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ASPECTS OF FRACTIONAL EXPONENT FUNCTORS

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ABSTRACT. We prove that certain categories arising from atoms in a Grothendieck topos are themselves Grothendieck toposes. We also investigate enrichments of these categories over the base topos; there are in fact often two distinct enrichments.

1. Introduction

In a cartesian closed category $\mathbf{E}$, some objects A may have the property that they are *atoms* in the sense that the exponent functor $(\ )^A : \mathbf{E} \to \mathbf{E}$ has a right adjoint $(\ )^{1/A}$ (an "amazing" right adjoint, in the terminology of Lawvere). Lawvere calls an endofunctor a *fractional exponent* $(\ )^{C/A}$ if it is of the form

$$\mathbf{E} \xrightarrow{(\ )^C} \mathbf{E} \xrightarrow{(\ )^{1/A}} \mathbf{E}.$$

In this note, we shall prove some category theoretic results concerning fractional exponents and atoms; these results were found in a response to questions, raised by Lawvere [16], on a certain category of second order differential equations in the context of Synthetic Differential Geometry (SDG). (It so happens that the infinitesimal objects like D, in models of SDG, are in fact atoms, cf. [11], essentially from [12].) We shall leave the applications to SDG for a future paper, and here concentrate on results of 'pure' category theory. The results are in two directions: on the one hand, we prove how atoms in a Grothendieck topos give rise to new Grothendieck toposes. They are essentially coalgebra toposes for fractional exponent endofunctors, and certain subcategories thereof. The constructions here will give an answer, in an abstract form, to the question raised by Lawvere, partly following his suggestion. On the other hand, we prove some results on the possible enrichment, or even indexing, of fractional exponent functors; this kind of structure turns out to be, in essence, equivalent to *points* of the atoms in question. The enrichment of the fractional exponent functors which we construct, will in general not correspond, under the adjointness, to the natural enrichment of the exponential functors themselves. This leads to two different enrichments (and two different ways of indexing) over the base topos of the categories constructed (e.g. two enrichments of the category of second order differential equations).

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2. Grothendieck toposes from atoms

We begin by some auxiliary results on equifiers.

Let $F, G : \mathbf{A} \to \mathbf{B}$ be functors, and let $\phi, \psi : F \to G$ natural transformations, so that the equifier $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ of ϕ and ψ (= the full subcategory of $\mathbf{A}$ given by those $X \in \mathbf{A}$ such that $\phi_X = \psi_X$) makes sense. Then

2.1. PROPOSITION. *If* $\mathbf{A}$ *has and* F *preserves a certain class of colimits, then* $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ *is closed under this class of colimits; if* $\mathbf{A}$ *has and* G *preserves a certain class of limits, then* $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ *is closed under this class of limits. And if* G *preserves monics, then* $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ *is closed under subobjects.*

PROOF. For the first assertion, we want to see that ϕ and ψ agree for an object of form $\lim_{\to} X_i$, provided they do so for the X_i's, so we want to prove that two arrows $F(\lim_{\to} X_i) \to G(\lim_{\to} X_i)$, are equal. But the domain here is by assumption a colimit of $F(X_i)$'s on which ϕ and ψ agree, and the rest then follows by naturality. The two other assertions are proved in a similar way. ∎

2.2. THEOREM. *With the notation of the previous Proposition, if* $\mathbf{A}$ *is a Grothendieck topos, and* F *is cocontinuous and* G *left exact, then the equifier* $\mathbf{E}(\phi, \psi)$ *is a Grothendieck topos, and the inclusion* $\mathbf{E}(\phi, \psi) \subseteq \mathbf{A}$ *is the inverse image functor of a surjective geometric morphism. If* G *preserves all limits, this geometric functor is even essential.*

PROOF. By the previous Proposition, the equifier subcategory $\mathbf{E}$ is closed in $\mathbf{A}$ under colimits, finite limits, and subobjects. The two first of these properties guarantee that $\mathbf{E}$ has the same exactness properties (involving these kinds of colimits/limits) as $\mathbf{A}$ does; so $\mathbf{E}$ satisfies the exactness properties a), b), c) of the Giraud characterization of Grothendieck toposes in [2] IV.1.1.2. To prove the last property d) (existence of a small generating family), take a small generating family K in $\mathbf{A}$ closed under quotients. Because the family is closed under quotients, every object X in $\mathbf{A}$ is covered by a family of *monics* with domain in K (take the *images* of the maps from objects of K to X). Since the equifier subcategory $\mathbf{E}$ is closed under subobjects, it follows that every $X \in \mathbf{E}$ is covered by a family of maps with their domains in the small family $K \cap \mathbf{E}$. (This proof of existence of a small generating subcategory for $\mathbf{E}$ was pointed out to us by Ieke Moerdijk.) So $\mathbf{E}$ is a Grothendieck topos. As a cocontinuous functor between Grothendieck toposes, the inclusion has a right adjoint (cf. [2] I.1.5). Also, by the Proposition, the inclusion is left exact since G is so. So we have a geometric morphism as asserted.

For the last assertion (existence of left adjoint for the inclusion), we first get from the Proposition that the inclusion $\mathbf{E} \subseteq \mathbf{A}$ preserves all limits. Also, it is accessible (by being a left adjoint), So the conclusion follows from the Adjoint Functor Theorem, (in the form of [1] 1.66, say). ∎

It is a well known fact in elementary topos theory that the category of coalgebras for a left exact comonad on an elementary topos again form an elementary topos, and that the forgetful functor is the inverse image of a surjective geometric morphism. There exists a similar result for Grothendieck toposes, where one needs to assume accessibility of the functor part of the comonad. We shall need some related results, not for comonads, but rather for mere endofunctors. Recall that if $G : \mathbf{E} \to \mathbf{E}$ is an endofunctor, then a coalgebra for it is a pair X, ξ where $X \in \mathbf{E}$ and $\xi : X \to G(X)$; and a morphism $(X, \xi) \to (X', \xi')$ is a map $f : X \to X'$ making the obvious square commute. A general investigation of coalgebras for endofunctors has been carried out by [6], and the following can also be obtained as a Corollary of their results.

2.3. Proposition. *Let $\mathbf{E}$ be a Grothendieck topos, and G a left exact accessible endofunctor on it. Then the category $Coalg(G)$ of coalgebras for G is a Grothendieck topos; the forgetful functor to $\mathbf{E}$ is the inverse image functor of a surjective geometric morphism.*

Proof. The comma category $\mathbf{E} \downarrow G$ is a Grothendieck topos, according to SGA 4, [2] IV.9.5 Theorem 4; they call it the topos obtained by *glueing* (recollement) along G. The forgetful functor $\mathbf{E} \downarrow G \to \mathbf{E} \times \mathbf{E}$ is the inverse image functor of a geometric morphism of Grothendieck toposes; this is likewise (implicit) in loc. cit. If we form the (strict) pull-back of categories of this forgetful functor along the diagonal $\mathbf{E} \to \mathbf{E} \times \mathbf{E}$, we obtain the category of G-coalgebras. But the functors in this pull-back diagram are all inverse image functors of geometric morphisms; this follows from [17], according to which colimits in the category of Grothendieck toposes and geometric morhisms is formed by forming limits of the inverse-image functors. ∎

2.4. Proposition. *Let $\mathbf{E}$ be a Grothendieck topos, and G a endofunctor on it which admits a left adjoint T. Then the category of coalgebras for G (which is $\cong$ to the category of T-algebras) is a Grothendieck topos; the forgetful functor to $\mathbf{E}$ is the inverse image functor of an essential and surjective geometric morphism.*

Proof. Any (left or right) adjoint functor between Grothendieck toposes is accessible, see [2] Proposition I.9.5. So by Proposition 2.3, the category of coalgebras for G is a Grothendieck topos, and the forgetful functor has a right adjoint. In particular, it is accessible. Using that G preserves limits, it is easy to see that the forgetful functor (creates and) preserves limits. But an accessible limit preserving functor between Grothendieck toposes has a left adjoint, by the Adjoint Functor Theorem (in the form of [1] 1.66 (p. 52), say). ∎

We consider now a fixed atom A in a Grothendieck topos $\mathbf{E}$, and denote by $(-)^{1/A}$ the right adjoint of $(-)^A$. Furthermore, we consider a fixed map $\alpha : A \to E$. (In the application to SDG, A will typically be D, and α will be the inclusion of D into D_2.) For any object X we have the map "restriction along α", $X^E \to X^A$. An *extension structure for* α on X we define to be a section of this map, i.e. a map $\xi' : X^A \to X^E$, which composes with the restriction map to the identity map on X^A; there is an obvious notion of morphism of extension structures.

The composite endofunctor G given by

$$X \mapsto X^{E/A}$$

(recall that $X^{E/A} := (X^E)^{1/A}$) is then left exact, it even has a left adjoint. So by Proposition 2.4, the category of coalgebras for G is a Grothendieck topos, and the forgetful functor to $\mathbf{E}$ is the inverse image functor p^* of an essential geometric surjection. Note that for a coalgebra (X, ξ),

$$\xi : X \to (X^E)^{1/A},$$

the structure map ξ corresponds by adjointness to a map

$$\xi' : X^A \to X^E.$$

The category of extension structures is equivalent to the full subcategory consisting of those coalgebras (X, ξ) for which ξ' has the property that

$$X^\alpha \circ \xi' = \text{identity map of } X^A$$

(i.e. ξ' restricts to the identity map along α). This subcategory is easily seen to be an equifier subcategory of the kind dealt with in Theorem 2.2. In fact, consider the functor $q = (-)^A \circ p^* : \mathrm{Coalg}(G) \to \mathbf{E}$ which takes (x, ξ) to X^A. We have two transformations from q to q; the one is the identity transformation, call it ϕ, the other, ψ is the one whose component at $(X, \xi) \in \mathrm{Coalg}(G)$ is $X^\alpha \circ \xi'$; and the equifier of these two is clearly the subcategory in question. Now since q preserves limits as well as colimits (by Proposition 2.4), it follows from Theorem 2.2 that the equifier subcategory inside the category of coalgebras is a Grothendieck topos, and that the inclusion to $\mathrm{Coalg}(G)$, and hence the forgetful functor to $\mathbf{E}$, is the inverse image of an essential geometric surjection.

We therefore have the following consequence:

2.5. Theorem. *Let $\alpha : A \to E$ be a map in a Grothendieck topos $\mathbf{E}$, and assume that A is an atom. Then the category $\mathbf{G}$ of "extension structures", i.e. the category whose objects are pairs (X, ξ') with $X \in \mathbf{E}$ and $\xi' : X^A \to X^E$ with*

$$X^A \xrightarrow{\xi'} X^E \xrightarrow{X^\alpha} X^A = id_{X^A}$$

(and evident morphisms), is a Grothendieck topos, and the forgetful functor $(X, \xi') \mapsto X$ is the inverse image functor of an essential geometric surjection $\mathbf{E} \to \mathbf{G}$.

A combination of these results, which likewise has applications in SDG, is concerned with the situation where we have a map $\alpha : A \to E$, as above (with A an atom), and where A and E are equipped with an action of a monoid $(R, \cdot)$ such that the map α is equivariant. Then we have induced actions of R on X^A and on X^E, and the restriction map $X^E \to X^A$ is R-equivariant. One may then consider extension structures $\xi : X^A \to X^E$ which likewise are R-equivariant. (For the case in SDG where $A \to E$ is the inclusion of D in D_2, and R is the number line acting by multipliation, the equivariant extension structures are the *sprays* on X.) We then have

2.6. THEOREM. *Let $\alpha : A \to E$ be as in the previous Theorem, and assume it is equivariant with respect to an action by a monoid R. Then the category of R-equivariant extension structures $\xi : X^A \to X^E$ is a Grothendieck topos, and the forgetful functor to $\mathbf{E}$ is the inverse image functor of a surjective geometric morphism.*

PROOF. The equivariance of ξ can be expressed by the equality of two maps (defined in tems of ξ and the R-actions) with domain $R \times X^A$ and codomain X^E. Taking exponential adjoints of these two maps, the equivariance of ξ can be expressed by equality of two maps with domain X^A and codomain $X^{E\times R}$. Taking, in turn, the adjoints of these two maps with respect to the adjointness $(-)^A \dashv (-)^{1/A}$, we get that the equivariance can be expressed in terms of equality of two maps

$$X \to (X^{E\times R})^{1/A},$$

both of which are constructed from ξ and hence clearly are natural in $(X, \xi) \in \mathbf{G}$, where $\mathbf{G}$ denotes the topos of the previous Theorem. They are both natural transformations from the forgetful functor

$$\mathbf{G} \to \mathbf{E} \text{ given by } (X, \xi) \mapsto X$$

to the functor

$$\mathbf{G} \to \mathbf{E} \text{ given by } (X, \xi) \mapsto (X^{E\times R})^{1/A},$$

and since the former preserves colimits and the latter limits, Theorem 2.2 applies. We thus get that the inclusion from the equifier of these two transformations to $\mathbf{G}$ is the inverse image of an essential geometric surjection. Composing this inclusion with the forgetful functor $\mathbf{G} \to \mathbf{E}$, which is likewise the inverse image functor of an essential geometric surjection, we get the result for equivariant extension structures, as claimed. ∎

In the context of SDG, there arise further variants on this theme, cf. [13], or our forthcoming [14].

3. Enrichment/strength of fractional exponents

We recall some notions from enriched category theory, cf. [4] or [7]. Recall that a cartesian closed category $\mathbf{E}$ is enriched in itself (i.e. is made into an $\mathbf{E}$-category) by means of Y^X as the object of maps from X to Y. Then an $\mathbf{E}$-enrichment of an endofunctor $G : \mathbf{E} \to \mathbf{E}$ consists of a family of maps

$$G_{X,Y} : Y^X \to G(Y)^{G(X)},$$

natural in X and Y, and which satisfies two equational conditions expressing the idea that G takes identity maps to identity maps, and preserves composition. Fixing one of the variables X or Y in the exponential functor Y^X gives an endofunctor canonically enriched in $\mathbf{E}$.

Recall from [9] Theorem 1.3 (or see [5] for a recent account) that an $\mathbf{E}$-enrichment ("strength") of an endofunctor $G : \mathbf{E} \to \mathbf{E}$ may be encoded equivalently in the form of a "tensorial strength", meaning a family of maps

$$t_{X,Y} : X \times G(Y) \to G(X \times Y),$$

natural in X and Y, and satisfying two equational conditions,

$$t_{1,Y} = 1_{GY} : 1 \times GY \to G(1 \times Y) \tag{1}$$

$$t_{U,V\times Y} \circ (U \times t_{V,Y}) = t_{U\times V,Y} : U \times V \times GY \to G(U \times V \times Y), \tag{2}$$

respectively, for all U, V, Y, (under the evident identifications like $1 \times GY = GY$ etc.).

We also recall that there is a notion for a natural transformation $\tau_X : G_1(X) \to G_2(X)$ between two $\mathbf{E}$-functors to be $\mathbf{E}$-*natural*, or strongly natural, see [4] 1.10; in terms of the tensorial form of enrichments (for endofunctors G_1 and G_2 on $\mathbf{E}$), this may be expressed simply as commutativity of all squares of the form

$$\begin{tikzcd} X \times G_1Y \arrow[r, "t^{(1)}_{X,Y}"] \arrow[d, "1 \times \tau_Y"'] & G_1(X \times Y) \arrow[d, "\tau_{X\times Y}"] \\ X \times G_2Y \arrow[r, "t^{(2)}_{X,Y}"'] & G_2(X \times Y) \end{tikzcd} \tag{3}$$

where $t^{(i)}$ denotes the enrichment of G_i $(i = 1, 2)$; cf. [9].

We shall use the the words *enrichment* and *strength* more or less synonymously, and similarly for *enriched* and *strong*.

Recall that any endofunctor of the form $(-)^A$ carries a canonical $\mathbf{E}$-enrichment (its tensorial strength is given below).

3.1. PROPOSITION. *Let $A \in \mathbf{E}$ be an object, and let $\phi_X : X^A \to X$ be natural in X. Then in order that ϕ be $\mathbf{E}$-natural, it is necessary and sufficient that, for all X, the composite*

$$X \xrightarrow{\Delta_X} X^A \xrightarrow{\phi_X} X$$

be the identity map on X, where $\Delta_X : X \to X^A$ denotes the 'diagonal'-map, exponential adjoint of the projection $X \times A \to X$.

PROOF. The commutative square corresponding to (3) reduces to the outer square in

$$\begin{tikzcd}
X \times Y^A \arrow[r, "\Delta_X \times 1"] \arrow[d, "1 \times \phi_Y"'] & X^A \times Y^A \cong (X \times Y)^A \arrow[d, "\phi_{X\times Y}"] \arrow[dr, phantom] \\
X \times Y \arrow[r, "1_{X\times Y}"'] & X \times Y
\end{tikzcd}$$

with the diagonal map $\phi_X \times \phi_Y : X^A \times Y^A \to X \times Y$,

where the upper map is the strength in monoidal form of $(-)^A$. Using the naturality of ϕ with respect to the two projections $X \times Y \to X$ and $X \times Y \to Y$, it is easy to see that under the identification $X^A \times Y^A \cong (X \times Y)^A$, $\phi_{X\times Y}$ becomes $\phi_X \times \phi_Y$, and then it is clear that $\phi_X \circ \Delta_X = 1_X$ implies commutativity of the square. The converse implication follows by taking $Y = 1$. ∎

REMARK. If $\mathbf{E}$ = Sets, all functors and transformations are $\mathbf{E}$-enriched. But for other toposes $\mathbf{E}$, there may exist A and natural transformations $\phi_X : X^A \to X$ which are not $\mathbf{E}$-natural; even for the case $A = 1$. Take e.g. $\mathbf{E} = \mathbf{Z}_2$-Sets, i.e. the topos of sets-with-an-involution, and let $\phi_X : X \to X$ be the involution on X.

3.2. PROPOSITION. *Let $F \dashv G$ be endofunctors on a cartesian closed category, and assume that F preserves finite products. Given a natural $\phi_X : F(X) \to X$, then the family of maps $\tau_{X,Y}$ given by*

$$F(X \times G(Y)) \cong F(X) \times F(G(Y)) \xrightarrow{\phi_X \times \epsilon_Y} X \times Y$$

(where ϵ is the counit of the adjunction $F \dashv G$) gives by transposition along $F \dashv G$ a family of maps

$$X \times G(Y) \xrightarrow{t_{X,Y}} G(X \times Y)$$

which is an enrichment (in the form of a tensorial strength) of the functor G.

PROOF. Given Y, the transpose of $t_{1,Y}$ is, by construction, $\phi_1 \times \epsilon_Y$, which under identifications of type $1 \times Y = Y$ is just ϵ_Y, the transpose of the identity map on GY, proving (1). Given U, V, Y, then the right hand side in (2) has for its transpose, under identifications of type $FX \times FY = F(X \times Y)$, the map $\phi_U \times \phi_V \times \epsilon_Y$, whereas the left hand side has transpose $U \times V \times \epsilon_Y \circ (\phi_{U\times V} \times FGY)$. But just by naturality of ϕ with respect to the two projections from $U \times V$, we conclude that $\phi_{U\times V} = \phi_U \times \phi_V$, and the required commutativity is then immediate. The Proposition is proved. ∎

REMARK. We may supplement the Proposition with a statement about natural transformations $f : F_1 \to F_2$; if such an f commutes with the "augmentations" $\phi_i : F_i(X) \to X$, then the mate of f, $G_2 \to G_1$, becomes a strongly natural transformation, with respect to the strengths obtained on the G_i's by virtue of the Proposition. In particular, the mate of ϕ itself, $X \to G(X)$, is strongly natural in X.

For any atom A, there is a canonical natural transformation $\pi_X : X^{1/A} \to X$, namely the composite

$$X^{1/A} \xrightarrow{\Delta} (X^{1/A})^A \xrightarrow{\epsilon_X} X,$$

where ϵ is the counit for the adjunction $(-)^A \dashv (-)^{1/A}$.

3.3. THEOREM. *Let A be an atom in $\mathbf{E}$. There is a bijective correspondence between the set of those $\mathbf{E}$-enrichments of the endofunctor $(-)^{1/A} : \mathbf{E} \to \mathbf{E}$, which make the transformation $\pi : (-)^{1/A} \to id_{\mathbf{E}}$ $\mathbf{E}$-natural, and the set of points $1 \to A$.*

PROOF. Assume given an enrichment, in the form of a tensorial strength, of the endofunctor $(-)^{1/A}$,

$$X \times (Y^{1/A}) \xrightarrow{t_{X,Y}} (X \times Y)^{1/A};$$

its transpose under the adjointness $(-)^A \dashv (-)^{1/A}$ consists then in maps

$$X^A \times (Y^{1/A})^A \xrightarrow{\tau_{X,Y}} X \times Y,$$

natural in X and Y. Using this naturality with respect to the maps $X \to 1$ and $Y \to 1$, it is easy to see that $\tau_{X,Y}$ must be of form $\phi_X \times \psi_Y$ with $\phi_X : X^A \to X$ natural in X and $\psi_Y : (Y^{1/A})^A \to Y$ natural in Y. From (1) it follows that ψ_Y must in fact be the counit ϵ_Y for the adjointness $(-)^A \dashv (-)^{1/A}$. On the other hand, for any natural family $\phi_X : X^A \to X$, the transposes $t_{X,Y}$ of the maps $\tau_{X,Y} := \phi_X \times \epsilon_Y$ will in fact provide an enrichment for $(-)^{1/A}$, without any assumptions on ϕ except naturality; this follows by taking $F = (-)^A$, $G = (-)^{1/A}$ in Proposition 3.2.

From the analysis made it follows that there is a bijective correspondence between strengths $t_{X,Y} : X \times (Y^{1/A}) \to (X \times Y)^{1/A}$ of the endofunctor $(-)^{1/A}$, and natural transformations $\phi_X : X^A \to X$.

Now compatibility of the strength t with π is easily seen, by transposition, to be equivalent to the normalization condition $\phi_X \circ \Delta_X = id_X$.

But by Proposition 3.1, this condition is equivalent to $\mathbf{E}$-naturality of $\phi_X : X^A \to X$. Finally, we invoke the enriched Yoneda Lemma, in the form of [7] 1.9 (therein called the *weak* Yoneda, since after all it talks about a bijection between two *sets*!). According to it, the set of $\mathbf{E}$-natural transformations $(-)^A \to (-)^B$ is in bijective correspondence with the set of maps $B \to A$. Now take $B = 1$. The Theorem is proved. ■

REMARK. On a cartesian closed category, any endofunctor $- \times A$ is the functor part of a comonad, with counit and comultiplication being $- \times (A \to 1)$ and $- \times \Delta_A$, respectively. It follows by mating that $(-)^A$ carries a canonical structure of monad. If A is an atom, then, again by mating, $(-)^{1/A}$ carries a canonical structure of comonad. The counit of this comonad structure is the π considered in Theorem 3.3. So if $(-)^{1/A}$ is supplied with a strength, by virtue of a point of A, as in the Theorem, π is strongly natural. One may ask whether also the comultiplication of the comonad is strongly natural, (so that the comonad becomes a strong one). The answer is yes. This follows in essence by the Remark after Proposition 3.2; for, a point o of A induces an augmentation of $((-)^A)^A$ (just evaluate twice at o), with which the multiplication of the monad $(-)^A$ is compatible. (Note that the monad $(-)^A$ carries a canonical strength, but that strength does not transfer to a strength on $(-)^{1/A}$, since the adjointness $(-)^A \dashv (-)^{1/A}$ is *not* strong, unless $A = 1$.)

The categories which we constructed in Section 2 always can be enriched in the base topos $\mathbf{E}$. In fact, we construct the categories as full subcategories of the category of G-coalgebras, and this category is equivalent to the category of T-algebras, where $T \dashv G$. Now a strength of either G or T will by quite standard procedure (to be recalled for the G-case) lead to an enrichment of $T - Alg \simeq G - Coalg$. Now, in the case at hand, T carries a standard enrichment; we shall also construct a non-standard enrichment on G, and these two enrichments do not correspond to each other under the adjointness $T \dashv G$. This leads to two distinct enrichments of $T - Alg \simeq G - Coalg$ (in fact two distinct indexings).

The non-standard enrichment which we shall construct on G depends on the atom A used for its construction being a *pointed* object. (This will be the case for the differential equations case, where $A = D$ which carries the point 0.)

We first describe how a strength on an endofunctor gives rise to an enrichment on its category of coalgebras.

For this purpose, it is better to have the $\mathbf{E}$-enrichment of G encoded not in tensorial form $X \times GY \to G(X \times Y)$, but rather in the classical form, as a family of maps $G_{X,Y}$: $[X, Y] \to [GX, GY]$ (where square brackets denote hom-objects, as in [7]). We shall write st (for "*strength of* G") rather than $G_{X,Y}$, to save subscripts. Also, when the bifunctor $[X, Y]$ is applied to a map ξ, say, we sometimes write $[\xi, 1]$ as ξ^* and $[1, \xi]$ as ξ_*; this is also standard mathematical usage for the contravariant, respectively covariant, aspect of hom-functors.

3.4. PROPOSITION. *Let* $G : \mathbf{E} \to \mathbf{E}$ *be a* $\mathbf{V}$*-functor, where* $\mathbf{V}$ *is a symmetric monoidal closed category with equalizers. Then the category* $Coalg(G)$ *of* G*-coalgebras carries a canonical* $\mathbf{V}$*-enrichment.*

PROOF. We are interested only in the case where $\mathbf{E} = \mathbf{V}$, and $\mathbf{V}$ is a topos, hence cartesian closed, and accordingly, we shall write $\times$ rather than $\otimes$ for the monoidal structure. But we shall write $[X, Y]$ rather than Y^X, for typographic convenience.

The construction is straightforward, in the spirit of [3] or [10] (in fact, we could probably read off the desired conclusion from either, — say from the proof of (2.5) in

[10], by suitable dualization and exponential adjoints). But we shall be more explicit: to construct the **V**-valued hom $[[X,Y]]$ for two G-coalgebras $X = (X,\xi)$ and $Y = (Y,\eta)$ (where $\xi : X \to GX$, $\eta : Y \to GY$), we take the equalizer object in the equalizer diagram

$$[[X,Y]] \xrightarrow{i_{X,Y}} [X,Y] \underset{\eta_*}{\overset{\xi^* \circ st_{X,Y}}{\rightrightarrows}} [X,GY].$$

(It is clear that $[[X,Y]]$ by $hom(1,-)$ goes to the set of coalgebra morphisms from X,ξ to Y,η.) If (Z,ζ) is a third G-coalgebra, we would like to prove that the composition map

$$[Y,Z] \times [X,Y] \xrightarrow{M_{XYZ}} [X,Z]$$

restricts to a map

$$[[Y,Z]] \times [[X,Y]] \longrightarrow [[X,Z]].$$

Now consider the following two maps $[Y,Z] \times [X,Y] \to [X,GZ]$; the first is

$$[Y,Z] \times [X,Y] \xrightarrow{1 \times \eta_*} [Y,Z] \times [X,GY] \xrightarrow{st \times 1} [GY,GZ] \times [X,GY] \xrightarrow{M} [X,GZ], \tag{4}$$

the second is

$$[Y,Z] \times [X,Y] \xrightarrow{st \times 1} [GY,GZ] \times [X,Y] \xrightarrow{\eta^* \times 1} [Y,GZ] \times [X,Y] \xrightarrow{M} [X,GZ]. \tag{5}$$

It is a consequence of the extraordinary naturality [7] of M with respect to η that these two maps are equal.

Now the strategy is to eliminate η in (4) in favour of ξ, using $i_{X,Y}$, and to eliminate η from (5) in favour of ζ, using $i_{Y,Z}$. For the first elimination, consider the map

$$[Y,Z] \times [X,Y] \xrightarrow{M} [X,Z] \xrightarrow{st} [GX,GZ] \xrightarrow{\xi^*} [X,GZ]. \tag{6}$$

We claim that the restrictions of (4) and (6) along $1 \times i_{X,Y}$ are equal. In (6), use $st \circ M = M \circ (st \times st)$, which is a general property of enrichments st of functors G ("G commutes with composition"). Also, naturality of M w.r.to ξ gives the first equality sign in

$$\begin{aligned} \xi^* \circ M \circ (st \times st) &= M \circ (1 \times \xi^*) \circ (st \times st) \\ &= M \circ st \times 1 \circ (1 \times \xi^*) \circ (1 \times st) \end{aligned}$$

(the second equality sign by bifunctorality of $\times$). When restricted along $1 \times i_{X,Y}$, the factor $(1 \times \xi^*) \circ (1 \times st)$ may be replaced by $1 \times \eta_*$, so that the total expression gets replaced by (4).

For the second elimination, consider the map

$$[Y,Z] \times [X,Y] \xrightarrow{M} [X,Z] \xrightarrow{\zeta_*} [X,GZ], \tag{7}$$

or, equivalently by naturality of M w.r.to ζ,

$$[Y,Z]\times[X,Y]\xrightarrow{\zeta_*\times 1}[Y,GZ]\times[X,Y]\xrightarrow{M}[X,GZ]. \tag{8}$$

When restricted along $i_{Y,Z}\times 1$, the composite (8) yields the same as does (5). We conclude that (7) and (6) have the same restriction along $i_{Y,Z}\times i_{X,Y}$. In formula

$$\zeta_*\circ M\circ i\times i=\xi^*\circ st\circ M\circ i\times i,$$

which is precisely the condition for $m\circ i\times i$ to factor across the equalizer $i_{X,Z}$ of ζ_* and $\xi^*\circ st$. ∎

Remark. It follows that for instance the category **G** of extension structures, as in Theorem 2.5, with A a *pointed* atom, carries two **E**-enrichments. Also, **G** is a topos, hence Cartesian closed, hence enriched in itself. Since the forgetful functor $\mathbf{G}\to\mathbf{E}$ preserves products (it has, in fact, adjoints on both sides, by the Theorem), it is a *closed* functor, in the sense of [4], hence transforms **G** enrichment into **E**-enrichment. The **E**-enriched category thus arising does not, so far we can see, have **G** for its underlying "ordinary" category, so cannot be compared with the enrichment of **G** we have described in the present section.

We shall postpone the comparison of our "non-standard" enrichment of $G-Coalg$ with the standard enrichment of the equivalent category $T-Alg$ until the following section, where it will be discussed in terms of indexed categories and functors.

4. Indexing

Recall (cf. e.g. [18]) that an *indexing* of an endofunctor $G:\mathbf{E}\to\mathbf{E}$ (where **E** is a topos, say), consists in a family of functors $G_I:\mathbf{E}/I\to\mathbf{E}/I$, one for each object I of **E**, commuting up to coherent isomorphisms with the pullback functors $f^*:\mathbf{E}/J\to\mathbf{E}/I$ induced by the morphisms $f:I\to J$ of **E** (with G itself being identified with G_1). An indexing on G implies in a canonical way a strength on G. Conversely if the endofunctor G is supplied with a strength *and* preserves pull-backs, there is canonically an indexing on it. This latter statement is an unpublished result due to Paré, which was dug out from oblivion by Johnstone [5]. Johnstone pointed out to us that the Paré Theorem immediately upgrades our result on strength to a result on indexing; we shall be explicit on this version for the case of an endofunctor G of the form $(-)^{1/A}$, (A an atom) where the strength on G comes about from a point $1\to A$ of the atom, by our recipe in Theorem 3.3.

For the case where G preserves all finite limits, (which surely is the case for $(-)^{1/A}$), the description of G_I of Paré-Johnstone may be presented as follows (cf. [5] Proposition 3.3). The strength supplies the endofunctor with a point, i.e. with a natural family of maps $t_I:I\to G(I)$, namely the composite

$$I\cong I\times G(1)\to G(I\times 1)\cong G(I),$$

(the middle map being the tensorial strength), and if $\xi : X \to I$ is an object of $\mathbf{E}/I$, $G_I(\xi)$ is the left hand edge in the pull-back diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & G(X) \arrow[d, "G(\xi)"] \\ I \arrow[r, "t_I"'] & G(I). \end{tikzcd}$$

For the case where $G = (-)^{1/A}$ and the strength of G is derived from a point of $0 : 1 \to A$, t_I is just the transpose of "evaluation at 0":$I^A \to I$.

Combining our result on strength with the Paré-Johnstone construction, we thus get:

4.1. THEOREM. *Let $0 : 1 \to A$ be a pointed atom in a Cartesian closed category with pullbacks. Then the endofunctor $G = (-)^{1/A}$ carries a canonical indexing, with G_I (or $((-)^{1/A})_I$) being the composite*

$$\mathbf{E}/I \xrightarrow{(-)^{1/A}} \mathbf{E}/(I^{1/A}) \xrightarrow{t_I^*} \mathbf{E}/I,$$

(with $t_I : I \to I^{1/A}$ being the transpose of evaluation at 0, $I^A \to I$).

Recall that $I^{A/A}$ denotes $(I^A)^{1/A}$, so that there is the unit of adjunction $I \to I^{A/A}$, here denoted u. One may give an alternative description of G_I (for $G = (-)^{1/A}$), namely as the threefold composite in

$$\mathbf{E}/I \xrightarrow{(ev_0)^*} \mathbf{E}/I^A \xrightarrow{(-)^{1/A}} \mathbf{E}/I^{A/A} \xrightarrow{u^*} \mathbf{E}/I;$$

for, t_I may be described as the composite

$$I \xrightarrow{u} I^{A/A} \xrightarrow{(ev_0)^{1/A}} I^{1/A}.$$

So the pulling back along t_I may be carried out in two stages, and utilizing that $(-)^{1/A}$ preserves pull-backs, one gets that equivalence (up to canonical isomorphism) of the two descriptions of G_I.

One early category theoretic investigation of the extra right adjoint functors arising from atoms was undertaken by Freyd and Yetter, [19]; a main result in [19] (attributed to Freyd) is that if A is an atom in an elementary topos $\mathbf{E}$, then A_I (i.e. the projection $A \times I \to I$) is an atom in the slice topos $\mathbf{E}/I$. The construction of the right adjoints in $\mathbf{E}/I$ of Freyd-Yetter utilizes the subobject classifier of $\mathbf{E}$; a simpler construction, which only depends on $\mathbf{E}$ being a locally cartesian closed category, was given by Johnstone, and quoted in Yetter's [20]; it will be recalled below. Let us denote the right adjoint of $(-)^{A_I} : \mathbf{E}/I \to \mathbf{E}/I$ by $(-)^{1/A_I}$. Yetter observed ([19] Theorem 2.4) that the family of these functors (as I ranges over $\mathbf{E}$) only in trivial cases provide an indexing of $(-)^{1/A}$, (in fact, more precisely, the square whose commutativity expresses compatibility of $(-)^{1/A}$ with $(-)^{1/A_I}$ (i.e. the square for index change along $I \to 1$) commutes (with the canonical 2-cell) precisely when I is *A-discrete* in the sense that the canonical $\Delta : I \to I^A$ is an isomorphism.) In particular, the right adjoint $(-)^{1/A_I}$ does not in general agree with our indexed G_I from the Theorem above. The following more precise comparison of these two functors was indicated to us by Peter Johnstone. He kindly consented to let us include it here. Consider the diagram

$$\mathbf{E}/I \underset{\Pi_\Delta}{\overset{ev_0^*}{\rightrightarrows}} \mathbf{E}/(I^A) \xrightarrow{(-)^{1/A}} \mathbf{E}/(I^{A/A}) \xrightarrow{u^*} \mathbf{E}/I$$

Then the bottom composite is, according to Yetter, [20] the (Johnstone-) description of the right adjoint witnessing atomicity of A_I in $\mathbf{E}/I$. (Here, Π_Δ denotes the right adjoint of pulling back along the canonical $\Delta : I \to I^A$.) The top composite is one of the equivalent descriptions of G_I which we gave above, i.e. expresses the indexed nature of the functor $(-)^{1/A}$.

There is a comparison 2-cell in the diagram, from the top composite to the bottom; in fact, there is a 2-cell $ev_0^* \Rightarrow \Pi_\Delta$: just precompose the adjointness unit $id_{\mathbf{E}/I^A} \Rightarrow \Pi_\Delta \circ \Delta^*$ with $ev_0^* : \mathbf{E}/I \to \mathbf{E}/I^A$ and use that $\Delta \circ ev_0$ is the identity map on I.

Now the way in which the indexed functor G_I gives rise to an indexed category of coalgebras is simply that the fibre over $I \in \mathbf{E}$ is the category of objects in $\mathbf{E}/I$ equipped with a coalgebra structure for the endofunctor G_I. For fixed I, the comparison 2-cell just described gives rise to a functor from the category of G_I-coalgebras to the category of coalgebras for the fractional exponent $(-)^{1/A_I}$. This latter is equivalent to the category of algebras for its left adjoint, i.e. to the category of algebras for the "fibrewise exponent" functor $(-)^{A_I}$. So an object in this category is simply an object $x : X \to I$ in $\mathbf{E}/I$, together with a structure of the following kind (expressed in "synthetic" or "set theoretic" terms): to each $i \in I$, a map $(X_i)^A \to X_i$. So the structure only accepts as inputs maps $A \to X$ which are "fibrewise" or "vertical with respect to $x : X \to I$".

This is to be contrasted with what a G_I structure on X means: it is a map in $\mathbf{E}/I$, $\xi : X \to G_I(X)$. Consider the pull-back diagram which defines G_I in the case of $G = (-)^{1/A}$,

$$\begin{tikzcd} G_I(X) \arrow[r] \arrow[d] & X^{1/A} \arrow[d, "x^{1/A}"] \\ I \arrow[r, "t_I"'] & I^{1/A}, \end{tikzcd}$$

and recall that t_I corresponds to $ev_0 : I^A \to I$ under the adjointness $(-)^A \dashv (-)^{1/A}$. Then we see that such a G_I-structure ξ is equivalent to a map $\xi' : X \to X^{1/A}$ making an evident square with codomain $I^{1/A}$ commute; passing to the transpose under the adjointness $(-)^A \dashv (-)^{1/A}$, such datum ξ' is in turn equivalent to a map $\xi'' : X^A \to X$

making the square

$$\begin{tikzcd} X^A \arrow[r, "\xi''"] \arrow[d, "x^A"'] & X \arrow[d, "x"] \\ I^A \arrow[r, "ev_0"'] & I \end{tikzcd}$$

commutative. Such ξ'' take value also on maps $A \to X$ which are not "vertical". Its value on vertical $A \to X$ provides X with the structure for the endofunctor $(-)^A$, giving an alternative description of the comparison functor between the two coalgebra categories, alluded to above. Hence it also provides a comparison between the two ways of indexing the coalgebra categories.

We hope to investigate the role of the "non-standard" indexing of the category of second order differential equations in [14].

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