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COMPARING COEQUALIZER AND EXACT COMPLETIONS

Dedicated to Joachim Lambek
on the occasion of his 75th birthday.

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ABSTRACT. We characterize when the coequalizer and the exact completion of a category $\mathscr{C}$ with finite sums and weak finite limits coincide.

Introduction

Our aim is to compare two well known completions: the coequalizer completion $\mathscr{C}_{\text{coeq}}$ of a small category $\mathscr{C}$ with finite sums (see [P]) and the exact completion of a small category $\mathscr{C}$ with weak finite limits (see [CV]). For a category $\mathscr{C}$ with finite sums and weak finite limits, $\mathscr{C}_{\text{ex}}$ is always a full subcategory of $\mathscr{C}_{\text{coeq}}$. We characterize when the two completions are equivalent - it turns out that this corresponds to a finiteness condition expressed in terms of reflexive and symmetric graphs in $\mathscr{C}$.

1. Two completions

For a small category $\mathscr{C}$ with finite sums, the *coequalizer completion* of $\mathscr{C}$ is a category $\mathscr{C}_{\text{coeq}}$ with finite colimits together with a finite sums preserving functor $G_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}_{\text{coeq}}$ such that, for any finite sums preserving functor $F : \mathscr{C} \to \mathscr{X}$ into a finitely cocomplete category, there is a unique finite colimits preserving functor $\overline{F} : \mathscr{C}_{\text{coeq}} \to \mathscr{X}$ with $\overline{F} \cdot G_{\mathscr{C}} = F$. This construction has been described by Pitts (cf. [BC]).

For a small category $\mathscr{C}$ with weak limits, the *exact completion* $E_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}_{\text{ex}}$ can be characterized by a universal property as well (see [CV]). In a special case when $\mathscr{C}$ has finite limits, $E_{\mathscr{C}}$ is a finite limits preserving functor into an exact category $\mathscr{C}_{\text{ex}}$ such that, for any finite limits preserving functor $F : \mathscr{C} \to \mathscr{X}$ into an exact category, there is a unique functor $F' : \mathscr{C}_{\text{ex}} \to \mathscr{X}$ which preserves finite limits and regular epimorphisms such that $F' \cdot E_{\mathscr{C}} = F$. Following [HT], $\mathscr{C}_{\text{ex}}$ can be described as a full subcategory of $\mathbf{Set}^{\mathscr{C}^{\text{op}}}$ and $E_{\mathscr{C}}$ as the codomain restriction of the Yoneda embedding $Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\text{op}}}$. To explain it, we recall that a functor $H : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ is *weakly representable* if it admits

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a regular epimorphism $\gamma : YC \to H$ from a representable functor. Then $\mathscr{C}_{\mathrm{ex}}$ consists of those weakly representable functors H admitting $\gamma : YC \to H$ whose kernel pair

$$K \underset{\beta}{\overset{\alpha}{\rightrightarrows}} YC \xrightarrow{\ \gamma\ } H \qquad (1)$$

has K weakly representable.

We will start by showing that $\mathscr{C}_{\mathrm{coeq}}$ can be presented as a full subcategory of $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ too, with $G_{\mathscr{C}}$ being the codomain restriction of Y. The full subcategory of $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ consisting of all finite products preserving functors will be denoted by $FP(\mathscr{C}^{\mathrm{op}})$. It is well known that $FP(\mathscr{C}^{\mathrm{op}})$ is a variety (see [AR] 3.17).

1.1. Lemma. *Let $\mathscr{C}$ be a category with finite sums. Then $\mathscr{C}_{\mathrm{coeq}}$ is equivalent to the full subcategory of $FP(\mathscr{C}^{\mathrm{op}})$ consisting of finitely presentable objects in $FP(\mathscr{C}^{\mathrm{op}})$.*

Proof. By the universal property of $\mathscr{C}_{\mathrm{coeq}}$, we get

$$FP(\mathscr{C}^{\mathrm{op}}) \approx \mathbf{Lex}((\mathscr{C}_{\mathrm{coeq}})^{\mathrm{op}})$$

where, on the right, there is the full subcategory of $\mathbf{Set}^{(\mathscr{C}_{\mathrm{coeq}})^{\mathrm{op}}}$ consisting of all finite limits preserving functors. The result thus follows from [AR] 1.46. ∎

1.2. Proposition. *Let $\mathscr{C}$ be a category with finite sums and weak finite limits. Then $\mathscr{C}_{\mathrm{ex}}$ is equivalent to a full subcategory of $\mathscr{C}_{\mathrm{coeq}}$.*

Proof. Let $H \in \mathscr{C}_{\mathrm{ex}}$ and consider the corresponding diagram (1). There is a regular epimorphism $\delta : YD \to K$ (because K is weakly representable) and we obtain a coequalizer

$$YD \underset{\overline{\beta}}{\overset{\overline{\alpha}}{\rightrightarrows}} YC \xrightarrow{\ \gamma\ } H \qquad (2)$$

where $\overline{\alpha} = \alpha\delta$ and $\overline{\beta} = \beta\delta$. Since YD is a regular projective in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$, the graph $(\overline{\alpha}, \overline{\beta})$ is reflexive (it means the existence of $\varphi : YC \to YD$ with $\overline{\alpha}\varphi = \overline{\beta}\varphi = \mathrm{id}_{Y(C)}$). Following [PW], (2) is a coequalizer in $FP(\mathscr{C}^{\mathrm{op}})$. Therefore, H is finitely presentable in $FP(\mathscr{C}^{\mathrm{op}})$ (cf. [AR] 1.3). Hence, using Lemma 1.1, H belongs to $\mathscr{C}_{\mathrm{coeq}}$. ∎

When $\mathscr{C}$ has finite sums, objects of $\mathscr{C}$ are precisely finitely generated free algebras in the variety $FP(\mathscr{C}^{\mathrm{op}})$. The condition of having weak finite limits too, is a very restrictive one. We give another formulation of it.

1.3. Proposition. *Let $\mathscr{C}$ have finite sums. Then $\mathscr{C}$ has weak finite limits iff finite limits of objects of $\mathscr{C}$ in $FP(\mathscr{C}^{\mathrm{op}})$ are finitely generated.*

Proof. Let $D : \mathscr{D} \to \mathscr{C}$ be a finite diagram and $(\delta_d : A \to YD_d)_{d \in \mathscr{D}}$ its limit in $FP(\mathscr{C}^{\mathrm{op}})$.

Assume that A is finitely generated. Then there is a regular epimorphism $\pi : YC \to A$ where $C \in \mathscr{C}$. Consider a cone $(f_d : X \to D_d)_{d \in \mathscr{D}}$ in $\mathscr{C}$. There is a unique $\varphi : YX \to A$ with $\delta_d\varphi = Yf_d$ for all $d \in \mathscr{D}$. Since π is a regular epimorphism (and YX regular projective) φ factorizes through π and therefore $(\delta_d\pi : YC \to YD_d)_{d\in\mathscr{D}}$ is a weak limit of YD in $Y(\mathscr{C})$. Hence D has a weak limit in $\mathscr{C}$.

Conversely, assume that D has a weak limit $(\gamma_d : C \to D_d)_{d\in\mathscr{D}}$ in $\mathscr{C}$. There is a unique $\pi : YC \to A$ with $\delta_d\pi = Y\gamma_d$ for all $d \in \mathscr{D}$. Consider $\varphi : YX \to A$, $X \in \mathscr{C}$. There exists $\varphi : X \to C$ such that $Y(\gamma_d\psi) = \delta_d\varphi$ for all $d \in \mathscr{D}$. Hence $\varphi = \pi\psi$, which implies that π is a regular epimorphism (because $YX, X \in \mathscr{C}$ are finitely generated free algebras in the variety $FP(\mathscr{C}^{\mathrm{op}})$). Hence A is finitely generated. ∎

2. When do they coincide?

2.1. Construction. Let $\mathscr{C}$ be a category with weak finite limits. Let $r_0, r_1 : C_1 \to C_0$ be a reflexive and symmetric graph in $\mathscr{C}$. It means that there are morphisms $d : C_0 \to C_1$ and $s : C_1 \to C_1$ with $r_0 d = r_1 d = \mathrm{id}_{C_0}$ and $r_1 s = r_0$, $r_0 s = r_1$. We form a weak pullback

$$\begin{tikzcd} & C_2 \arrow[dl, "\bar{r}_0"'] \arrow[dr, "\bar{r}_1"] & \\ C_1 \arrow[dr, "r_1"'] & & C_1 \arrow[dl, "r_0"] \\ & C_0 & \end{tikzcd} \tag{3}$$

By taking $r_i^2 = r_i\bar{r}_i$, $i = 0, 1$, we get the graph

$$C_2 \overset{r_0^2}{\underset{r_1^2}{\rightrightarrows}} C_1$$

This graph is reflexive: $d^2 : C_0 \to C_2$ is given by $\bar{r}_0 d^2 = \bar{r}_1 d^2 = d$. It is also symmetric: $s^2 : C_2 \to C_2$ is given by $\bar{r}_0 s^2 = s\bar{r}_1$ and $\bar{r}_1 s^2 = s\bar{r}_0$. By iterating this procedure, we get reflexive and symmetric graphs

$$C_n \overset{r_0^n}{\underset{r_1^n}{\rightrightarrows}} C_1$$

for $n = 1, 2, \ldots$.

2.2. Definition. *Let $\mathscr{C}$ have weak finite limits. We say that a reflexive and symmetric graph $r_0, r_1 : C_1 \to C_0$ has a bounded transitive hull if there is $n \in \mathbb{N}$ such that, for any $m > n$, there exists $f_m : C_m \to C_n$ with $r_0^n f_m = r_0^m$ and $r_1^n f_m = r_1^m$.*

It is easy to check that the definition does not depend on the choice of weak finite limits. Evidently, if $\mathscr{C}$ has finite limits, Definition 2.2 means that the pseudoequivalence generated by the graph $r_0, r_1 : C_1 \to C_0$ is equal to $r_0^n, r_1^n : C_n \to C_0$.

2.3. Theorem. *Let $\mathscr{C}$ have finite sums and weak finite limits. Then $\mathscr{C}_{\mathrm{ex}}$ is equivalent to $\mathscr{C}_{\mathrm{coeq}}$ iff any reflexive and symmetric graph in $\mathscr{C}$ has a bounded transitive hull.*

PROOF. I. Let $r_0, r_1 : C_1 \to C_0$ be a reflexive and symmetric graph in $\mathscr{C}$. Consider the coequalizer

$$\begin{tikzcd} YC_1 \arrow[r, shift left, "Yr_0"] \arrow[r, shift right, "Yr_1"'] & YC_0 \arrow[r, "\gamma"] & H \end{tikzcd}$$

in $\mathbf{Set}^{\mathscr{C}^{op}}$. Put $H_1 = YC_1$, $H_0 = YC_0$ and $\gamma_i = Yr_i$ for $i = 0, 1$. Let γ_0^n, $\gamma_1^n : H_n \to H_0$ be iterations of the graph (γ_0, γ_1) constructed as before, by using pullbacks in $\mathbf{Set}^{\mathscr{C}^{op}}$. There are morphisms $\xi_n : YC_n \to H_n$ such that $\xi_1 = \mathrm{id}_{H_0}$ and $\bar{\gamma}_i^n \xi_{n+1} = \xi_n Y(\bar{r}_i^n)$ for $i = 1, 2$ and $n = 0, 1, \ldots$:

$$\begin{tikzcd} & & YC_{n+1} \arrow[dll, "Y\bar{r}_0^n"'] \arrow[d, "\xi_{n+1}"] \arrow[drr, "Y\bar{r}_1^n"] & & \\ YC_n \arrow[dr, "\xi_n"'] & & H_{n+1} \arrow[dl, "\bar{\gamma}_0^n"'] \arrow[dr, "\bar{\gamma}_1^n"] & & YC_n \arrow[dl, "\xi_n"] \\ & H_n \arrow[dr, "\gamma_1^n"'] & & H_n \arrow[dl, "\gamma_0^n"] & \\ & & H_0 & & \end{tikzcd}$$

By induction, we will prove that ξ_n are regular epimorphisms in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Assume that ξ_n is a regular epimorphism and consider the pullback

$$\begin{tikzcd} & G \arrow[dl, "\rho_0"'] \arrow[dr, "\rho_1"] & \\ YC_n \arrow[dr, "\gamma_1^n \xi_n"'] & & YC_n \arrow[dl, "\gamma_0^n \xi_n"] \\ & H_0 & \end{tikzcd}$$

in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. There are morphisms $\varphi : YC_{n+1} \to G$ and $\psi : G \to H_{n+1}$ such that $\psi\varphi = \xi_{n+1}$, $\varrho_i \varphi = Y\bar{r}_i^n$ and $\bar{\gamma}_i^n \psi = \xi_n \varrho_i$ for $i = 0, 1$. Since ξ_n is a regular epimorphism in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$, ψ is a regular epimorphism in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Furthermore, it follows from the proof of Proposition 1.3 that φ is a regular epimorphism in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ too. Hence ξ_{n+1} is a regular epimorphism in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$.

Let I be the relation in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ determined by the graph (γ_0, γ_1). It means that I is given by the regular epi-monopair factorization

$$\begin{tikzcd} H_1 \arrow[rr, shift left, "\gamma_0"] \arrow[rr, shift right, "\gamma_1"'] \arrow[dr, "\iota"'] & & H_0 \\ & I \arrow[ur, shift left, "\iota_0"] \arrow[ur, shift right, "\iota_1"'] & \end{tikzcd}$$

in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Let $\iota_0^n, \iota_1^n : I^n \to H_0$ be the composition of n copies of I. Then I^n is the relation generated by the graph $\gamma_0^n, \gamma_1^n : H_n \to H_0$. Since $\xi_n : YC_n \to H_n$ is a regular epimorphism, I^n is also the relation generated by the graph $Yr_0^n, Yr_1^n : YC_n \to YC_0 = H_0$.

II. Now, assume that $\mathscr{C}$ has bounded transitive hulls of reflexive and symmetric graphs. We are going to prove that $\mathscr{C}_{\mathrm{ex}} \approx \mathscr{C}_{\mathrm{coeq}}$. Let $H : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ belong to $\mathscr{C}_{\mathrm{coeq}}$. Following Proposition 1.2, it suffices to prove that H belongs to $\mathscr{C}_{\mathrm{ex}}$. Since $FP(\mathscr{C}^{\mathrm{op}})$ is a variety and, following Lemma 1.1, H is finitely presentable, H is presented by a coequalizer

$$H_1 \overset{\gamma_0}{\underset{\gamma_1}{\rightrightarrows}} H_0 \xrightarrow{\gamma} H \tag{4}$$

of a reflexive and symmetric graph in $FP(\mathscr{C}^{\mathrm{op}})$ where H_1 and H_0 are free algebras in $FP(\mathscr{C}^{\mathrm{op}})$ over finitely many generators (cf. [AR], Remark 3.13). Since finitely presentable free algebras in $FP(\mathscr{C}^{\mathrm{op}})$ are precisely finite sums of representable functors and $G_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}_{\mathrm{coeq}}$ preserves finite sums, the functors H_0 and H_1 are representable, $H_i = YC_i$, $i = 0, 1$. Hence we get a reflexive and symmetric graph $r_0, r_1 : C_1 \to C_0$ in $\mathscr{C}$ such that $\gamma_i = Yr_i$, $i = 0, 1$. Since (4) is a coequalizer in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ too (by [PW]), it suffices to show that the kernel pair

$$K \overset{\alpha}{\underset{\beta}{\rightrightarrows}} YC_0 \xrightarrow{\gamma} H$$

has K weakly representable.

Since the graph (r_0, r_1) has a bounded transitive hull, there is n such that, for any $m > n$, there exists a graph morphism $Yf_m : YC_m \to YC_n$ from the graph (Yr_0^m, Yr_1^m) to the graph (Yr_0^n, Yr_1^n). Following I., they induce morphisms $I^m \to I^n$ of the corresponding relations. Hence I^n is an equivalence relation and, consequently, it yields a kernel pair of γ

$$I^n \overset{\iota_0^n}{\underset{\iota_1^n}{\rightrightarrows}} H_0 \xrightarrow{\gamma} H$$

Hence $K \cong I^n$ and since I^n is a quotient of YC_n, K is weakly representable.

III. Conversely, let $\mathscr{C}_{\mathrm{ex}} \approx \mathscr{C}_{\mathrm{coeq}}$ and consider a reflexive and symmetric graph $r_0, r_1 : C_1 \to C_0$ in $\mathscr{C}$. Take a coequalizer

$$YC_1 \overset{Yr_0}{\underset{Yr_1}{\rightrightarrows}} YC_0 \xrightarrow{\gamma} H$$

in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Following [PW], it is a coequalizer in $FP(\mathscr{C}^{\mathrm{op}})$ as well and, using Lemma 1.1, we get that $H \in \mathscr{C}_{\mathrm{coeq}}$. Hence $H \in \mathscr{C}_{\mathrm{ex}}$ and therefore the kernel pair

$$\begin{tikzcd} K \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & YC_0 \arrow[r, "\gamma"] & H \end{tikzcd}$$

has K weakly representable. Hence K is finitely generated in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ (see [AR] 1.69). Since K is a union of the chain of compositions I^n, $n = 0, 1, \ldots$, there is n such that $K \cong I^n$. Hence $I^m \cong I^n$ for all $m \geq n$. Following I., I^m is the relation generated by the graph (Yr_0^m, Yr_1^m). Since YC_m are regular projectives, there are graph morphisms $YC_m \to YC_n$ for all $m > n$. Hence, there are graph morphisms $f_m : C_m \to C_n$ for all $m > n$. We have proved that $\mathscr{C}$ has bounded transitive hulls of reflexive and symmetric graphs. ∎

2.4. Example. 1) Let $\mathscr{V}$ be a variety in which finitely generated algebras are closed under finite products and subalgebras (like sets, vector spaces or abelian groups). Let $\mathscr{C}$ be the full subcategory of $\mathscr{V}$ consisting of finitely generated free algebras. Then $\mathscr{C}_{\mathrm{ex}} \approx \mathscr{C}_{\mathrm{coeq}}$.

At first, following [AR] 3.16, $\mathscr{V} \cong FP(\mathscr{C}^{\mathrm{op}})$ and $Y : \mathscr{C} \to FP(\mathscr{C}^{\mathrm{op}})$ corresponds to the inclusion $\mathscr{C} \subseteq \mathscr{V}$. Consider a reflexive and symmetric graph $r_0, r_1 : C_1 \to C_0$ in $\mathscr{C}$. The equivalence relation $K \subseteq C_0 \times C_0$ determined by it is finitely generated (as a subalgebra of $C_0 \times C_0$). Following III. of the proof of 2.3, the graph (r_0, r_1) has a bounded transitive hull.

Remark that $\mathscr{C}$ has weak finite limits (following Proposition 1.3).

2) On the other hand, it is easy to find examples of a small category $\mathscr{C}$ such that $\mathscr{C}_{\mathrm{ex}} \approx \mathscr{C}_{\mathrm{coeq}}$ does not hold. It suffices to consider the category $\mathscr{C}$ of countable sets (and the infinite path as a reflexive and symmetric graph in it) and to use Theorem 2.3.

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