Theory and Applications of Categories, Vol. 5, No. 1, 1999, pp. 1–11.

A NOTE ON DISCRETE CONDUCHÉ FIBRATIONS

PETER JOHNSTONE
Transmitted by Ieke Moerdijk

ABSTRACT. The class of functors known as discrete Conduché fibrations forms a common generalization of discrete fibrations and discrete opfibrations, and shares many of the formal properties of these two classes. F. Lamarche [7] conjectured that, for any small category $\mathcal{B}$, the category $\mathbf{DCF}/\mathcal{B}$ of discrete Conduché fibrations over $\mathcal{B}$ should be a topos. In this note we show that, although for suitable categories $\mathcal{B}$ the discrete Conduché fibrations over $\mathcal{B}$ may be presented as the 'sheaves' for a family of coverings on a category $\mathcal{B}_{\mathrm{tw}}$ constructed from $\mathcal{B}$, they are in general very far from forming a topos.

Introduction

J. Giraud [4] was the first to investigate the question of exponentiability in slice categories of **Cat**; his results were rediscovered some years later by F. Conduché [3], and the class of functors concerned is now commonly called the class of Conduché fibrations. A functor $p: \mathcal{C} \to \mathcal{B}$ is called a *Conduché fibration* if, given $f: c \to d$ in $\mathcal{C}$ and a factorization $pf = u \circ v$ in $\mathcal{B}$, there exists a factorization $f = g \circ h$ in $\mathcal{C}$ with $pg = u$ and $ph = v$, and this factorization is unique up to equivalence, where two factorizations $g \circ h = g' \circ h'$ are declared equivalent if there exists k: cod $h \to$ dom g' such that $k \circ h = h'$, $g' \circ k = g$ and pk is an identity morphism. (That is, equivalence of factorizations is the equivalence relation generated by the set of all pairs for which there exists k as above.) The Giraud–Conduché result asserts that a functor $p: \mathcal{C} \to \mathcal{B}$ between small categories is exponentiable as an object of $\mathbf{Cat}/\mathcal{B}$ if and only if it is a Conduché fibration.

In this note we shall be concerned entirely with the class of *discrete Conduché fibrations*, which are those in which the lifting of a factorization is unique 'on the nose', and not just up to equivalence. Equivalently, they are the Conduché fibrations which reflect identity morphisms. (To see that a discrete Conduché fibration necessarily has the latter property, observe that if pf is an identity morphism but f is not then the factorization $pf = pf \circ pf$ has (at least) two distinct liftings $f = f \circ 1 = 1 \circ f$.) Just as the class of all Conduché fibrations includes all (Grothendieck) fibrations and opfibrations, so the discrete Conduché fibrations include all discrete fibrations and discrete opfibrations. Moreover, they have similar factorization properties to discrete fibrations: given a

Received by the editors 1998 August 13 and, in revised form, 1998 December 18.
Published on 1999 January 8.
1991 Mathematics Subject Classification: Primary 18A22, Secondary 18B25.
Key words and phrases: .

commutative triangle

$$\begin{tikzcd} \mathcal{D} \arrow[r, "q"] \arrow[dr, "r"'] & \mathcal{C} \arrow[d, "p"] \\ & \mathcal{B} \end{tikzcd}$$

of categories and functors, where p is a discrete Conduché fibration, then q is a discrete Conduché fibration if and only if r is. (The proofs are immediate from the definition.)

It follows that small categories and discrete Conduché fibrations form a lluf subcategory **DCF** of **Cat** (that is, one containing all the objects but only some of the morphisms), and that $\mathbf{DCF}/\mathcal{B}$ is a full subcategory of $\mathbf{Cat}/\mathcal{B}$ for any small category $\mathcal{B}$. Since $\mathbf{DCF}/\mathcal{B}$ further contains full subcategories (of discrete opfibrations and discrete fibrations respectively) equivalent to the functor categories $[\mathcal{B}, \mathbf{Set}]$ and $[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$, it is of some potential interest as a 'single universe' in which both covariant and contravariant functors on $\mathcal{B}$ enjoy equal status — and since its objects are all exponentiable in $\mathbf{Cat}/\mathcal{B}$, it is at least reasonable to hope that it might be cartesian closed (though we shall eventually see that it is not, in general). In fact F. Lamarche [7] conjectured that it might be a topos. A proof of this conjecture was recently announced by M. Bunge and S. Niefield, but turned out to contain an error. The present note began life (in a preprint version written in July 1998) as a simple presentation of a counterexample to Lamarche's conjecture; but — prompted in part by the comments of two anonymous referees — we have since extended it to contain a proof of the conjecture in a special case, which appears to be best possible, as well as a second counterexample to show that in general $\mathbf{DCF}/\mathcal{B}$ is not even representable as a category of sheaves. (The conjecture has also been proved in the special case by Bunge and Niefield in the revised version of their paper [2], and by Bunge and Fiore [1]; but their proof is less direct than ours.)

1. Discrete Conduché Fibrations

We begin with some simple observations about the categories **DCF** and $\mathbf{DCF}/\mathcal{B}$, which may also be found in [2].

1.1. Lemma. *The inclusion* $\mathbf{DCF} \to \mathbf{Cat}$ *creates simply connected limits.*

Proof. By [9], it suffices to consider wide pullbacks. Let $(p_i \colon \mathcal{C}_i \to \mathcal{B} \mid i \in I)$ be a family of discrete Conduché fibrations with common codomain, and let $p \colon \mathcal{C} \to \mathcal{B}$ be their wide pullback (i.e. their product in $\mathbf{Cat}/\mathcal{B}$). Given a morphism $f = (f_i \mid i \in I)$ in $\mathcal{C}$ and a factorization $pf = u \circ v$ in $\mathcal{B}$, since $pf = p_i f_i$ for all i we can uniquely lift the factorization to $f_i = g_i \circ h_i$ in $\mathcal{C}_i$ for each i, and then $g = (g_i \mid i \in I)$ and $h = (h_i \mid i \in I)$ give the required unique factorization of f. Thus p is a discrete Conduché fibration. It now follows from Lemma 1.1(ii) that the projections $\mathcal{C} \to \mathcal{C}_i$ are all in **DCF**; and a similar application of Lemma 1.1 shows that a cone over the original diagram in **Cat** lies in **DCF** if and only if its unique factorization through $\mathcal{C}$ does so. Thus the limiting cone in **Cat** is still a limit in **DCF**. ■

It follows that, for any $\mathcal{B}$, $\mathbf{DCF}/\mathcal{B}$ is closed under arbitrary limits in $\mathbf{Cat}/\mathcal{B}$, and in particular that it is complete. It also has a generating set: given a morphism f in $\mathcal{B}$, let $[\![f]\!]$ be the category whose objects are all factorizations of $f = g \circ h$ in $\mathcal{B}$ and whose morphisms $(g, h) \to (g', h')$ are morphisms $k\colon \mathrm{cod}\ h \to \mathrm{dom}\ g'$ in $\mathcal{B}$ satisfying $k \circ h = h'$ and $g' \circ k = g$. There is an obvious functor $[\![f]\!] \to \mathcal{B}$ sending an object (g, h) to $\mathrm{dom}\ g = \mathrm{cod}\ h$ and a morphism k to itself, and this is clearly a discrete Conduché fibration. Moreover, for any discrete Conduché fibration $p\colon \mathcal{C} \to \mathcal{B}$, morphisms $[\![f]\!] \to \mathcal{C}$ over $\mathcal{B}$ are easily seen to correspond to morphisms $\tilde{f}$ of $\mathcal{C}$ satisfying $p\tilde{f} = f$, from which it follows easily that the collection of all $[\![f]\!]$ forms a generating set for $\mathbf{DCF}/\mathcal{B}$.

Since the inclusion $\mathbf{DCF} \to \mathbf{Cat}$ preserves and reflects monomorphisms, we know that monomorphisms in **DCF** are just those injective functors which are (necessarily discrete) Conduché fibrations. Up to isomorphism, the subobjects of $\mathcal{B}$ in **DCF** thus correspond to *Conduché subcategories* of $\mathcal{B}$, that is subcategories which are closed under factorization. In particular, we see that **DCF** (and hence $\mathbf{DCF}/\mathcal{B}$ for any $\mathcal{B}$) is well-powered. Rather more alarmingly, we have

1.2. Lemma. *The category* **DCF** *is not well-copowered.*

Proof. Let $\mathcal{C}$ denote the category with four objects a, b, c, d and just two non-identity morphisms $a \to b$, $c \to d$. For each ordinal $\alpha > 0$, let $\mathcal{D}_\alpha$ denote the poset obtained from the ordinal $\alpha + 1$ by adding a new element $*$ satisfying $0 \leq * \leq \alpha$ (but incomparable with all other elements of $\alpha + 1$). Let $f_\alpha\colon \mathcal{C} \to \mathcal{D}_\alpha$ be the order-preserving map sending a, b, c, d to $0, *, *, \alpha$ respectively: it is readily checked that f_α is a discrete Conduché fibration (since there are no nontrivial factorizations to be lifted), and that it is an epimorphism in **DCF** — in fact a strong epimorphism, since its image is not contained in any proper Conduché subcategory of $\mathcal{D}_\alpha$. But it is clear that $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ are not isomorphic unless $\alpha = \beta$; so $\mathcal{C}$ has a proper class of non-isomorphic quotients of this form. ■

However, this problem disappears when we pass from **DCF** to $\mathbf{DCF}/\mathcal{B}$. The point is that, if a category $\mathcal{C}$ admits a discrete Conduché fibration to $\mathcal{B}$, each morphism in $\mathcal{C}$ has exactly as many factorizations as its image in $\mathcal{B}$, and thus the phenomenon of 'unboundedly many factorizations' which we observed in the proof of Lemma 1.2 cannot occur. So it is possible to put a bound on the cardinalities of categories which appear as epimorphic images of a given object of $\mathbf{DCF}/\mathcal{B}$, and hence this category is well-copowered. Similarly, we may show that, for any functor $p\colon \mathcal{C} \to \mathcal{B}$, there is an upper bound on the cardinalities of categories $\widehat{\mathcal{C}}$ which can appear in diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "r"] \arrow[dr, "p"'] & & \widehat{\mathcal{C}} \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

such that q is a discrete Conduché fibration and r does not factor through any proper Conduché subcategory of $\widehat{\mathcal{C}}$; thus we obtain a solution-set which allows us to conclude

1.3. Proposition. *For any small category $\mathcal{B}$, $\mathbf{DCF}/\mathcal{B}$ is a reflective subcategory of $\mathbf{Cat}/\mathcal{B}$.* ∎

In the original version of [2], it was claimed that $\mathbf{DCF}/\mathcal{B}$ is also coreflective in $\mathbf{Cat}/\mathcal{B}$. As we shall see in 3.4 below, that claim is incorrect.

2. Discrete Conduché Fibrations as Sheaves

Given a category $\mathcal{B}$, the *twisted morphism category* $\mathcal{B}_{\mathrm{tw}}$ has the morphisms of $\mathcal{B}$ as its objects, and morphisms $f \to g$ in $\mathcal{B}_{\mathrm{tw}}$ are pairs of morphisms ($u\colon \mathrm{dom}\ g \to \mathrm{dom}\ f$, $v\colon \mathrm{cod}\ f \to \mathrm{cod}\ g$) such that

$$\begin{tikzcd} \bullet \arrow[d, "f"'] & \bullet \arrow[l, "u"'] \arrow[d, "g"] \\ \bullet \arrow[r, "v"] & \bullet \end{tikzcd}$$

commutes. (Note that this is the dual of the twisted morphism category as defined in [2]: the direction which we have chosen for its morphisms is more convenient for our purposes.) We remark in passing that if $\mathcal{B}$ is a partially ordered set, then so is $\mathcal{B}_{\mathrm{tw}}$, and the latter may be identified with the set of (nonempty) *intervals* in $\mathcal{B}$ (that is, subsets $[a, b] = \{c \in \mathcal{B} \mid a \leq c \leq b\}$ with $a \leq b$ in $\mathcal{B}$), ordered by inclusion.

It is not hard to see that the construction $(-)_{\mathrm{tw}}$ is functorial on $\mathbf{Cat}$, and that it preserves limits as a functor $\mathbf{Cat} \to \mathbf{Cat}$ — in fact it is easy to construct a left adjoint for it.

Of particular interest to us is the following result, first observed by Lamarche [7]:

2.1. Lemma. *A functor $p\colon \mathcal{C} \to \mathcal{B}$ is a discrete Conduché fibration if and only if $p_{\mathrm{tw}}\colon \mathcal{C}_{\mathrm{tw}} \to \mathcal{B}_{\mathrm{tw}}$ is a discrete fibration.*

Proof. Suppose p is a discrete Conduché fibration; let f be an object of $\mathcal{C}_{\mathrm{tw}}$, and let $(u, v)\colon g \to pf$ a morphism of $\mathcal{B}_{\mathrm{tw}}$ with codomain pf. Then the factorization $pf = v \circ g \circ u$ lifts uniquely to a factorization $f = \tilde{v} \circ \tilde{g} \circ \tilde{u}$, yielding a unique morphism $(\tilde{u}, \tilde{v})\colon \tilde{g} \to f$ lying over (u, v). So p_{tw} is a discrete fibration. Conversely, if p_{tw} is a discrete fibration, we observe first that p must reflect identity morphisms; for if pf is an identity morphism but f is not, we have (at least) three different liftings of $(pf, pf)\colon pf \to pf$ with codomain f. Now, given an arbitrary morphism f of $\mathcal{C}$ and a factorization $pf = v \circ u$, the unique lifting of $(u, v)\colon 1_{\mathrm{dom}\ v} \to pf$ with codomain f yields the required factorization of f in $\mathcal{C}$. ∎

It is easy to see that $(-)_{\rm tw}$ is a faithful functor $\mathbf{Cat}/\mathcal{B} \to \mathbf{Cat}/\mathcal{B}_{\rm tw}$: it is not full in general, but it becomes so when restricted to the subcategory $\mathbf{DCF}/\mathcal{B}$. For if $p\colon \mathcal{C} \to \mathcal{B}$ and $q\colon \mathcal{D} \to \mathcal{B}$ are discrete Conduché fibrations and we are given a functor $r\colon \mathcal{C}_{\rm tw} \to \mathcal{D}_{\rm tw}$ over $\mathcal{B}_{\rm tw}$, then the effect of r on objects of $\mathcal{C}_{\rm tw}$ defines a mapping $r_0\colon \mathrm{mor}\,\mathcal{C} \to \mathrm{mor}\,\mathcal{D}$; we need to verify that this is functorial. But if (g, f) is a composable pair of morphisms in $\mathcal{C}$, then applying r to the morphisms $(f, 1)\colon g \to g \circ f$ and $(1, g)\colon f \to g \circ f$ (and recalling that q reflects identity morphisms) yields two factorizations $r_0(g \circ f) = r_0 g \circ x = y \circ r_0 f$ both lying over the factorization $p(g \circ f) = pg \circ pf$; so these must coincide, and $r_0(g \circ f) = r_0 g \circ r_0 f$. Similar arguments show that r_0 preserves identity morphisms, domains and codomains, and that the effect of r on an arbitrary morphism of $\mathcal{C}_{\rm tw}$ coincides with that of $(r_0)_{\rm tw}$.

Thus we may identify $\mathbf{DCF}/\mathcal{B}$ with a full subcategory of the category of discrete fibrations over $\mathcal{B}_{\rm tw}$, or equivalently of functors $\mathcal{B}_{\rm tw}^{\rm op} \to \mathbf{Set}$; it is closed under limits, since $(-)_{\rm tw}$ preserves them, and in fact it is not hard to see, by arguments similar to those at the end of the last section, that it is reflective in $[\mathcal{B}_{\rm tw}^{\rm op}, \mathbf{Set}]$. We note also that this subcategory contains all the representable functors, since the discrete fibration corresponding to $\mathcal{B}_{\rm tw}(-, f)$ is just $[\![f]\!]_{\rm tw} \to \mathcal{B}_{\rm tw}$ in the notation of the previous section.

Given an arbitrary discrete fibration $q\colon \mathcal{E} \to \mathcal{B}_{\rm tw}$, it thus becomes of interest to ask whether it can be 'untwisted' to a discrete Conduché fibration $p\colon \mathcal{C} \to \mathcal{B}$. If so, then it is easy to see that we may take $\mathrm{ob}\,\mathcal{C}$ to be

$$\{e \in \mathrm{ob}\,\mathcal{E} \mid qe \text{ is an identity morphism in } \mathcal{B}\}\ ,$$

and $\mathrm{mor}\,\mathcal{C}$ to be the set of all objects of $\mathcal{E}$. Moreover, if e is an object of $\mathcal{E}$ (lying over an object f of $\mathcal{B}_{\rm tw}$, say), the domain and codomain of e as a morphism of $\mathcal{C}$ may be obtained by lifting $(1, f)\colon 1_{\mathrm{dom}\,f} \to f$ and $(f, 1)\colon 1_{\mathrm{cod}\,f} \to f$ respectively to morphisms of $\mathcal{E}$ with codomain e, and taking their domains. The problem is thus to define composition in $\mathcal{C}$. Suppose d and e are objects of $\mathcal{E}$ which (with the above definitions of domain and codomain) should be composable as morphisms of $\mathcal{C}$, and let $qd = f, qe = g$. In $\mathcal{B}_{\rm tw}$, we have a commutative square

$$\begin{tikzcd} 1 \arrow[r, "{(f,1)}"] \arrow[d, "{(1,g)}"] & f \arrow[d, "{(1,g)}"] \\ g \arrow[r, "{(f,1)}"] & g \circ f \end{tikzcd} \tag{1}$$

which is easily verified to be a pullback (and a pushout); the assertion that d and e are composable means, in terms of the functor $E\colon \mathcal{B}_{\rm tw}^{\rm op} \to \mathbf{Set}$ corresponding to $\mathcal{E}$, that they are elements of $E(f)$ and $E(g)$ which agree when restricted to $1_{\mathrm{cod}\,f}$, and we wish to say that under these circumstances there should be a unique element of $E(g \circ f)$ restricting to d and e. This is almost the assertion that E should satisfy the sheaf axiom for the cover generated by the morphisms

$$g \xrightarrow{(f,1)} g \circ f \qquad \text{and} \qquad f \xrightarrow{(1,g)} g \circ f \ ; \tag{2}$$

the only problem is that these two morphisms need not be monomorphisms in $\mathcal{B}_{\mathrm{tw}}$, so the assertion that d and e generate a compatible family relative to this cover would impose further restrictions on d and e, which we do not want to do.

However, there is a class of categories for which this difficulty disappears. We shall say that $\mathcal{B}$ is *factorization preordered* if, whenever we have a diagram

$$\bullet \xrightarrow{f} \bullet \overset{g}{\underset{h}{\rightrightarrows}} \bullet \xrightarrow{k} \bullet$$

in $\mathcal{B}$ with $g \circ f = h \circ f$ and $k \circ g = k \circ h$, we necessarily have $g = h$. Equivalently, for every morphism l of $\mathcal{B}$, the category $[\![l]\!]$ defined in the previous section is a preorder. Note that this holds either if all morphisms of $\mathcal{B}$ are monic or if all morphisms are epic; in particular, it holds whenever $\mathcal{B}$ is a preorder. Then we have

2.2. Lemma. *Let $\mathcal{B}$ be a factorization preordered category. Then a morphism $(u,v)\colon f \to g$ of $\mathcal{B}_{\mathrm{tw}}$ is monic provided either u is epic or v is monic in $\mathcal{B}$.*

Proof. Suppose u is epic, and suppose we have two morphisms (w_1, x_1) and $(w_2, x_2)\colon h \to f$ having equal composites with (u, v). Then $w_1 \circ u = w_2 \circ u$, so $w_1 = w_2 = w$ say. Now we have $v \circ x_1 = v \circ x_2$ and $x_1 \circ h \circ w = f = x_2 \circ h \circ w$, so $x_1 = x_2$. The argument when v is monic is similar. ∎

2.3. Proposition. *Let $\mathcal{B}$ be a factorization preordered category. Then $\mathbf{DCF}/\mathcal{B}$ is equivalent to the full subcategory of $[\mathcal{B}_{\mathrm{tw}}^{\mathrm{op}}, \mathbf{Set}]$ consisting of functors which satisfy the sheaf axiom for the coverings generated by* (2)*, for all composable pairs (f,g) of morphisms of $\mathcal{B}$.*

Proof. Given a discrete fibration $q\colon \mathcal{E} \to \mathcal{B}_{\mathrm{tw}}$, corresponding to a functor $E\colon \mathcal{B}_{\mathrm{tw}}^{\mathrm{op}} \to \mathbf{Set}$, it is now easy to verify that the 'untwisting' of $\mathcal{E}$, as defined above, forms a category $\mathcal{C}$ precisely when E satisfies the sheaf axiom for the indicated covers, since the morphisms $(f,1)\colon g \to g \circ f$ and $(1,g)\colon f \to g \circ f$ are both monic by Lemma 2.2, and their pullback is as given in (1). (The associativity of composition comes from the observation that, if (f,g,h) is a composable triple of morphisms of $\mathcal{B}$, then E must satisfy the sheaf axiom for the cover of $h \circ g \circ f$ generated by $(g \circ f, 1)$, (f, h) and $(1, h \circ g)$, since this may be obtained by composing two covers of the form (2).) Further, if this occurs, then the obvious functor $\mathcal{C} \to \mathcal{B}$ is a discrete Conduché fibration, and $\mathcal{E}$ is isomorphic over $\mathcal{B}_{\mathrm{tw}}$ to $\mathcal{C}_{\mathrm{tw}}$. ∎

2.4. Example. To show that the 'factorization preordered' condition in Proposition 2.3 cannot be omitted, consider the following example. The category $\mathcal{B}$ has as objects all ordinals less than $\omega + 2$: there are no morphisms $\alpha \to \beta$ in $\mathcal{B}$ if $\alpha > \beta$, and there is one morphism $n \to m$ for each finite n and m with $n \le m$. There are 2^n morphisms $n \to \omega$

for each n; we label these as $f_{(n,a)}$ where a ranges over all subsets of n. The composite of $n \to m$ with $f_{(m,b)}$ is $f_{(n,b\cap n)}$; thus each $f_{(n,a)}$ has exactly two factorizations through $n \to n+1$. Finally, $\omega+1$ is a (strict) terminal object of $\mathcal{B}$.

Next, we define a directed graph $\mathcal{C}$ and a graph morphism $p\colon \mathcal{C} \to \mathcal{B}$. $\mathcal{C}$ has just one vertex lying over each of ω and $\omega+1$, and 2^n vertices lying over each finite n (as before, we label these by pairs (n,a)). There is one arrow $(n,a) \to (m,b)$ iff $n \le m$ and $a = b \cap n$, and one arrow $(n,a) \to \omega$ (whose image under p is $f_{(n,a)}$) for each (n,a). $\mathcal{C}$ also has one arrow $\omega \to \omega+1$ (as well as 'identity' arrows on all its vertices), but no arrows $(n,a) \to \omega+1$ for any (n,a). Clearly, $\mathcal{C}$ cannot be made into a category, since we cannot define the composite of $(n,a) \to \omega$ and $\omega \to \omega+1$. However, if we make $\mathcal{C}$ into a 'partial category' by defining composites wherever possible, we can form $\mathcal{C}_{\mathrm{tw}}$ as usual, and it is an honest category; moreover, p_{tw} becomes a discrete fibration $\mathcal{C}_{\mathrm{tw}} \to \mathcal{B}_{\mathrm{tw}}$ satisfying the sheaf axiom for the system of covers (2). The point is that the arrows $(n,a) \to \omega$ and $\omega \to \omega+1$ in $\mathcal{C}$, considered as objects of $\mathcal{C}_{\mathrm{tw}}$, do not determine a compatible family relative to the covering generated by

$$
\begin{tikzcd}
n \arrow[d, "f_{(n,a)}"'] & n \arrow[l, "1"'] \arrow[d] \\
\omega \arrow[r] & \omega+1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\omega \arrow[d] & n \arrow[l, "f_{(n,a)}"'] \arrow[d] \\
\omega+1 \arrow[r, "1"] & \omega+1
\end{tikzcd}
$$

because the two morphisms

$$
\begin{tikzcd}
n \arrow[d] & n \arrow[l, "1"'] \arrow[d, "f_{(n,a)}"] \\
n+1 \arrow[r, shift left, "f_{(n+1,a)}"] \arrow[r, shift right, "f_{(n+1,a\cup\{n\})}"'] & \omega
\end{tikzcd}
$$

have the same composite with the first of these, but the 'restrictions' of $(n,a) \to \omega$ along them are different.

3. Stability Under Pullback

It appears from Proposition 2.3 that we have 'almost' proved $\mathbf{DCF}/\mathcal{B}$ to be a topos, in the case when $\mathcal{B}$ is factorization preordered. All we lack is the information that the system of covers (2) is stable under pullback in $\mathcal{B}_{\mathrm{tw}}$. However, this is unfortunately not true, even in the case when $\mathcal{B}$ is a poset; and, as soon as it fails, $\mathbf{DCF}/\mathcal{B}$ loses almost all the familiar properties of a topos. We now embark on the investigation of the minimal counterexample.

For this, we shall take the base category $\mathcal{B}$ to be the 'generic commutative square' $\mathcal{Q}$, i.e. the category whose objects and non-identity morphisms are represented by the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, "l"'] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

and the above notation will be standard for the rest of this section. We begin by observing

3.1. Lemma. *For the above category $\mathcal{Q}$, the system of covers* (2) *is not stable under pullback. Moreover, its closure under pullback includes covers with respect to which the 'untwistable' discrete fibrations do not necessarily satisfy the sheaf axiom.*

Proof. Consider the cover of $l = g \circ f$ generated by $(f, 1)\colon g \to l$ and $(1, g)\colon f \to l$. It is easy to see that the pullback of this cover along $(h, k)\colon 1_c \to l$ is empty; but the empty cover on 1_c does not belong to the given system. Moreover, the initial discrete fibration $\mathbf{0} \to \mathcal{Q}_{\mathrm{tw}}$ is untwistable, and it does not satisfy the sheaf axiom for any empty cover. ∎

Next, we note a result which shows that $\mathbf{DCF}/\mathcal{Q}$ is very far from being a topos.

3.2. Lemma. *The lattice of Conduché subcategories of $\mathcal{Q}$ is not modular.*

Proof. For any set S of morphisms, let $\langle S \rangle$ denote the smallest Conduché subcategory containing S (i.e. the intersection of all such subcategories). We note that $\langle f, g \rangle$ is the whole of $\mathcal{Q}$, since it contains $l = g \circ f$ and hence also h and k; but $\langle f, h \rangle$ is just the union of $\langle f \rangle$ and $\langle h \rangle$, since the latter is a Conduché subcategory. It now follows easily that the five subcategories $\mathcal{Q}$, $\langle f, h \rangle$, $\langle f \rangle$, $\langle g \rangle$ and $\langle 1_b \rangle$ form a copy of the non-modular lattice N_5 (cf. [5], p. 59). ∎

3.3. Corollary.

(i) $\mathbf{DCF}/\mathcal{Q}$ *is not cartesian closed.*

(ii) $\mathbf{DCF}/\mathcal{Q}$ *does not have a subobject classifier.*

Proof. (i) In any cartesian closed category, the subobjects of 1 (being closed under product and exponentiation in the ambient category) form a cartesian closed meet-semilattice. Hence, if they actually form a lattice, it must be distributive. But we observed earlier that the subobjects of 1 in $\mathbf{DCF}/\mathcal{Q}$ (that is, the subobjects of $\mathcal{Q}$ in $\mathbf{DCF}$) correspond to Conduché subcategories of $\mathcal{Q}$; and we have just seen that these form a non-distributive lattice.

(ii) Similarly, in any category with finite limits and a subobject classifier, the subobjects of any object form a cartesian closed poset. (The proof is similar to that in a topos — cf. [6], 3.51 — which does not use the cartesian closed structure.) Once again, we have seen that this is not the case in $\mathbf{DCF}/\mathcal{Q}$. ∎

Next, we fulfil a promise made in section 1:

3.4. Lemma. *The category* $\mathbf{DCF}/\mathcal{Q}$ *is not closed under coequalizers in* $\mathbf{Cat}/\mathcal{Q}$*; in particular, it is not coreflective.*

Proof. Let $\mathcal{C}$ be the disjoint union of the Conduché subcategories $\langle f \rangle$ and $\langle g \rangle$ of $\mathcal{Q}$: then $\mathcal{C} \to \mathcal{Q}$ is clearly a discrete Conduché fibration. But if we form the coequalizer in $\mathbf{Cat}/\mathcal{Q}$ of the two morphisms $\mathbf{1} \rightrightarrows \mathcal{C}$ which send the unique object of $\mathbf{1}$ to the codomain of f and the domain of g respectively, we obtain the subcategory of $\mathcal{Q}$ consisting of f, g, l and the appropriate identity morphisms, which is not a Conduché subcategory. ■

For completeness, we also note

3.5. Scholium. *The category* $\mathbf{DCF}/\mathcal{Q}$ *is not regular.*

Proof. The coequalizer of the two morphisms $\mathbf{1} \rightrightarrows \mathcal{C}$ in $\mathbf{DCF}/\mathcal{Q}$ exists, of course; it is just the reflection in $\mathbf{DCF}/\mathcal{Q}$ of the coequalizer described above. But this is the whole of $\mathcal{Q}$; so the morphism $\mathcal{C} \to \mathcal{Q}$ is regular epic in $\mathbf{DCF}/\mathcal{Q}$. However, if we pull it back along the inclusion $\langle h \rangle \to \mathcal{Q}$, we obtain a proper monomorphism; so regular epimorphisms are not stable under pullback in $\mathbf{DCF}/\mathcal{Q}$. ■

Finally, we show that $\mathcal{Q}$ is indeed a minimal counterexample. We shall say that a category $\mathcal{B}$ is *factorization strongly connected* if, for each morphism l of $\mathcal{B}$, the category $[\![l]\!]$ is strongly connected; that is, for every commutative square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

in $\mathcal{B}'$, we can find either $x\colon b \to c$ satisfying $x \circ f = h$ and $k \circ x = g$ or $y\colon c \to b$ satisfying $y \circ h = f$ and $g \circ y = k$.

3.6. Proposition. *Let* $\mathcal{B}$ *be a factorization strongly connected category. Then the system of covers* (2) *is stable under pullback in* $\mathcal{B}_{\mathrm{tw}}$*. If* $\mathcal{B}$ *is also factorization preordered, then* $\mathbf{DCF}/\mathcal{B}$ *is a topos.*

Proof. Consider the cover of an object $l = g \circ f$ of $\mathcal{B}_{\mathrm{tw}}$ generated by $(f, 1)\colon g \to l$ and $(1, g)\colon f \to l$, and a morphism $(u, v)\colon h \to l$. The argument splits into three cases:

If there exists x satisfying $x \circ f = u$ and $v \circ h \circ x = g$, then (u, v) factors through $(f, 1)$, so the pullback of the cover along (u, v) contains the identity morphism on h.

Otherwise, there exists y satisfying $y \circ u = f$ and $v \circ h = g \circ y$. If now we have a morphism z satisfying $g \circ z = v$ and $z \circ h = y$, then (u, v) factors through $(1, g)$, so again the pullback of the cover contains 1_h.

In the remaining case, we have y as before and a morphism w satisfying $v \circ w = g$ and $w \circ y = h$. Now the composites $(u, v) \circ (y, 1) \colon w \to l$ and $(u, v) \circ (1, w) \colon y \to l$ factor through $(f, 1)$ and $(1, g)$ respectively, so the pullback of the cover contains the cover of h generated by the factorization $h = w \circ y$.

This completes the proof of the first assertion; the second follows from it and Proposition 2.3. ■

We remark that the revised version of [2] also contains a proof, albeit by substantially different methods, that $\mathbf{DCF}/\mathcal{B}$ is a topos whenever $\mathcal{B}$ is 'factorization totally ordered' (i.e. satisfies the two hypotheses of Proposition 3.6, plus the requirement that each $[\![l]\!]$ should be skeletal). More recently, Bunge and Fiore [1] have extended this proof to cover the case when $\mathcal{B}$ satisfies a condition (their (CFI)) equivalent to the conjunction of our two conditions; they have also observed that this conjunction is equivalent to the 'interval glueing' condition considered by Lawvere [8].

In particular, if $\mathcal{B}$ is any poset in which each interval is totally ordered, then $\mathbf{DCF}/\mathcal{B}$ is a topos. (However, a category satisfying the conditions of Proposition 3.6 need not be a preorder: an interesting example, exploited by Lawvere [8], is provided by the monoid of non-negative real numbers under addition.) On the other hand, over any poset containing an interval which is not totally ordered (equivalently, containing a copy of $\mathcal{Q}$ as a full subcategory), we can reproduce all the negative results of this section.

References

[1] M.C. Bunge and M.P. Fiore, Unique factorization lifting functors and categories of processes, preprint, October 1998.

[2] M.C. Bunge and S.B. Niefield, Exponentiability and single universes, preprint, June 1998; revised version to appear in *J. Pure Appl. Alg.*

[3] F. Conduché, Au sujet de l'existence d'adjoints à droite aux foncteurs "image réciproque" dans la catégorie des catégories, *C. R. Acad. Sci. Paris* 275 (1972), A891–894.

[4] J. Giraud, Méthode de la descente, *Bull. Soc. Math. France, Mémoire* 2 (1964).

[5] G. Grätzer, *General Lattice Theory*, Birkhäuser Verlag (1978).

[6] P.T. Johnstone, *Topos Theory*, L.M.S. Monographs no. 10, Academic Press (1977).

[7] F. Lamarche, 'A new kind of fibration?', talk at Peripatetic Seminar on Sheaves and Logic, Utrecht, October 1996.

[8] F.W. Lawvere, State categories and response functors, preprint, 1986.

[9] R. Paré, Simply connected limits, *Canad. J. Math.* 42 (1990), 731–746.

Department of Pure Mathematics,
University of Cambridge, England
Email: `ptj@pmms.cam.ac.uk`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/1999/n1/n1.{dvi,ps}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

Subscription information. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi and Postscript format. Details will be e-mailed to new subscribers and are available by `WWW/ftp`. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

Information for authors. The typesetting language of the journal is TeX, and LaTeX is the preferred flavour. TeX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at URL `http://www.tac.mta.ca/tac/` or by anonymous ftp from `ftp.tac.mta.ca` in the directory `pub/tac/info`. You may also write to `tac@mta.ca` to receive details by e-mail.