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# CLOSED MODEL CATEGORIES FOR $[n, m]$-TYPES

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ABSTRACT. For $m \geqslant n > 0$, a map $f$ between pointed spaces is said to be a weak $[n, m]$-equivalence if $f$ induces isomorphisms of the homotopy groups $\pi_k$ for $n \leqslant k \leqslant m$ . Associated with this notion we give two different closed model category structures to the category of pointed spaces. Both structures have the same class of weak equivalences but different classes of fibrations and therefore of cofibrations. Using one of these structures, one obtains that the localized category is equivalent to the category of $n$-reduced $CW$-complexes with dimension less than or equal to $m+1$ and $m$-homotopy classes of cellular pointed maps. Using the other structure we see that the localized category is also equivalent to the homotopy category of $(n-1)$-connected $(m+1)$-coconnected $CW$-complexes.

## Introduction.

D. Quillen [19] introduced the notion of closed model category and proved that the categories of spaces and of simplicial sets have the structure of a closed model category. This structure gives you some advantages. For instance, you can use sequences of homotopy fibres or homotopy cofibres associated to a map. In many cases, you can also compare two closed model categories by using a pair of adjoint functors. For example, you can prove that the localized categories of spaces and of simplicial sets are equivalent. In other cases, the cofibrant (or fibrant) approximation of an object gives objects and canonical maps with certain universal properties or can be used to construct derived functors.

In this paper, for $m \geqslant n > 0$ , we take as weak equivalences those maps of $\mathrm{Top}_\star$ which induce isomorphisms on the homotopy group functors $\pi_k$ for $m \geqslant k \geqslant n$ . A map $f$ with this property is said to be a weak $[n, m]$-equivalence. We complete this class of weak equivalences with fibrations and cofibrations in two different ways:

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In the first structure, we use $[n,m]$-fibrations and $[n,m]$-cofibrations to obtain a closed model category structure such that all the pointed spaces are $[n,m]$-fibrant and all $n$-reduced $CW$-complexes with dimension less than or equal to $m+1$ are $[n,m]$-cofibrant. Using this structure one has that the localized category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ is equivalent to the $m$-homotopy category of $n$-reduced $CW$-complexes with dimension less than or equal to $m+1$ . Recall that two cellular pointed maps $f,g\colon X \longrightarrow Y$ are $m$-homotopic if there is a cellular pointed homotopy $F\colon X\otimes 0 \cup sk_m X \otimes I \cup X \otimes 1 \longrightarrow Y$ such that $F\partial_0 = f$ and $F\partial_1 = g$ , where $\partial_0$ and $\partial_1$ are the usual canonical inclusions and $\mathrm{sk}_m$ denotes the standard $m$-skeleton of a $CW$-complex.

In the second structure (the structure "prime"), we use new classes of $[n,m]'$-fibrations and $[n,m]'$-cofibrations to give a distinct closed model category structure such that a $[n,m]'$-cofibrant space is weak equivalent to a $n$-reduced $CW$-complex and a pointed space $X$ is $[n,m]'$-fibrant if and only if $X$ is $(m+1)$-coconnected. Therefore the $[n,m]'$-cofibrant $[n,m]'$-fibrant spaces are weak equivalent to $n$-reduced $(m+1)$-coconnected $CW$-complexes. In this case we have a different homotopical interpretation of the localized category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$. One has that the localized category is equivalent to standard homotopy category of $n$-reduced $(m+1)$-coconnected $CW$-complexes.

We remark that the equivalence between these two different homotopical interpretations of the localized category are topological versions of the well known $(m+1)$-skeleton and $(m+1)$-coskeleton functors.

We point out that the category of $(n-1)$-connected $(m+1)$-coconnected spaces is not closed by finite limits and colimits. This implies that is not possible to develop some standard homotopy constructions in this category. Nevertheless, the category of $(n-1)$-connected $(m+1)$-coconnected spaces is closed under the homotopy fibres and loops given by the new closed model category $\mathrm{Top}_\star^{[n,m]'}$ .

In order to have a shorter paper we have mainly developed questions related to closed model category structures. However, we briefly mention the following aspects:

There are equivalences of categories with the corresponding $[n,m]$-types of pointed simplicial sets and $[n-1,m-1]$-types of simplicial groups. We refer the reader to [4] , [12] for some closed model categories for $[n-1,m-1]$-types of simplicial groups.

In the stable range $m \leqslant 2n-2$, we have a natural equivalence of categories

$$\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]}) \simeq \mathrm{Ho}(\mathrm{Top}_\star^{[n+1,m+1]})$$

Therefore, for each "length" $r$, we only have to study a finite number of categories of this form, exactly the categories: $\mathrm{Ho}(\mathrm{Top}_\star^{[0,r]})$ , $\mathrm{Ho}(\mathrm{Top}_\star^{[1,r+1]})$ , $\cdots$ , $\mathrm{Ho}(\mathrm{Top}_\star^{[r+2,2r+2]})$ . For the study of some stable algebraic models for spaces with two consecutive non trivial homotopy groups, we refer the reader to [3] , [6] , [10], [12] .

We also remark that using the fibrant approximations of a space $X$ in the model categories $\mathrm{Top}_\star^{[n,m]'}$ , when $m \to \infty$ , we obtain the well known Postnikov decomposition of $X$. We have included a reformulation of the Postnikov theory to describe how the Grothendieck integration of the cohomological functor $H^{m+1}(-;\pi_m(-))$ on the category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'}) \times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})^{op}$ is equivalent to the category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$ .

We note that for $(n-1)$-connected pointed spaces $X$ , $Y$ the following exact sequence gives an interesting relation between the set of pointed homotopy classes from $X$ to $Y$, and the hom-sets of the categories $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$

$$0 \to \lim_m^1 \mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})(\Sigma X, Y) \to \mathrm{Ho}(\mathrm{Top}_\star)(X,Y) \to \lim_m \mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})(X,Y) \to 0$$

This implies that the family of categories $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ gives a good approach to the total homotopy type of pointed spaces. This formula has been used to work with phantom maps. A map $f: X \to Y$ is said to be a phantom map it its restriction to each skeleton is inessential. Results about the existence of phantom maps have been proved by B.I. Gray [13] and for the study of spaces of different type but with the same $n$-type for all $n \geqslant 0$ we refer the reader to [13], [16], [21].

One of the techniques to study the types and $n$-types of spaces is the construction of algebraic models for some particular class of spaces. Recall the notion of homotopy system, introduced by J.H.C. Whitehead, which is an algebraic model for the types and $n$-types of connected CW-complexes whose homotopy groups are isomorphic to the homology groups of the corresponding universal covering spaces. Brown-Higgings [1] have developed the notion of crossed complex which generalizes the homotopy system for non-connected spaces.

Brown-Golasinski [2] have proved that the category of crossed complexes admits the structure of a closed model category. There are also a truncated version for $n$-types of crossed complexes and for pro-crossed complexes given by Hernández-Porter [14] .

There are other many algebraic models for $n$-types, for example the notion of cat$^n$-group introduced by J.-L. Loday [15], the crossed $n$-cubes analyzed by T. Porter [18] and the hypercrossed complexes studied by Cegarra-Carrasco [5]. For the case of $[n, n+1]$-types one has the categories of cat$^1$-groups, braided cat$^1$-groups and symmetric cat$^1$-groups. We want to mention that some of these models can be adapted for the equivariant setting, Moerdijk-Svensson [17] have given models for equivariant 2-types and Garzón-Miranda [11] have developed a technique to give models for higher dimensions.

We think that our study of closed model categories for $[n, m]$-types of spaces suggests that the study of algebraic models for $[n, m]$-types and stable $[n, m]$-types can be developed by using closed model category structures. An equivalence of closed model categories is stronger than an equivalence of categories. The existence of an equivalence between a model category of spaces and an algebraic model category permits that some homotopy constructions can be developed by using algebraic techniques.

## 1. Preliminaries.

In this section we recall some definitions and notations which will be used later.

1.1 DEFINITION. *A closed model category $\mathcal{C}$ is a category endowed with three distinguished families of maps called cofibrations, fibrations and weak equivalences satisfying the axioms $CM1$–$CM5$ below:*

*$CM1$. $\mathcal{C}$ is closed under finite projective and inductive limits.*
*$CM2$. If $f$ and $g$ are maps such that $gf$ is defined then if two of these $f, g$ and $gf$ are weak equivalences then so is the third.*

*Recall that the maps in $\mathcal{C}$ form the objects of a category $Maps(\mathcal{C})$ having commutative squares for morphisms. We say that a map $f$ in $\mathcal{C}$ is a retract of $g$ if there are morphisms $\varphi: f \longrightarrow g$ and $\psi: g \longrightarrow f$ in $Maps(\mathcal{C})$ such that $\psi\varphi = id_f$.*

*A map which is a weak equivalence and a fibration is said to be a trivial fibration and, similarly, a map which is a weak equivalence and a cofibration is said to be a trivial cofibration.*
*$CM3$. If $f$ is a retract of $g$ and $g$ is a fibration, cofibration or weak equivalence then so is $f$.*
*$CM4$. (Lifting.) Given a solid arrow diagram*

$$(*) \qquad \begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

*the diagonal arrow from $B$ to $X$ exists in either of the following situations:*
*(i) $i$ is a cofibration and $p$ is a trivial fibration,*
*(ii) $i$ is a trivial cofibration and $p$ is a fibration.*
*$CM5$. (Factorization.) Any map $f$ may be factored in two ways:*
*(i) $f = pi$ where $i$ is a cofibration and $p$ is a trivial fibration,*
*(ii) $f = qj$ where $j$ is a trivial cofibration and $q$ is a fibration.*

We say that a map $i: A \longrightarrow B$ in a category has the left lifting property (LLP) with respect to another map $p: X \longrightarrow Y$ and $p$ is said to have the right lifting property RLP with respect to $i$ if the dotted arrow exists in any diagram of the form $(*)$.

The initial object of $\mathcal{C}$ is denoted by $\emptyset$ and the final object by $\star$. An object $X$ of $\mathcal{C}$ is said to be fibrant if the morphism $X \longrightarrow \star$ is a fibration and it is said cofibrant if $\emptyset \longrightarrow X$ is a cofibration.

Let $\mathrm{Top}_\star$ be the category of pointed topological spaces and $\mathrm{SS}_\star$ the category of pointed simplicial sets.

The following functors will be used:

$\mathrm{Sing}: \mathrm{Top}_\star \longrightarrow \mathrm{SS}_\star$, the "singular" functor which is right adjoint to the "realization" functor $|\ |: \mathrm{SS}_\star \longrightarrow \mathrm{Top}_\star$ .

$\mathrm{cosk}_q: \mathrm{SS}_\star \longrightarrow \mathrm{SS}_\star$, the " $q$–coskeleton" functor which is right adjoint to the "$q$–skeleton" functor $\mathrm{sk}_q: \mathrm{SS}_\star \longrightarrow \mathrm{SS}_\star$ .

$R_n: \mathrm{SS}_\star \longrightarrow \mathrm{SS}_\star$ the "$n$-reduction" functor defined as follows: Given a pointed simplicial set $X$, the $n$-reduction $R_n(X)$ is the simplicial subset of $X$ of those simplices of $X$ whose $q$-faces for $q < n$ are degeneracies of the base 0-simplex. The left adjoint of $R_n$ is the functor $(\ )_{(n)}: \mathrm{SS}_\star \longrightarrow \mathrm{SS}_\star$ defined by $X_{(n)} = X/\,\mathrm{sk}_{n-1} X$.

We shall use the following notation: For each integer $n \geqslant 0$, $\triangle[n]$ denotes the "standard $n$-simplex", and for $n > 0$, $\dot{\triangle}[n]$ (resp. $\mathrm{V}(n,k)$ for $0 \leqslant k \leqslant n$) denotes the simplicial subset of $\triangle[n]$ which is the union of the $i$-faces of $\triangle[n]$ for $0 \leqslant i \leqslant n$ (resp. $0 \leqslant i \leqslant n$, $i \neq k$).

In this paper the following closed model categories given by Quillen [19] ,[20] will be considered:

(1) $\mathrm{Top}_\star$ denotes the category of pointed topological spaces with the following structure: Given a map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$, $f$ is said to be a fibration if it is a fibre map in the sense of Serre; $f$ is a weak equivalence if $f$ induces isomorphisms $\pi_q(f)$ for $q \geqslant 0$ and for any choice of base point and $f$ is a cofibration if it has the LLP with respect to all trivial fibrations.

(2) $\mathrm{SS}_\star$ denotes the category of pointed simplicial sets with the following structure: A map $f\colon X \longrightarrow Y$ in $\mathrm{SS}_\star$ is said to be a fibration if $f$ is a fibre map in the sense of Kan; $f$ is a weak equivalence if its geometric realization, $|f|$, is a homotopy equivalence and $f$ is a cofibration if it has the LLP with respect to any trivial fibration.

(3) $\mathrm{SS}_n$ denotes the category of the $n$-reduced simplicial sets. A pointed simplicial set $X$ is said to be $n$-reduced if $\mathrm{sk}_{n-1} X$ is isomorphic to the simplicial subset generated by the base 0-simplex of $X$ . We write $\mathrm{SS}_n$ for the full subcategory of $\mathrm{SS}_\star$ determined by all the $n$-reduced simplicial sets. A map $f\colon X \longrightarrow Y$ in $\mathrm{SS}_n$ is said to be a cofibration in $\mathrm{SS}_n$ if $f$ is injective, $f$ is a weak equivalence if it is a weak equivalence in $\mathrm{SS}_\star$ and $f$ is a fibration if it has the RLP with respect to the trivial cofibrations in $\mathrm{SS}_n$ .

We also need the closed model structures given in [7] and [8] .

(4) $\mathrm{Top}_\star^{n]}$ , $n \geqslant 0$, denotes the category of pointed topological spaces with the following structure: A map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$ is said to be an $n]$-fibration if $f$ has in Top the RLP with respect to the maps of the family $V^{p-1} \longrightarrow I^p$, $0 < p \leqslant n+1$, and $V^{n+1} \longrightarrow \dot{I}^{n+2}$, where $I^q$ denotes the $q$-dimensional unit cube; $\dot{I}^q$ is the union of all the $(q-1)$-faces of $I^q$ (if $q = 0$, $\dot{I}^q = \emptyset$) and $V^{q-1} = cl(\dot{I}^q - (I^{q-1} \times \{1\}))$ is the space obtained to removing the face $I^{q-1} \times \{1\}$ of $\dot{I}^q$. A map $f$ is said to be a weak $n]$-equivalence if, for $k = 0, 1, \cdots, n$ and $x \in X$, the induced map $\pi_q(f)\colon \pi_q(X,x) \longrightarrow \pi_q(Y, f(x))$ is an isomorphism. An $n]$-fibration which is also a weak $n]$-equivalence is said to be a trivial $n]$-fibration, and a map $f$ is an $n]$-cofibration if $f$ has the LLP with respect to each trivial $n]$-fibration.

(5) $\mathrm{SS}_\star^{n]}$ , $n \geqslant 0$, denotes the category of pointed simplicial sets with the following structure: A map $f\colon X \longrightarrow Y$ in $\mathrm{SS}_\star$ is said to be a weak $n]$-equivalence if $|f|\colon |X| \longrightarrow |Y|$ is a weak $n]$-equivalence in $\mathrm{Top}_\star$ , $f$ is said to be an $n]$-fibration if $f$ has the RLP with respect to $\mathrm{V}(p,k) \longrightarrow \triangle[p]$ for $0 < p \leqslant n+1$, $0 \leqslant k \leqslant p$ and $\mathrm{V}(n+2,k) \longrightarrow \dot{\triangle}[n+2]$, $0 \leqslant k \leqslant n+2$. A map $f$ which is a weak $n]$-equivalence and an $n]$-fibration is said to be a trivial $n]$-fibration, and a map $f$ is an $n]$-cofibration if $f$ has the LLP with respect to any trivial $n]$-fibration.

(6) $\mathrm{Top}_\star^{[n}$ , $n > 0$, denotes the category of pointed topological spaces with the following structure: A map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$ is said to be a weak $[n$-equivalence if the induced map $\pi_q(f)\colon \pi_q(X) \longrightarrow \pi_q(Y)$ is an isomorphism for each $q \geqslant n$; $f$ is an $[n$-fibration if it has the RLP with respect to the inclusions in $\mathrm{Top}_\star$

$$|\,\mathrm{V}(p,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(p,k)| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$$

for every $p > n$ and $0 \leqslant k \leqslant p$. If $f$ is both an $[n$-fibration and a weak $[n$-equivalence is said to be a trivial $[n$-fibration. And $f$ is an $[n$-cofibration if it has the LLP with respect to any trivial $[n$-fibration.

Let $\mathrm{Ho}(\mathrm{Top}_\star)$, $\mathrm{Ho}(\mathrm{SS}_\star)$, $\mathrm{Ho}(\mathrm{SS}_n)$, $\mathrm{Ho}(\mathrm{Top}_\star^{n]})$ $\mathrm{Ho}(\mathrm{SS}_\star^{n]})$ and $\mathrm{Ho}(\mathrm{Top}_\star^{[n})$ denote the corresponding localized categories obtained by formal inversion of the respective families of weak equivalences defined above.

## 2. The categories $\mathrm{Top}_\star^{[n,m]}$ and $\mathrm{Top}_\star^{[n,m]'}$.

In the category of pointed topological spaces and continuous maps, $\mathrm{Top}_\star$, for each pair of integers $n, m$ such that $0 < n \leqslant m$, we consider the following families of maps:

2.1 Definition. *Let $f : X \longrightarrow Y$ be a map in* $\mathrm{Top}_\star$ ,

*(i) $f$ is a weak $[n,m]$-equivalence if the induced map $\pi_q(f)\colon \pi_q(X) \longrightarrow \pi_q(Y)$ is an isomorphism for every $q$ such that $n \leqslant q \leqslant m$.*

*(ii) $f$ is an $[n,m]$-fibration if it has the RLP with respect to the inclusions*

$$|\,\mathrm{V}(p,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(p,k)| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$$

*for every $p$ such that $n < p \leqslant m+1$ and $0 \leqslant k \leqslant p$, and*

$$|\,\mathrm{V}(m+2,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(m+2,k)| \longrightarrow |\,\dot\triangle[m+2]\,/\,\mathrm{sk}_{n-1}\,\dot\triangle[m+2]\,|$$

*for $0 \leqslant k \leqslant m+2$ .*

*A map which is both an $[n,m]$-fibration and a weak $[n,m]$-equivalence is said to be a trivial $[n,m]$-fibration.*

*(iii) $f$ is an $[n,m]$-cofibration if it has the LLP with respect to any trivial $[n,m]$-fibration.*

*A map which is both an $[n,m]$-cofibration and a weak $[n,m]$-equivalence is said to be a trivial $[n,m]$-cofibration.*

*A pointed space $X$ is said to be $[n,m]$-fibrant if the map $X \longrightarrow \star$ is an $[n,m]$-fibration, and $X$ is said to be $[n,m]$-cofibrant if the map $\star \longrightarrow X$ is an $[n,m]$-cofibration.*

Remark. We note that the homotopy group $\pi_q(X)$ only depends on the path component $C$ of the given base point of $X$ . Therefore the inclusion $C \longrightarrow X$ is always a weak $[n,m]$-equivalence. On the other hand, the objects $|\,\mathrm{V}(p,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(p,k)|$, $|\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$, $|\,\mathrm{V}(m+2,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(m+2,k)|$ and $|\,\dot\triangle[m+2]\,/\,\mathrm{sk}_{n-1}\,\dot\triangle[m+2]\,|$

used in the definition of $[n, m]$-fibration are considered as pointed spaces. It is also clear that all objects in $\mathrm{Top}_\star$ are $[n, m]$-fibrant.

Using the same class of "weak equivalences", we introduce new classes of "fibrations" and "cofibrations" that will give a different structure to $\mathrm{Top}_\star$ . The new class of fibrations is a subclass of the fibrations given in Definition 2.1. We distinguish the new structure by using the notation "prime".

2.1' DEFINITION. *Let $f : X \longrightarrow Y$ be a map in $\mathrm{Top}_\star$ ,*
*(i) $f$ is a weak $[n, m]'$-equivalence if $f$ is a weak $[n, m]$-equivalence.*
*(ii) $f$ is an $[n, m]'$-fibration if $f$ is an $[n, m]$-fibration and it has the RLP with respect to the inclusions*

$$| \dot{\triangle}[p] \,/\, \mathrm{sk}_{n-1} \, \dot{\triangle}[p] \,| \longrightarrow | \,\triangle[p] \,/\, \mathrm{sk}_{n-1} \, \triangle[p] \,|$$

*for any $p \geqslant m + 2$.*

*In a similar way to Definition 2.1, we define the corresponding notions of trivial $[n, m]'$-fibration, $[n, m]'$-cofibration,trivial $[n, m]'$-cofibration and $[n, m]'$-fibrant or $[n, m]'$-cofibrant object.*

In this paper, with the definitions given above we will prove the following results:

2.2 THEOREM. *For each pair of integers $n, m$, such that $0 < n \leqslant m$, the category $\mathrm{Top}_\star$ together with the families of $[n, m]$-fibrations, $[n, m]$-cofibrations and weak $[n, m]$-equivalences, has the structure of a closed model category.*

2.2' THEOREM. *Analogous to Theorem 2.2 writing $[n, m]'$ instead of $[n, m]$.*

We shall denote by $\mathrm{Top}_\star^{[n,m]}$ the closed model category $\mathrm{Top}_\star$ with the distinguished families of $[n, m]$-fibrations, $[n, m]$-cofibrations and weak $[n, m]$-equivalences. When $n = m$ we shall denote by $\mathrm{Top}_\star^{[n]}$ the category $\mathrm{Top}_\star^{[n,n]}$. Similarly, we will use the notation $\mathrm{Top}_\star^{[n,m]'}$ , $\mathrm{Top}_\star^{[n]'}$ .

It is well know that Axiom $CM1$ is satisfied by $\mathrm{Top}_\star$, Axiom $CM2$ is an immediate consequence of the properties of the isomorphisms of groups, and the definition of $[n, m]$-cofibration ($[n, m]'$-cofibration) implies obviously Axiom $CM4$ (i). Then, we will complete the proof of the Theorem 2.2 and Theorem 2.2' as a consequence of the results below.

2.3 LEMMA. *If a map $f$ is a retract of a map $g$ and $g$ has the RLP (resp LLP) with respect to another map $h$, then $f$ has also this property.*

2.4 PROPOSITION. *(Axiom $CM3$) In $\mathrm{Top}_\star$ if a map $f$ is a retract of a map $g$ and $g$ is an $[n, m]$-fibration, $[n, m]$-cofibration or weak $[n, m]$-equivalence, then so is $f$.*

2.4' PROPOSITION. *Analogous to Proposition 2.4 writing* $[n, m]'$ *instead of* $[n, m]$.

2.5 PROPOSITION. *Let* $f$ *be a map in* $\mathrm{Top}_\star$, *then*

*(i)* $f$ *is an* $[n, m]$*-fibration if and only if* $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ *is a fibration in* $\mathrm{SS}_n$,

*(ii)* $f$ *is a weak* $[n, m]$*-equivalence if and only if* $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ *is a weak equivalence in* $\mathrm{SS}_n$,

*(iii)* $f$ *is a trivial* $[n, m]$*-fibration if and only if* $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ *is a trivial fibration in* $\mathrm{SS}_n$.

PROOF. (i) Taking into account that the functors Sing and $R_n$ are right adjoints to the functors $|\ |$ and $(\ )_{(n)}$ respectively, we have, for a map $f$ in $\mathrm{Top}_\star$, that $f$ is an $[n, m]$-fibration if and only if $R_n \operatorname{Sing} f$ has the RLP with respect to the inclusions

$$\mathrm{V}(m+2, k) \longrightarrow \dot{\triangle}[m+2]$$

for $0 \leqslant k \leqslant m+2$, and

$$\mathrm{V}(p, k) \longrightarrow \triangle[p]$$

for $n < p \leqslant m+1$, $0 \leqslant k \leqslant p$.

Note that if $p > m+2$, then $\operatorname{sk}_{m+1} \mathrm{V}(p, k) \longrightarrow \operatorname{sk}_{m+1} \triangle[p]$ is an isomorphism; if $p = m+2$, then $\operatorname{sk}_{m+1} \mathrm{V}(m+2, k) \longrightarrow \operatorname{sk}_{m+1} \triangle[m+2]$ is isomorphic in Maps(SS) to $\mathrm{V}(m+2, k) \longrightarrow \dot{\triangle}[m+2]$, and if $p < m+2$, then $\operatorname{sk}_{m+1} \mathrm{V}(p, k) \longrightarrow \operatorname{sk}_{m+1} \triangle[p]$ is isomorphic to $\mathrm{V}(p, k) \longrightarrow \triangle[p]$ .

Therefore, $f$ is an $[n, m]$-fibration if and only if $R_n \operatorname{Sing} f$ has the RLP with respect to the inclusions $\operatorname{sk}_{m+1} \mathrm{V}(p, k) \longrightarrow \operatorname{sk}_{m+1} \triangle[p]$ for each $p > n$ , $0 \leqslant k \leqslant p$ .

Now, applying that the functor $\operatorname{sk}_{m+1}$ is left adjoint to the functor $\operatorname{cosk}_{m+1}$ the above condition is equivalent to $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ has the RLP with respect to the family of inclusions $\mathrm{V}(p, k) \longrightarrow \triangle[p]$, $n < p$.

Because $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ is a map in $\mathrm{SS}_n$ and for any pointed space $X$, $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} X$ is a Kan complex, we can apply the Proposition 2.12 of [20] to conclude that $f$ is an $[n, m]$-fibration in $\mathrm{Top}_\star$ if and only if $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ is a fibration in $\mathrm{SS}_n$.

(ii) Since for any pointed space X, $\operatorname{Sing} X$ is a Kan simplicial set, then $R_n \operatorname{Sing} X$ is the $n$-Eilenberg subcomplex of $\operatorname{Sing} X$. Therefore, for each $q \geqslant n$, we have the isomorphisms

$$\pi_q(R_n \operatorname{Sing} X) \cong \pi_q(\operatorname{Sing} X) \cong \pi_q(X).$$

On the other hand, for each pointed simplicial set L, the natural map $\eta\colon L \longrightarrow \operatorname{cosk}_{m+1} L$ induces the isomorphisms $\pi_q(\eta)$, for $q \leqslant m$.

Taking into account that for any pointed space $X$, $R_n \operatorname{Sing} X$ is a Kan simplicial set, then $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} X$ is an $(m+1)$-coconnected $n$-reduced simplicial set. Therefore we obtain $\pi_q(\operatorname{cosk}_{m+1} R_n \operatorname{Sing} X) = 0$ for $q < n$ or $q > m$ and the isomorphisms $\pi_q(\operatorname{cosk}_{m+1} R_n \operatorname{Sing} X) \cong \pi_q(X)$ for $n \leqslant q \leqslant m$.

So, for a map $f$ in $\mathrm{Top}_\star$, $f$ is a weak $[n, m]$-equivalence if and only if $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} f$ is a weak equivalence in $\mathrm{SS}_n$.

(iii) It is an immediate consequence of (i) and (ii). ∎

2.6 Proposition. *For a map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$, the following statements are equivalent:*

*(i) $f$ is a trivial $[n,m]$-fibration,*

*(ii) $f$ has the RLP with respect to the inclusions*

$$|\,\dot{\triangle}[p]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[p]\,| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$$

*for $n \leqslant p \leqslant m+1$.*

Proof. Let $f$ be a map in $\mathrm{Top}_\star$. By Proposition 2.5 and Proposition 2.3 of [20] , $f$ is a trivial $[n,m]$-fibration if and only if $\mathrm{cosk}_{m+1} R_n \,\mathrm{Sing}\, f$ has the RLP with respect to the inclusions $\dot{\triangle}[p] \longrightarrow \triangle[p]$ for each integer $p > 0$.

Because $\mathrm{sk}_{m+1}$ and $\mathrm{cosk}_{m+1}$ are adjoints, the above condition is equivalent to $R_n \,\mathrm{Sing}\, f$ has the RLP with respect to the inclusions

$$\mathrm{sk}_{m+1}\,\dot{\triangle}[p] \longrightarrow \mathrm{sk}_{m+1}\,\triangle[p]$$

for each $p > 0$.

We note that if $p \geqslant m+2$, the map $\mathrm{sk}_{m+1}\,\dot{\triangle}[p] \longrightarrow \mathrm{sk}_{m+1}\,\triangle[p]$ is an isomorphism and, for $p \leqslant m+1$, the inclusion $\mathrm{sk}_{m+1}\,\dot{\triangle}[p] \longrightarrow \mathrm{sk}_{m+1}\,\triangle[p]$ is isomorphic in Maps(SS) to the inclusion $\dot{\triangle}[p] \longrightarrow \triangle[p]$. Then, $f$ is a trivial $[n,m]$-fibration if and only if $R_n \,\mathrm{Sing}\, f$ has the RLP with respect to the inclusions $\dot{\triangle}[p] \longrightarrow \triangle[p]$ for $0 < p \leqslant m+1$.

Now, using the adjointness of $R_n \,\mathrm{Sing}$ and $|\ |(\ )_{(n)}$, we obtain that the above condition is equivalent to $f$ has the RLP with respect to

$$|\,\dot{\triangle}[p]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[p]\,| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|,\ \ n \leqslant p \leqslant m+1.$$

■

2.6' Proposition. *For a map $f\colon X \longrightarrow Y$ in $\mathrm{Top}_\star$, the following statements are equivalent:*

*(i) $f$ is a trivial $[n,m]'$-fibration,*

*(ii) $f$ has the RLP with respect to the inclusions*

$$|\,\dot{\triangle}[p]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[p]\,| \longrightarrow |\,\triangle[p]\,/\,\mathrm{sk}_{n-1}\,\triangle[p]\,|$$

*for every $p \geqslant n$.*

Remark. Note that since (ii) is the characterization of the trivial $[n$-fibrations (see [8]), then the family of the $[n,m]'$-cofibrations agree with the family of the $[n$-cofibrations.

2.7 Proposition. *(Axiom CM5) Let $f\colon X \longrightarrow Y$ be a map in $\mathrm{Top}_\star$, then $f$ can be factored in two ways:*

*(i) $f = pi$, where $i$ is an $[n,m]$-cofibration and $p$ is a trivial $[n,m]$-fibration,*

*(ii)* $f = qj$*, where* $j$ *is a weak* $[n,m]$*-equivalence having the LLP with respect to all* $[n,m]$*-fibrations and* $q$ *is an* $[n,m]$*-fibration.*

PROOF. Given a class $\mathcal{F}$ of maps, denote by $\mathcal{F}'$ the class of maps which have the RLP with respect to the maps of $\mathcal{F}$.

(i) Consider the family $\mathcal{F}$ of inclusions

$$|\,\dot{\triangle}[r]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[r]\,| \longrightarrow |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|,\ \ n \leqslant r \leqslant m+1.$$

By Proposition 2.6, $\mathcal{F}'$ is the class of trivial $[n,m]$-fibrations.

Now, we can use the "small object argument", in a similar way to Lemma 3 of ch II, §3 of Quillen [19] to factor $f\colon X \longrightarrow Y$ as $f = pi$ where $p$ is in $\mathcal{F}'$ and $i$ has the LLP with respect to the maps of $\mathcal{F}'$. Then, $p$ is a trivial $[n,m]$-fibration and $i$ is an $[n,m]$-cofibration.

(ii) Consider the following family $\mathcal{F}$ of maps which is the union of the following $\mathcal{F}_1$ and $\mathcal{F}_2$:

$\mathcal{F}_1$ is the family of inclusions

$$|\,\mathrm{V}(r,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(r,k)| \longrightarrow |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|,\ \ n < r \leqslant m+1, 0 \leqslant k \leqslant r,$$

and $\mathcal{F}_2$ is the family

$$|\,\mathrm{V}(m+2,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(m+2,k)| \longrightarrow |\,\dot{\triangle}[m+2]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[m+2]\,|,\ \ 0 \leqslant k \leqslant m+2.$$

In this case, by Definition 2.1, $\mathcal{F}'$ is the class of $[n,m]$-fibrations. Analogously to (i), we can factor $f = qj$ where $q$ is an $[n,m]$-fibration and $j$ has the LLP with respect to all $[n,m]$-fibrations.

Now, we note that for any map $|\,\mathrm{V}(m+2,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(m+2,k)| \longrightarrow X,\ 0 \leqslant k \leqslant m+2$, in $\mathrm{Top}_\star$ , the inclusion map

$$h\colon X \longrightarrow X \bigcup_{|\,\mathrm{V}(m+2,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(m+2,k)|} |\,\dot{\triangle}[m+2]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[m+2]\,|$$

induces isomorphisms $\pi_q(h)$ for $q \leqslant m$; and for any map $|\,\mathrm{V}(r,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(r,k)| \longrightarrow X,\ r > n,\ 0 \leqslant k \leqslant r$, in $\mathrm{Top}_\star$ , the inclusion

$$h'\colon X \longrightarrow X \bigcup_{|\,\mathrm{V}(r,k)/\,\mathrm{sk}_{n-1}\,\mathrm{V}(r,k)|} |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|$$

is a trivial cofibration.

Using these facts one can check that the map $j$ is a weak $[n,m]$-equivalence. ∎

2.7' PROPOSITION. *Analogous to Proposition 2.7, writing* $[n,m]'$ *instead of* $[n,m]$.

PROOF. (i) This decomposition is the same that the given for the $[n$-structure. See [8].

(ii) In a similar way to the proof of Proposition 2.7 (ii) and taking into account that for any map

$$|\,\dot{\triangle}[r]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[r]\,| \longrightarrow X,\ \ r \geqslant m+2.$$

in $\mathrm{Top}_\star$, the inclusion map

$$h''\colon X \longrightarrow X \bigcup_{|\,\dot{\triangle}[r]\,/\,\mathrm{sk}_{n-1}\,\dot{\triangle}[r]\,|} |\,\triangle[r]\,/\,\mathrm{sk}_{n-1}\,\triangle[r]\,|$$

induces isomorphisms $\pi_q(h'')$ for $q \leqslant m$. ∎

REMARK. Note that if $X=\star$, the $[n,m]'$-cofibrant space $Z$ , constructed in the proof of Proposition 2.7' (i) for the decomposition $\star \longrightarrow Z \longrightarrow Y$ , is $(n-1)$-connected. And if $Y=\star$, the $[n,m]'$-fibrant space $W$ , constructed in the proof of Proposition 2.7' (ii) for the decomposition $X \longrightarrow W \longrightarrow \star$ , is $(m+1)$-coconnected. Then for any pointed space $X$, we can construct in a functorial way, an object of $\mathrm{Top}_\star$, denoted by $X^{[n,m]'}$, which is $[n,m]'$-fibrant and $[n,m]'$-cofibrant, weak $[n,m]'$-equivalent to $X$, and $X^{[n,m]'}$ is $(n-1)$-connected and $(m+1)$-coconnected.

2.8 COROLLARY. *(Axiom* $CM4$ *(ii)) Any trivial* $[n,m]$*-cofibration has the LLP with respect to all* $[n,m]$*-fibrations.*

PROOF. Let $i\colon A \longrightarrow B$ be a trivial $[n,m]$-cofibration. By Proposition 2.7 we have a commutative diagram in $\mathrm{Top}_\star$

$$\begin{array}{ccc} A & \xrightarrow{j} & W \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ B & \xrightarrow[id]{} & B \end{array}$$

where $q$ is an $[n,m]$-fibration and $j$ is a weak $[n,m]$-equivalence which has the LLP with respect to any $[n,m]$-fibration.

Because Axiom $CM2$ is verified, $q$ is a trivial $[n,m]$-fibration. Therefore, there is a lifting $h\colon B \longrightarrow W$ for the diagram above, and the map $i$ is a retract of $j$. Applying Lemma 2.3, it follows that $i$ has the LLP with respect to all $[n,m]$-fibrations. ∎

2.8' COROLLARY. *Analogous to Corollary 2.8 writing* $[n,m]'$ *instead of* $[n,m]$.

REMARK. (i) In Definition 2.1 we have considered classes of cofibrations, fibrations and weak equivalences to define the $[n,m]$-structure for integers $n,m$ such that $0<n\leqslant m$ . Obviously we can extend Definition 2.1 for the case $m=\infty$. In order to extend this

for the case $n = 0$ , we proceed as follows: Note that for any simplicial set $K$, when $n > 0$, $|K/\operatorname{sk}_{n-1} K|$ is the pushout $|K| \bigcup\limits_{|\operatorname{sk}_{n-1} K|} \star$ . If we take $\operatorname{sk}_{-1}(\ ) = \emptyset$ , then, for $n = 0$, this pushout is homeomorphic to $|K^+|$ , where $K^+$ denotes the disjoint union of $K$ and a point. Note that the class of $[n,m]$-fibrations given in Definition 2.1 and the class of trivial $[n,m]$-fibrations are characterized by the RLP with respect to the family $\mathcal{F}[n,m]$ given in Definition 2.1 and the family $\mathcal{T}[n,m]$ given in Proposition 2.6. With this notation, for $n = 0$ , the RLP with respect to $\mathcal{F}[n,m]$ induces the class of $[0,m]$-fibrations and the RLP with respect to $\mathcal{T}[n,m]$ produces the class of trivial $[0,m]$-fibrations. We can use the LLP with respect these classes to define the classes of trivial $[0,m]$-cofibrations and $[0,m]$-cofibrations. Finally, we can define the weak $[0,m]$-equivalences how those morphisms $f$ that can be factored as $f = pi$ , where $i$ is a trivial $[0,m]$-cofibration and $p$ is a trivial $[0,m]$-fibration. This $[0,m]$-structure is just the $m]$-structure given in §1 (4), that have been analyzed in [7] . For $m = \infty$ , $n > 0$ , one obtains the category $\mathrm{Top}_\star^{[n}$ (see [8] ) and for the case $n = 0$, $m = \infty$ we have the structure of closed model category $\mathrm{Top}_\star$ given by Quillen for pointed spaces.

(ii) We can give an equivalent definition of the notion of $[n,m]$-fibration if we change the family of inclusions used in Definition 2.1 by the following family:

$$CS^k \otimes 0 \bigcup S^k \otimes I \longrightarrow CS^k \otimes I \ ; \quad n-1 \leqslant k \leqslant m-1$$

and

$$CS^m \otimes 0 \bigcup S^m \otimes I \longrightarrow \partial(CS^m \otimes I)$$

where $\partial$ denotes the standard boundary.

Then, the trivial $[n,m]$-fibrations are characterized by the RLP with respect to the inclusions $\star \longrightarrow S^n$ and $S^k \longrightarrow D^{k+1}$ for every $k$ such that $n \leqslant k \leqslant m$.

(iii) Similarly, for the $[n,m]'$-structure, we can give an equivalent definition of the notion of $[n,m]'$-fibration by adding to the family of inclusions given in (ii) the maps

$$S^k \longrightarrow D^{k+1} \ , \quad k \geqslant m+1.$$

Then, the trivial $[n,m]'$-fibrations are characterized by the RLP with respect to the inclusions $\star \longrightarrow S^n$ and $S^k \longrightarrow D^{k+1}$ for every $k \geqslant n$.

## 3. The category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$.

Let $n, m$ be integers such that $0 < n \leqslant m$. Let $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ denote the localized category obtained by formal inversion of the family of weak $[n,m]$-equivalences. Note that $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]}) = \mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$.

We shall compare $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ with the localized category $\mathrm{Ho}(\mathrm{SS}_n)$ :

Consider the adjoint functors

$$\mathrm{SS}_n \underset{\operatorname{cosk}_{m+1} R_n \operatorname{Sing}}{\overset{|\ |\mathrm{sk}_{m+1}}{\rightleftarrows}} \mathrm{Top}_\star^{[n,m]}$$

Note that, by Proposition 2.5, the functor $\operatorname{cosk}_{m+1} R_n \operatorname{Sing}$ preserves fibrations and weak equivalences. We also have that every object of $\mathrm{Top}_\star^{[n,m]}$ is $[n,m]$-fibrant. On the other hand, all objects of $\mathrm{SS}_n$ are cofibrant and the functor $|\ |\mathrm{sk}_{m+1}$ verifies the following properties:

3.1 PROPOSITION. *Let $f\colon X \longrightarrow Y$ be a map in $\mathrm{SS}_n$. Then:*
*(i) If $f$ is a weak equivalence in $\mathrm{SS}_n$, then $|\,\mathrm{sk}_{m+1} f|$ is a weak $[n,m]$-equivalence.*
*(ii) If $f$ is a cofibration in $\mathrm{SS}_n$, then $|\,\mathrm{sk}_{m+1} f|$ is an $[n,m]$-cofibration.*

PROOF. (i) For any pointed simplicial set $K$, the natural map $\mu\colon \mathrm{sk}_{m+1} K \longrightarrow K$ induces an isomorphism $\pi_q(\mu)$ for each $q \leqslant m$.

So, given a map $f$ in $\mathrm{SS}_n$ such that $\pi_q(f)$ is an isomorphism for every $q$, the maps $\pi_q(|\,\mathrm{sk}_{m+1} f|)$ are isomorphism for $q \leqslant m$. Obviously, $|\,\mathrm{sk}_{m+1} f|$ is a weak $[n,m]$-equivalence.

(ii) It is an immediate consequence of the Proposition 2.5 (iii) and the fact that the functors $|\ |\mathrm{sk}_{m+1}$ and $\operatorname{cosk}_{m+1} R_n \operatorname{Sing}$ are adjoints. ∎

Recall that if $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a functor between closed model categories, and $F$ carries a weak equivalence between cofibrant objects of $\mathcal{A}$ into a weak equivalence of $\mathcal{B}$, there exists a left derived functor $F^L\colon \mathrm{Ho}(\mathcal{A}) \longrightarrow \mathrm{Ho}(\mathcal{B})$ defined by $F^L(X) = F(LX)$, where $LX \longrightarrow X$ is a trivial fibration and $LX$ is a cofibrant object in $\mathcal{A}$. In a dual context one has right derived functors $G^R$.

In our case, by Proposition 2.5 and Proposition 3.1, it follows that the functors $|\ |\mathrm{sk}_{m+1}$ and $\operatorname{cosk}_{m+1} R_n \operatorname{Sing}$ induce the adjoint functors $(|\ |\mathrm{sk}_{m+1})^L = |\ |\mathrm{sk}_{m+1}$ and $(\operatorname{cosk}_{m+1} R_n \operatorname{Sing})^R = \operatorname{cosk}_{m+1} R_n \operatorname{Sing}$ between the localized categories:

$$\mathrm{Ho}(\mathrm{SS}_n) \underset{\operatorname{cosk}_{m+1} R_n \operatorname{Sing}}{\overset{|\ |\mathrm{sk}_{m+1}}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$$

Remember that for any pointed space $X$, $\operatorname{cosk}_{m+1} R_n \operatorname{Sing} X$ is an $(m+1)$-coconnected $n$-reduced simplicial set. Let $\mathrm{Ho}(\mathrm{SS}_n)|_{(m+1)-coco}$ the full subcategory of $\mathrm{Ho}(\mathrm{SS}_n)$ determined by the $(m+1)$-coconnected $n$-reduced simplicial sets. Then, we have:

3.2 THEOREM. *The pair of adjoint functors $|\ |\mathrm{sk}_{m+1}$ , $\operatorname{cosk}_{m+1} R_n \operatorname{Sing}$ induce an equivalence of categories*

$$\mathrm{Ho}(\mathrm{SS}_n)|_{(m+1)\text{-coco}} \underset{\operatorname{cosk}_{m+1} R_n \operatorname{Sing}}{\overset{|\ |\mathrm{sk}_{m+1}}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$$

PROOF. It suffices to check that for any $(m+1)$-coconnected object $X$ of $\mathrm{SS}_n$, the unit $X \longrightarrow \operatorname{cosk}_{m+1} R_n \operatorname{Sing} |\,\mathrm{sk}_{m+1} X|$ of the adjunction is an isomorphism of $\mathrm{Ho}(\mathrm{SS}_n)$.

And for every object $Y$ of $\mathrm{Top}_\star$, the counit $|\mathrm{sk}_{m+1}\,\mathrm{cosk}_{m+1}\,R_n\,\mathrm{Sing}\,Y| \longrightarrow Y$ is an isomorphism of $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$. ∎

REMARKS. The category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ is related with the closed model categories given in §1 as follows:

(i) Consider the equivalences of categories given in [8]:

$$\mathrm{Ho}(\mathrm{SS}_n) \underset{R_n\,\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star^{[n}) \underset{Id}{\overset{(Id)^L}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-co}}$$

where $\mathrm{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-co}}$ denotes the full subcategory of $\mathrm{Ho}(\mathrm{Top}_\star)$ determined by the $(n-1)$-connected spaces.

These functors induce between the respective full subcategories determined by the $(m+1)$-coconnected objects the equivalences:

$$\mathrm{Ho}(\mathrm{SS}_n)|_{(m+1)\text{-coco}} \underset{R_n\,\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star^{[n})|_{(m+1)\text{-coco}} \underset{Id}{\overset{(Id)^L}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-co};(m+1)\text{-coco}}$$

On the other hand, consider the equivalences of categories given in [7]:

$$\mathrm{Ho}(\mathrm{SS}_\star)|_{(m+1)\text{-coco}} \underset{(\mathrm{cosk}_{m+1})^R}{\overset{\mathrm{sk}_{m+1}}{\rightleftarrows}} \mathrm{Ho}(\mathrm{SS}_\star^{m]}) \underset{\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star^{m]})$$

which induce in the respective subcategories determined by the $(n-1)$-connected objects the equivalences:

$$\mathrm{Ho}(\mathrm{SS}_\star)|_{(n-1)\text{-co};(m+1)\text{-coco}} \underset{(\mathrm{cosk}_{m+1})^R}{\overset{\mathrm{sk}_{m+1}}{\rightleftarrows}} \mathrm{Ho}(\mathrm{SS}_\star^{m]})|_{(n-1)\text{-co}} \underset{\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Top}_\star^{m]})|_{(n-1)\text{-co}}$$

Tacking into account that the equivalence between the localized categories

$$\mathrm{Ho}(\mathrm{Top}_\star) \underset{\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{SS}_\star)$$

induces an equivalence of categories:

$$\mathrm{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-co};(m+1)\text{-coco}} \underset{\mathrm{Sing}}{\overset{|\ |}{\rightleftarrows}} \mathrm{Ho}(\mathrm{SS}_\star)|_{(n-1)\text{-co};(m+1)\text{-coco}}$$

we have that the category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ is also equivalent to the categories $\mathrm{Ho}(\mathrm{Top}_\star)|_{(n-1)\text{-co};(m+1)\text{-coco}}$, $\mathrm{Ho}(\mathrm{SS}_\star)|_{(n-1)\text{-co};(m+1)\text{-coco}}$, $\mathrm{Ho}(\mathrm{SS}_\star^{m]})|_{(n-1)\text{-co}}$, $\mathrm{Ho}(\mathrm{Top}_\star^{m]})|_{(n-1)\text{-co}}$, $\mathrm{Ho}(\mathrm{Top}_\star^{[n})|_{(m+1)\text{-coco}}$.

(ii) Let $\pi\,\mathrm{CW}_{[n,m]}$ denote the category of pointed CW-complexes with dimension $\leqslant m+1$ whose $(n-1)$-skeleton consists just of one 0-cell and the morphisms are given by pointed cellular $m$-homotopy classes of pointed cellular maps. Then, the functors given in (i) induce an equivalence between the categories $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ and $\pi\,\mathrm{CW}_{[n,m]}$.

(iii) Let $\pi\,\mathrm{CW}_{[n,m]'}$ denote the category of pointed CW-complexes $(m+1)$-coconnected whose $(n-1)$-skeleton consists just of one 0-cell and the morphisms are given by pointed cellular homotopy classes of pointed cellular maps. Then, since $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]}) = \mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$, we can use the $[n,m]'$-structure to check that the categories $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ and $\pi\,\mathrm{CW}_{[n,m]'}$ are equivalent.

(iv) It is well known that $\mathrm{Ho}(\mathrm{Top}_\star^{[0]})$ is equivalent to the category of pointed sets, $\mathrm{Ho}(\mathrm{Top}_\star^{[1]})$ is equivalent to the category of groups and for $k \geqslant 2$, $\mathrm{Ho}(\mathrm{Top}_\star^{[k]})$ is equivalent to the category of abelian groups. For two consecutive non trivial homotopy groups, we have that $\mathrm{Ho}(\mathrm{Top}_\star^{[0,1]})$ is equivalent to a localization of pointed groupoids, $\mathrm{Ho}(\mathrm{Top}_\star^{[1,2]})$ is equivalent to a localization of cat-groups, $\mathrm{Ho}(\mathrm{Top}_\star^{[2,3]})$ is equivalent to a localization of braided cat-groups and for $k \geqslant 3$, $\mathrm{Ho}(\mathrm{Top}_\star^{[k,k+1]})$ is equivalent to a localization of symmetric cat-groups (see [6], [12] ).

(v) The $[n,m]$-structures and $[n,m]'$-structures developed for pointed spaces are connected with the closed model structures developed in [12] . In particular we have the usual equivalence of categories $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]})$ with categories of $[n-1,m-1]$-types of simplicial groups.

## 4. Integration of the singular cohomology.

Let $n,m$ integers such that $1 < n < m$. In this section, we shall prove that the localized category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$ is the category of elements of $P$, where $P$ is an adequate functor from the category $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})^{\mathrm{op}} \times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})$ to the category of sets.

Recall that if $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow Sets$ is a functor, where $\mathcal{C}^{\mathrm{op}}$ denotes the opposite category of a category $\mathcal{C}$, then the category of elements of $P$, denoted by $\int_{\mathcal{C}} P$, is defined as follows: Its objects are all pairs $(C,p)$ where $C$ is an object of $\mathcal{C}$ and $p \in P(C)$. Its morphisms $(C',p') \longrightarrow (C,p)$ are those morphisms $u\colon C' \longrightarrow C$ of $\mathcal{C}$ for which $P(u)p = p'$. These morphisms are composed by composing the underlying arrows $u$ of $\mathcal{C}$.

We consider the functor

$$P\colon \mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})^{\mathrm{op}} \times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'}) \longrightarrow Sets$$

defined by

$$P(A,B) = H^{m+1}(A', \pi_m B)$$

where $A'$ denotes the object $A^{[n,m-1]'}$ , which is the $[n,m-1]'$-cofibrant and $[n,m-1]'$-fibrant approximation of $A$ in the $[n,m-1]'$-structure (see the Remark after Proposition 2.7'.)

Now, if

$$\int_{\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'}) \times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})^{\mathrm{op}}} H^{m+1}((\ )', \pi_m(\ ))$$

is the category of elements of $P$, we have the following result:

4.1 THEOREM. *The category* $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$ *is equivalent to the category*

$$\int_{\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})\times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})^{\mathrm{op}}} H^{m+1}((\ )', \pi_m(\ ))$$

PROOF. We can check that the categories above are equivalent by using the functors

$$\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'}) \underset{am}{\overset{de}{\rightleftarrows}} \int_{\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})\times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})^{\mathrm{op}}} H^{m+1}((\ )', \pi_m(\ ))$$

defined as follows:

Let $(A,B;p)$ be an object of $\int_{\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})\times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})^{\mathrm{op}}} H^{m+1}((\ )', \pi_m(\ ))$. We can suppose that $A$ is an $[n,m-1]'$-fibrant $[n,m-1]'$-cofibrant object in $\mathrm{Top}_\star^{[n,m-1]'}$ and $B$ is an $[m]'$-fibrant $[m]'$-cofibrant object in $\mathrm{Top}_\star^{[m]'}$ . Then $A$ is an $(n-1)$connected and $m$-coconnected pointed topological space and $B$ is an $(m-1)$-connected and $(m+1)$-coconnected pointed topological space. In this case we have that $A' = A$ .

We note that if $p \in H^{m+1}(A', \pi_m B) = [A, K(\pi_m B, m+1)]$ , then we define $am(A,B;p)$ as the amplification of $A$ by $p$; in others words, $am(A,B;p)$ is the homotopy fibre of $p$, defined by the pull-back

$$\begin{tikzcd} am(A,B;p) \arrow[r] \arrow[d, "q"'] & P \arrow[d, "\alpha"] \\ A \arrow[r, "p"'] & K(\pi_m B, m+1) \end{tikzcd}$$

where $\alpha\colon P \longrightarrow K(\pi_m B, m+1)$ is a fibration of Serre and $P$ is weak equivalent to a point. We note that $q\colon am(A,B;p) \longrightarrow A$ is a fibration of Serre whose fibre is an Eilenberg–Mac Lane space $K(\pi_m B, m)$. Observe that $am(A,B;p)$ is an $(n-1)$-connected, $(m+1)$-coconnected pointed space which is isomorphic to $A$ in $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})$ and isomorphic to $B$ in $\mathrm{Ho}(\mathrm{Top}_\star^{[m]'})$.

Conversely, let $X$ be an object of $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$. We can suppose that $X$ is an $(n-1)$-connected and $(m+1)$-coconnected pointed topological space. Let $\mathcal{P} = \{X^q, f_q, k^{q+1}\}$ a fibred Postnikov system for $X$. We may assume that $X^q = X^0 = \star$ for all $q \leqslant n-1$ and $X^q = X^m$ for all $q > m$. Note that $X^m$ and $X$ are isomorphic in $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$.

Then, we define $de(X)$ as the object $(X^{m-1}, K; k^{m+1})$, where $K$ denotes the fibre of the map $f_m\colon X^m \longrightarrow X^{m-1}$ which is an Eilenberg–Mac Lane space $\mathrm{K}(\pi_m X, m)$ and $k^{m+1}$ is the $(m+1)$-invariant of Postnikov.

By the properties of the Postnikov invariants it is obvious that the functors $am$ and $de$ give an equivalence of categories.

■

REMARKS.

(i) If we apply consecutively the functor $de$

$$\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'}) \simeq \int_{\mathrm{Ho}(\mathrm{Top}_\star^{[n,m-1]'})\times \mathrm{Ho}(\mathrm{Top}_\star^{[m]'})^{\mathrm{op}}} H^{m+1}((\ )', \pi_m(\ )) \quad , \quad m > n.$$

we obtain the Postnikov decomposition of an object of $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$ . That is, for $X$ in $\mathrm{Ho}(\mathrm{Top}_\star^{[n,m]'})$ , one has

$$deX = (X^{m-1}, K(\pi_m X, m); k^{m+1})$$

$$deX^{m-1} = (X^{m-2}, K(\pi_{m-1}X, m-1); k^m)$$

$$\cdots$$

$$deX_{n+1} = (X^n, K(\pi_{n+1}X, n+1); k^{n+2})$$

(ii) Using the following equivalences of categories

$$\mathrm{Ho}(\mathrm{Top}_\star^{[k]'}) \simeq \mathrm{Ab}, \quad k \geqslant 2$$

$$\mathrm{Ho}(\mathrm{Top}_\star^{[2,3]'}) \simeq \mathrm{Ho}(\mathrm{Bcat}(\mathrm{Gr}))$$

$$\mathrm{Ho}(\mathrm{Top}_\star^{[3,4]'}) \simeq \mathrm{Ho}((\mathrm{Scat}(\mathrm{Gr}))$$

where Ab is the category of abelian groups, Bcat(Gr) the category of braided cat-groups, and Scat(Gr) is the category of symmetric cat-groups, we have the induced equivalences of categories

$$\mathrm{Ho}(\mathrm{Bcat}(\mathrm{Gr})) \simeq \int_{Ab\times Ab^{\mathrm{op}}} H^4((\ )', \pi_3(\ ))$$

$$\mathrm{Ho}((\mathrm{Scat}(\mathrm{Gr})) \simeq \int_{Ab\times Ab^{\mathrm{op}}} H^5((\ )', \pi_4(\ ))$$

(iii) For the case of $[n, n+2]$-types, we have

$$\mathrm{Ho}(\mathrm{Top}_\star^{[2,4]'}) \simeq \int_{\mathrm{Ho}(\mathrm{Bcat}(\mathrm{Gr}))\times Ab^{\mathrm{op}}} H^5((\ )', \pi_4(\ ))$$

$$\mathrm{Ho}(\mathrm{Top}_\star^{[3,5]'}) \simeq \int_{\mathrm{Ho}(\mathrm{Scat}(\mathrm{Gr}))\times Ab^{\mathrm{op}}} H^6((\ )', \pi_5(\ ))$$

$$\mathrm{Ho}(\mathrm{Top}_\star^{[4,6]'}) \simeq \int_{\mathrm{Ho}(\mathrm{Scat}(\mathrm{Gr}))\times Ab^{\mathrm{op}}} H^7((\ )', \pi_6(\ ))$$

We leave the reader the work of writing down the equivalences above with two integrals and we propose the study of the possible topological interpretation of Fubini Theorems of this theory of integration of functors on a product of categories.

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