec{u}} \arrow[u, "\kappa_{\vec{u}}"] \arrow[r, shift left, "d_{\vec{u};n+1}"] \arrow[r, shift right, "c_{\vec{u};n+1}"'] & G_{\mathrm{tr}_{(n+1)}(\vec{u})} \arrow[u, "\kappa_{\mathrm{tr}_{(l)}(\vec{u})}"'] \arrow[r, "d_{\mathrm{tr}_{(n+1)}(\vec{u});n}"'] & G_{\mathrm{tr}_{(n)}(\vec{u})} \arrow[u, "\kappa_{\mathrm{tr}_{(n)}(\vec{u})}"']
\end{tikzcd}$$

the right square commutes and the left square commutes serially, by definition of d and c. Thus the equality $d \circ d = d \circ c$ reduces to

$$d_{\mathrm{tr}_{(l)}(\vec{u});n} \circ d_{\vec{u};n+1} = d_{\mathrm{tr}_{(l)}(\vec{u});n} \circ c_{\vec{u};n+1} \tag{50}$$

for any $n \in \omega$ and $\vec{u} \in \mathrm{UD}$. We prove (50) by induction on length $\vec{u}$.

If $\mathrm{lh}(\vec{u}) = 1$ then (50) reduces to $d_n \circ d_{n+1} = d_n \circ c_{n+1}$ which holds in any ω-graph.

Assume that $\mathrm{lh}(\vec{u}) > 1$, $z = \min(\vec{u})$ and that $\vec{u} = \vec{u'}, z, \vec{u''}$. If $z > n$, using inductive hypothesis and Lemma 6.5, we have

$$d_{\mathrm{tr}_{(n+1)}(\vec{u});n} \circ d_{\vec{u};n+1} = d_n \circ d_{\mathrm{tr}_{(n+1)}(\vec{u'});n} \circ \pi_{0;\vec{u'}} =$$

$$d_n \circ c_{\mathrm{tr}_{(n+1)}(\vec{u'});n} \circ \pi_{0;\vec{u'}} = d_n \circ d_{\mathrm{tr}_{(n+1)}(\vec{u''});n} \circ \pi_{1;\vec{u''}} =$$

$$d_n \circ c_{\mathrm{tr}_{(n+1)}(\vec{u''});n} \circ \pi_{1;\vec{u''}} = d_{\mathrm{tr}_{(l)}(\vec{u});n} \circ c_{\vec{u};n+1}$$

If $z = n$, using inductive hypothesis, we have

$$d_{\mathrm{tr}_{(n+1)}(\vec{u});n} \circ d_{\vec{u};n+1} = d_n \circ \pi_0 \circ (d_{\vec{u'};n+1} \times d_{\vec{u''};n+1}) =$$

$$d_n \circ \pi_0 \circ d_{\vec{u'};n+1} \circ \pi_{0;\vec{u'}} = d_n \circ \pi_0 \circ c_{\vec{u'};n+1} \circ \pi_{0;\vec{u'}} =$$

$$d_n \circ \pi_0 \circ (c_{\vec{u'};n+1} \times c_{\vec{u''};n+1}) = d_{\mathrm{tr}_{(n+1)}(\vec{u});n} \circ c_{\vec{u};n+1}$$

If $z < n$, using inductive hypothesis, we have

$$d_{\mathrm{tr}_{(n+1)}(\vec{u});n} \circ d_{\vec{u};n+1} =$$

$$(d_{\mathrm{tr}_{(n+1)}(\vec{u'});n} \times d_{\mathrm{tr}_{(n+1)}(\vec{u''});n}) \circ (d_{\vec{u'};n+1} \times d_{\vec{u''};n+1}) =$$

$$(c_{\mathrm{tr}_{(n+1)}(\vec{u'});n} \times c_{\mathrm{tr}_{(n+1)}(\vec{u''});n}) \circ (c_{\vec{u'};n+1} \times c_{\vec{u''};n+1}) =$$

$$d_{\mathrm{tr}_{(n+1)}(\vec{u});n} \circ c_{\vec{u};n+1}$$

So (50) holds.

The equality $c^{[G]} \circ d^{[G]} = c^{[G]} \circ c^{[G]}$ can be proved similarly.

Ad (vi).1. Let $n_1 \langle n_0, n_2$ and $< \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$. Then in the diagram

$$\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[rrr, "1"] \arrow[dddd, "\pi_{0;\vec{u}}"'] \arrow[dr, "\kappa_{\vec{u};\vec{v}}"] & & & G_{[\vec{u},n_1,\vec{v}]} \arrow[dl, "\kappa_{[\vec{u},n_1,\vec{v}]}"'] \arrow[dddd, "d_{[\vec{u},n_1,\vec{v}];n_1}"] \\
& {[G]_{n_0,n_1,n_2}} \arrow[r, "m"] \arrow[dd, "\pi_0"'] & {[G]_n} \arrow[dd, "d_{(n_1)}"] & \\
& & & \\
& {[G]_{n_0}} \arrow[r, "d_{(n_1)}"'] & {[G]_{n_1}} & \\
G_{\vec{u}} \arrow[rrr, "d_{\vec{u};n_1}"'] \arrow[ur, "\kappa_{\vec{u}}"] & & & G_{\mathrm{tr}_{(n_1)}(\vec{u})} \arrow[ul, "\kappa_{\mathrm{tr}_{(l)}(\vec{u})}"']
\end{tikzcd}$$

the four side squares commute. Thus the commutation of the inner square reduces to the commutation of the outer square for any pair $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$. The commutation of the outer square can be proved by induction on n_1-size of $\vec{u}$. From this (vi).1 follows.

To show (vi).2, let $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$, $l \leq \max(n_0, n_2) = n$, $n'_0 = \min(l, n_0)$, $n'_1 =$

$\min(l, n_1)$, and consider the diagram

$$\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[rrrr, "1"] \arrow[dr, "\kappa_{\vec{u};\vec{v}}"] \arrow[ddd, "d_{\vec{u};l} \times d_{\vec{v};l}"'] & & & & G_{[\vec{u},n_1,\vec{v}]} \arrow[dl, "\kappa_{[\vec{u},n_1,\vec{v}]}"'] \arrow[ddd, "d_{[\vec{u},n_1,\vec{v}];l}"] \\
& {[G]_{n_0,n_1,n_2}} \arrow[rr, "m"] \arrow[d, "d_{(l)} \times d_{(l)}"'] & & {[G]_n} \arrow[d, "d_{(l)}"] & \\
& {[G]_{n_0',n_1,n_2'}} \arrow[rr, "m"'] & & {[G]_l} & \\
G_{[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} \arrow[ur, "\kappa_{\mathrm{tr}_{(l)}(\vec{u});\mathrm{tr}_{(l)}(\vec{v})}"'] \arrow[rrrr, "1"'] & & & & G_{[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} \arrow[ul, "\kappa_{\mathrm{tr}_{(l)}(\vec{u});\mathrm{tr}_{(l)}(\vec{v})}"]
\end{tikzcd}$$

As the four side squares commute, then again the commutation of the inner square reduces to the commutation of the outer square for any pair $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$. This is left as an exercise.

The commutation of (vi).3 and (vi).4 can be proved similarly. The fact that $[G]$ satisfies (vii), (viii), (ix), and (x), essentially follows from the equalities on ud-vectors proved in Lemma 2.4.

We end the verification that $[G]$ is an ω-category in $\mathcal{C}$ by showing that it satisfies Middle Exchange Law (xi). We need to show that for, $l < n_1 < n_0, n_2$, $n = \max(n_0, n_2)$, the diagram

$$\begin{tikzcd}
{[G]_{n_0,n_1,n_2}} \arrow[rr, "\langle m(d_{(l)} \times 1), m(1 \times d_{(l)}) \rangle"] \arrow[drr, "m"'] & & {[G]_{n_2,l,n_0}} \arrow[d, "m"] \\
& & {[G]_n}
\end{tikzcd} \tag{51}$$

commutes. Let $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$. By Lemma 2.4.13, we have $[\vec{u}, n_1, \vec{v}] = [[\vec{u}, n_1, \mathrm{tr}_{(l)}(\vec{v})], l, [\mathrm{tr}_{(l)}(\vec{u}), n_1, \vec{v}]]$. Moreover the square

$$\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "1 \times c_{\vec{v};l}"] \arrow[d, "d_{\vec{u};l} \times 1"'] & G_{[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]} \arrow[d, "d_{\vec{u};l} \times 1"] \\
G_{[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} \arrow[r, "1 \times c_{\vec{v};l}"'] & G_{[\mathrm{tr}_{(l)}(\vec{u}),n_1,\mathrm{tr}_{(l)}(\vec{v})]}
\end{tikzcd}$$

is a pullback. Thus by Lemma 6.5

$$d_{\vec{v};l} \times id = \pi_{0;[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} : G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}],l,[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]]} \longrightarrow G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} \tag{52}$$

$$id \times c_{\vec{v};l} = \pi_{1;[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]} : G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}],l,[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]]} \longrightarrow G_{[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]} \tag{53}$$

Thus to show that (51) commutes , it is enough to show that the following three squares

$$\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "1"] \arrow[d, "\kappa_{\vec{u};\vec{v}}"'] & G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}],l,[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]]} \arrow[d, "\kappa_{[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}];[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]}"] \\
G_{n_0,n_1,n_2} \arrow[r, "\langle m(d_{(l)} \times 1), m(1 \times d_{(l)})\rangle"'] & G_{n_2,l,n_0}
\end{tikzcd} \tag{54}$$

$$\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "\kappa_{\vec{u};\vec{v}}"] \arrow[d, "1"'] & {[G]_{n_0,n_1,n_2}} \arrow[d, "m_{n_0,n_1,n_2}"] \\
G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "\kappa_{[\vec{u}),n_1,\vec{v}]}"'] & {[G]_n}
\end{tikzcd}$$

$$\begin{tikzcd}
{[G]_{n_2,l,n_0}} \arrow[d, "m"'] & G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}],l,[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]]} \arrow[l, "\kappa_{[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}];[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]}"'] \arrow[d, "1"] \\
{[G]_n} & G_{[\vec{u},n_1,\vec{v}]} \arrow[l, "\kappa_{\vec{u};\vec{v}}"]
\end{tikzcd}$$

commute. The second and the third squares commute by definition of m_{n_0,n_1,n_2} and m_{n_2,l,n_0}, respectively. Thus it remains to show that (54) commutes, as well. As $[G]_{n_0,n_1,n_2}$ is a pullback, we shall show that the square (54) commutes, when composed with the projections

$$\pi_0 : [G]_{n_2,l,n_0} \longrightarrow [G]_{n_2} \qquad \pi_1 : [G]_{n_2,l,n_0} \longrightarrow [G]_{n_0}.$$

Composing (54) with π_0, we get the diagram

$$
\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[rr, "1"] \arrow[d, "\kappa_{\vec{u};\vec{v}}"'] \arrow[dddr, bend right=60, "d_{\vec{u}\times 1}"'] & & G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}],l,[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]]} \arrow[d] \arrow[dddl, bend left=60, "\pi_{0;[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]}"] \\
{[G]_{n_0,n_1,n_2}} \arrow[rr, "{\langle m(d_{(l)}\times 1), m(1\times d_{(l)})\rangle}"'] \arrow[d, "d_{(l)}\times 1"'] & & {[G]_{n_2,l,n_0}} \arrow[d, "\pi_0"] \\
{[G]_{l,n_1,n_2}} \arrow[rr, "m"'] & & {[G]_{n_2}} \\
& G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

where the unmarked morphisms are the obvious embeddings into coproducts. The outer triangle commutes as it is (52). The left and right squares commute, by definition of $d_{(l)}$ and π_0, respectively. The triangle at the bottom commutes, by definition of m_{l,n_1,n_2}. Hence (54) followed by π_0 commutes.

Composing (54) with π_1, we get the diagram

$$
\begin{tikzcd}
G_{[\vec{u},n_1,\vec{v}]} \arrow[rr, "1"] \arrow[d, "\kappa_{\vec{u};\vec{v}}"'] \arrow[dddr, bend right=60, "1\times c_{\vec{u}}"'] & & G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}],l,[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]]} \arrow[d] \arrow[dddl, bend left=60, "\pi_{1;[\vec{u},n_1,\mathrm{tr}_{(l)}(\vec{v})]}"] \\
{[G]_{n_0,n_1,n_2}} \arrow[rr, "{\langle m(d_{(l)}\times 1), m(1\times d_{(l)})\rangle}"'] \arrow[d, "1\times c_{(l)}"'] & & {[G]_{n_2,l,n_0}} \arrow[d, "\pi_1"] \\
{[G]_{n_0,n_1,l}} \arrow[rr, "m"'] & & {[G]_{n_0}} \\
& G_{[[\mathrm{tr}_{(l)}(\vec{u}),n_1,\vec{v}]} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

which by similar reasons as before shows that (54) followed by π_1 commutes. Therefore (54) commutes as well, and $[G]$ satisfy (xi). In this way we have proved that $[G]$ is an ω-category in $\mathcal{C}$.

Now we show that $\eta : G \longrightarrow [G]$ is a universal morphism. So assume that $\overline{F}$: $[G] \longrightarrow A$ is an ω-graph morphism into an ω-category A. We construct an ω-functor $\overline{F} : [G] \longrightarrow A$, extending F.

Since F is an ω-graph morphism, it preserves both domains and codomains, and hence it induces a morphism between pullbacks

$$F_{\vec{u}} : G_{\vec{u}} \longrightarrow A_{\vec{u}}$$

for any ud-vector $\vec{u}$. Then, for $m \in \omega$, we define the morphism

$$\overline{F}_k : [G]_k \longrightarrow A_k$$

so that, for any ud-vector $\vec{u} \in \mathrm{UD}_k$, its composition with the embedding $\kappa_{\vec{u}} : G_{\vec{u}} \longrightarrow [G]_k$ is the following morphism:

$$\begin{tikzcd} G_{\vec{u}} \arrow[r, "F_{\vec{u}}"] & A_{\vec{u}} \arrow[r, "m^A_{\vec{u}}"] & A_{\mathrm{ht}(\vec{u})} \arrow[r, "\iota^A_{(k)}"] & A_k \end{tikzcd}$$

To see that $\overline{F}$ so defined preserves domains, it is enough to note that, for any $l \leq n$, $\vec{u} \in \mathrm{UD}_n$, the following three squares

$$\begin{tikzcd} G_{\vec{u}} \arrow[r, "F_{\vec{u}}"] \arrow[d, "d_{\vec{u};l}"'] & A_{\vec{u}} \arrow[r, "m"] \arrow[d, "d_{\vec{u};l}"'] & A_u \arrow[r, "\iota_{(n)}"] \arrow[d, "d_{(l')}"] & A_n \arrow[d, "d_{(l)}"] \\ G_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[r, "F_{\mathrm{tr}_{(l)}(\vec{u})}"'] & A_{\mathrm{tr}_{(l)}(\vec{u})} \arrow[r, "m"'] & A_{l'} \arrow[r, "\iota_{(l)}"'] & A_l \end{tikzcd}$$

commute, where $u = \mathrm{ht}(\vec{u})$ and $l' = \min(l, u)$. The commutations of these squares can be easily shown by induction on l-size of $\vec{u}$.

The fact that $\overline{F}$ preserves codomains and identities is left for the reader.

To prove that $\overline{F}$ preserves compositions, we need to show that for $n_1 < n_0, n_2$, the diagram

$$\begin{tikzcd} {[G]_{n_0,n_1,n_2}} \arrow[r, "\overline{F}_{n_0} \times \overline{F}_{n_2}"] \arrow[d, "m"'] & A_{n_0,n_1,n_2} \arrow[d, "m"] \\ {[G]_n} \arrow[r, "\overline{F}_n"'] & A_n \end{tikzcd}$$

commutes. But this reduces to the commutation of the following diagram

$$\begin{tikzcd} G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "F_{\vec{u}} \times F_{\vec{v}}"] \arrow[d, "1"'] & A_{[\vec{u},n_1,\vec{v}]} \arrow[r, "m_{\vec{u}} \times m_{\vec{v}}"] \arrow[d, "1"] & A_{u,n_1,v} \arrow[r, "\iota_{(n_0)} \times \iota_{(n_2)}"] \arrow[d, "m"] & A_{n_0,n_1,n_2} \arrow[d, "m"] \\ G_{[\vec{u},n_1,\vec{v}]} \arrow[r, "F_{[\vec{u},n_1,\vec{v}]}"'] & A_{[\vec{u},n_1,\vec{v}]} \arrow[r, "m_{[\vec{u},n_1,\vec{v}]}"'] & A_k \arrow[r, "\iota_{(n)}"'] & A_n \end{tikzcd}$$

for any $\langle \vec{u}, \vec{v} \rangle \in \mathrm{UD}_{n_0,n_1,n_2}$, where $u = \mathrm{ht}(\vec{u})$, $u = \mathrm{ht}(\vec{v})$, $k = \max(u, v)$, $n = \max(n_0, n_2)$. The commutation of the left square is trivial, the commutation of the middle square follows from Lemma , and the commutation of the right square follows from axioms (x).1 and (x).3. of the definition of the ω-category. Thus $\overline{F}$ is an ω-functor.

The uniqueness of $\overline{F}$ can be proved by a similar easy inductive argument. ∎

Having the explicit description of the free internal ω-categories we can easily show the following corollary.

6.10. Corollary. *The functor*

$$[-] : \omega\mathrm{Gr}(\mathcal{C}) \longrightarrow \omega\mathrm{Cat}(\mathcal{C})$$

preserves pullbacks.

Remark. This result appears in [S]; see the Proposition in Section 5 in [S].

Proof. Suppose that

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"'] & H \arrow[d, "g"] \\ G \arrow[r, "f"'] & K \end{tikzcd}$$

is a pullback in $\omega\mathrm{Gr}(\mathcal{C})$. Thus for $n \in \omega$, the square

$$\begin{tikzcd} P_n \arrow[r, "k_n"] \arrow[d, "h_n"'] & H_n \arrow[d, "g_n"] \\ G_n \arrow[r, "f_n"'] & K_n \end{tikzcd}$$

is a pullback in $\mathcal{C}$. Since limits commutes with limits and k, h, f, g preserves domains and codomains, for any ud-vector $\vec{u}$, the square

$$\begin{tikzcd} P_{\vec{u}} \arrow[r, "k_{\vec{u}}"] \arrow[d, "h_{\vec{u}}"'] & H_{\vec{u}} \arrow[d, "g_{\vec{u}}"] \\ G_{\vec{u}} \arrow[r, "f_{\vec{u}}"'] & K_{\vec{u}} \end{tikzcd}$$

is a pullback in $\mathcal{C}$, as well. Then, since coproducts are commutes with pullbacks in $\mathcal{C}$, the coproduct of pullbacks

$$\begin{tikzcd}
\coprod_{\vec{u}\in \mathrm{UD}_n} P_{\vec{u}} \arrow[r, "\coprod_{\vec{u}\in \mathrm{UD}_n} k_{\vec{u}}"] \arrow[d, "\coprod_{\vec{u}\in \mathrm{UD}_n} h_{\vec{u}}"'] & \coprod_{\vec{u}\in \mathrm{UD}_n} H_{\vec{u}} \arrow[d, "\coprod_{\vec{u}\in \mathrm{UD}_n} g_{\vec{u}}"] \\
\coprod_{\vec{u}\in \mathrm{UD}_n} G_{\vec{u}} \arrow[r, "\coprod_{\vec{u}\in \mathrm{UD}_n} f_{\vec{u}}"'] & \coprod_{\vec{u}\in \mathrm{UD}_n} K_{\vec{u}}
\end{tikzcd} \tag{55}$$

is again a pullback in $\mathcal{C}$. But the above square is just

$$\begin{tikzcd}
[P]_n \arrow[r, "[k]_n"] \arrow[d, "[h]_n"'] & [H]_n \arrow[d, "[g]_n"] \\
[G]_n \arrow[r, "[f]_n"'] & [K]_n
\end{tikzcd}$$

and hence $[-]$ preserves pullbacks.endproof

7. Notation

The numbers in square brackets indicate the page or pages where the notation is first introduced.

Conventions:

1. The ud-vectors are invariants for finite trees, finite disks, simple ω-graphs, and simple ω-categories. For a given ud-vector $\vec{u}$ we define a specific finite tree $\theta_{\vec{u}}$ [123, 158], a specific finite disk $\gamma_{\vec{u}}$ [146, 156] (ud-vector in subscript), a specific simple ω-graph $\alpha^{\vec{u}}$ [175, 176], and a specific simple ω-category $\delta^{\vec{u}}$ [184] (ud-vector in superscript). All the above structures are 'graded' structures. The levels of trees and of disks are marked in superscripts, and the levels of ω-graphs, and of simple ω-categories are marked in subscripts. Thus, for example, D^n is the n-th level of the disk D, whereas A_n is the set of n-cells of the ω-category A. Adopting this convention, we have for any $n, s \geq 0$ that

$$\gamma_n^s = \delta_n^s$$

   i.e. the s-th level of γ_n is equal to the n-th level of δ^s.

2. The operations in (specific) structures are not in bold-face, whereas the operations on (specific) structures are in **bold**-face. Thus, for example, for $s \geq s'$ and any

natural n, the projection inside a disk γ_n from level s to level s' is marked $p : \gamma_n^s \longrightarrow \gamma_n^{s'}$, and the projection morphism between the simple categories δ^s to $\delta^{s'}$ is marked $\mathbf{p} : \delta^s \longrightarrow \delta^{s'}$. Then the some specific instances of bold and not bold morphisms coincide, e.g.

$$p_n^s : \gamma_n^{s+1} \longrightarrow \gamma_n^s \;=\; \mathbf{p}_n^s : \delta_n^{s+1} \longrightarrow \delta_n^s.$$

7.1. ud-vectors.

1. $\vec{u}$: a ud-vector [128];
2. Some functions defined on ud-vectors:
   (a) $\mathrm{lt}(\vec{u})$: the length of a ud-vector $\vec{u}$ [128];
   (b) $\mathrm{ht}(\vec{u})$: the height of a ud-vector $\vec{u}$ [128];
   (c) $\mathrm{size}_{(l)}(\vec{u})$: the l-size of $\vec{u}$ [129];
3. Some sets of ud-vectors:
   (a) UD_n: the set of ud-vectors of height less or equal $n \in \omega$ [131];
   (b) $\mathrm{UD}_{n_0,n_1,n_2}$: the set of pairs of n_1-compatible ud-vectors of height less or equal n_0 and n_2, respectively [131];
4. Some operations defined on ud-vectors:
   (a) $\mathrm{tr}_{(l)}(\vec{u})$: the l-truncation of $\vec{u}$ [129];
   (b) $[\vec{u}, l, \vec{v}]$: the l-amalgam of l-compatible pair ud-vectors $\vec{u}$ and $\vec{v}$ [129].

7.2. Trees, bundles, and disks.

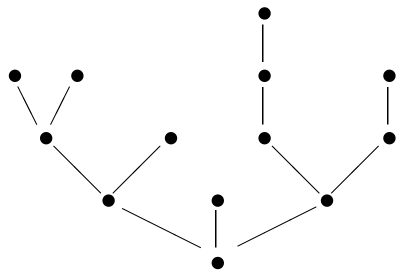
1. A tree T [123]:

$$\cdots \xrightarrow{p^{s+1}} T^{s+1} \xrightarrow{p^s} T^s \quad \cdots \quad T^1 \xrightarrow{p^0} T^0 \cong 1$$

2. Some special trees:
   (a) $\theta_{\vec{u}}$: the tree corresponding to a ud-vector $\vec{u}$ [123, 158];
   (b) The tree θ_3 can be drawn as [123]:

$$\begin{array}{c} \mathbf{3} \\ | \\ \mathbf{2} \\ | \\ \mathbf{1} \\ | \\ \mathbf{0} \end{array}$$

(c) The tree $\theta_{3,2,3,1,2,0,1,0,4,1,3}$ can be drawn as [158]:



3. A bundle of intervals over B [123]:

$$\begin{tikzcd} R \arrow[r, shift left=2, "r^0"] \arrow[r, shift right=2, "r^1"'] & X \arrow[r, "p"] & B \arrow[l, shift right=2, "b"'] \arrow[l, shift left=2, "t"] \end{tikzcd}$$

4. A (finite) disk D [123]:

$$\begin{tikzcd} \cdots \arrow[r, "p^{s+1}"] & D^{s+1} \arrow[l, shift right=3, "b^{s+1}"'] \arrow[l, shift left=3, "t^{s+1}"] \arrow[r, "p^s"] & D^s \arrow[l, shift right=3, "b^s"'] \arrow[l, shift left=3, "t^s"] & \cdots & D^1 \arrow[r, "p^0"] & D^0 \cong 1 \arrow[l, shift right=3, "b^0"'] \arrow[l, shift left=3, "t^0"] \end{tikzcd}$$

(a) $\partial(D^n) = b(D^{n-1}) \cup c(D^{n-1})$: the boundary of D^n [124];

(b) $\imath(D^n) = D^n \setminus \partial(D^n)$: the interior of D^n [124];

(c) $\imath(D)$: the internal tree of the disk D [124];

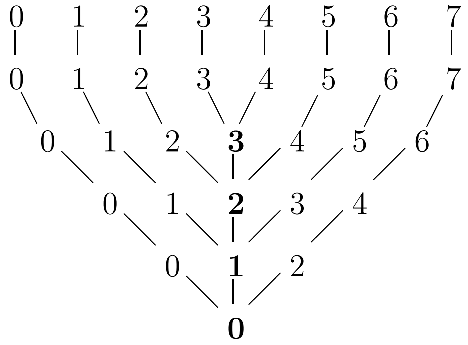
(d) $x \lhd y$: the successor in a fiber relation, $x, y \in D_n$, $n \in \omega$ [145];

(e) $\mu^D_{x,y} = \max\{l : p^{(l)}(x) = p^{(l)}(y)\}$: the level on which the nodes x and y of D do match [145].

5. Some special disks:

(a) $\gamma_{\vec{u}}$: the disk corresponding to a ud-vector $\vec{u}$ [156];

(b) The first six levels of γ_3 (the inner nodes are marked bold) [146]:



(c) A bundle in a disk γ_n over γ_n^s, $s \geq 0$,$n \geq 0$ [192]:

$$\begin{tikzcd}
\leq_n^{s+1} \arrow[r, shift left=2, "\rho_n^{0;s+1}"] \arrow[r, shift right=2, "\rho_n^{1;s+1}"'] & \gamma_n^{s+1} \arrow[r, "p_n^s" description] & \gamma_n^s \arrow[l, shift right=3, "b_n^{s+1}"'] \arrow[l, shift left=3, "t_n^{s+1}"]
\end{tikzcd}$$

6. Some special disk morphisms:

   (a) $\mathbf{d} = \mathbf{d}_{(s)} = \mathbf{d}_{(n;s)}$, $c = c_{(s)} = c_{(n;s)} : \gamma_n \longrightarrow \gamma_s$: the domain and the codomain morphisms [148];

   (b) $\mathbf{m}_{\vec{u}} : \gamma_{\vec{u}} \longrightarrow \gamma_{\mathrm{ht}(\vec{u})}$: the compositions [164];

   (c) $\boldsymbol{\iota} = \boldsymbol{\iota}_{(l)} = \boldsymbol{\iota}_{(s;l)} : \gamma_s \longrightarrow \gamma_l$: the identity morphisms, for $l \geq s$ [164];

   (d) in the diagram: $l \leq n$

$$\begin{tikzcd}
\gamma_n^s \arrow[r, shift left=3, "\mathbf{d}_{(l)}^s"] \arrow[r, shift right=3, "\mathbf{c}_{(l)}^s"'] & \gamma_l^s \arrow[l, "\boldsymbol{\iota}_{(n)}^s" description]
\end{tikzcd}$$

   (e) $\boldsymbol{\pi}_i : \gamma_{\vec{u}} \longrightarrow \gamma_{u_{2i}}$: the projections, for $i \in \mathrm{lh}(\vec{u})$ [156]:

$$\begin{tikzcd}[column sep=small]
 & & & \gamma_{\vec{u}} \arrow[dlll, "\boldsymbol{\pi}_0"'] \arrow[dl, "\boldsymbol{\pi}_2"'] \arrow[dr, "\boldsymbol{\pi}_{2k-1}"] \arrow[drrr, "\boldsymbol{\pi}_{2k}"] & & & \\
\gamma_{u_0} \arrow[dr, "c_{u_1}"'] & & \gamma_{u_2} \arrow[dl, "\mathbf{d}_{u_1}"] & \ldots & \gamma_{u_{2k-2}} \arrow[dr, "c_{u_{2k-1}}"'] & & \gamma_{u_{2k}} \arrow[dl, "\mathbf{d}_{u_{2k}}"] \\
 & \gamma_{u_1} & & & & \gamma_{u_{2k-1}} &
\end{tikzcd}$$

(f) n-cuts of disks (outer morphisms into γ_n) [168]:
$\overline{x} : D \longrightarrow \gamma_n$: if $x \in D^n$ is a leaf in D [158];
$\overline{x;y} : D \longrightarrow \gamma_{n_1}$: if $x \neq y$, $n_1 = \mu_{x,y}$, $p^{n_1+1}(x) \lhd p^{n_1+1}(y)$ [158];

(g) the canonical factorization of a morphism f in $\mathcal{D}$ [166]:

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\overset{\circ}{f}"'] & \gamma_n \\ \gamma_{\vec{u}} \arrow[r, "\mathbf{m}_{\vec{u}}"'] & \gamma_u \arrow[u, "\boldsymbol{\iota}_{(n)}"'] \end{tikzcd}$$

with $\overset{\circ}{f}$ outer part of f, $\boldsymbol{\iota}_{(n)}$ inner mono, $\mathbf{m}_{\vec{u}}$ inner epi;

(h) $\mathrm{Cut}(D)$: the ω-graph of cuts of D [168];

(i) A (fragment of) an internal ω-category $\mathbf{C}$ in $\mathcal{D}$ at level s, $s \geq 0$, $n \geq 0$ [167]:

$$\begin{tikzcd} \gamma^s_{n+1,n,n+1} \arrow[r, shift left=2, "\boldsymbol{\pi}^s_{0,n+1}"] \arrow[r, "\mathbf{m}^s_{n+1,n,n+1}" description] \arrow[r, shift right=2, "\boldsymbol{\pi}^s_{1,n+1}"'] & \gamma^s_{n+1} \arrow[r, shift left=2, "\mathbf{d}^s_n"] \arrow[r, shift right=2, "\mathbf{c}^s_n"'] & \gamma^s_n \arrow[l, "\boldsymbol{\iota}^s_{n+1}" description] \end{tikzcd}$$

7.3. ω-graphs and ω-categories.

1. A general ω-graph G [125, 210]:

$$\cdots \quad G_{n+1} \underset{c_n}{\overset{d_n}{\rightrightarrows}} G_n \quad \cdots$$

(a) $d^G_{\vec{u};l}, c^G_{\vec{u};l} : G_{\vec{u}} \longrightarrow G_{\mathrm{tr}_{(l)}(\vec{u})}$: the multi-domain and the multi-codomain morphisms [136];

(b) $G_{n+1}(x,y)$: hom-set of $n+1$-cells with domain x and codomain y in G [125];

(c) $x \rhd y$: predecessor relation [125];

(d) $\max(G_n)$: the set of maximal elements in G_n [127];

(e) $\max(G_{n+1}, x, y)$: the maximal element of $G_{n+1}(x,y)$ [127];

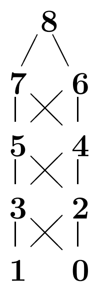
(f) $x \perp y$: x is comparable with y [145];

(g) $\nu^G_{e,e'} = \max\{l : d_{(l)}(e) \perp d_{(l)}(e')\}$: the level on which domains (and codomains) of cells e and e' are comparable [175].

2. Some special simple ω-graphs:

(a) $\alpha^{\vec{u}}$: the ω-graph corresponding to a ud-vector $\vec{u}$ [176];

(b) α^4 can be pictured as follows [175]:



3. Some special morphisms of simple ω-graphs:

(a) $\mathbf{b},\ \mathbf{t} : \alpha^n \longrightarrow \alpha^{n'}$: the bottom and the top morphisms [176];

(b) $\mathbf{b}^{\vec{u};l},\ \mathbf{t}^{\vec{u};l} : \alpha^{\mathrm{tr}_{(l)}(\vec{u})} \longrightarrow \alpha^{\vec{u}}$: the multi-bottom and the multi-top morphisms [177];

(c) $\boldsymbol{\kappa}^i : \alpha^{u_{2i}} \longrightarrow \alpha^{\vec{u}}$: the coprojections into $\alpha^{\vec{u}}$ [176]:

$$\begin{tikzcd}
 & & & \alpha^{\vec{u}} & & & \\
\alpha^{u_0} \arrow[urrr, "\boldsymbol{\kappa}^0"] & & \alpha^{u_2} \arrow[ur, "\boldsymbol{\kappa}^2"] & \cdots & \alpha^{u_{2k-2}} \arrow[ul, "\boldsymbol{\kappa}^{2k-2}"'] & & \alpha^{u_{2k}} \arrow[ulll, "\boldsymbol{\kappa}^{2k}"'] \\
 & \alpha^{u_1} \arrow[ul, "\mathbf{b}^{u_0}"] \arrow[ur, "\mathbf{t}^{u_2}"'] & & & & \alpha^{u_{2k-1}} \arrow[ul, "\mathbf{b}^{u_{2k-2}}"] \arrow[ur, "\mathbf{t}^{u_{2k}}"'] &
\end{tikzcd}$$

(d) $\boldsymbol{\kappa}^{0;\vec{u}} : \alpha^{\vec{u}} \longrightarrow \alpha^{[\vec{u},l,\vec{v}]}$, $\quad \boldsymbol{\kappa}^{1;\vec{v}} : \alpha^{\vec{v}} \longrightarrow \alpha^{[\vec{u},l,\vec{v}]}$: the coprojections into $\alpha^{[\vec{u},l,\vec{v}]}$ [177].

4. A general ω-category A [125, 210]:

(a) $d^A_{(s)} = d^A_{n;s}, c^A_{(s)} = c^A_{n;s} : A_n \longrightarrow A_s$: the domain and the codomain morphisms [210];

(b) $d^A_{\vec{u};l}, c^A_{\vec{u};l} : A_{\vec{u}} \longrightarrow A_{\mathrm{tr}_{(l)}(\vec{u})}$: the multi-domain and the multi-codomain morphisms [136];

(c) $\iota^A_{(n,l)} = \iota^A_{(l)} = \iota^A : A_n \longrightarrow A_l$: the identities [211];

(d) $m^A_{n_0,n_1,n_2} = m^A_{n_1} = m^A : A_{n_0,n_1,n_2} \longrightarrow A_{\max(n_0,n_2)}$: the compositions [211];

(e) $m^A_{\vec{u}} : A_{\vec{u}} \longrightarrow A_{\mathrm{ht}(\vec{u})}$: the canonical composition morphism [138];

(f) $\pi^A_{i..j} : A_{\vec{u}} \longrightarrow A_{u_{2i},\ldots,u_{2j}}$: the projection $i \leq j$ [136];

(g) $\pi^A_{o;\vec{u}} : A_{[\vec{u},l,\vec{v}]} \longrightarrow A_{\vec{u}}$, $\quad \pi^A_{1;\vec{v}} : A_{[\vec{u},l,\vec{v}]} \longrightarrow A_{\vec{v}}$: the projections [137].

5. The free ω-categories:

   (a) $[G]$: the free ω-category on a ω-graph G [179, 225];

   (b) $[G]_n = \coprod_{\vec{u}\in \mathrm{UD}_n} G_{\vec{u}}$: the set/object of n-cells in $[G]$ [225];

   (c) $\kappa_{\vec{u}} = \kappa_{\vec{u};n} : G_{\vec{u}} \longrightarrow [G]_n$: the coprojections [225];

   (d) $[G]_{n_0,n_1,n_2} = \coprod_{(\vec{u},\vec{v})\in \mathrm{UD}_{n_0,n_1,n_2}} G_{[\vec{u},n_1,\vec{v}]}$: the set/object of n_1-compatible pairs of n_0- and n_2-cells in $[G]$ [226];

   (e) $\kappa_{\vec{u},\vec{v}} = \kappa_{\vec{u},\vec{v};n_0,n_1,n_2} : G_{[\vec{u},n_1,\vec{v}]} \longrightarrow [G]_{n_0,n_1,n_2}$: the coprojections [226]; ;

   (f) $\delta^{\vec{u}} = [\alpha^{\vec{u}}]$: the simple ω-category corresponding to a ud-vector $\vec{u}$ [184];

   (g) A fragment of a simple ω-category δ^s in $\mathcal{S}$, $s \geq 0$, $n \geq 0$ [184]:

$$\begin{tikzcd} \delta^s_{n+1} \times_{\delta^s_n} \delta^s_{n+1} \arrow[r, shift left=2, "\pi^s_{0,n+1}"] \arrow[r, "m^s_{n+1,n,n+1}"] \arrow[r, shift right=2, "\pi^s_{1,n+1}"'] & \delta^s_{n+1} \arrow[r, shift left=2, "d^s_n"] \arrow[r, shift right=2, "c^s_n"'] & \delta^s_n \arrow[l, "\iota^s_{n+1}"'] \end{tikzcd}$$

   (h) $l \leq n$

$$\begin{tikzcd} \delta^s_n \arrow[r, shift left=2, "d^s_{(l)}"] \arrow[r, shift right=2, "c^s_{(l)}"'] & \delta^s_l \arrow[l, "\iota^s_{(n)}"'] \end{tikzcd}$$

   the composite morphisms;

   (i) mac_G and mac_S: the maximal cells in $[G]$ and S, respectively (if exist) [126].

6. Some special morphisms of simple ω-categories:

   (a) $\mathbf{b},\ \mathbf{t} : \delta^s \longrightarrow \delta^{s'}$: the bottom and the top morphisms [185];

   (b) $\mathbf{b}^{\vec{s};l},\ \mathbf{t}^{\vec{s};l} : \delta^{\mathrm{tr}_{(l)}(\vec{s})} \longrightarrow \delta^{\vec{s}}$: the multi-bottom and the multi-top morphisms [185];

   (c) $\boldsymbol{\kappa}^{0;\vec{u}} : \delta^{\vec{u}} \longrightarrow \delta^{[\vec{u},l,\vec{v}]}$ $\quad\boldsymbol{\kappa}^{1;\vec{v}} : \delta^{\vec{v}} \longrightarrow \delta^{[\vec{u},l,\vec{v}]}$ the embeddings into the coproduct [185];

   (d) $l \leq s$

$$\begin{tikzcd} \delta^s_n \arrow[r, "\mathbf{p}^{(l)}_n"] & \delta^l_n \arrow[l, shift right=2, "\mathbf{t}^{(s)}_n"'] \arrow[l, shift left=2, "\mathbf{b}^{(s)}_n"] \end{tikzcd}$$

   the composite morphisms;

   (e) $\downarrow : [G] \longrightarrow \delta^0$, $\quad e{\downarrow}e' : [G] \longrightarrow \delta^{s+1}$: the morphisms into δ^n, $e \rhd e'$, $e, e' \in G_s$ [186];

(f) $\mathbf{p}^s : \delta^{s+1} \longrightarrow \delta^s$: the projection [187];

(g) $\boldsymbol{\rho}^{0;s+1}, \boldsymbol{\rho}^{1;s+1} : \delta^{s+1,s,s+1} \longrightarrow \delta^{s+1}, \qquad \boldsymbol{\rho}^{s+1} : \delta^{s+1} \longrightarrow \delta^{s+1,s,s+1}$: the order on δ^{s+1} [187];

(h) The (a fragment of) the internal disk $\mathbf{D}$ in $\mathcal{S}$ at level n, $s \geq 0$, $n \geq 0$ [187]; :

$$\begin{tikzcd} \delta_n^{s+1,s,s+1} \arrow[r, shift left=2, "\boldsymbol{\rho}_n^{0;s+1}"] \arrow[r, shift right=2, "\boldsymbol{\rho}_n^{1;s+1}"'] & \delta_n^{s+1} \arrow[r, "\mathbf{p}_n^s"] & \delta_n^s \arrow[l, shift right=4, "\mathbf{b}_n^s"'] \arrow[l, shift left=4, "\mathbf{t}_n^s"] \end{tikzcd}$$

(i) $\boldsymbol{\varrho}^{\vec{u}} : \delta^u \longrightarrow \delta^{\vec{u}}$: the unique essential morphism from δ^u to $\delta^{\vec{u}}$ [201].

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