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ON SIFTED COLIMITS AND GENERALIZED VARIETIES

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ABSTRACT. Filtered colimits, i.e., colimits over schemes $\mathcal{D}$ such that $\mathcal{D}$-colimits in **Set** commute with finite limits, have a natural generalization to sifted colimits: these are colimits over schemes $\mathcal{D}$ such that $\mathcal{D}$-colimits in **Set** commute with finite products. An important example: reflexive coequalizers are sifted colimits. Generalized varieties are defined as free completions of small categories under sifted-colimits (analogously to finitely accessible categories which are free filtered-colimit completions of small categories). Among complete categories, generalized varieties are precisely the varieties. Further examples: category of fields, category of linearly ordered sets, category of nonempty sets.

Introduction

Filtered colimits belong, no doubt, to the most basic concepts of category theory. Let us just recall the notion of a finitely presentable object as one whose hom-functor preserves filtered colimits. (This, in every variety of algebras, is equivalent to the usual – less elegant – algebraic definition.)

Now, filtered colimits are characterized as colimits with domains (or diagram schemes) $\mathcal{D}$ such that $\mathcal{D}$-colimits commute in **Set** with finite limits. In the present paper we work with a wider class of colimits: sifted colimits, i.e., colimits of diagrams whose domain $\mathcal{D}$ is such that $\mathcal{D}$-colimits commute with finite products in **Set**. Important example: reflexive coequalizers (i.e., coequalizers of pairs $f_1, f_2 : A \to B$ for which $d : B \to A$ exists with $f_1 d = f_2 d = id$). We call an object A strongly finitely presentable if its hom-functor preserves sifted colimits. This implies, of course, that A is finitely presentable. But, due to the reflexive coequalizers, A is also a regular projective. In a variety[1], strongly finitely presentable algebras are precisely the finitely presentable regular projectives (i.e., precisely the retracts of free algebras on finitely many generators). This is what H.-E. Porst calls varietal generator (see [P]) and M.-C. Pedicchio and R. Wood call effective projective in [PW].

Recall the concept of a finitely accessible category of C. Lair [L_1] and M. Makkai and R. Paré [MP]: it is a category $\mathcal{K}$ with filtered colimits and a set of finitely presentable objects whose closure under filtered colimits is all of $\mathcal{K}$. We introduce the natural restriction by substituting "filtered" by "sifted": We call a category $\mathcal{K}$ a *generalized variety* if it has sifted colimits and a set of strongly finitely presentable objects whose closure under sifted colimits is all of $\mathcal{K}$. Every variety has this property, and among complete categories,

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[1]Throughout the paper by a variety we mean a category of finitary algebras (possibly many-sorted) presented by equations.

varieties are the only ones. But there are other interesting examples: for categories with connected limits, to be a generalized variety is equivalent to being multialgebraic in the sense of Y. Diers (e.g., the category of fields and homomorphisms and the category of linearly ordered sets and order-preserving maps are multialgebraic).

An example of a generalized variety which is not multialgebraic is the category of all nonempty sets and functions.

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1. Sifted colimits

1.1. DEFINITION. *A small category $\mathcal{D}$ is called* sifted *if colimits over $\mathcal{D}$ commute in* **Set** *with finite products.*

1.2. REMARK. (a) Explicitly, $\mathcal{D}$ is sifted iff it is nonempty (and thus, colimits over $\mathcal{D}$ commute with empty product) and given diagrams $D_1, D_2 : \mathcal{D} \to \mathbf{Set}$, then the canonical map

$$colim(D_1 \times D_2) \to colim D_1 \times colim D_2$$

is an isomorphism.

(b) By *sifted colimits* we mean colimits over sifted categories.

(c) Filtered colimits are sifted, of course. In fact, small filtered categories $\mathcal{D}$ can be *defined* by the property that colimits over $\mathcal{D}$ commute in **Set** with finite limits.

(d) Every category $\mathcal{D}$ with finite coproducts is sifted. This follows from Theorem 1.5 below.

(e) Commutation of $\mathcal{D}$-colimits with products of pairs has been studied by C. Lair in [L$_2$]. There he introduced the concept of a *tamisante category* $\mathcal{D}$ as a category such that for every pair (A, B) of objects the category of all cospans with domains A, B is connected. And he proved that every tamisante category $\mathcal{D}$ has the property that $\mathcal{D}$-colimits commute in **Set** with products of pairs. We prove that this sufficient condition is in fact also necessary. We present a full proof of the necessity below based on a concept of "morphism of zig-zags", but the main idea of that proof has been taken over from [L$_2$].

Explicitly: the category of cospans with domains A, B, denoted by $(A, B) \downarrow \mathcal{D}$, has as objects all cospans

$$A \xleftarrow{\varphi_0} Z \xrightarrow{\varphi_1} B$$

in $\mathcal{D}$ and as morphisms all commutative diagrams

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & & B \arrow[ul, "\varphi_1"'] \arrow[dl, "\varphi_1'"] \\ & Z' & \end{tikzcd}$$

in $\mathcal{D}$.

1.3. Example. Reflexive coequalizers are sifted colimits: they are easily seen to be precisely the colimits over the category $\mathcal{D}$ obtained from the following graph

$$\begin{tikzcd} A \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & B \arrow[l, "d"'] \end{tikzcd}$$

modulo the equations

$$f_1 d = f_2 d = id \ .$$

In **Set**, reflexive coequalizers commute with finite products. In fact, given $D_i : \mathcal{D} \to \mathbf{Set}$, a coequalizer of $D_i f_1$ and $D_i f_2$ is the canonical map $c_i : DB \to DB/\sim_i$ where for two elements $x, y \in DB$ we have $x \sim_i y$ iff x can be connected with y by a $(D_i f_1, D_i f_2)$-zig-zag. The reflexivity of the pair f_1, f_2 guarantees that given $x \sim_1 y$ and $u \sim_2 v$ we can choose those two zig-zags to be of the same type and hence, to render a $(D_1 f_1 \times D_2 f_1, D_1 f_2 \times D_2 f_2)$-zig-zag between (x, u) and (y, v).

1.4. Remark. (a) In the following theorem we work with zig-zags in a category $\mathcal{D}$, i.e., diagrams of the following form

(1) $$Z_0 \xrightarrow{\varphi_0} Z_1 \xleftarrow{\varphi_1} Z_2 \xrightarrow{\varphi_2} \cdots Z_{n-1} \xleftarrow{\varphi_{n-1}} Z_n$$

(where $n = 0$ represents Z_0 as an "empty zig-zag" and $n = 2$ is just a (Z_0, Z_2)-cospan).

(b) A *zig-zag morphism* from the zig-zag (1) to the following zig-zag

(2) $$Z_0' \xrightarrow{\varphi_0'} Z_1' \xleftarrow{\varphi_1'} Z_2' \xrightarrow{\varphi_2'} \cdots Z_{m-1}' \xleftarrow{\varphi_{n-1}'} Z_m'$$

is a collection $h = (h_i)_{i=0}^m$ of morphisms

$$h_i = Z_{r(i)} \longrightarrow Z_i' \quad \text{for } i = 0, \ldots, m$$

where $r(0) \le r(1) \le \ldots \le \ldots r(m)$ and the following holds each $i = 0, \ldots, m-1$

(3) either $r(i+1) = r(i)$ and the triangle composed by h_{i+1}, h_i, φ_i' commutes, or $r(i+1) = r(i) + 1$ and the diagram composed of h_{i+1}, h_i, φ_i' and $\varphi_{r(i)}$ commutes.

Example of a zig-zag morphism:

$$\begin{tikzcd}
& Z_0 \arrow[r] & Z_1 \arrow[dll, "h_0"'] \arrow[dl, "h_1"] \arrow[d, "h_2"] & Z_2 \arrow[l] \arrow[d, "h_3"] \arrow[dr, "h_4"] & \\
Z'_0 \arrow[r] & Z'_1 & Z'_2 \arrow[l] \arrow[r] & Z'_3 & Z'_4 \arrow[l]
\end{tikzcd}$$

(c) Composition of zig-zag morphisms as well as identity morphisms are defined component-wise.

1.5. Notation. (a) $ZZ(\mathcal{D})$ denotes the category of zig-zags in $\mathcal{D}$ and zig-zag morphisms.

(b) For a functor $F : \mathcal{A} \longrightarrow \mathbf{Set}$ the category $el\,F$ of elements of F has objects (A, a) where $A \in \mathcal{A}$ and $a \in FA$; morphisms $f : (A, a) \to (A', a')$ are $\mathcal{A}$-morphisms $f : A \to A'$ with $Ff(a) = a'$. The forgetful functor $U : el\,F \to \mathcal{A}$ is given by $U(A, a) = A$.

1.6. Theorem. *The following conditions on a small, nonempty category $\mathcal{D}$ are equivalent:*

(1) $\mathcal{D}$ is sifted,

(2) the category $(A, B) \downarrow \mathcal{D}$ of all (A, B)-cospans is connected for every pair A, B of objects of $\mathcal{D}$

and

(3) every pair of zig-zags of $\mathcal{D}$ has a cospan in the category $ZZ(\mathcal{D})$ of zig-zags.

Remark. Explicitly, (2) states that for every pair (A, B) of objects of $\mathcal{D}$ (i) an (A, B)-cospan exists and (ii) two (A, B)-cospans are always connected by a zig-zag in $(A, B) \downarrow \mathcal{D}$.

Proof. $1 \to 2$: Consider the diagrams $\mathcal{D}(A, -)$, $\mathcal{D}(B, -) : \mathcal{D} \to \mathbf{Set}$. They obviously both have a colimit isomorphic to 1 in **Set**, therefore, the diagram

$$D = \mathcal{D}(A, -) \times \mathcal{D}(B, -)$$

has a colimit isomorphic to $1 \times 1 = 1$. This means that the category $el\,D$ of elements D is connected. It is easily seen that $el(D) \cong (A, B) \downarrow \mathcal{D}$.

$2 \to 3$: Assuming (2) we prove the following strengthening of (3):

(3^*) Given zig-zags

Z – connecting A and B,

Z' – connecting A' and B',

then for every pair of cospans

$$A \xrightarrow{p} A^* \xleftarrow{p'} A' \qquad \text{and} \qquad B \xrightarrow{q} B^* \xleftarrow{q'} B'$$

there exists a zig-zag

$$Z^* - \text{ connecting } A^* \text{ and } B^*$$

and two zig-zag morphisms:

$h : Z \to Z^*$ with the first component p and the last one q,
$h' : Z' \to Z^*$ with the first component p' and the last one q'.
It is clear that (2) and (3^*) imply (3).

We proceed by induction on the sum of the lengths of Z and Z'. Denote these lengths by n and m, resp.

Initial step: $n + m \leqq 2$. Thus, $n \leqq 2$ and $m \leqq 2$ and the initial data can be presented as follows:

$$\begin{tikzcd}
A \arrow[r, "\varphi_0"] \arrow[d, "p"'] & Z_1 \arrow[d, "r"'] & B \arrow[l, "\varphi_1"'] \arrow[d, "q"] \\
A^* & C^* & B^* \\
A' \arrow[r, "\varphi_0'"'] \arrow[u, "p'"] & Z_1' \arrow[u, "r'"] & B' \arrow[l, "\varphi_1'"] \arrow[u, "q'"']
\end{tikzcd}$$

The upper horizontal line is Z (if $n = 0$, put $\varphi_0 = \varphi_1 = id_A$), the lower horizontal line is Z' (if $m = 0$, put $\varphi_0' = \varphi_1' = id_{A'}$). The (Z_1, Z_1')-cospan in the middle has been chosen arbitrarily, applying (2) to $(Z_1, Z_1') \downarrow \mathcal{D}$.

Now we have two (A, A')-cospans above: (p, p') and $(r\varphi_0, r'\varphi_0')$. By applying (2) to $(A, A') \downarrow \mathcal{D}$ we connect these two cospans by a zig-zag whose objects are $(h_0, h_0') \stackrel{def}{=} (p, p'), (h_1, h_1'), \ldots, (h_k, h_k') \stackrel{def}{=} (r\varphi_0, r'\varphi_0')$. Analogously, we have two (B, B')-cospans (q, q') and $(r\varphi_1, r'\varphi_1')$ which can be connected by a zig-zag in $(B, B') \downarrow \mathcal{D}$, say, with objects $(h_k, h_k') = (r\varphi_1, r', \varphi_1')$, $(h_{k+1}, h_{k+1}'), \ldots, (h_l, h_l') \stackrel{def}{=} (q, q')$. This defines a zig-zag Z^* with the desired properties indicated by the middle horizontal line in the following diagram:

$$\begin{tikzcd}
A \arrow[rrrr, "\varphi_0"] \arrow[d, "p=h_0"'] \arrow[dr] \arrow[drr] \arrow[drrr, "h_{k-1}"] & & & & Z_1 \arrow[d, "h_k"', "=r"] & & & & B \arrow[llll, "\varphi_1"'] \arrow[d, "q=h_1"] \arrow[dl] \arrow[dll] \arrow[dlll, "h_{k+1}"'] \\
A^* = Z_0^* \arrow[r] & Z_1^* \arrow[r] & Z_2^* \arrow[r] & \cdots Z_{k-1}^* & Z_k^* \arrow[l] \arrow[r] & Z_{k+1}^* \cdots & Z_{l-2}^* \arrow[l] & Z_{l-1}^* \arrow[l] & Z_l^* = B^* \arrow[l] \\
A' \arrow[rrrr, "\varphi_0'"'] \arrow[u, "p'=h_0'"] \arrow[ur] \arrow[urr] \arrow[urrr] & & & & Z_1' \arrow[u, "h_k'", "=r'"'] & & & & B' \arrow[llll, "\varphi_1'"] \arrow[u, "q'=h_1'"'] \arrow[ul] \arrow[ull] \arrow[ulll, "h_{k+1}'"]
\end{tikzcd}$$

Induction step: $n + m > 2$. Suppose e.g. $n > 2$ and apply the induction hypothesis first on the pair of zig-zags

$$\widetilde{Z} \equiv A \xrightarrow{\varphi_0} Z_1 \xleftarrow{\varphi_1} Z_2 \quad \text{(the initial segment of } Z\text{)}$$

and A' (zig-zag of length 0) and the given cospan $A \xrightarrow{p} A^* \xleftarrow{p'} A'$ together with an arbitrarily chosen (Z_2, A')-cospan $Z_2 \xrightarrow{r} C^* \xleftarrow{r'} A'$. We obtain a zig-zag $\widetilde{Z}^*$ connecting A^* with C^* and zig-zag morphisms

$$\widetilde{h} : \widetilde{Z} \to \widetilde{Z}^* \text{ and } \widetilde{h}' : A' \to \widetilde{Z}^*$$

with the specified first and last components.

Next, we apply the induction hypothesis on the rest $\widehat{Z}$ of Z (a zig-zag connecting Z_2 with B) and the zig-zag Z' using the following cospans:

$$Z_2 \xrightarrow{\quad r \quad} C^* \xleftarrow{\quad r' \quad} A' \qquad\qquad \text{and} \qquad\qquad B \xrightarrow{\quad q \quad} B^* \xleftarrow{\quad q' \quad} B'$$

We obtain a pair of zig-zag morphisms

$$\widehat{h} : \widehat{Z} \to \widehat{Z}^* \quad \text{and} \quad \widehat{h}' : Z' \to \widehat{Z}^* .$$

Here $\widehat{Z}^*$ is a zig-zag connecting C^* with B^*, and by gluing $\widetilde{Z}^*$ together with $\widehat{Z}^*$ at C^* (the end-object of the first one and the start object of the latter one), we obtain a zig-zag Z^* connecting A^* with B^*. Also, the first component of $\widehat{h}$ is r, equal to the last component of $\widetilde{h}$, thus, we obtain a morphism of zig-zags

$$h : Z \to Z^*$$

by using first the components of $\widetilde{h}$ and then those of $\widehat{h}$ (except that the first component of $\widehat{h}$ is not repeated, of course). Analogously with

$$h' : Z' \to Z^* .$$

$3 \to 1$: It suffices to prove that $3^* \to 1$. We first observe that (3^*) implies the following

(4) Let Z and Z' be zig-zags connecting A and B. Then there exists a zig-zag Z^* connecting A and B and zig-zag morphisms $h : Z \to Z^*$, $h' : Z' \to Z^*$ which both have the first component id_A and the last one id_B.

In fact, apply (3^*) to $p = p' = id_A$ and $q = q' = id_B$.

Let $D, D' : \mathcal{D} \to \mathbf{Set}$ be diagrams and suppose colimits of D, D' and $D \times D'$ are given as follows:

$$\begin{aligned}(Dd \overset{c_d}{\to} C) &= colim D\\ (D'd \overset{c'_d}{\to} C') &= colim D'\end{aligned}$$

and

$$(Dd \times D'd \overset{c^*_d}{\to} C^*) = colim(D \times D')\,.$$

We prove that the canonical map

$$f : C^* \to C \times C', \qquad f \cdot c^*_d = c_d \times c'_d$$

is a bijection.

(a) f is surjective. In fact, given $(x, x') \in C \times C'$ there exist $d, d' \in \mathcal{D}$ such that (x, x') lies in the image of $c_d \times c'_d$. By (4) there exists a (d, d')-cospan in $\mathcal{D}$, say, with codomain d^*. Then c_d factors through c_{d^*} and $c'_{d'}$ through c'_{d^*}, thus, (x, x') lies in the image of $c_{d^*} \times c'_{d^*} = f \cdot c^*_{d^*}$ - thus, f is surjective.

(b) f is injective. We prove that if two elements $u, \bar{u} \in C^*$ fulfill

$$f(u) = f(\bar{u}) \tag{5}$$

then $u = \bar{u}$. We can express u and $\bar{u}$ in the following form:

$$\begin{aligned} u &= c^*_d(v, v^*) && \text{for } d \in \mathcal{D},\ v \in Dd \text{ and } v' \in D'd\\ \bar{u} &= c^*_{\bar{d}}(w, \bar{w}) && \text{for } \bar{d} \in \mathcal{D},\ w \in D\bar{d} \text{ and } w' \in D'\bar{d}\end{aligned}$$

Then (5) is equivalent to

$$c_d(v) = c_{\bar{d}}(w) \quad \text{and} \quad c'_{\bar{d}}(v') = c_{\bar{d}}(w')\,. \tag{6}$$

By the well known description of colimits in **Set**, this means that the elements (d, v) and $(\bar{d}, w)$ can be connected by a zig-zag Z in the category $el\,D$ of elements of D, and the elements (d, v') and $(\bar{d}, w')$ can be connected by a zig-zag Z' in $el\,D'$.

The forgetful functor

$$U : el\,D \to \mathcal{D}\,, \qquad (d, x) \longmapsto d$$

can be applied component-wise to obtain a functor on zig-zags

$$ZZ(U) : ZZ(el\,D) \to ZZ(\mathcal{D})\,.$$

Analogously for $ZZ(U')$.

Case 1: $ZZ(U)$ maps the zig-zag Z to the same (underlying) zig-zag Z_0 in $\mathcal{D}$ as $ZZ(U')$ maps Z'. In this case immediately combine Z and Z' to a zig-zag connecting $(d, (v, v'))$ with $(\bar{d}, (w, w'))$ in $el(D \times D')$ (and having underlying zig-zag Z_0). This proves $c^*_d(v, v') = c^*_{\bar{d}}(w, w')$, in other words, $u = \bar{u}$.

Case 2: the underlying zig-zags of Z and Z' are different. Denote by $D[d, \bar{d}]$ the subcategory of $ZZ(\mathcal{D})$ of all zig-zags connecting d with $\bar{d}$ and all zig-zag morphisms with first component $id_{\,d}$ and last one $id_{\,\bar{d}}$. Analogously $el\,D[(d, v), (\bar{d}, w)]$ and $el\,D'[(d, v'), (\bar{d}, w')]$.

The forgetful functor $U : el\,D \to \mathcal{D}$ is a cofibration. It is easy to verify that, then, $ZZ(U)$ is a cofibration too, and so is the domain-codomain restriction

$$\widetilde{U} : el\,D[(d,v),(\bar{d},w)] \to \mathcal{D}[d,\bar{d}]\,,$$

Analogously, the cofibration U' leads to a cofibration

$$\widetilde{U}' : el\,D'[(d,v'),(\bar{d},w')] \to \mathcal{D}[d,\bar{d}]\,.$$

Applying (4) to the zig-zags $\widetilde{U}(Z)$, $\widetilde{U}'(Z')$, we obtain a zig-zag Z_0 in $\mathcal{D}(d,\bar{d})$ and zig-zag morphisms

$$h : \widetilde{U}(Z) \to Z_0 \quad \text{and} \quad h' : \widetilde{U}(Z') \to Z_0$$

in $\mathcal{D}(d,\bar{d})$. Since $\widetilde{U}$ is a cofibration, it lifts Z_0 to a zig-zag $\bar{Z}$ and it lifts h to a zig-zag morphism $\bar{h} : Z \to \bar{Z}$ with $\widetilde{U}(\bar{h}) = h$; analogously we obtain $\bar{h}' : Z' \to \bar{Z}'$. Now $\bar{Z}$ and $\bar{Z}'$ have the same underlying zig-zag, Z_0, and we can apply Case 1. ∎

2. The Completion Sind

2.1. Analogously to the free completion

$$\mathit{Ind}\,\mathcal{A}$$

of $\mathcal{A}$ under filtered colimits introduced by Grothendieck [AGV], we study a free completion

$$\mathit{Sind}\,\mathcal{A}$$

of $\mathcal{A}$ under sifted colimits. For example, if $\mathcal{A}$ is a small category with finite colimits, then

$$\mathit{Ind}\,\mathcal{A} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{lex}}$$

is the category of all presheaves on $\mathcal{A}^{\mathrm{op}}$ preserving finite limits, see [AGV], and, as we will show,

$$\mathit{Sind}\,\mathcal{A} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{fp}}$$

is the category of all presheaves on $\mathcal{A}^{\mathrm{op}}$ preserving finite products. There are many more analogies between *Ind* and *Sind* .

2.2. Definition. *For every category $\mathcal{A}$ we denote by*

$$\eta_{\mathcal{A}} : \mathcal{A} \to \mathit{Sind}\,\mathcal{A}$$

a free completions of $\mathcal{A}$ under sifted colimits. That is, $\eta_{\mathcal{A}}$ is a full embedding into a category $\mathit{Sind}\,\mathcal{A}$ with sifted colimits with the following universal property:

> *For every category $\mathcal{B}$ with sifted colimits the functor category $[\mathcal{A},\mathcal{B}]$ is equivalent to the category $[\mathit{Sind}\,\mathcal{A},\mathcal{B}]_{\mathrm{sift}}$ of all functors from $\mathit{Sind}\,\mathcal{A}$ to $\mathcal{B}$ preserving sifted colimits via the functor*

$$(-)\cdot\eta_{\mathcal{A}} : [\mathit{Sind}\,\mathcal{A},\mathcal{B}]_{\mathrm{sift}} \to [\mathcal{A},\mathcal{B}]\,.$$

2.3. Examples. (1) If $\mathcal{A}$ is a poset then

$$Ind\,\mathcal{A} = Sind\,\mathcal{A}$$

is the ideal completion of $\mathcal{A}$, i.e., the poset of all ideals (directed down-sets) ordered by inclusion.

(2) Let $\mathcal{A}$ be a category with finite coproducts. Denote by $\mathcal{A}^*$ a free completion of $\mathcal{A}$ under reflexive coequalizers (this has been described explicitly by A. Pitts, see [BC]). Then

$$Sind\,\mathcal{A} = Ind\,\mathcal{A}^*\,,$$

as proved in 2.8 below.

(3) For the category $\mathcal{A}$:

$$\begin{tikzcd} A \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & B \arrow[l, "d"'] \end{tikzcd}$$

with the free composition modulo

(1) $\qquad f_1 d = f_2 d = id$

we see that $\mathcal{A}$ is finite and has split idempotents, thus,

$$Ind\,\mathcal{A} = \mathcal{A}\,.$$

However, $Sind\,\mathcal{A}$ contains a coequalizer $c : B \to C$ of the reflexive pair f_1, f_2. In fact, $Sind\,\mathcal{A}$ is the extension of $\mathcal{A}$ as follows

$$\begin{tikzcd} A \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & B \arrow[l, "d"'] \arrow[r, "c"] & C \end{tikzcd}$$

with free composition modulo (1) and $cf_1 = cf_2$.

(4) In general, the equation $Sind\,\mathcal{A} = Ind\,\mathcal{A}^*$ is not true.

Consider the category $\mathcal{A}$ given by the following graph

$$
\begin{tikzcd}
\vdots & \vdots \\
A_2 \arrow[r, shift left, "p_2"] \arrow[r, shift right, "q_2"'] & B_2 \\
A_1 \arrow[u, "a_1"'] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "q_1"'] & B_1 \arrow[u, "b_1"] \arrow[ul, "d_1"'] \\
A_0 \arrow[u, "a_0"'] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "q_0"'] & B_0 \arrow[u, "b_0"] \arrow[ul, "d_0"']
\end{tikzcd}
$$

and the following commutativity conditions for all $n \in \omega$:

$$p_{n+1}a_n = b_n p_n\,, \quad q_{n+1}a_n = b_n q_n\,, \quad a_{n+1}d_n = d_{n+1}b_n\,,$$

and

$$b_n = p_{n+1}d_n = q_{n+1}d_n\,.$$

Since $\mathcal{A}$ has no reflexive pairs, $\mathcal{A} = \mathcal{A}^*$. The category $Ind\,\mathcal{A}$ is obtained by adding to $\mathcal{A}$ a colimit A_ω of the chain $(A_n)_{n\in\omega}$, a colimit B_ω of the chain $(B_n)_{n\in\omega}$, and three morphisms $p_\omega = colim p_n$, $q_\omega = colim q_n$, $d_\omega = colim d_n$. Since $p_\omega d_\omega = q_\omega d_\omega = id$, we obtain a reflexive pair without a coequalizer in $Ind\,\mathcal{A}$, thus, $Ind\,\mathcal{A} \neq Sind\,\mathcal{A}$.

2.4. Remark. Let $\mathcal{D}$ be a small category. Recall that a presheaf D in $\mathbf{Set}^{\mathcal{D}}$ is called *flat* if it is a filtered colimit of hom-functors. Or, equivalently, if the dual of the category $el\,D$ of elements of D is filtered. The completion $Ind\,\mathcal{A}$ can be, for $\mathcal{A}$ small, described as the category of all flat presheaves on $\mathcal{A}^{\mathrm{op}}$, see [B].

Recall further that if $\mathcal{D}$ is small, than F is flat iff $Lan_Y F$ preserves finite limits; here $Lan_Y F$ denotes a left Kan extension of F along the Yoneda embedding $Y : \mathcal{D} \to \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$. We now generalize this to sifted colimits.

2.5. Definition. *A functor $D : \mathcal{D} \to \mathbf{Set}$ is called* sifted-flat *provided that it is a sifted colimit of hom-functors.*

2.6. Theorem. *The following conditions on a functor $F : \mathcal{D} \to \mathbf{Set}$, $\mathcal{D}$ small, are equivalent:*

(i) F is sifted-flat,

(ii) the dual of the category of elements of F is sifted,

(iii) F lies in the (iterated) closure of hom-functors under sifted colimits,

(iv) $Lan_Y F$ preserves finite products.

REMARK. (iv) can be weakened to
(iv)* $Lan_Y F$ preserves finite products of hom-functors.

(iv)* →(ii):
PROOF. For two hom-functors $\hom(-, A_1)$, $\hom(-, A_2)$ put

$$D = \hom(-, A_1) \times \hom(-, A_2).$$

Then $(Lan_Y F)(D)$ is a colimit of $F \cdot E_D$ (where $E_D : el\, D \to \mathcal{D}$ is the diagram of elements of D; observe that objects of $el\, D$ are spans $A_1 \overset{x_1}{\leftarrow} X \overset{x_2}{\rightarrow} A_2$). Denote by

$$c(x_1, x_2) : FX \to Lan_Y F(D)$$

a colimit cocone of $F \cdot E_D$ in **Set**. Now (iv)* states that for arbitrary objects A_1 and A_2 the maps

$$\pi_i : Lan_Y F(D) \to FA_i \qquad (i = 1, 2)$$

defined by

$$\pi_i \cdot c(x_1, x_2) = Fx_i \qquad \text{for all spans } (x_1, x_2)$$

form a product in **Set**. We will prove that this implies that the dual of the category $el\, F$ of elements of F is sifted.

For arbitrary elements (A_1, a_1) and (A_2, a_2) of F (i.e., objects of $el\, F$) we know that $(a_1, a_2) \in FA_1 \times FA_2$ has the form $(\pi_1(\bar{a}), \pi_2(\bar{a}))$ for some $\bar{a} \in Lan_Y F(D)$. And since $Lan_Y F(D) = colim F E_D$, $\bar{a}$ has the form $\bar{a} = c(x_1, x_2)(a)$ for some span $A_1 \overset{x_1}{\leftarrow} X \overset{x_2}{\rightarrow} A$ and some $a \in FA$. Thus $a_i = \pi_i \cdot c(x_1, x_2)(\bar{a}) = Fx_i(a)$ and we proved the existence of a cospan in $(el\, F)^{\mathrm{op}}$

$$(A_1, a_1) \overset{x_1}{\rightarrow} (X, a) \overset{x_2}{\leftarrow} (A_2, a_2)\,.$$

Let another cospan

$$(A_1, a_1) \overset{x_1'}{\rightarrow} (X', a') \overset{x_2'}{\leftarrow} (A_2, a_2)$$

be given. Then

$$\pi_i c(x_1, x_2)(a) = a_i = \pi_i c(x_1', x_2')(a') \qquad \text{for } i = 1, 2$$

implies (since π_1, π_2 are projections of a product) that

$$c(x_1, x_2)(a) = c(x_1', x_2')(a')\,.$$

By construction of colimits in **Set**, the latter means that the two elements of FE_D, $((x_1, x_2), a)$ and $((x_1', x_2'), a')$, are connected by a zig-zag in $(el(FE_D))^{\mathrm{op}}$. Now we have an obvious forgetful functor $(el(FE_D))^{\mathrm{op}} \to (el\, F)^{\mathrm{op}}$ which maps that zig-zag onto a zig-zag connecting (x, a) with (x', a') in $(A_1, A_2) \downarrow \mathcal{D}$.

(ii) $\to$ (i) $\to$ (iii) is trivial

(iii) $\to$ (iv) Since $Lan_Y(-)$ preserves colimits, (iii) implies that $Lan_Y F$ is obtained as an iterated sifted colimit from the set of functors $Lan_Y \mathcal{D}(-, A)$. The last functor preserves finite products (since this is just the evaluation-at-A functor from $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ to $\mathbf{Set}$). And a sifted colimit of functors preserving finite products preserves finite products too. This proves (iv). ∎

2.7. COROLLARY. *For every small category $\mathcal{A}$ we can describe Sind $\mathcal{A}$ as the full subcategory of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ of all sifted-flat functors (with respect to the codomain restriction of the Yoneda embedding $Y : \mathcal{A} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$).*

PROOF. Following 2.6 (iii), the full subcategory $\bar{\mathcal{A}}$ of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ consisting of all sifted-flat functors has sifted colimits. Following 2.6 (ii), a left Kan extension $Lan_Y H : \bar{\mathcal{A}} \to \mathcal{B}$ exists for each functor $H : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has sifted colimits. Since $Lan_Y H$ clearly preserves sifted colimits, $\bar{\mathcal{A}} \approx Sind\,\mathcal{A}$. ∎

2.8. COROLLARY. *If $\mathcal{A}$ is a small category with finite coproducts then*

$$Sind\,\mathcal{A} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{fp}}$$

PROOF. Since $Y : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{A}}$ preserves finite products, any sifted-flat functor $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ preserves finite products following 2.6 (iv). Conversely, if F preserves finite products then $Lan_Y F$ preserves finite products of hom-functors because

$$\begin{aligned}
(Lan_Y F)(\hom(-, A_1) \times \hom(-, A_2)) &\cong (Lan_Y F)(\hom(-, A_1 \times A_2)) \\
&\cong F(A_1 \times A_2) \\
&\cong F(A_1) \times F(A_2) \\
&\cong Lan_Y(F)(\hom(-, A_1)) \times (Lan_Y F)(\hom(-, A_1)).
\end{aligned}$$

Hence F is sifted-flat following 2.6 (iv)*. ∎

REMARK. The last corollary proves the claim of Example 2.3 (2): the categories $[(\mathcal{A}^*)^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{lex}}$ and $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{fp}}$ are equivalent.

2.9. REMARK. Corollary 2.8 immediately generalizes to small categories $\mathcal{A}$ with finite multicoproducts. (A multicoproduct of a finite set $A_1, \ldots, A_n$ is a set of cocones $(c_{ij} : A_j \to C_i)_{j=1,\ldots,n}$, $i \in I$ such that every cocone of $A_1, \ldots, A_n$ factors through precisely one of these, and the factorization is unique). A functor $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is said to *preserve multiproducts* provided that for each finite set $A_1, \ldots, A_n$ of objects in $\mathcal{A}$ the cone

$$\pi_j : \coprod_{i \in I} FC_i \to FA_j \qquad (j = 1, \ldots, n)$$

where π_j has components Fc_{ij}, is a product in $\mathbf{Set}$. Then

$$Sind\,\mathcal{A} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{fmp}}$$

is the full subcategory of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ formed by all functors preserving finite multiproducts.

In fact, $Y : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}^{\mathcal{A}}$ preserves finite multiproducts. Since $Lan_Y F$ preserves for any $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ arbitary coproducts, any sifted-flat functor F preserves finite multiproducts following 2.6.(iv). Conversely, if F preserves finite multiproducts then $Lan_Y F$ preserves finite products because

$$\begin{aligned}
(Lan_Y F)(\prod_{j=1}^{n} \hom(-, A_j)) &\cong (Lan_Y F) \coprod_{i \in I} \hom(-, C_i)) \\
&\cong \coprod_{i \in I} (Lan_Y F)(\hom(-, C_i)) \\
&\cong \coprod_{i \in I} F C_i \\
&\cong \prod_{j=1}^{n} F A_j \\
&\cong \prod_{j=1}^{n} (Lan_Y F) \hom(-, A_j).
\end{aligned}$$

3. Generalized Varieties

3.1. Recall from [L_1] and [MP] the fruitful concept of a finitely accessible category, i.e., a category $\mathcal{K}$ such that

(a) $\mathcal{K}$ has filtered colimits

and

(b) $\mathcal{K}$ has a (small) set $\mathcal{A}$ of finitely presentable objects such that every object of $\mathcal{K}$ is a filtered colimit of objects in $\mathcal{A}$.

Moreover, finitely presentable objects are, of course, precisely those whose hom-functors preserve filtered colimits. We now substitute "filtered" by "sifted" and obtain the following concepts.

3.2. Definition. *An object of a category is called* strongly finitely presentable *provided that its hom-functor preserves sifted colimits.*

3.3. Lemma. *Let $\mathcal{K}$ be a category with kernel pairs. An object which is strongly finitely presentable is*

(i) finitely presentable

and

(ii) a regular projective.

If $\mathcal{K}$ is a variety of finitary algebras, (i) and (ii) are equivalent to strong finite presentability.

PROOF. If K is strongly finitely presentable, then $\hom(K,-)$ preserves coequalizers of kernel pairs, since these are reflexive coequalizers (see 1.3 (1)), thus, K is a regular projective.

Suppose $\mathcal{K}$ is a variety. Then for every object K with (i) and (ii), the functor $\hom(K,-): \mathcal{K} \to \mathbf{Set}$ preserves limits, filtered colimits and regular epimorphisms. Hence it is a completely exact functor, which implies that it preserves sifted colimits, see [ALR]. ■

3.4. EXAMPLES. (1) In **Set**, finitely presentable = strongly finitely presentable, the same holds in the category K-**Vec** of vector spaces over a field K.

(2) In **Ab** the category of Abelian groups, strongly finitely presentable objects are precisely the free Abelian groups on finitely many generators.

(3) In **Pos** the category of posets, strongly finitely presentable objects are the finite discretely ordered ones (=finitely presentable regular projectives).

3.5. REMARK. (i) A finite coproduct of strongly finitely presentable objects is strongly finitely presentable.

(ii) For finite colimits this is no longer true (see 3.4.2)

(iii) A finite multicoproduct of strongly finitely presentable objects has all components strongly finitely presentable.

In fact, let A_j, $j = 1, \ldots, n$ be strongly finitely presentable objects in $\mathcal{K}$ and $(c_{ij} : A_j \to C_i)$, $i \in I$ be a multicoproduct in $\mathcal{K}$. Let $(s_t : X_t \to X)_{t\in T}$ be a sifted colimit in $\mathcal{K}$. Then

$$\begin{aligned}
\coprod_{i\in I} \hom(C_i, colim_{t\in T}\, X_t) &\cong \prod_{j=1}^{n} \hom(A_j, colim_{t\in T}, X_t) \\
&\cong colim_{t\in T} \prod_{j=1}^{n} \hom(A_j, X_t) \\
&\cong colim_{t\in T} \coprod_{i\in I} \hom(C_i, X_t) \\
&\cong \coprod_{i\in I} colim_{t\in T}, \hom(C_i, X_t)
\end{aligned}$$

and this canonical isomorphism has canonical components

$$\hom(C_i, colim_{t\in T}\, X_t) \cong colim_{t\in T}\, \hom(C_i, X_t)$$

for each $i \in I$. Hence C_i is strongly finitely presentable (for each $i \in I$).

3.6. DEFINITION. *By a* generalized variety *is meant a category which has*

(a) sifted colimits

and

(b) a (small) set $\mathcal{A}$ of strongly finitely presentable objects such that every object is a sifted colimit of objects in $\mathcal{A}$.

3.7. Examples. (1) Every variety $\mathcal{V}$ is a generalized variety. In fact, choose a set $\mathcal{A}$ of representatives for finitely presentable regular projectives. Every object K is a canonical colimit of the forgetful functor $\mathcal{A} \downarrow K \to \mathcal{V}$ (since $\mathcal{A}$ is dense) and since $\mathcal{A}$ has finite coproducts, so does $\mathcal{A} \downarrow K$, thus, the canonical colimits are sifted (1.3 (2)).

(2) In the next section we will see other examples of generalized varieties: fields, linearly ordered sets, sets and injective functions. These are generalized varieties with connected limits.

(3) The category of non-empty sets and functions is, obviously, a generalized variety (and does not have connected limits).

(4) Let us mention an interesting non-example: the category $\mathbf{Gra}_c$ of connected graphs and graphs homomorphisms. This category has sifted colimits because it is closed under sifted colimits in $\mathbf{Gra}$, the category of all graphs. (In fact, let $\mathcal{D}$ be a small category such that every pair of objects has a cospan, then a $\mathcal{D}$-colimit of connected graphs in $\mathbf{Gra}$ is connected). It is easy to see that the "obvious" dense set $\mathcal{A}$ in $\mathbf{Gra}$, consisting of (a) a single vertex (no edges) and (b) a single edge, lies in $\mathbf{Gra}_c$ and forms a strong generator of strongly finitely presentable objects of $\mathbf{Gra}_c$. Nevertheless, $\mathbf{Gra}_c$ fails to be a generalized variety. In fact, the only regular projectives in $\mathbf{Gra}_c$ are the graphs with a single vertex, and they do not generate other graphs by sifted colimits in $\mathbf{Gra}_c$.

(5) The category $\mathbf{Gra}_c^*$ of connected graphs and injective graph homomorphisms is a generalized variety. The existence of sifted colimits is clear, and strongly finitely presentable objects are precisely finite connected graphs. Any connected graph is a sifted union of finite connected graphs.

3.8. Lemma. *In every generalized variety $\mathcal{K}$ the collection $\mathcal{K}_0$ of all strongly finitely presentable objects is essentially small and dense, and the comma-categories $\mathcal{K}_0 \downarrow K$ are sifted for all objects K of $\mathcal{K}$.*

Proof. Let $\mathcal{A}$ be set as in 3.5 (b). Let $K_0 \in \mathcal{K}_0$ be expressed as a sifted colimit $(A_i \xrightarrow{a_i} K_0)_{i\in I}$ with $A_i \in \mathcal{A}$ for each $i \in I$. Since $\hom(K_0, -)$ preserves that colimit, id_{K_0} factors through some A_i. Thus, $\mathcal{K}_0$ consists of retracts of objects in $\mathcal{A}$. Since $\mathcal{A}$ is small, this proves that $\mathcal{K}_0$ is essentially small.

Let $K \in \mathcal{K}$ be an arbitrary object and express K as a sifted colimit $(A_i \xrightarrow{a_i} K)_{i\in I}$ with $A_i \in \mathcal{A}$. Every morphism $f : K_0 \to K$, $K_0 \in \mathcal{K}_0$, factors through some a_i and given two such factorizations:

$$f = a_i \cdot f' = a_j \cdot f'' \qquad (i, j \in I)$$

then they are connected by a zig-zag in $\mathcal{A} \downarrow K$ – this follows from the fact that $\hom(K, -)$ preserves the above colimit. We conclude that K is a canonical colimit of the diagram $\mathcal{K}_0 \downarrow K \to \mathcal{K}$. And that diagram is sifted because the original sifted diagram of all $A_i \xrightarrow{a_i} K$ is a cofinal subdiagram in it. ∎

3.9. Remark. (1) Finitely accessible categories are precisely the categories $Ind\,\mathcal{A}$, $\mathcal{A}$ small. Quite analogously: generalized varieties are precisely the categories

$$Sind\,\mathcal{A}, \qquad \mathcal{A} \text{ small}.$$

In fact, *Sind* $\mathcal{A}$ is a generalized variety because, by 2.6, every object of *Sind* $\mathcal{A}$ is a sifted colimit of objects of $\mathcal{A}$ (or, the corresponding hom-functors), and since *Sind* $\mathcal{A}$ is closed under sifted colimits in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$, and hom-functors are strongly finitely presentable in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$, they are also strongly finitely presentable in *Sind* $\mathcal{A}$.

Conversely, if $\mathcal{K}$ is a generalized variety, let $\mathcal{A}$ be a full subcategory representing all strongly finitely presentable objects. By Lemma 3.8, all objects of $\mathcal{K}$ are canonical sifted colimits of $\mathcal{A}$-objects. This implies $\mathcal{K} \approx \mathit{Sind}\,\mathcal{A}$ quite analogously to the proof of $\mathcal{K} \approx \mathit{Ind}\,\mathcal{A}$ in case $\mathcal{K}$ is finitely accessible (see e.g. [AR_1], Theorem 2.26).

(2) Recall that among complete or cocomplete categories, finitely accessible ones are precisely the locally finitely presentable categories of Gabriel and Ulmer. This has a direct analogy:

3.10. Theorem. *A cocomplete category is a generalized variety iff it is equivalent to a variety.*

Remark. In fact, every generalized variety with finite coproducts is a variety: from the existence of sifted colimits follow all coproducts (=filtered colimits of finite coproducts) and reflexive coequalizers – thus, cocompleteness.

Proof. Any variety is a cocomplete generalized variety. Let $\mathcal{K}$ be a cocomplete generalized variety. Then the full subcategory $\mathcal{K}_0$ representing all strongly finitely presentable objects of $\mathcal{K}$ has finite coproducts (following Remark 3.5(i)). Using Remark 3.9 and Corollary 2.8, we get that

$$\mathcal{K} \approx \mathit{Sind}\,\mathcal{K}_0 \approx [\mathcal{K}_0^{\mathrm{op}}, \mathbf{Set}]_{\mathrm{fp}}$$

and $\mathcal{K}$ is therefore equivalent to a variety. ∎

4. Multialgebraic categories

4.1. Since the dissertation of F.W.Lawvere [La] it is well known that varieties are precisely categories sketchable by FP-sketches. That is, given an *FP-sketch* $\mathcal{S}$, i.e., a small category $\mathcal{A}$ with chosen finite discrete cones, we form the category

$$\mathbf{Mod}\mathcal{S} \subseteq \mathbf{Set}^{\mathcal{A}}$$

of all functors turning the given cones into (finite) products in $\mathbf{Set}$. Then

(1) $\mathbf{Mod}\mathcal{S}$ is equivalent to a variety

and

(2) every variety is equivalent to some $\mathbf{Mod}\mathcal{S}$.

(In [La] the case of one-sorted varieties and FP-sketches generated by a single object is treated. See [AR_1] for the many-sorted case.)

Y. Diers presented in [D] a generalization to sketches using finite multiproducts, and called the categories of models *multialgebraic categories*. In [AR_2] we have shown that instead of the (non-standard) multiproducts the following standard concept can be used:

by an FPC-sketch (for "finite products and [arbitrary] coproducts") $\mathcal{S}$ is meant a small category $\mathcal{A}$ with chosen

(a) discrete finite cones

and

(b) discrete cocones.

We denote by

$$\mathbf{Mod}\mathcal{S} \subseteq \mathbf{Set}^{\mathcal{A}}$$

the category of all *models* of $\mathcal{S}$, i.e., the full subcategory of $\mathbf{Set}^{\mathcal{A}}$ of all functors turning the given cones into (finite) products and the given cocones to coproducts. The following concept is then identical with that of Y. Diers:

DEFINITION. *A category is called* multialgebraic *if it is FPC-sketchable, i.e., equivalent to the category of models of an FPC-sketch.*

CHARACTERIZATION THEOREM (Y. Diers, [D]). *A category is multialgebraic iff it has*

(i) *multicolimits,*

(ii) *filtered colimits,*

(iii) *effective equivalence relations,*

(iv) *a regular generator formed by finitely presentable regular projectives.*

REMARK. Y. Diers has another condition, viz, the existence of kernel pairs, but it follows from (i)-(iii) that all connected limits exist (see [AR_1], 4.30).

4.2. REMARK. Multialgebraic categories have, in contrast to varieties, no kind of "equational presentation". In [AR_2] we have introduced multivarieties: these are classes of algebras presented by exclusive-or's of equations. Every multivariety with effective equivalence relations is multialgebraic, and vice versa. An example of a multivariety which is not multialgebraic is the category of unary algebras on one injective operations.

The following examples demonstrate how natural the syntax via FPC-sketches is for important multialgebraic theories.

4.3. EXAMPLES. (1) Fields.

Let $\mathcal{S}_0$ be the usual FP-sketch for rings. We thus have, among others, morphisms

$$\begin{aligned} +, * &: X \times X \to X \\ \pi_1, \pi_2 &: X \times X \to X \\ 0 &: 1 \to X \end{aligned}$$

and others needed to express the ring equations. We now add a new object Y (representing all non-zero elements of X) and morphisms

$$\begin{aligned} e &: Y \to X \quad \text{(embedding)} \\ i &: Y \to X \quad \text{(multiplication inverse)} \end{aligned}$$

and denote by $\langle e, i\rangle : Y \to X \times X$ the corresponding pair. Next we put one commutativity condition

$$\begin{tikzcd} Y \arrow[r, "\langle e,i\rangle"] \arrow[dr, "e"'] & X \times X \arrow[d, "*"] \\ & X \end{tikzcd}$$

and one cocone

$$\begin{tikzcd} 1 \arrow[dr, "0"'] & & Y \arrow[dl, "e"] \\ & X & \end{tikzcd}$$

The resulting sketch $\mathcal{S}$ has the category of fields and field homomorphisms as $\mathbf{Mod}\mathcal{S}$.

(3) Linearly ordered sets (see [AR$_1$], 2.57).

Let $\mathcal{S}_0$ be the usual FP-sketch for sup-semilattices. We thus have, among others, morphisms

$$\begin{aligned} \vee &: X \times X \to X \\ \pi_1, \pi_2 &: X \times X \to X \\ \Delta &: X \to X \times X \end{aligned}$$

and others expressing the commutativity, associativity, and idempotency of $\vee$. We now add to $\mathcal{S}_0$ two new objects E and $\bar{E}$ and morphisms

$$e : E \to X \times X \quad \text{and} \quad \bar{e} : \bar{E} \to X \times X$$

subject to

$$\pi_1 e = \pi_2 \bar{e} \quad \text{and} \quad \pi_2 e = \pi_1 \bar{e}$$

as well as

$$\pi_2 e = \sigma e.$$

Finally, we add a cocone

$$\begin{tikzcd} E \arrow[dr, "e"'] & X \arrow[d, "\Delta"] & \bar{E} \arrow[dl, "\bar{e}"] \\ & X \times X & \end{tikzcd}$$

In a model M, we have a relation ME on MX whose inverse relation $M\bar{E}$ fulfills: $MX \times MX = ME \cup M\bar{E} \cup \Delta_{MX}$ and which, due to $\pi_2 e = \sigma e$, is just the strict order relation of the semilattice $(MX, M\vee)$. Thus, we obtain a sketch $\mathcal{S}$ whose category $\mathbf{Mod}\mathcal{S}$ is the category of linearly ordered sets and order-preserving mappings.

(5) Sets and injective functions.
Let $\mathcal{S}$ be the sketch with objects

$$X, X \times X, Y \quad (= \text{complement of the diagonal})$$

and morphisms

$$\begin{aligned} \pi_1, \pi_2 \,:\, X \times X \to X\,, \\ \Delta \,:\, X \to X \times X \end{aligned}$$

and

$$y : Y \to X \times X$$

subject to

$$\pi_1 \Delta = \pi_2 \Delta = id \ .$$

There is one product specification

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

and one coproduct specification

$$\begin{tikzcd} X \arrow[dr, "\Delta"'] & & Y \arrow[dl, "y"] \\ & X \times X & \end{tikzcd}$$

It is obvious that $\mathbf{Mod}\mathcal{S}$ is equivalent to the category of all sets and injective functions.

4.4. Theorem. *A category with multicolimits is a generalized variety iff it is multialgebraic.*

Proof. This is completely analogous to that of Theorem 3.10 (we use 3.5(iii) and 2.9 instead of 3.5(i) and 2.8). ∎

4.5. Remark. Thus, we see that in the presence of multicolimits, generalized varieties are precisely the categories which are (finite product, coproduct)-sketchable. For generalized varieties without completeness assumptions we know at least one implication in case coproducts are substituted by colimits:

4.6. Proposition. *Every generalized variety can be sketched by a (finite product, colimit)-sketch.*

PROOF. Let $\mathcal{K} = Sind\,\mathcal{A}$ be a generalized variety. By 2.7 and Remark 2.6, $Sind\,\mathcal{A}$ consists of those functors $F \in \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ such that $Lan_Y F$ preserves finite products of hom-functors. Denote by $\mathcal{B}$ a small full subcategory of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ representing finite products of hom-functors. Let $\mathcal{S}$ be the sketch on $\mathcal{B}$ whose cones are the finite products of hom-functors, and whose cocones represent every finite product of hom-functors as a colimit of hom-functors. Since $Lan_Y F$ preserves all colimits, it immediately follows from 2.6 (iv)* that

$$\mathbf{Mod}\mathcal{S} \approx Sind\,\mathcal{A}.$$

■

4.7. EXAMPLE. The converse to 4.6 does not hold. For example, consider the sketch $\mathcal{S}$ with one object and one endomorphism specified to be epi. This is a ($\emptyset$, colimit)-sketch whose category of models is the category of unary algebras on one surjective operation. This category is not a generalized variety.

4.8. REMARKS. (1) Following Proposition 4.6, any generalized variety $\mathcal{K}$ is accessible. Hence $\mathcal{K}$ is complete iff it is cocomplete and $\mathcal{K}$ has connected limits iff it has multicolimits (cf. [AR_1], 2.47 and 4.30). This may be added to the characterization theorems 3.10 and 4.4.

(2) In fact, every generalized variety is ω_1-accessible: by 3.8 (1) it has the form $\mathcal{K} = Sind\,\mathcal{A}$, $\mathcal{A}$ small, i.e., $\mathcal{K}$ is the closure of hom-functors under sifted colimits in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ (see 2.7). For every sifted category $\mathcal{D}$ it is clear that $\mathcal{D}$ is an ω_1-directed union of its full, countable, sifted subcategories (see 1.6), thus, objects of $\mathcal{K}$ are ω_1-directed colimits of ω_1-presentable objects (since a countable colimit of hom-functors is ω_1-presentable in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$, thus, in $\mathcal{K}$ too).

4.9 OPEN PROBLEMS. (1) Is every generalized variety finitely accessible?

(2) Is there a full description of generalized varieties by means of sketches?

References

[AGV] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas,* Lect. Notes in Math. 269, Springer-Verlag, Berlin 1972.

[ALR] J. Adámek, F.W. Lawvere and J. Rosický, *On the duality between varieties and algebraic functors,* to appear.

[AR_1] J. Adámek and J. Rosický, *Locally presentable and accessible categories,* Cambridge University Press, Cambridge 1994.

[AR_2] J. Adámek and J. Rosický, *On multivarieties and multialgebraic categories,* accepted in Jour. Pure Appl. Alg.

[B] F. Borceux, *Handbook of categorical algebra1,* Cambridge Univ. Press 1994.

[BC] M. Bunge and A. Carboni, *The symmetric topos* , Jour. Pure Appl. Algebra 105 (1995), 233–249.

[D] Y. Diers, *Catégories multialgébriques,* Arch. Math. (Basel) 34 (1980), 193–209.

[La] F. W. Lawvere, *Functorial semantics of algebraic theories,* Dissertation, Columbia University 1963.

[L_1] C. Lair, *Catégories modelables et catégories esquissables,* Diagrammes 6, (1981) 1–20.

[L_2] C. Lair, *Sur le genre d'esquissibilité des catégories modelables (accessibles) possédant les produits de deux,* Diagrammes 35 (1996), 25–52.

[MP] M. Makkai and R. Paré, *Accessible categories: the foundations of categorical model theory,* Contemp. Math. 104, Amer. Math. Soc., Providence 1989.

[P] H.-E. Porst, *Minimal generators in varieties,* preprint 1997.

[PW] M. C. Pedicchio and R. J. Wood, *A simple characterization of theories of varieties,* Journal of Algebra 233 (2000), 483–501.

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