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A LOGIC FOR CATEGORIES

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Abstract. We present a doctrinal approach to category theory, obtained by abstracting from the indexed inclusion (via discrete fibrations and opfibrations) of left and of right actions of $X \in \mathbf{Cat}$ in categories over X. Namely, a "weak temporal doctrine" consists essentially of two indexed functors with the same codomain such that the induced functors have both left and right adjoints satisfying some exactness conditions, in the spirit of categorical logic.

The derived logical rules include some adjunction-like laws involving the truth-values-enriched hom and tensor functors, which condense several basic categorical properties and display a nice symmetry. The symmetry becomes more apparent in the slightly stronger context of "temporal doctrines", which we initially treat and which include as an instance the inclusion of lower and upper sets in the parts of a poset, as well as the inclusion of left and right actions of a graph in the graphs over it.

1. Introduction

Let X be a set endowed with an equivalence relation $\sim$, and let $\mathcal{V}X$ be the poset of closed parts, that is those subsets V of X such that $x \in V$ and $x \sim y$ implies $y \in V$. A part $P \in \mathcal{P}X$ has both a "closure" $\Diamond P$ and an "interior" $\Box P$, that is the inclusion $i : \mathcal{V}X \to \mathcal{P}X$ has both a left and a right adjoint:

$$\Diamond \dashv i \dashv \Box : \mathcal{P}X \to \mathcal{V}X$$

Thus the (co)reflection maps (inclusions) $\varepsilon_P : i\Box P \to P$ and $\eta_P : P \to i\Diamond P$ induce bijections (between 0-elements or 1-elements sets):

$$\frac{\mathcal{P}X(iV, i\Box P)}{\mathcal{P}X(iV, P)} \quad ; \quad \frac{\mathcal{P}X(i\Diamond P, iV)}{\mathcal{P}X(P, iV)}$$

By taking $\Box(P \Rightarrow Q)$ as a $\mathcal{V}X$-enrichment of $\mathcal{P}X(P, Q)$, it turns out that the above adjunctions are also enriched in $\mathcal{V}X$ giving isomorphisms:

$$\frac{\Box(iV \Rightarrow i\Box P)}{\Box(iV \Rightarrow P)} \quad ; \quad \frac{\Box(i\Diamond P \Rightarrow iV)}{\Box(P \Rightarrow iV)} \tag{1}$$

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We also have the related laws:

$$\frac{i\Diamond(iV \times iW)}{iV \times iW} \quad ; \quad \frac{i\Box(iV \Rightarrow iW)}{iV \Rightarrow iW} \quad ; \quad \frac{\Diamond(i\Diamond P \times iV)}{\Diamond(P \times iV)} \tag{2}$$

the first two of them saying roughly that closed parts are closed with respect to product (intersection) and exponentiation (implication). Given a groupoid X, the same laws hold for the inclusion of the actions of X in the groupoids over X (via "covering groupoids"). The above situation will be placed in the proper general context in sections 2 and 3, where we develop some technical tools concerning enriched adjunctions and apply them to hyperdoctrines [Lawvere, 1970].

Now, let us drop the symmetry condition on $\sim$, that is suppose that X is a poset; then we have the poset of lower-closed parts $\mathcal{L}X$ and that of upper-closed parts $\mathcal{R}X$. Again, the inclusions $i^\ell : \mathcal{L}X \to \mathcal{P}X$ and $i^r : \mathcal{R}X \to \mathcal{P}X$ have both a left and a right adjoint:

$$\Diamond^\ell \dashv i^\ell \dashv \Box^\ell : \mathcal{P}X \to \mathcal{L}X \qquad ; \qquad \Diamond^r \dashv i^r \dashv \Box^r : \mathcal{P}X \to \mathcal{R}X$$

While the first ones of (1) and (2) hold "on each side":

$$\frac{\Box^\ell(i^\ell V \Rightarrow i^\ell \Box^\ell P)}{\Box^\ell(i^\ell V \Rightarrow P)} \quad ; \quad \frac{\Box^r(i^r V \Rightarrow i^r \Box^r P)}{\Box^r(i^r V \Rightarrow P)} \tag{3}$$

$$\frac{i^\ell \Diamond^\ell(i^\ell V \times i^\ell W)}{i^\ell V \times i^\ell W} \quad ; \quad \frac{i^r \Diamond^r(i^r V \times i^r W)}{i^r V \times i^r W} \tag{4}$$

the other ones hold only in a mixed way:

$$\frac{\Box^r(i^\ell \Diamond^\ell P \Rightarrow i^\ell V)}{\Box^r(P \Rightarrow i^\ell V)} \quad ; \quad \frac{\Box^\ell(i^r \Diamond^r P \Rightarrow i^r V)}{\Box^\ell(P \Rightarrow i^r V)} \tag{5}$$

$$\frac{i^\ell \Box^\ell(i^r V \Rightarrow i^\ell W)}{i^r V \Rightarrow i^\ell W} \quad ; \quad \frac{i^r \Box^r(i^\ell V \Rightarrow i^r W)}{i^\ell V \Rightarrow i^r W} \tag{6}$$

$$\frac{\Diamond^\ell(i^\ell \Diamond^\ell P \times i^r V)}{\Diamond^\ell(P \times i^r V)} \quad ; \quad \frac{\Diamond^r(i^r \Diamond^r P \times i^\ell V)}{\Diamond^r(P \times i^\ell V)} \tag{7}$$

The laws (3) through (7) hold also for the inclusion of the left and the right actions of a category (or a reflexive graph [Lawvere, 1989]) X in categories (or graphs) over X (via discrete fibrations and opfibrations):

$$i^\ell : \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Cat}/X \qquad ; \qquad i^r : \mathbf{Set}^X \to \mathbf{Cat}/X$$

and, when they make sense, also for the inclusion of open and closed parts in the parts of a topological space (or, more generally, of local homeomorphisms and proper maps to a space X in spaces over X; see [Pisani, 2009]).

Abstracting from these situations, we may define a "temporal algebra" as a cartesian closed category with two reflective and coreflective full subcategories satisfying the above laws (in fact, it is enough to assume either (5) or (6) or (7)). A "temporal doctrine" is then essentially an indexed temporal algebra $\langle i^\ell_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^r_X\, ;\, X \in \mathcal{C}\rangle$ such that the inclusions i^ℓ_1 and i^r_1 over the terminal object $1 \in \mathcal{C}$ are isomorphic and each substitution functor $f^* : \mathcal{P}Y \to \mathcal{P}X$ has a left adjoint Σ_f and a right adjoint Π_f. Temporal doctrines and their basic properties are presented in sections 4 and 5.

In Section 6 we show how the "truth-values" $\mathcal{L}1 \cong \mathcal{R}1$ serve as values for an enriching of $\mathcal{P}X$, $\mathcal{L}X$ and $\mathcal{R}X$ in which the adjunctions

$$\Sigma_f \dashv f^* \dashv \Pi_f : \mathcal{P}X \to \mathcal{P}Y$$

$$\begin{array}{ccc} \Diamond^\ell_X \dashv i^\ell_X \dashv \Box^\ell_X : \mathcal{P}X \to \mathcal{L}X & ; & \Diamond^r_X \dashv i^r_X \dashv \Box^r_X : \mathcal{P}X \to \mathcal{R}X \\ \exists^\ell_f \dashv f^\ell \dashv \forall^\ell_f : \mathcal{L}X \to \mathcal{L}Y & ; & \exists^r_f \dashv f^r \dashv \forall^r_f : \mathcal{R}X \to \mathcal{R}Y \end{array} \tag{8}$$

are also enriched (where, for $f : X \to Y$ in $\mathcal{C}$, $\exists^\ell_f M$ and $\forall^\ell_f M$ can be obtained as $\Diamond^\ell_Y \Sigma_f i^\ell_X M$ and $\Box^\ell_Y \Pi_f i^\ell_X M$ respectively). For example, the temporal doctrine of posets is two-valued while that of reflexive graphs is set-valued, by identifying sets with discrete graphs.

If $\mathcal{C} = \mathbf{Cat}$, the functors Π_f are not always available, and the above mentioned enrichment is only partially defined. This weaker situation will be axiomatized in Section 10, where we will see that (8) can be still enriched giving:

$$\frac{\mathbf{nat}^\ell_X(L, f^\ell M)}{\mathbf{nat}^\ell_Y(\exists^\ell_f L, M)} \quad ; \quad \frac{\mathbf{nat}^\ell_X(f^\ell M, L)}{\mathbf{nat}^\ell_Y(M, \forall^\ell_f L)} \tag{9}$$

where

$$\mathbf{nat}^\ell_X(L, M) := \mathbf{end}_X(i^\ell_X L \Rightarrow i^\ell_X M) := \forall^\ell_X \Box^\ell_X (i^\ell_X L \Rightarrow i^\ell_X M)$$

(and similarly for "right actions" or "right closed parts" in $\mathcal{R}$). In a somewhat dual way one also obtains:

$$\frac{\mathbf{ten}_X(f^\ell M, N)}{\mathbf{ten}_Y(M, \exists^r_f N)} \quad ; \quad \frac{\mathbf{ten}_X(L, f^r N)}{\mathbf{ten}_Y(\exists^\ell_f L, N)} \tag{10}$$

where one defines the tensor product by

$$\mathbf{ten}_X(M, N) := \mathbf{coend}_X(i^\ell_X M \times i^r_X N) := \exists^\ell_X \Diamond^\ell_X (i^\ell_X M \times i^r_X N) \cong \exists^r_X \Diamond^r_X (i^\ell_X M \times i^r_X N)$$

(where the last isomorphism is mediated by $\mathcal{L}1 \cong \mathcal{R}1$).

In Section 7 we show how the laws (9) and (10) allow one to derive in an effective and transparent way several basic facts of category theory, in particular concerning (co)limits, the Yoneda lemma, Kan extensions and final functors. In sections 8 and 9 other "classical" properties are obtained exploiting also a "comprehension" axiom, relating $\mathcal{P}X$ and $\mathcal{C}/X$.

This approach also offers a new perspective on duality: we do not assume the coexistence of a (generalized) category X and its dual X^{op} (which in fact is not so obvious as it may seem at a first sight). Rather, we capture the interplay between left and right "actions" or "parts" of a category or "space" X by the above-sketched axioms concerning the inclusion of both of them in a category of more general "labellings" or "parts".

It is remarkable that while (6) is equivalent to (5) and to (7), they underlie seemingly unrelated items. On the one hand, for a truth value V in $\mathcal{L}1 \cong \mathcal{R}1$ the "$V-$complement" of $M \in \mathcal{L}X$

$$\neg(M, V) \ := \ i^\ell_X M \Rightarrow i^r_X X^r V$$

"is valued" in $\mathcal{R}X$, that is factors by (6) through i^r_X (and conversely). This generalizes the open-closed duality via complementation in topology (and in particular the upper-lower-sets duality for a poset) which is given by $\neg(M, 0)$, for the initial truth value 0.

On the other hand, if we denote by $\{x\} := \Sigma_x 1$ the "part" in $\mathbf{Cat}/X$ corresponding to the object $x : 1 \to X$, then $\Diamond^\ell_X\{x\} = X/x$ corresponds to the presheaf represented by x and for $N \in \mathcal{R}X$ we can prove that $\mathbf{ten}_X(X/x, M) \cong x^r N$ using (7) as follows:

$$\mathbf{ten}_X(X/x, N) \cong \exists^r_X \Diamond^r_X (i^\ell_X \Diamond^\ell_X \{x\} \times_X i^r_X N) \cong \exists^r_X \Diamond^r_X (\{x\} \times_X i^r_X N) \cong$$

$$\cong \Diamond^r_1 \Sigma_X (\Sigma_X 1 \times_X i^r_X N) \cong \Diamond^r_1 \Sigma_X \Sigma_x (1 \times x^* i^r_X N) \cong \Diamond^r_1 i^r_1 (x^r N) \cong x^r N$$

While in any temporal doctrine we can similarly derive $\mathbf{hom}_X(X/x, M) \cong x^\ell M$ using (5), in $\mathbf{Cat}$ such a proof of the Yoneda lemma stumbles against the lack of Π_x and the related non-exponentiability of $\{x\}$ (whenever x is a non-trivial retract in X). In this case, or in any (weak) temporal doctrine, one can use directly the first of (9):

$$\mathbf{nat}^\ell_X(X/x, M) \cong \mathbf{nat}^\ell_X(\exists^\ell_x 1, M) \cong \mathbf{nat}^\ell_1(1, x^\ell M) \cong x^\ell M$$

Similarly, using (10) one gets again:

$$\mathbf{ten}_X(N, X/x) \cong \mathbf{ten}_X(N, \exists^\ell_x 1) \cong \mathbf{ten}_1(x^r N, 1) \cong x^r N$$

The present paper is a development of previous works on "balanced category theory" (see in particular [Pisani, 2008] and [Pisani, 2009]); the doctrinal approach adopted here emphasizes the logical aspects and suggests a wider range of applications.

2. Enriching adjunctions

In this section, we make some remarks that will be used in the sequel. Along with ordinary adjunctions, Kan defined and studied what are now known as adjunctions with parameter and enriched adjunctions. In particular, we will use the following result from [Kan, 1958]:

2.1. Lemma. *Given functors $F, F' : \mathcal{C} \times \mathcal{P} \to \mathcal{V}$ and $R, R' : \mathcal{P}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{C}$ such that there are adjunctions, for any $P \in \mathcal{P}$,*

$$F(-, P) \dashv R(P, -) \qquad ; \qquad F'(-, P) \dashv R'(P, -)$$

the natural transformations $F \to F'$ correspond bijectively to the ones $R' \to R$, and this correspondence restricts to natural isomorphisms. In particular $F \cong F'$ iff $R \cong R'$. ■

The next proposition implies that a geometric morphism is naturally enriched in its codomain:

2.2. Proposition. *Let $F \dashv R : \mathcal{C} \to \mathcal{V}$ be an adjunction between cartesian closed categories, with F preserving finite products. Then $\mathcal{C}$ is enriched in $\mathcal{V}$ via*

$$\mathbf{hom}_{\mathcal{C}}^{\mathcal{V}}(X, Y) := R(X \Rightarrow_{\mathcal{C}} Y) \tag{11}$$

and there are natural isomorphisms:

$$\mathbf{hom}_{\mathcal{C}}^{\mathcal{V}}(FV, X) \cong \mathbf{hom}_{\mathcal{V}}(V, RX) \tag{12}$$

(where $\mathbf{hom}_{\mathcal{V}}(V, W) := V \Rightarrow_{\mathcal{V}} W$ is the internal hom of $\mathcal{V}$) that is the adjunction $F \dashv R$ is itself enriched in $\mathcal{V}$. Furthermore, the natural transformations given by the arrow mappings of F and R are also enriched:

$$\mathbf{hom}_{\mathcal{V}}(V, W) \to \mathbf{hom}_{\mathcal{C}}^{\mathcal{V}}(FV, FW) \qquad ; \qquad \mathbf{hom}_{\mathcal{C}}^{\mathcal{V}}(X, Y) \to \mathbf{hom}_{\mathcal{V}}(RX, RY)$$

Proof. For the first part, we have

$$\mathcal{V}(1, R(X \Rightarrow Y)) \cong \mathcal{C}(F1, X \Rightarrow Y) \cong \mathcal{C}(1, X \Rightarrow Y) \cong \mathcal{C}(X, Y)$$

(In fact, more generally, R transfers any enriching in $\mathcal{C}$ to an enriching in $\mathcal{V}$.) For the second part, since $F(V \times W) \cong FV \times FW$, we can apply Lemma 2.1 to the adjunctions:

$$F(- \times W) \dashv W \Rightarrow R(-) \qquad ; \qquad F(-) \times FW \dashv R(FW \Rightarrow -)$$

For the third part, the chain of natural transformations:

$$\mathcal{V}(U, V \Rightarrow W) \cong \mathcal{V}(U \times V, W) \to \mathcal{C}(F(U \times V), FW) \cong$$
$$\cong \mathcal{C}(FU \times FV, FW) \cong \mathcal{C}(FU, FV \Rightarrow FW) \cong \mathcal{V}(U, R(FV \Rightarrow FW))$$

yields the desired natural transformation, which is easily seen to enrich the arrow mapping of F. For R we similarly have:

$$\mathcal{V}(U, R(X \Rightarrow Y)) \cong \mathcal{C}(FU, X \Rightarrow Y) \cong \mathcal{C}(FU \times X, Y) \to$$
$$\to \mathcal{C}(F(U \times RX), Y) \cong \mathcal{V}(U \times RX, RY) \cong \mathcal{V}(U, RX \Rightarrow RY)$$

where the non-isomorphic step is induced by the canonical

$$\langle Fp, \varepsilon Fq \rangle : F(U \times RX) \to FU \times X \tag{13}$$

Thus, R is fully faithful, also as an enriched functor, iff (13) is an iso, that is $F \dashv R$ satisfies the Frobenius law. Since here we have not used the fact that F preserves all finite products, but only the terminal object (in order to obtain an enrichment of R) we get in particular a proof of Corollary 1.5.9 (i) in [Johnstone, 2002]. ■

If F has a further left adjoint $L : \mathcal{C} \to \mathcal{V}$, then it is left exact and the above proposition applies. We now show that in this case the adjunction $L \dashv F$ is also enriched in $\mathcal{V}$ iff it satisfies the Frobenius reciprocity law:

2.3. Proposition. *Suppose that $\mathcal{C}$ and $\mathcal{V}$ are cartesian closed and that*

$$L \dashv F \dashv R : \mathcal{C} \to \mathcal{V}$$

Then the existence of the following natural isomorphisms are equivalent:

1. $LX \times_{\mathcal{V}} V \cong L(X \times_{\mathcal{C}} FV)$
2. $F(V \Rightarrow_{\mathcal{V}} W) \cong FV \Rightarrow_{\mathcal{C}} FW$
3. $\mathbf{hom}^{\mathcal{V}}_{\mathcal{C}}(X, FV) \cong \mathbf{hom}_{\mathcal{V}}(LX, V)$

Proof. By applying Lemma 2.1 to the adjunctions:

$$L(-) \times V \dashv F(V \Rightarrow -) \qquad ; \qquad L(- \times FV) \dashv FV \Rightarrow F(-)$$

one gets the well-known equivalence of 1) and 2) (see [Lawvere, 1970] or [Johnstone, 2002]); similarly, the adjunctions:

$$LX \times - \dashv LX \Rightarrow - \qquad ; \qquad L(X \times F(-)) \dashv R(X \Rightarrow F(-))$$

give the equivalence of 1) and 3). ■

2.4. Remark. It is well known that given adjunctions $L \dashv F \dashv R : \mathcal{C} \to \mathcal{V}$, with F fully faithful, if $\mathcal{C}$ is cartesian closed then so is also $\mathcal{V}$; in fact, products in $\mathcal{V}$ can be calculated by

$$1_{\mathcal{V}} := R1_{\mathcal{C}} \qquad ; \qquad V \times_{\mathcal{V}} W := R(FV \times_{\mathcal{C}} FW)$$

or also by

$$1_{\mathcal{V}} := L1_{\mathcal{C}} \qquad ; \qquad V \times_{\mathcal{V}} W := L(FV \times_{\mathcal{C}} FW)$$

and exponentials by

$$V \Rightarrow_{\mathcal{V}} W := R(FV \Rightarrow_{\mathcal{C}} FW) \tag{14}$$

Note that, following Proposition 2.2, (14) can be written

$$\mathbf{hom}_{\mathcal{V}}(V, W) \cong \mathbf{hom}^{\mathcal{V}}_{\mathcal{C}}(FV, FW)$$

indicating that F is fully faithful as an enriched functor, so that (12) becomes

$$\mathbf{hom}^{\mathcal{V}}_{\mathcal{C}}(FV, X) \cong \mathbf{hom}^{\mathcal{V}}_{\mathcal{C}}(FV, FRX) \tag{15}$$

Note also that, in this case, the equivalent conditions of Proposition 2.3 can be rewritten as follows:

$$L(X \times FV) \cong L(FLX \times FV) \tag{16}$$

$$FV \Rightarrow FW \cong FR(FV \Rightarrow FW) \tag{17}$$

$$\mathbf{hom}^{\mathcal{V}}_{\mathcal{C}}(X, FV) \cong \mathbf{hom}^{\mathcal{V}}_{\mathcal{C}}(FLX, FV) \tag{18}$$

where the isomorphisms are induced by the unit of $L \dashv F$ (the first and the third ones) and by the counit of $F \dashv R$ (the second one).

3. The logic of hyperdoctrines

We now show how some of the results of Section 2 apply to hyperdoctrines [Lawvere, 1970], giving interesting consequences. Recall that a hyperdoctrine is an indexed category $\langle \mathcal{P}X\,;\, X \in \mathcal{C}\rangle$ such that $\mathcal{C}$ and all the categories $\mathcal{P}X$ are cartesian closed, and such that each substitution functor $f^* : \mathcal{P}Y \to \mathcal{P}X$ has both a left and a right adjoint $\Sigma_f \dashv f^* \dashv \Pi_f$ for any $f : X \to Y$ in $\mathcal{C}$. The logical significance of hyperdoctrines, and in particular the role of the adjoints to the substitution functors as existential and universal quantification, and that of $\mathcal{P}1$ as "sentences" or "truth values", are clearly illustrated in [Lawvere, 1970] and in other papers by the same author.

Here we also assume that the adjunctions $\Sigma_f \dashv f^*$ satisfy the Frobenius reciprocity law, that is that (13) is an iso. On the other hand, we do not need to assume that $\mathcal{C}$ is cartesian closed but only that it has a terminal object.

3.1. Corollary. *Let $\langle \mathcal{P}X\,;\, X \in \mathcal{C}\rangle$ be a hyperdoctrine and define*

$$\mathbf{hom}_X(P,Q) := \Pi_X(P \Rightarrow Q) : (\mathcal{P}X)^{\mathrm{op}} \times \mathcal{P}X \to \mathcal{P}1$$

$$\mathbf{meets}_X(P,Q) := \Sigma_X(P \times Q) : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}1$$

where the quantification indexes denote the map $X \to 1$. Then $\mathbf{hom}_X$ enriches $\mathcal{P}X$ in $\mathcal{P}1$ and, for any map $f : X \to Y$, there are natural isomorphisms:

$$\mathbf{hom}_X(f^*Q,P) \cong \mathbf{hom}_Y(Q,\Pi_f P) \quad ; \quad \mathbf{hom}_X(P,f^*Q) \cong \mathbf{hom}_Y(\Sigma_f P,Q)$$

$$\mathbf{meets}_X(f^*Q,P) \cong \mathbf{meets}_Y(Q,\Sigma_f P) \quad ; \quad \mathbf{meets}_X(P,f^*Q) \cong \mathbf{meets}_Y(\Sigma_f P,Q)$$

Proof. Propositions 2.2 and 2.3 (applied to $\Sigma_f \dashv f^* \dashv \Pi_f$) and the Frobenius law itself respectively give:

$$\Pi_X(f^*Q \Rightarrow P) \cong \Pi_Y \Pi_f(f^*Q \Rightarrow P) \cong \Pi_Y(Q \Rightarrow \Pi_f P)$$

$$\Pi_X(P \Rightarrow f^*Q) \cong \Pi_Y \Pi_f(P \Rightarrow f^*Q) \cong \Pi_Y(\Sigma_f P \Rightarrow Q)$$

$$\Sigma_X(P \times f^*Q) \cong \Sigma_Y \Sigma_f(P \times f^*Q) \cong \Sigma_Y(\Sigma_f P \times Q)$$

■

Say that a map $f : X \to Y$ is "surjective" if $\Sigma_f \top_X \cong \top_Y$, where $\top_X$ denotes a terminal object of $\mathcal{P}X$.

3.2. Corollary. *If $f : X \to Y$ is surjective map then*

$$\Pi_X(f^*Q) \cong \Pi_Y Q \quad ; \quad \Sigma_X(f^*Q) \cong \Sigma_Y Q$$

Proof. For the first one we have:

$$\Pi_X(f^*Q) \cong \mathbf{hom}_X(\top_X, f^*Q) \cong \mathbf{hom}_Y(\Sigma_f \top_X, Q) \cong \mathbf{hom}_Y(\top_Y, Q) \cong \Pi_Y Q$$

The proof of the second one follows the same pattern:

$$\Sigma_X(f^*Q) \cong \mathbf{meets}_X(\top_X, f^*Q) \cong \mathbf{meets}_Y(\Sigma_f \top_X, Q) \cong \mathbf{meets}_Y(\top_Y, Q) \cong \Sigma_Y Q$$

■

Note that for $\langle \mathcal{P}X\,;\, X \in \mathbf{Set}\rangle$, Corollary 3.2 becomes the fact that the inverse image functor along a surjective mapping $f : X \to Y$ preserves non-emptiness and reflects maximality: if $P \subseteq Y$ is non-empty so it is $f^{-1}P$ and if $f^{-1}P = X$ then $P = Y$.

There are three canonical ways to get a "truth value" in $\mathcal{P}1$ from $P \in \mathcal{P}X$, namely quantifications along $X : X \to 1$ and evaluation at a point $x : 1 \to X$:

$$\Pi_X P \qquad ; \qquad \Sigma_X P \qquad ; \qquad x^* P$$

In the proof of Corollary 3.2 we have used the fact that quantifications along X are "represented" (by $\top_X$):

$$\Pi_X P \;\cong\; \mathbf{hom}_X(\top_X, P) \qquad ; \qquad \Sigma_X P \;\cong\; \mathbf{meets}_X(\top_X, P)$$

Now we show that the same is true for evaluation; namely, evaluation at x is "represented" by the "singleton":

$$\{x\} := \Sigma_x \top_1$$

3.3. Corollary. *Given a point $x : 1 \to X$ there are isomorphisms*

$$x^* P \;\cong\; \mathbf{hom}_X(\{x\}, P) \qquad ; \qquad x^* P \;\cong\; \mathbf{meets}_X(\{x\}, P)$$

natural in $P \in \mathcal{P}X$.

Proof. $\mathbf{hom}_X(\Sigma_x \top_1, P) \;\cong\; \mathbf{hom}_1(\top_1, x^* P) \;\cong\; \Pi_1(x^* P) \;\cong\; x^* P$

$$\mathbf{meets}_X(\Sigma_x \top_1, P) \;\cong\; \mathbf{meets}_1(\top_1, x^* P) \;\cong\; \Sigma_1(x^* P) \;\cong\; x^* P$$

(Note that the quantification index 1 is the identity on $1 \in \mathcal{C}$.) ∎

3.4. Remark. Suppose that $\mathcal{C}$ has pullbacks, so that we also have the doctrine $\langle \mathcal{C}/X\,;\, X \in \mathcal{C}\rangle$, with $f_! \dashv f^{-1} : \mathcal{C}/Y \to \mathcal{C}/X$ for $f : X \to Y$. Suppose also that $\langle \mathcal{P}X\,;\, X \in \mathcal{C}\rangle$ satisfies the comprehension axiom [Lawvere, 1970] $c_X \dashv k_X : \mathcal{P}X \to \mathcal{C}/X$. Then the set-valued "external evaluation" of $P \in \mathcal{P}X$ at $x : 1 \to X$ can be expressed in various ways:

$$\mathcal{P}1(\top_1, x^{-1}P) \;\cong\; \mathcal{P}X(\{x\}, P) \;\cong\; \mathcal{P}X(c_X x, P) \;\cong$$

$$\cong\; \mathcal{C}/X(x, k_X P) \;\cong\; \mathcal{C}/X(x_! 1, k_X P) \;\cong\; \mathcal{C}(1, x^{-1} k_X P)$$

3.5. Corollary. [formulas for quantifications] *Given $P \in \mathcal{P}X$, a map $f : X \to Y$ and a point $y : 1 \to Y$, there are isomorphisms*

$$y^* \Pi_f P \;\cong\; \mathbf{hom}_X(f^*\{x\}, P) \qquad ; \qquad y^* \Sigma_f P \;\cong\; \mathbf{meets}_X(f^*\{x\}, P)$$

natural in $P \in \mathcal{P}X$.

Proof. $y^* \Pi_f P \;\cong\; \mathbf{hom}_Y(\{y\}, \Pi_f P) \;\cong\; \mathbf{hom}_X(f^*\{y\}, P)$

$$y^* \Sigma_f P \;\cong\; \mathbf{meets}_Y(\{y\}, \Sigma_f P) \;\cong\; \mathbf{meets}_X(f^*\{y\}, P)$$ ∎

Note that for $\langle \mathcal{P}X\,;\,X\in\mathbf{Set}\rangle$, Corollary 3.5 gives the classical formula for the coimage of a part along a mapping f, and a (less classical) formula for the image: y is in the image $\Sigma_f P$ iff its inverse image meets P.

4. Temporal doctrines

A **temporal doctrine** $\langle i^\ell_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^r_X\,;\,X\in\mathcal{C}\rangle$ consists of two indexed functors with the same codomain, satisfying the axioms listed below.

We denote the substitution functors along a map $f : X\to Y$ in $\mathcal{C}$ by

$$f^\ell : \mathcal{L}Y \to \mathcal{L}X \qquad ; \qquad f^r : \mathcal{R}Y \to \mathcal{R}X \qquad ; \qquad f^* : \mathcal{P}Y \to \mathcal{P}X$$

Thus we have (coherent) isomorphisms:

$$(gf)^\ell \cong f^\ell g^\ell \qquad ; \qquad (gf)^r \cong f^r g^r \qquad ; \qquad (gf)^* \cong f^* g^*$$

(and similarly for identities) and also

$$i^\ell_X f^\ell \cong f^* i^\ell_Y \qquad ; \qquad i^r_X f^r \cong f^* i^r_Y$$

We denote by $\mathcal{B}X$ the indexed pullback $\mathcal{L}X \times_{\mathcal{P}X} \mathcal{R}X$, by j^ℓ_X and j^r_X its indexed projections to $\mathcal{L}X$ and $\mathcal{R}X$ respectively, and

$$b_X := i^\ell_X j^\ell_X = i^r_X j^r_X : \mathcal{B}X \to \mathcal{P}X$$

The first group of axioms requires the existence of some adjoint functors:

1. The indexing category has a terminal object: $1\in\mathcal{C}$.

2. The categories of $\mathcal{P}X$ are cartesian closed. Thus, for any $X\in\mathcal{C}$, we have a terminal object $1_X\in\mathcal{P}X$, products $P\times_X Q$ and exponentials $P\Rightarrow_X Q$.

3. The substitution functors $f^* : \mathcal{P}Y \to \mathcal{P}X$ have both left and right adjoints:
$$\Sigma_f \dashv f^* \dashv \Pi_f$$

4. The functors $i^\ell_X : \mathcal{L}X \to \mathcal{P}X$ and $i^r_X : \mathcal{R}X\to\mathcal{P}X$ have both left and right adjoints:
$$\Diamond^\ell_X \dashv i^\ell_X \dashv \Box^\ell_X \qquad ; \qquad \Diamond^r_X \dashv i^r_X \dashv \Box^r_X$$

5. The doctrine $\mathcal{P}X$ satisfies the comprehension axiom [Lawvere, 1970]: the canonical functors $c_X : \mathcal{C}/X \to \mathcal{P}X$ (sending $f : T\to X$ to $\Sigma_f 1_T$) have right adjoints:
$$c_X \dashv k_X : \mathcal{P}X \to \mathcal{C}/X$$

The second group of axioms imposes some exactness condition on these functors:

1. The functors i^ℓ_X and i^r_X are fully faithful:

$$\diamondsuit^\ell_X i^\ell_X \cong \mathrm{id}_{\mathcal{L}X} \qquad ; \qquad \Box^\ell_X i^\ell_X \cong \mathrm{id}_{\mathcal{L}X}$$

(and similarly for i^r_X).

2. The doctrine $\mathcal{P}X$ satisfies the Frobenius law:

$$\Sigma_f P \times_Y Q \cong \Sigma_f (P \times_X f^* Q) \tag{19}$$

for any $f : X \to Y$ (naturally in $P \in \mathcal{P}X$ and $Q \in \mathcal{P}Y$).

3. The adjunctions $\diamondsuit^\ell_X \dashv i^\ell_X$ and $\diamondsuit^r_X \dashv i^r_X$ satisfy the "mixed Frobenius laws", that is their units induce isomorphisms

$$\diamondsuit^\ell_X (P \times_X i^r_X N) \cong \diamondsuit^\ell_X (i^\ell_X \diamondsuit^\ell_X P \times_X i^r_X N) \tag{20}$$

$$\diamondsuit^r_X (P \times_X i^\ell_X M) \cong \diamondsuit^r_X (i^r_X \diamondsuit^r_X P \times_X i^\ell_X M) \tag{21}$$

(natural in $P \in \mathcal{P}X$, $N \in \mathcal{R}X$ and $M \in \mathcal{L}X$).

4. The projections $j^\ell_1 : \mathcal{B}1 \to \mathcal{L}1$ and $j^r_1 : \mathcal{B}1 \to \mathcal{R}1$ are isomorphisms.

5. The comprehension functors $k_X : \mathcal{P}X \to \mathcal{C}/X$ are fully faithful:

$$c_X k_X P = \Sigma_{k_X P} 1_{X_!(k_X P)} \cong P \tag{22}$$

(where we use the notations of Remark 3.4, so that $X_!$ is the domain projection $\mathcal{C}/X \to \mathcal{C}$. Note that the index $k_X P$ of Σ is an object of $\mathcal{C}/X$, so that it should be more exactly be replaced by $X_!(k_X P)$, where now $k_X P$ denote the map to the terminal in $\mathcal{C}/X$).

4.1. Examples.

1. Any hyperdoctrine $\langle \mathcal{P}X\,;\, X \in \mathcal{C}\rangle$ (see Section 3) with a fully faithful comprehension functor gives rise to a (rather trivial) temporal doctrine:

$$\langle\, \mathrm{id} : \mathcal{P}X \to \mathcal{P}X \leftarrow \mathcal{P}X : \mathrm{id}\,;\, X \in \mathcal{C}\,\rangle$$

Thus the results of Section 3 can be seen as particular cases of those we will obtain for temporal doctrines.

2. $\langle i^\ell_X : \mathcal{L}X \to \mathcal{P}X \leftarrow \mathcal{R}X : i^r_X\,;\, X \in \mathbf{Pos}\rangle$, where $\langle \mathcal{P}X\,;\, X \in \mathbf{Pos}\rangle$ is the doctrine of all the parts of a poset, while $\mathcal{L}X$ and $\mathcal{R}X$ are the subdoctrines of lower-closed and upper-closed parts of X.

3. $\langle i^\ell_X : \mathcal{L}X \to \mathbf{Grph}/X \leftarrow \mathcal{R}X : i^r_X\,;\, X \in \mathbf{Grph}\rangle$, where $\mathbf{Grph}$ is the category of reflexive graphs, while $\mathcal{L}X$ and $\mathcal{R}X$ are the categories of left and right actions of X (or of the free category generated by it; see [Lawvere, 1989]).

4. Sets endowed with an equivalence relation give rise, as sketched in the introduction, to a "symmetrical" temporal doctrine: all the projections j^ℓ_X and j^r_X are isomorphisms (so that $\mathcal{L}X \cong \mathcal{R}X \cong \mathcal{B}X$ as subobjects of $\mathcal{P}X$). Note that, since the axioms are symmetrical, each temporal doctrine has a dual obtained by exchanging the left and the right side (that is i^ℓ and i^r); while a symmetrical temporal doctrine is clearly self-dual (that is isomorphic to its own dual) the same is true also for **Grph**, via the "opposite" functor $\mathbf{Grph} \to \mathbf{Grph}$.

5. Any strong balanced factorization category $\langle \mathcal{C}\,; \mathcal{E}, \mathcal{M}\,; \mathcal{E}', \mathcal{M}' \rangle$ (see [Pisani, 2008] and [Pisani, 2009]) such that $\mathcal{C}$ is locally cartesian closed and $\mathcal{M}/X$ and $\mathcal{M}'/X$ are coreflective in $\mathcal{C}/X$, gives rise to the temporal doctrine
$$\langle i^\ell_X : \mathcal{M}/X \to \mathcal{C}/X \leftarrow \mathcal{M}'/X : i^r_X\,;\, X \in \mathcal{C}\rangle$$

6. Given a temporal doctrine on a category $\mathcal{C}$ and any subcategory $\mathcal{C}'$ of $\mathcal{C}$ such that $1 \in \mathcal{C}'$, one gets by restriction another temporal doctrine on $\mathcal{C}'$.

4.2. Remark. The name "temporal doctrine" is clearly suggested by the functors $\Diamond^\ell$, $\Box^\ell$, $\Diamond^r$ and $\Box^r$, which can be seen as modal operators acting in the two directions of time.

A categorical approach to modal and tense logic was developed in the eighties by Ghilardi and Meloni and independently by Reyes et al. Not being here specifically concerned with these logics, we just note that the temporal doctrine of posets mentioned in the examples above is also an instance of temporal doctrine in the sense of [Ghilardi & Meloni, 1991].

Let me also acknowledge that it was Prof. Giancarlo Meloni, the supervisor of my PhD thesis, who introduced me to categorical logic showing in particular how adjunctions can be an effective tool for doing calculations.

5. Basic properties

5.1. terminology. Since a (weaker form of) temporal doctrine is mainly intended to model the situation $\langle \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Cat}/X \leftarrow \mathbf{Set}^X\,;\, X \in \mathbf{Cat}\rangle$, the objects of $\mathcal{C}$ should be thought of as generalized categories. In fact in the sequel we will freely borrow terminology from category theory, whenever opportune. However, the interior ($\Box^\ell$ and $\Box^r$) and closure ($\Diamond^\ell$ and $\Diamond^r$) operators suggest that it also make sense to consider the objects of $\mathcal{C}$ as a sort of spaces, so that we will also borrow some terminology from topology; in fact, the links with that subject can be taken quite seriously as sketched in [Pisani, 2009], where it is discussed also the significance of the "closure" reflection in "open parts" (or "local homeomorphisms"). Anyway, if X is a topological space and i^ℓ_X and i^r_X are the inclusion of open and closed parts respectively in $\mathcal{P}X$, the mixed Frobenius laws (and their equivalent ones in Proposition 5.2 below) hold true when they make sense, that is when only the operators $\Box^\ell_X$ and $\Diamond^r_X$ are involved.

Thus we sometimes refer to objects and arrows of $\mathcal{C}$ as "spaces" and "maps"; to the objects of $\mathcal{P}X$ as "parts" of X and to those of $\mathcal{L}X$ and $\mathcal{R}X$ as "left closed" and "right closed" parts of X, respectively. The reflections $\Diamond^\ell_X$ and $\Diamond^r_X$ are the "left" and "right" closure operators respectively, while $\Box^\ell_X$ and $\Box^r_X$ are the "left" and "right" interior operators.

To summarize the definition, apart from the axioms concerning the comprehension adjunctions, a temporal doctrine is a hyperdoctrine $\mathcal{P}X$ (in the sense of Section 3) with two reflective and coreflective indexed subcategories $\mathcal{L}X$ and $\mathcal{R}X$ such that $\mathcal{L}1$ and $\mathcal{R}1$ are isomorphic as subcategories of $\mathcal{P}1$; furthermore, and most importantly, we assume the mixed Frobenius laws (20) and (21), which are rich of important consequences. The reason of their name follows by Remark 2.4: they look like the Frobenius laws for $\Diamond^\ell_X \dashv i^\ell_X$ and $\Diamond^r_X \dashv i^r_X$, except that i^ℓ_X and i^r_X are exchanged in the second factors. In fact, we have the corresponding mixed equivalent conditions:

5.2. Proposition. *The following laws hold in a temporal doctrine, and each of them can be used in the definition in place of* (20) *and* (21)*:*

$$i^r_X N \Rightarrow i^\ell_X M \;\cong\; i^\ell_X \Box^\ell_X (i^r_X N \Rightarrow i^\ell_X M) \;\;;\;\; i^\ell_X M \Rightarrow i^r_X N \;\cong\; i^r_X \Box^r_X (i^\ell_X M \Rightarrow i^r_X N) \tag{23}$$

$$\Box^r_X (P \Rightarrow i^\ell_X M) \;\cong\; \Box^r_X (i^\ell_X \Diamond^\ell_X P \Rightarrow i^\ell_X M) \;\;;\;\; \Box^\ell_X (P \Rightarrow i^r_X N) \;\cong\; \Box_X (i^r_X \Diamond^r_X P \Rightarrow i^r_X N) \tag{24}$$

Furthermore

$$\Box^\ell_X (i^\ell_X M \Rightarrow P) \;\cong\; \Box^\ell_X (i^\ell_X M \Rightarrow i^\ell_X \Box^\ell_X P) \;\;;\;\; \Box^r_X (i^r_X N \Rightarrow P) \;\cong\; \Box^r_X (i^r_X N \Rightarrow i^r_X \Box^r_X P)$$

Proof. The isomorphism (20) induce two families of natural isomorphisms,

$$\Diamond^\ell_X (- \times_X i^r_X N) \;\cong\; \Diamond^\ell_X (i^\ell_X \Diamond^\ell_X (-) \times_X i^r_X N) : \mathcal{P}X \to \mathcal{L}X \tag{25}$$

$$\Diamond^\ell_X (P \times_X i^r_X (-)) \;\cong\; \Diamond^\ell_X (i^\ell_X \Diamond^\ell_X P \times_X i^r_X (-)) : \mathcal{R}X \to \mathcal{L}X \tag{26}$$

Then, as in Proposition 2.3, their right adjoint with parameter give the first ones of (23) and (24); the other ones clearly follow symmetrically from (21). For the last statement, recall (15). ■

From (23) we immediately get:

5.3. Corollary. *If the part $P \in \mathcal{P}X$ is left closed and $Q \in \mathcal{P}X$ is right closed (that is $P \cong i^\ell_X M$ and $Q \cong i^r_X N$) then $P \Rightarrow Q$ is itself right closed.* ■

As already mentioned in the Introduction, we so have an "explanation" of the fact that the complement of an upper-closed part of a poset is lower-closed (and conversely).

5.4. Corollary. *The categories $\mathcal{B}X$ are themselves cartesian closed, with the "same" exponential as $\mathcal{P}X$:*

$$b_X B \Rightarrow b_X C \;\cong\; b_X (B \Rightarrow_{\mathcal{B}X} C)$$

■

5.5. Proposition. *$\langle \mathcal{L}X\,;\,X\in\mathcal{C}\rangle$ and $\langle \mathcal{R}X\,;\,X\in\mathcal{C}\rangle$ are themselves hyperdoctrines, with a fully faithful comprehension adjoint.*

Proof.

1. As in Remark 2.4, the categories $\mathcal{L}X$ and $\mathcal{R}X$ are cartesian closed, with exponentials given by
$$\Box^\ell_X(i^\ell_X L \Rightarrow i^\ell_X M) \qquad ; \qquad \Box^r_X(i^r_X N \Rightarrow i^r_X O)$$
We denote products in $\mathcal{L}X$ and $\mathcal{R}X$ by
$$\top^\ell_X \qquad ; \qquad L\wedge^\ell_X M \qquad ; \qquad \top^r_X \qquad ; \qquad N\wedge^r_X O$$

2. The substitution functors for left and right closed parts have both left and right adjoints:
$$\exists^\ell_f \dashv f^\ell \dashv \forall^\ell_f \qquad ; \qquad \exists^r_f \dashv f^r \dashv \forall^r_f$$
where
$$\exists^\ell_f \;\cong\; \Diamond^\ell_Y \Sigma_f\, i^\ell_X \qquad ; \qquad \forall^\ell_f \;\cong\; \Box^\ell_Y \Pi_f\, i^\ell_X \tag{27}$$
(and similarly for $\exists^r_f$ and $\forall^r_f$). Note that these satisfy:
$$\exists^\ell_f \Diamond^\ell_X \;\cong\; \Diamond^\ell_Y \Sigma_f \qquad ; \qquad \forall^\ell_f \Box^\ell_X \;\cong\; \Box^\ell_Y \Pi_f \tag{28}$$
(and similarly for $\exists^r_f$ and $\forall^r_f$).

3. The canonical functors $\mathcal{C}/X \to \mathcal{L}X$ send $f:T\to X$ to
$$\exists^\ell_f \top^\ell_T \;\cong\; \Diamond^\ell_X \Sigma_f\, i^\ell_X \top^\ell_T \;\cong\; \Diamond^\ell_X \Sigma_f 1_T \;\cong\; \Diamond^\ell_X c_X f$$
that is factor through the corresponding ones for $\mathcal{P}X$. Thus, they have the functors $k_X i^\ell_X : \mathcal{L}X \to \mathcal{C}/X$ as fully faithful right adjoints (and similarly for $\mathcal{R}X$; we leave it to the reader to check the above factorization for the arrow mapping).

The fact that the adjunctions $\Diamond^\ell_X \dashv i^\ell_X$ and $\Diamond^r_X \dashv i^r_X$ satisfy the mixed Frobenius laws implies a restricted form of the Frobenius law for each of them and also for $\exists^\ell_f \dashv f^\ell$ and $\exists^r_f \dashv f^r$, which will be used in the sequel:

5.6. Proposition. [restricted Frobenius laws] *For any $X\in\mathcal{C}$, there are natural isomorphisms:*
$$\Diamond^\ell_X(P\times_X i^\ell_X j^\ell_X B) \;\cong\; \Diamond^\ell_X P \wedge^\ell_X j^\ell_X B$$
For any $f:X\to Y$ in $\mathcal{C}$, there are natural isomorphisms:
$$\exists^\ell_f(M\wedge^\ell_X f^\ell j^\ell_Y B) \cong \exists^\ell_f M \wedge^\ell_Y j^\ell_Y B \qquad ; \qquad \exists^r_f(N\wedge^r_X f^r j^r_Y B) \cong \exists^r_f N \wedge^r_Y j^r_Y B$$

PROOF. For the first one, by the mixed Frobenius law we get:

$$\Diamond_X^\ell(P \times_X i_X^\ell j_X^\ell B) \cong \Diamond_X^\ell(P \times_X i_X^r j_X^r B) \cong \Diamond_X^\ell(i_X^\ell \Diamond_X^\ell P \times_X i_X^r j_X^r B) \cong$$

$$\cong \Diamond_X^\ell(i_X^\ell \Diamond_X^\ell P \times_X i_X^\ell j_X^\ell B) \cong \Diamond_X^\ell i_X^\ell(\Diamond_X^\ell P \wedge_X^\ell j_X^\ell B) \cong \Diamond_X^\ell P \wedge_X^\ell j_X^\ell B$$

For the second one, we then have:

$$\exists_f^\ell(M \wedge_X^\ell f^\ell j_Y^\ell B) \cong \Diamond_Y^\ell \Sigma_f i_X^\ell(M \wedge_X^\ell f^\ell j_Y^\ell B) \cong$$

$$\cong \Diamond_Y^\ell \Sigma_f(i_X^\ell M \times_X i_X^\ell f^\ell j_Y^\ell B) \cong \Diamond_Y^\ell \Sigma_f(i_X^\ell M \times_X f^* i_Y^\ell j_Y^\ell B) \cong$$

$$\cong \Diamond_Y^\ell(\Sigma_f i_X^\ell M \times_Y i_Y^\ell j_Y^\ell B) \cong (\Diamond_Y^\ell \Sigma_f i_X^\ell M) \wedge_Y^\ell j_Y^\ell B \cong \exists_f^\ell M \wedge_Y^\ell j_Y^\ell B$$

■

6. Functors valued in truth values

In the sequel, a major role will be played by the "truth values" category $\mathcal{B}1$. We denote by `true` its terminal object, so that

$$j_1^\ell \texttt{true} \cong \top_1^\ell \qquad ; \qquad j_1^r \texttt{true} \cong \top_1^r$$

The functors $X^* b_1 : \mathcal{B}1 \to \mathcal{P}X$ can be factorized in various ways:

$$i_X^\ell X^\ell j_1^\ell = X^* i_1^\ell j_1^\ell = X^* i_1^r j_1^r = i_X^r X^r j_1^r$$

(where X denotes also the map $X \to 1$). Thus their left and right adjoints can be factorized as:

$$(j_1^\ell)^{-1} \exists_X^\ell \Diamond_X^\ell \cong (j_1^\ell)^{-1} \Diamond_1^\ell \Sigma_X \cong (j_1^r)^{-1} \Diamond_1^r \Sigma_X \cong (j_1^r)^{-1} \exists_X^r \Diamond_X^r \tag{29}$$

$$(j_1^\ell)^{-1} \forall_X^\ell \Box_X^\ell \cong (j_1^\ell)^{-1} \Box_1^\ell \Pi_X \cong (j_1^r)^{-1} \Box_1^r \Pi_X \cong (j_1^r)^{-1} \forall_X^r \Box_X^r \tag{30}$$

We refer to (any one of) these as the "coend" and "end" functors at X, respectively:

$$\mathbf{coend}_X \dashv X^* b_1 \dashv \mathbf{end}_X : \mathcal{P}X \to \mathcal{B}1$$

This terminology is justified by the fact that, for a bifunctor $H : X^{\mathrm{op}} \times X \to \mathbf{Set}$, one can easily construct an object h of $\mathbf{Cat}/X$ such that $\mathbf{end}_X h$ gives the usual end of H, while $\mathbf{coend}_X h$ gives the coend of H in the sense of strong dinaturality, which in most relevant cases reduces to the usual one as well (see [Pisani, 2007]).

Next we define the functors

$$\mathbf{meets}_X : \mathcal{P}X \times \mathcal{P}X \to \mathcal{B}1 \qquad ; \qquad \mathbf{hom}_X : (\mathcal{P}X)^{\mathrm{op}} \times \mathcal{P}X \to \mathcal{B}1$$

$$\mathbf{meets}_X(P, Q) := \mathbf{coend}_X(P \times Q) \qquad ; \qquad \mathbf{hom}_X(P, Q) := \mathbf{end}_X(P \Rightarrow Q)$$

and their restrictions

$$\mathbf{ten}_X : \mathcal{R}X \times \mathcal{L}X \to \mathcal{B}1$$

$$\mathbf{nat}^\ell_X : (\mathcal{L}X)^{\mathrm{op}} \times \mathcal{L}X \to \mathcal{B}1 \qquad ; \qquad \mathbf{nat}^r_X : (\mathcal{R}X)^{\mathrm{op}} \times \mathcal{R}X \to \mathcal{B}1$$

$$\mathbf{ten}_X(M,N) := \mathbf{meets}_X(i^\ell_X M, i^r_X N)$$

$$\mathbf{nat}^\ell_X(L,M) := \mathbf{hom}_X(i^\ell_X L, i^\ell_X M) \qquad ; \qquad \mathbf{nat}^r_X(N,O) := \mathbf{hom}_X(i^r_X N, i^r_X O)$$

For instance, in the temporal doctrine of posets $\mathcal{B}1 \cong \mathcal{P}1 \cong \{\mathtt{false} \vdash \mathtt{true}\}$ and

$$\mathbf{meets}_X(P,Q) = \mathtt{true}$$

iff P and Q have a non-empty intersection (and similarly for $\mathbf{ten}_X(M,N)$). Of course, $\mathbf{hom}_X(P,Q) = \mathtt{true}$ iff $P \subseteq Q$ (and similarly for $\mathbf{nat}^\ell_X(L,M)$ and $\mathbf{nat}^r_X(N,O)$). In the temporal doctrine of reflexive graphs, $\mathcal{P}1 \cong \mathbf{Grph}$ while $\mathcal{B}1 \cong \mathbf{Set}$; the coend of $P : T \to X$ in $\mathbf{Grph}/X$ is given by the components of T, while its end is the set of sections $X \to T$.

Note that while $\mathbf{hom}_X$ and $\mathbf{meets}_X$ are valued in $\mathcal{B}1$ rather than in $\mathcal{P}1$ as in Section 3, the notation becomes consistent for the first example in 4.1.

6.1. The enriched adjunction laws. In the following proposition we show that the adjunctions which define a temporal doctrine can be internalized, that is they are enriched in the truth values category $\mathcal{B}1$. Furthermore, some of them have an exact counterpart in a similar law, with the "meets" or "tensor" functors in place of the "hom" or "nat" functors; the proofs are also nicely symmetrical.

6.2. Proposition. *The functors $\mathbf{hom}_X$, $\mathbf{nat}^\ell_X$ and $\mathbf{nat}^r_X$ enrich $\mathcal{P}X$, $\mathcal{L}X$ and $\mathcal{R}X$ respectively in $\mathcal{B}1$ and, for any space $X \in \mathcal{C}$ or map $f : X \to Y$, there are natural isomorphisms:*

$$\mathbf{hom}_X(f^*Q,P) \cong \mathbf{hom}_Y(Q,\Pi_f P) \quad ; \quad \mathbf{hom}_X(P,f^*Q) \cong \mathbf{hom}_Y(\Sigma_f P,Q) \tag{31}$$

$$\mathbf{meets}_X(f^*Q,P) \cong \mathbf{meets}_Y(Q,\Sigma_f P) \quad ; \quad \mathbf{meets}_X(P,f^*Q) \cong \mathbf{meets}_Y(\Sigma_f P,Q) \tag{32}$$

$$\mathbf{nat}^\ell_X(M,\Box^\ell_X P) \cong \mathbf{hom}_X(i^\ell_X M,P) \quad ; \quad \mathbf{nat}^r_X(N,\Box^r_X P) \cong \mathbf{hom}_X(i^r_X N,P) \tag{33}$$

$$\mathbf{nat}^\ell_X(\Diamond^\ell_X P,M) \cong \mathbf{hom}_X(P,i^\ell_X M) \quad ; \quad \mathbf{nat}^r_X(\Diamond^r_X P,N) \cong \mathbf{hom}_X(P,i^r_X N) \tag{34}$$

$$\mathbf{ten}_X(\Diamond^\ell_X P,N) \cong \mathbf{meets}_X(P,i^r_X N) \quad ; \quad \mathbf{ten}_X(M,\Diamond^r_X P) \cong \mathbf{meets}_X(i^\ell_X M,P) \tag{35}$$

$$\mathbf{nat}^\ell_X(f^\ell M,L) \cong \mathbf{nat}^\ell_Y(M,\forall^\ell_f L) \quad ; \quad \mathbf{nat}^r_X(f^r O,N) \cong \mathbf{nat}^r_Y(O,\forall^r_f N) \tag{36}$$

$$\mathbf{nat}^\ell_X(L,f^\ell M) \cong \mathbf{nat}^\ell_Y(\exists^\ell_f L,M) \quad ; \quad \mathbf{nat}^r_X(N,f^r O) \cong \mathbf{nat}^r_Y(\exists^r_f N,O) \tag{37}$$

$$\mathbf{ten}_X(L,f^r N) \cong \mathbf{ten}_Y(\exists^\ell_f L,N) \quad ; \quad \mathbf{ten}_X(f^\ell M,N) \cong \mathbf{ten}_Y(M,\exists^r_f N) \tag{38}$$

Proof. For the first part, see Proposition 2.2. For (31) and (32), recall Corollary 3.1 and note that the present "hom" and "meets" functors factor through the ones there.

Equations (33), (34) and (35) follow from Proposition 5.2 and the other factorizations of the coend and the end functors in (29) and (30). Recalling (27), we obtain the remaining ones by composition of (enriched) adjoints. Alternatively, one can explicitly derive them as we exemplify for (37) and (38):

$$\begin{aligned}
(j_1^\ell)^{-1}\Box_1^\ell\Pi_X(i_X^\ell L \Rightarrow i_X^\ell f^\ell M) &\cong (j_1^\ell)^{-1}\Box_1^\ell\Pi_Y\Pi_f(i_X^\ell L \Rightarrow f^* i_Y^\ell M)\\
&\cong (j_1^\ell)^{-1}\Box_1^\ell\Pi_Y(\Sigma_f i_X^\ell L \Rightarrow i_Y^\ell M) \cong (j_1^r)^{-1}\forall_Y^r\Box_Y^r(\Sigma_f i_X L \Rightarrow i_Y^\ell M)\\
&\cong (j_1^r)^{-1}\forall_Y^r\Box_Y^r(i_Y^\ell\Diamond_Y^\ell\Sigma_f i_X^\ell L \Rightarrow i_Y^\ell M)
\end{aligned}$$

$$\begin{aligned}
(j_1^\ell)^{-1}\Diamond_1^\ell\Sigma_X(i_X^\ell f^\ell M \times i_X^r N) &\cong (j_1^\ell)^{-1}\Diamond_1^\ell\Sigma_Y\Sigma_f(f^* i_Y^\ell M \times i_X^r N)\\
&\cong (j_1^\ell)^{-1}\Diamond_1^\ell\Sigma_Y(i_Y^\ell M \times \Sigma_f i_X^r N) \cong (j_1^r)^{-1}\exists_Y^r\Diamond_Y^r(i_Y^\ell M \times \Sigma_f i_X^r N)\\
&\cong (j_1^r)^{-1}\exists_Y^r\Diamond_Y^r(i_Y^\ell M \times i_Y^r\Diamond_Y^r\Sigma_f i_X^r N)
\end{aligned}$$

■

7. Limits, colimits and Yoneda properties

As we will see in Section 10, the laws in Proposition 6.2 still hold for weak temporal doctrines (at least to the extent that **hom** is defined), which include the motivating instance $\langle \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Cat}/X \leftarrow \mathbf{Set}^X\,;\, X \in \mathbf{Cat}\rangle$. Thus, with the same technique exploited in Section 3, we begin to draw some consequences which in fact hold in the weaker context as well. Accordingly, we mainly maintain the policy of using terms which reflect the case $\mathcal{C} = \mathbf{Cat}$ just mentioned.

We define the (internally enriched) "limit" and "colimit" functors by restricting the end and the coend functors to (left or right) closed parts:
$\mathbf{lim}_X^\ell := \mathbf{end}_X i_X^\ell \cong (j_1^\ell)^{-1}\forall_X^\ell : \mathcal{L}X \to \mathcal{B}1$
$\mathbf{lim}_X^r := \mathbf{end}_X i_X^r \cong (j_1^r)^{-1}\forall_X^r : \mathcal{R}X \to \mathcal{B}1$
$\mathbf{colim}_X^\ell := \mathbf{coend}_X i_X^\ell \cong (j_1^\ell)^{-1}\exists_X^\ell : \mathcal{L}X \to \mathcal{B}1$
$\mathbf{colim}_X^r := \mathbf{coend}_X i_X^r \cong (j_1^r)^{-1}\exists_X^r : \mathcal{R}X \to \mathcal{B}1$

Say that $f : X \to Y$ is "final" (resp. "initial") if $\exists_f^\ell \top_X^\ell \cong \top_Y^\ell$ (resp. $\exists_f^r \top_X^r \cong \top_Y^r$).

7.1. Corollary. [final maps preserve (co)limits] *If $f : X \to Y$ is final then*

$$\mathbf{lim}_X^\ell(f^\ell M) \cong \mathbf{lim}_Y^\ell M \quad ; \quad \mathbf{colim}_X^r(f^r N) \cong \mathbf{colim}_Y^r N \tag{39}$$

and if $f : X \to Y$ is initial then

$$\mathbf{colim}_X^\ell(f^\ell M) \cong \mathbf{colim}_Y^\ell M \quad ; \quad \mathbf{lim}_X^r(f^r N) \cong \mathbf{lim}_Y^r N$$

PROOF. For the first one of (39), using (37) we have:

$$\mathbf{lim}^\ell_X(f^\ell M) \cong \mathbf{nat}^\ell_X(\top^\ell_X, f^\ell M) \cong \mathbf{nat}^\ell_Y(\exists^\ell_f \top^\ell_X, M) \cong \mathbf{nat}^\ell_Y(\top^\ell_Y, M) \cong \mathbf{lim}^\ell_Y M$$

For the second one of (39), we follow exactly the same pattern using (38) instead:

$$\mathbf{colim}^r_X(f^r N) \cong \mathbf{ten}_X(\top^\ell_X, f^r N) \cong \mathbf{ten}_Y(\exists^\ell_f \top^\ell_X, N) \cong \mathbf{ten}_Y(\top^\ell_Y, N) \cong \mathbf{colim}^r_Y N$$

The other ones are proved in the same way. ∎

There are three canonical ways of obtaining a truth value in $\mathcal{B}1$ from a closed part, namely the limit or the colimit functors and "evaluation" at a point $x : 1 \to X$:

$$\mathbf{lim}^\ell_X M \quad ; \quad \mathbf{colim}^\ell_X M \quad ; \quad (j^\ell_1)^{-1}(x^\ell M)$$

$$\mathbf{lim}^r_X N \quad ; \quad \mathbf{colim}^r_X N \quad ; \quad (j^r_1)^{-1}(x^r N)$$

In the proof of Corollary 7.1 we have used the fact that limits and colimits over X are "represented" (by $\top^\ell_X$ or $\top^r_X$). Now we show that the same is true for evaluation; namely, evaluation at x is "represented" by the left and right "slices":

$$X/x := \exists^\ell_x \top^\ell_1 \quad ; \quad x\backslash X := \exists^r_x \top^r_1 \tag{40}$$

Note that slices can be obtained as the (left or right) closure of the "singletons" $\{x\} = \Sigma_x 1_1 = c_X x$ of Section 3:

$$X/x = \Diamond^\ell_X \{x\} \quad ; \quad x\backslash X = \Diamond^r_X \{x\}$$

7.2. COROLLARY. [Yoneda properties] *Given a point $x : 1 \to X$ in $\mathcal{C}$, there are isomorphisms*

$$(j^\ell_1)^{-1}(x^\ell M) \cong \mathbf{nat}^\ell_X(X/x, M) \quad ; \quad (j^\ell_1)^{-1}(x^\ell M) \cong \mathbf{ten}_X(M, x\backslash X)$$

natural in $P \in \mathcal{P}X$ (and dually for right closed parts).

PROOF.

$$\mathbf{nat}^\ell_X(\exists^\ell_x \top^\ell_1, M) \cong \mathbf{nat}^\ell_1(\top^\ell_1, x^\ell M) \cong \mathbf{lim}^\ell_1(x^\ell M) \cong (j^\ell_1)^{-1}\forall^\ell_1(x^\ell M) \cong (j^\ell_1)^{-1}(x^\ell M)$$

$$\mathbf{ten}_X(M, \exists^r_x \top^r_1) \cong \mathbf{ten}_1(x^\ell M, \top^r_1) \cong \mathbf{colim}^\ell_1(x^\ell M) \cong (j^\ell_1)^{-1}\exists^\ell_1(x^\ell M) \cong (j^\ell_1)^{-1}(x^\ell M)$$

∎

The evaluation at points of quantified parts can be obtained as follows:

7.3. COROLLARY. [formulas for quantifications, interior and closure] *Given a map $f : X \to Y$ and a point $y : 1 \to Y$ in $\mathcal{C}$, there are isomorphisms*

$$(j_1^\ell)^{-1}(y^\ell \forall_f^\ell M) \cong \mathbf{nat}_X^\ell(f^\ell Y/y, M) \qquad ; \qquad (j_1^\ell)^{-1}(y^\ell \exists_f^\ell M) \cong \mathbf{ten}_X(M, f^r y\backslash Y)$$

natural in $M \in \mathcal{L}X$ (and dually for right closed parts). There are isomorphisms

$$(j_1^\ell)^{-1}(x^\ell \Box_X^\ell P) \cong \mathbf{hom}_X(i_X^\ell X/x, P) \qquad ; \qquad (j_1^\ell)^{-1}(x^\ell \Diamond_X^\ell P) \cong \mathbf{meets}_X(P, i_X^r x\backslash X)$$

natural in $P \in \mathcal{P}X$ (and dually for right closure).

PROOF. Using Corollary 7.2 and (36), (38), (33) and (35) respectively, we get:

$(j_1^\ell)^{-1}(y^\ell \forall_f^\ell M) \cong \mathbf{nat}_Y^\ell(Y/y, \forall_f^\ell M) \cong \mathbf{nat}_X^\ell(f^\ell Y/y, M)$
$(j_1^\ell)^{-1}(y^\ell \exists_f^\ell M) \cong \mathbf{ten}_Y(\exists_f^\ell M, y\backslash Y) \cong \mathbf{ten}_X(M, f^r y\backslash Y, M)$
$(j_1^\ell)^{-1}(x^\ell \Box_X^\ell P) \cong \mathbf{nat}_X^\ell(X/x, \Box_X^\ell P) \cong \mathbf{hom}_X(i_X^\ell X/x, P)$
$(j_1^\ell)^{-1}(x^\ell \Diamond_X^\ell P) \cong \mathbf{ten}_X(\Diamond_X^\ell P, x\backslash X) \cong \mathbf{meets}_X(P, i_X^r x\backslash X)$ ∎

8. Exploiting comprehension

In this section and in the next one we present some consequences of the comprehension adjunction $c_X \dashv k_X : \mathcal{P}X \to \mathcal{C}/X$ and of the assumption that it is fully faithful.

8.1. THE COMPONENTS FUNCTOR. We define the "components" functor $\pi_0 : \mathcal{C} \to \mathcal{B}1$ by:

$$\pi_0 X := \mathbf{coend}_X 1_X = \mathbf{colim}_X^\ell \top_X^\ell = \mathbf{colim}_X^r \top_X^r$$

8.2. REMARKS. Note that X is connected, that is $\pi_0 X \cong \mathtt{true}$, iff $X \to 1$ is final (or initial). Note also that the components functor

$$\pi_0 X = \mathbf{coend}_X 1_X = (j_1^\ell)^{-1} \Diamond_1^\ell \Sigma_X 1_X = (j_1^\ell)^{-1} \Diamond_1^\ell c_1 X = (j_1^r)^{-1} \Diamond_1^r c_1 X$$

is left adjoint to the full inclusion $k_1 b_1 : \mathcal{B}1 \to \mathcal{C}$. Coherently, we say that a space $X \in \mathcal{C}$ is "discrete" if $X \cong k_1 b_1 V$, for a truth value $V \in \mathcal{B}1$, so that π_0 yields in fact the reflection in discrete spaces.

Conversely, the coend functor can be reduced to components or to a colimit by

$$\mathbf{coend}_X P \cong \pi_0 X_! k_X P \cong \mathbf{colim}_{X_! k_X P}^\ell \top_{X_! k_X P}^\ell \tag{41}$$

Indeed we have:

$$\mathbf{coend}_X P \cong (j_1^\ell)^{-1} \Diamond_1^\ell \Sigma_X P \cong (j_1^\ell)^{-1} \Diamond_1^\ell \Sigma_X c_X k_X P \cong (j_1^\ell)^{-1} \Diamond_1^\ell \Sigma_X \Sigma_{k_X P} 1_{X_! k_X P} \cong$$

$$\cong (j_1^\ell)^{-1} \Diamond_1^\ell \Sigma_{X_! k_X P} 1_{X_! k_X P} \cong \mathbf{coend}_{X_! k_X P} 1_{X_! k_X P} \cong \pi_0 X_! k_X P$$

8.3. The limit and colimit formulas for nat and ten. Since $\mathbf{coend}_X$ can be reduced to a colimit by (41), the same is true for the functors $\mathbf{meets}_X$ and $\mathbf{ten}_X$. In fact we can do better, by reducing $\mathbf{ten}_X(M,N)$ to a colimit over the space corresponding to $i^r_X N$ (or to $i^\ell_X M$), rather than to $i^r_X N \times_X i^\ell_X M$; similarly, $\mathbf{nat}_X$ can be reduced to a limit:

8.4. Proposition. [(co)limit formulas for ten and nat]

$$\mathbf{meets}_X(P,Q) \cong \mathbf{coend}_{X_! k_X Q}(k_X Q)^* P \cong \mathbf{coend}_{X_! k_X P}(k_X P)^* Q \tag{42}$$

$$\mathbf{ten}_X(M,N) \cong \mathbf{colim}^\ell_{X_! k_X i^r_X N}(k_X i^r_X N)^\ell M \cong \mathbf{colim}^r_{X_! k_X i^\ell_X M}(k_X i^\ell_X M)^r N \tag{43}$$

$$\mathbf{hom}_X(P,Q) \cong \mathbf{end}_{X_! k_X Q}(k_X Q)^* P \cong \mathbf{end}_{X_! k_X P}(k_X P)^* Q \tag{44}$$

$$\mathbf{nat}^\ell_X(L,M) \cong \mathbf{lim}^\ell_{X_! k_X i^\ell_X L}(k_X i^\ell_X L)^\ell M \;\; ; \;\; \mathbf{nat}^r_X(N,O) \cong \mathbf{lim}^r_{X_! k_X i^r_X N}(k_X i^r_X N)^r O \tag{45}$$

Proof. For (42), by applying (32) of Proposition 6.2 we get

$$\mathbf{meets}_X(P,Q) \cong \mathbf{meets}_X(P, \Sigma_{k_X Q} 1_{X_! k_X Q}) \cong$$

$$\cong \mathbf{meets}_{X_! k_X Q}((k_X Q)^* P, 1_{X_! k_X Q}) \cong \mathbf{coend}_{X_! k_X Q}(k_X Q)^* P$$

For (43), recalling Proposition 5.5 and using (38) of Proposition 6.2 we get:

$$\mathbf{ten}_X(M,N) \cong \mathbf{ten}_X(M, \exists_{k_X i^r_X N} \top_{X_! k_X i^r_X N}) \cong$$

$$\cong \mathbf{ten}_X((k_X i^r_X N)^\ell M, \top_{X_! k_X i^r_X N}) \cong \mathbf{colim}^\ell_{X_! k_X i^r_X N}(k_X i^r_X N)^\ell M$$

Similarly, the last two equations are obtained by applying (31) and (37):

$$\mathbf{hom}_X(P,Q) \cong \mathbf{hom}_X(\Sigma_{k_X P} 1_{X_! k_X P}, Q) \cong$$

$$\cong \mathbf{hom}_{X_! k_X P}(1_{X_! k_X P}, (k_X P)^* Q) \cong \mathbf{end}_{X_! k_X P}(k_X P)^* Q$$

$$\mathbf{nat}^\ell_X(L,M) \cong \mathbf{nat}^\ell_X(\exists_{k_X i^\ell_X L} \top_{X_! k_X i^\ell_X L}, M) \cong$$

$$\cong \mathbf{nat}^\ell_X(\top_{X_! k_X i^\ell_X L}, (k_X i^\ell_X L)^\ell M) \cong \mathbf{lim}^\ell_{X_! k_X i^\ell_X L}(k_X i^\ell_X L)^\ell M$$

■

By applying (43) or (45) to the formulas for quantifications along a map $f : X \to Y$ of Corollary 7.3, we obtain a colimit and a limit formula for evaluation at $y : 1 \to Y$ of $\exists^\ell_f M$ (or $\exists^r_f N$) and $\forall^\ell_f M$ (or $\forall^r_f N$), respectively.

Say that a part $P \in \mathcal{P}X$ is "left dense" if its left closure is terminal: $\Diamond^\ell_X P \cong \top^\ell_X$; right density is of course defined dually.

8.5. Proposition. *A part $P \in \mathcal{P}X$ is left dense iff $k_X P$ is final. A map $f : X \to Y$ in $\mathcal{C}$ is final iff $c_Y f$ is left dense in Y. A space $X \in \mathcal{C}$ is connected iff $\Sigma_X 1_X$ is dense.*

Proof. (Note that we implicitly use the canonical bijection between the objects of $\mathcal{C}/X$ and maps in $\mathcal{C}$ with codomain X.) For the first one we have:

$$\Diamond^\ell_X P \cong \Diamond^\ell_X \Sigma_{k_X P} 1_{X_! k_X P} \cong \Diamond^\ell_X \Sigma_{k_X P} i^\ell_{X_! k_X P} \top^\ell_{X_! k_X P} \cong \exists^\ell_{k_X P} \top^\ell_{X_! k_X P}$$

and for the second one: $\exists^\ell_f \top^\ell_X \cong \Diamond^\ell_Y \Sigma_f 1_X \cong \Diamond^\ell_Y c_Y f$

For the last one, see Remarks 8.2. ■

9. The sup and inf reflections

For $X \in \mathcal{C}$, let $\overline{X}^{\ell}$ (resp. $\overline{X}^{r}$) be the full subcategory of $\mathcal{L}X$ (resp. $\mathcal{R}X$) generated by the left (resp. right) slices (40), and denote the inclusion functors by

$$h^{\ell}_X : \overline{X}^{\ell} \to \mathcal{L}X \qquad ; \qquad h^{r}_X : \overline{X}^{r} \to \mathcal{R}X$$

The partially defined left adjoints to $i^{\ell}_X h^{\ell}_X$, $i^{r}_X h^{r}_X$, $k_X i^{\ell}_X h^{\ell}_X$ and $k_X i^{r}_X h^{r}_X$ are denoted respectively by:

$$\sup_X : \mathcal{P}X \to \overline{X}^{\ell} \ ; \ \inf_X : \mathcal{P}X \to \overline{X}^{r} \qquad ; \qquad \mathrm{Sup}_X : \mathcal{C}/X \to \overline{X}^{\ell} \ ; \ \mathrm{Inf}_X : \mathcal{C}/X \to \overline{X}^{r}$$

9.1. Remark. Of course, $\mathrm{Sup}_X f$ exists iff $\sup_X c_X f$ does, and in that case they are the same. Note also that the sup (resp. inf) of a part depends only on its left (resp. right) closure: $\sup_X P$ exists iff $\Diamond^{\ell}_X P$ has a reflection in $\overline{X}^{\ell}$.

9.2. Proposition. *The following are equivalent for a space $X \in \mathcal{C}$:*

1. *X has a final point $x : 1 \to X$;*
2. *there is a left dense part of X with a sup;*
3. *there is a final map $T \to X$ with a Sup;*
4. *$\mathrm{id}_X : X \to X$ has a Sup.*

Proof. By Proposition 8.5 and Remark 9.1, the last three conditions are equivalent. Since any point $x : 1 \to X$ has the slice X/x as its Sup, 1) implies 3).

Suppose conversely that a left dense part $P \in \mathcal{P}X$ has a sup; then by Remark 9.1 $\Diamond^{\ell}_X P \cong \top^{\ell}_X$ has a reflection X/x in $\overline{X}^{\ell}$. By general well-known facts about reflections, it follows that also $X/x \cong \top^{\ell}_X$, that is the point $x : 1 \to X$ is final. ∎

9.3. Proposition. [final maps preserve sups] *Let $t : S \to T$ be a final map in $\mathcal{C}$; then $f : T \to X$ has a* Sup *iff $ft : S \to X$ does, and in that case they coincide.*

Proof. By Remark 9.1, it is enough to show that $\Diamond^{\ell}_X c_X ft \cong \Diamond^{\ell}_X c_X f$:

$$\Diamond^{\ell}_X c_X ft \cong \exists^{\ell}_{ft} \top^{\ell}_S \cong \exists^{\ell}_f \exists^{\ell}_t \top^{\ell}_S \cong \exists^{\ell}_f \top^{\ell}_T \cong \Diamond^{\ell}_X c_X f$$

∎

10. The logic of categories

Now we weaken the axioms for temporal doctrines so to include the motivating instance

$$\langle \mathbf{Set}^{X^{\mathrm{op}}} \to \mathbf{Cat}/X \leftarrow \mathbf{Set}^{X}\, ;\, X \in \mathbf{Cat} \rangle$$

and show that most of the laws in Proposition 6.2 still hold. By restricting to groupoids we get another important instance of (symmetrical) weak temporal doctrine.

10.1. Weak temporal doctrines. Weak temporal doctrines are defined as temporal doctrines except that:

1. We do not require the existence of the Π_f, right adjoint to f^*; instead we do require, for any map $f : X \to Y$, the existence of
$$\forall_f^\ell : \mathcal{L}X \to \mathcal{L}Y \qquad ; \qquad \forall_f^r : \mathcal{R}X \to \mathcal{R}Y$$
right adjoint to f^ℓ and f^r respectively.
2. We do not assume that the categories of parts $\mathcal{P}X$ are cartesian closed, but only that they are cartesian and that the left or right closed parts are exponentiable therein, that is
$$i_X^\ell M \Rightarrow P \qquad ; \qquad i_X^r N \Rightarrow P$$
always exist in $\mathcal{P}X$.

In a weak temporal doctrine we still have the $\mathbf{coend}_X$ and $\mathbf{end}_X$ functors as in Section 6, left and right adjoint to:

$$i_X^\ell X^\ell j_1^\ell = X^* i_1^\ell j_1^\ell = X^* i_1^r j_1^r = i_X^r X^r j_1^r$$

with the difference that, among the factorizatons of $\mathbf{end}_X$ in (30), only

$$(j_1^\ell)^{-1} \forall_X^\ell \Box_X^\ell \;\cong\; (j_1^r)^{-1} \forall_X^r \Box_X^r : \mathcal{P}X \to \mathcal{B}1 \qquad (46)$$

are always available. We also define

$$\mathbf{meets}_X : \mathcal{P}X \times \mathcal{P}X \to \mathcal{B}1 \qquad ; \qquad \mathbf{hom}_X : (\mathcal{P}X)^{\mathrm{op}} \times \mathcal{P}X \to \mathcal{B}1$$

$$\mathbf{ten}_X : \mathcal{L}X \times \mathcal{R}X \to \mathcal{B}1$$

$$\mathbf{nat}_X^\ell : (\mathcal{L}X)^{\mathrm{op}} \times \mathcal{L}X \to \mathcal{B}1 \qquad ; \qquad \mathbf{nat}_X^r : (\mathcal{R}X)^{\mathrm{op}} \times \mathcal{R}X \to \mathcal{B}1$$

as in Section 6, but now the $\mathbf{hom}_X$ may be only partially defined, depending on whether the exponential $P \Rightarrow Q$ exists or does not exist. On the contrary, the above axiom on exponentials assures that the $\mathbf{nat}_X$ are always defined.

The functors $\mathbf{lim}_X^\ell$, $\mathbf{lim}_X^r$, $\mathbf{colim}_X^\ell$ and $\mathbf{colim}_X^r$ are defined as in Section 7, and $\sup_X$, Sup_X, $\inf_X$ and Inf_X as in Section 9.

That $\langle \mathbf{Cat}/X; X \in \mathbf{Cat} \rangle$, with the subdoctrines of discrete fibrations and discrete opfibrations (or, equivalently, of left and right actions), form indeed a weak temporal doctrine follows in particular from the following facts:

- Discrete (op)fibrations over $X \in \mathbf{Cat}$ are both reflective and coreflective in $\mathbf{Cat}/X$ (see [Pisani, 2007]).
- The reflections satisfy the mixed Frobenius laws, as a consequence of the reciprocal stability of the comprehensive factorization systems (as explained in [Pisani, 2008] and [Pisani, 2009]).

In this case, the sup (resp. inf) reflections give the colimit (resp. limit) of the corresponding functor. Thus Proposition 9.3 can be seen as the "external" correspective of Corollary 7.1.

10.2. PROPOSITION. *The functors $\mathbf{nat}^\ell_X$ and $\mathbf{nat}^r_X$ enrich $\mathcal{L}X$ and $\mathcal{R}X$ respectively in $\mathcal{B}1$ and, for any space $X \in \mathcal{C}$ or map $f : X \to Y$, there are natural isomorphisms:*

$$\mathbf{meets}_X(f^*Q, P) \cong \mathbf{meets}_Y(Q, \Sigma_f P) \quad ; \quad \mathbf{meets}_X(P, f^*Q) \cong \mathbf{meets}_Y(\Sigma_f P, Q) \tag{47}$$

$$\mathbf{nat}^\ell_X(M, \Box^\ell_X P) \cong \mathbf{hom}_X(i^\ell_X M, P) \quad ; \quad \mathbf{nat}^r_X(N, \Box^r_X P) \cong \mathbf{hom}_X(i^r_X N, P) \tag{48}$$

$$\mathbf{ten}_X(\Diamond^\ell_X P, N) \cong \mathbf{meets}_X(P, i^r_X N) \quad ; \quad \mathbf{ten}_X(M, \Diamond^r_X P) \cong \mathbf{meets}_X(i^\ell_X M, P) \tag{49}$$

$$\mathbf{nat}^\ell_X(f^\ell M, L) \cong \mathbf{nat}^\ell_Y(M, \forall^\ell_f L) \quad ; \quad \mathbf{nat}^r_X(f^r O, N) \cong \mathbf{nat}^r_Y(O, \forall^r_f N) \tag{50}$$

$$\mathbf{nat}^\ell_X(L, f^\ell M) \cong \mathbf{nat}^\ell_Y(\exists^\ell_f L, M) \quad ; \quad \mathbf{nat}^r_X(N, f^r O) \cong \mathbf{nat}^r_Y(\exists^r_f N, O) \tag{51}$$

$$\mathbf{ten}_X(L, f^r N) \cong \mathbf{ten}_Y(\exists^\ell_f L, N) \quad ; \quad \mathbf{ten}_X(f^\ell M, N) \cong \mathbf{ten}_Y(M, \exists^r_f N) \tag{52}$$

PROOF. The proofs of (47), (48), (49) and (52) are as in Proposition 6.2. For the other ones one can not use the technique of Proposition 6.2, since the functors Π_f are not always available. We get (50) by applying Proposition 2.2 to $f^\ell \dashv \forall^\ell_f$ and recalling that exponentials in $\mathcal{L}X$ are given by $\Box^\ell_X(i^\ell_X L \Rightarrow i^\ell_X M)$:

$$\mathbf{nat}^\ell_X(f^\ell M, L) \cong \mathbf{end}_X(i^\ell_X f^\ell M \Rightarrow i^\ell_X L) \cong (j^\ell_1)^{-1}\forall^\ell_X \Box^\ell_X(i^\ell_X f^\ell M \Rightarrow i^\ell_X L) \cong$$

$$\cong (j^\ell_1)^{-1}\forall^\ell_Y \forall^\ell_f \Box^\ell_X(i^\ell_X f^\ell M \Rightarrow i^\ell_X L) \cong (j^\ell_1)^{-1}\forall^\ell_Y \Box^\ell_Y(i^\ell_Y M \Rightarrow i^\ell_Y \forall^\ell_f L) \cong \mathbf{nat}^\ell_Y(M, \forall^\ell_f L)$$

For (51), since we have adjunctions (with parameter $L \in \mathcal{L}X$)

$$\exists^\ell_f(L \wedge^\ell_X X^\ell j^\ell_1(-)) \dashv \mathbf{nat}^\ell_X(L, f^\ell(-)) \quad ; \quad \exists^\ell_f L \wedge^\ell_Y Y^\ell j^\ell_1(-) \dashv \mathbf{nat}^\ell_Y(\exists^\ell_f L, -)$$

and since, by the restricted Frobenius law of Proposition 5.6,

$$\exists^\ell_f(L \wedge^\ell_X X^\ell j^\ell_1 V) \cong \exists^\ell_f(L \wedge^\ell_X f^\ell Y^\ell j^\ell_1 V) \cong \exists^\ell_f L \wedge^\ell_Y Y^\ell j^\ell_1 V$$

the result follows from Lemma 2.1. ∎

10.3. Proposition. *Whenever they are defined, there are natural isomorphisms:*

$$\mathbf{hom}_X(f^*Q, P) \cong \mathbf{hom}_Y(Q, \Pi_f P) \quad ; \quad \mathbf{hom}_X(P, f^*Q) \cong \mathbf{hom}_Y(\Sigma_f P, Q) \quad (53)$$

$$\mathbf{nat}^\ell_X(\diamondsuit^\ell_X P, M) \cong \mathbf{hom}_X(P, i^\ell_X M) \quad ; \quad \mathbf{nat}^r_X(\diamondsuit^r_X P, N) \cong \mathbf{hom}_X(P, i^r_X N) \quad (54)$$

Proof. We proceed as for (51), by showing that their left adjoints are isomorphic; for instance, for the second one of (53) we have adjunctions (with parameter $P \in \mathcal{P}X$):

$$\Sigma_f(P \times i^\ell_X X^\ell j^\ell_1(-)) \dashv (j^\ell_1)^{-1}\forall^\ell_X \Box^\ell_X(P \Rightarrow f^*(-))\ ;\ \Sigma_f P \times i^\ell_Y Y^\ell j^\ell_1(-) \dashv (j^\ell_1)^{-1}\forall^\ell_Y \Box^\ell_Y(\Sigma_f P \Rightarrow -)$$

and using the Frobenius law:

$$\Sigma_f(P \times i^\ell_X X^\ell j^\ell_1 V) \cong \Sigma_f(P \times X^* i^\ell_1 j^\ell_1 V) \cong \Sigma_f(P \times f^* Y^* i^\ell_1 j^\ell_1 V) \cong \Sigma_f P \times i^\ell_Y Y^\ell j^\ell_1 V$$

Similarly, for (54) one uses the restricted Frobenius law. ∎

10.4. Conclusions. All we have proved in sections 7, 8 and 9 depended only on the comprehension axiom and on the laws in Proposition 6.2. In propositions 10.2 and 10.3 we have seen that the same laws hold for weak temporal doctrines as well, as far as the functors $\mathbf{hom}_X$ involved are defined.

Thus, except for (44) of Proposition 8.4, in which $\mathbf{hom}_X(P, Q)$ may be not defined, those results hold in any weak temporal doctrine. (Note that the **hom** in the interior formula of Corollary 7.3 do exist, since the required exponentials are provided by the axioms for weak temporal doctrines).

In particular we get, for "generalized categories", the Yoneda properties, the formulas for quantifications (or "Kan extensions"), the formulas for the (co)reflection in (left or right) "closed parts" (or "actions"), and the properties of final or initial maps with respect to (co)limits, both "internally" (Corollary 7.1) and "externally" (Proposition 9.3).

Thus we maintain that the logic of weak temporal doctrines well deserves to be called "a logic for categories", in fact

1. being summarized by a few adjunction-like laws, it lends itself to effective and transparent calculations; furthermore, along with the obvious "left-right" symmetry, there is a far more interesting sort of duality: in many cases laws and proofs on the "hom-side" correspond exactly to those on the "tensor-side";

2. this calculus allows one to easily derive some basic non-trivial categorical facts;

3. it is "autonomous", providing its own truth values;

4. suitable natural strengthenings or weakenings can be considered, so to obtain more refined properties or a wider range of applications; some of them will be considered in a forthcoming work.

References

S. Ghilardi, G. Meloni (1991), *Relational and Topological Semantics for Temporal and Modal Predicative Logics*, Atti del Congresso: Nuovi problemi della Logica e della Filosofia della Scienza (Viareggio), CLUEB Bologna, 59-77.

P.T. Johnstone (2002), Sketches of an Elephant: A Topos Theory Compendium, Oxford Science Publications.

D.M. Kan (1958), *Adjoint Functors*, Trans. Amer. Math. Soc. **87**, 294-329.

F.W. Lawvere (1970), *Equality in Hyperdoctrines and the Comprehension Scheme as an Adjoint Functor*, Proceedings of the AMS Symposium on Pure Mathematics, XVII, 1-14.

F.W. Lawvere (1989), *Qualitative Distinctions between some Toposes of Generalized Graphs*, Proceedings of the AMS Symposium on Categories in Computer Science and Logic, Contemporary Mathematics, vol. 92, 261-299.

C. Pisani (2007), Components, Complements and the Reflection Formula, *Theory and Appl. Cat.* **19**, 19-40.

C. Pisani (2008), Balanced Category Theory, *Theory and Appl. Cat.* **20**, 85-115.

C. Pisani (2009), Balanced Category Theory II, preprint, arXiv:math.CT/0904.1790v3.

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