Theory and Applications of Categories, Vol. 19, No. 3, 2007, pp. 41–49.

AXIOMATIC COHESION

F. WILLIAM LAWVERE

ABSTRACT. The nature of the spatial background for classical analysis and for modern theories of continuum physics requires more than the partial invariants of locales and cohomology rings for its description. As Maxwell emphasized, this description has various levels of precision depending on the needs of investigation. These levels correspond to different categories of space, all of which have intuitively the feature of cohesion. Our aim here is to continue the axiomatic study of such categories, which involves the following aspects:

I. Categories of space as cohesive backgrounds
II. Cohesion versus non-cohesion; quality types
III. Extensive quality; intensive quality in its rarefied and condensed aspects; the canonical qualities form and substance
IV. Non-cohesion within cohesion via constancy on infinitesimals
V. The example of reflexive graphs and their atomic numbers
VI. Sufficient cohesion and the Grothendieck condition
VII. Weak generation of a subtopos by a quotient topos

I look forward to further work on each of these aspects, as well as development of categories of dynamical laws, constitutive relations, and other mathematical structures that naturally live in cohesive categories.

I. Categories of space as cohesive backgrounds for mathematical structures

An explicit science of cohesion is needed to account for the varied background models for dynamical mathematical theories. Such a science needs to be sufficiently expressive to explain how these backgrounds are so different from other mathematical categories, and also different from one another and yet so united that they can be mutually transformed. An everyday example of such mutual transformation is the weatherman's application of the finite element method (which can be viewed as analysis in a combinatorial topos) to equations of continuum thermomechanics (which can be viewed as analysis in a smooth topos, where smooth functions and distributions live).

II. Cohesion versus non-cohesion; quality types

I analyze cohesion by contrasting it with non-cohesion. In that I follow Cantor, who approached his Mengen by negating them into Kardinalen; the latter are (not isomorphism

Received by the editors 2007-01-16 and, in revised form, 2007-05-31.
Transmitted by P. T. Johnstone. Published on 2007-06-05.
2000 Mathematics Subject Classification: 18A40, 18B25, 18B30, 74A60, 74A99.
Key words and phrases: Cohesion, qualities, graphs, nature of space.

classes but) spaces so devoid of internal cohesion and variation that they satisfy his general "continuum" hypothesis. Not only have those very abstract sets served as a background for algebraic structures, but also as a background for models of cohesion itself. Thus by negation of the negation, the initial nebulous notion of Mengen becomes a 2-category of mutually transforming extensive categories; reasonably closed such categories have map-spaces and thus are distributive for two reasons, and indeed many but not all examples are toposes. Before trying to make that 2-category precise, we must make explicit another negation of the negation that was emphasized by Grothendieck: the contrast of cohesion E with non-cohesion S can be expressed by geometric morphisms

$$p : E \longrightarrow S$$

but that contrast can be made relative, so that S itself may be an "arbitrary" topos. More exactly, S can be one that is appropriate for negating E in this spirit. For example, in a case E of algebraic geometry wherein spaces of all dimensions exist, S is usefully taken as a corresponding category of zero-dimensional spaces such as the Galois topos (of Barr-atomic sheaves on finite extensions of the ground field). As the example illustrates, the Grothendieck relativization within the realm of toposes means that the Cantorian negation can be applied, not only to mathematics as a whole, but (even better) to specific branches.

A topos morphism

$$p : E \longrightarrow S$$

can express a contrast between cohesion and non-cohesion (as made more precise below). Such a morphism can also express a contrast between variation and non-variation (where E is a "generalized space" parameterizing families of S-objects). The generalized space conception can be a useful guide even for all toposes: Top/S is an extensive category that contains an object-classifier R so that sheaves are encoded as R-valued functions; these functions can be integrated with respect to distributions [1]; the 2-category structure yields a notion of weak equivalence; and so on. However, as I have argued in [4] and [5], our elephant carries instruments that can also

(1) clarify the distinction between cohesion and variation by contrasting relevant positive properties of each;

(2) show how cohesion can serve as a background for motion and variation via dynamical laws and variable quantities. In particular a double negation of the classical notion of sheaf should give, to each space X in a topos $E \longrightarrow S$ of cohesive spaces, an assignment of its topos of "variable sets" $E(X) \longrightarrow S$ ("smaller" than its "gros" topos E/X); for example, from a "gros étalé" E there emerge the "petit etale" $E(X)$ that are neither localic nor groupoidal, yet quite special as toposes.

In the present work I further study positive properties of cohesion, and in particular, the class of categories having sufficient cohesion, as well as the contrasting class whose very special cohesion deserves the name of "quality". These two classes still need to be fully related with the class of categories of pure variation; all known examples of pure variation have the positive property that there are enough objects A with no idempotent endomaps, i.e., two maps $X \longrightarrow Y$ are equal if they are equal on all figures $A \longrightarrow X$ of those special shapes A.

DEFINITION 1. *A functor $q^* : S \longrightarrow F$ (between extensive categories) which is full and faithful and which is both reflective and coreflective by a single functor*

$$q_! = q_*$$

makes F a quality type *over S.*

PROPOSITION 1. *A quality type has a minimal central idempotent (a central idempotent is any natural endomorphism θ of the identity functor of F, defined over S, such that $\theta\theta = \theta$). The subcategory consists of those objects Y of F for which θ_Y is 1_Y, whereas the adjunction maps for $q_! = q_*$ split the idempotents θ_Y for all Y in F.*

PROPOSITION 2. *If $E \longrightarrow S$ over sets is either*
localic
groupoidal
petit étale
an étendue in Grothendieck's sense
locally separable (qd in Johnstone's sense),

then it is not a quality type unless $E \longrightarrow S$ is an equivalence.

PROOF. For any such E there is a subcategory that is separating and has no non-trivial idempotents at all. If θ is a central idempotent, then for every map $x : A \longrightarrow X$ with A in such a subcategory

$$\theta_X x = x\theta_A = x = 1_X x$$

hence $\theta_X = 1_X$. ∎

DEFINITION 2. *A cartesian-closed extensive category E is a* category of cohesion *relative to another such category S if it is equipped with an adjoint string of four functors*

$$p_! \dashv p^* \dashv p_* \dashv p^!$$

having the further properties:
(a) $p_!$ preserves finite products and $p^!$ is full and faithful. (Thus for toposes we would say that p is "connected surjective" and "local" (see Johnstone and Moerdijk [3]), $p^!$ is a subtopos, and p^* is an exponential ideal.)
(b) $p_!$ preserves S-parameterized powers in the sense that

$$p_!(X^{p^* W}) = p_!(X)^W$$

is a natural isomorphism for all X in E and W in S. (This "continuity" property (b) follows from (a) if all hom sets in S are finite; it also holds if the contrast with S is determined as in IV below by an infinitely divisible interval in E.)

(c) The canonical map $p_* \longrightarrow p_!$ in S is epimorphic (I refer to this property as the "Nullstellensatz"; it holds iff the other canonical map $p^* \longrightarrow p^!$ in E is monomorphic).

The two downward functors express the opposition between "points" and "pieces". The two upward ones oppose pure cohesion ("codiscrete") and pure anti-cohesion ("discrete"); these two are identical in themselves with S but united by the points concept p^* that uniquely places them as full subcategories in E. Steve Schanuel has pointed out that the Nullstellensatz by itself implies that the comparison map in (a), mapping pieces of a product to pairs of pieces, is at least epimorphic. The case $W = 2$ of (b) implies (a).

A cartesian-closed quality type is a category of cohesion in one extreme sense because if (c) is an isomorphism, then (a) and (b) follow. An opposite extreme is "sufficient cohesion", as discussed in VI below.

Recall that in the classical, essentially localic, account of cohesion there is no left adjoint "pieces" functor whose values have the same degree of non-cohesion as the pure points. Many determinations of cohesion are needed for various mathematical situations. I speak informally of connectedness, etc., referring to the behavior of objects with respect to $p_!$.

III. Extensive quality; intensive quality in its rarefied and condensed aspects; the canonical qualities form and substance

Definition 3. *An* extensive quality *on a category $p : E \longrightarrow S$ of cohesion is a functor h such that*

h preserves finite coproducts
the codomain of h is a quality type $q : F \longrightarrow S$
$q_! h = p_!$

i.e., an extensive quality of X has the same number of connected pieces as X.

Theorem 1. (Hurewicz) *Any category of cohesion has a canonical extensive quality h defined by $F(X, Y) = p_!(Y X)$ with $h =$ identity on objects. Moreover, h preserves finite products and exponentiation.*

Proof. Using cartesian closure and clause (a), a category F is constructed; it is actually extensive and itself cartesian closed. Taking q^* to be hp^*, the property $q_* = q_!$ follows from the "continuity" clause (b) and the fact (a) that p^* is an exponential ideal, because both are essentially the same $p_!$. ■

Gabriel and Zisman's work [2] showed that even without the continuity clause (indeed for the example of the topos E of simplicial sets), one could obtain an extensive quality by forming a category of fractions of the Hurewicz F.

The canonical extensive quality could be called "form" (it seems to neglect substance). By contrast, the canonical intensive quality defined below is called "substance" and seems to neglect form. (This contrast is related to the contrast between "in the large" and "in the small" in traditional analysis. The Poincaré conjecture expresses the idea that the two canonical qualities could jointly reflect isomorphisms.) Homology is another extensive quality that depends on form and partially measures it; it often even preserves products if valued in commutative coalgebras.

DEFINITION 4. *An* intensive quality *on a category E of cohesion is a functor s_* from E that*

preserves finite products and finite coproducts

has a quality type $q : L \longrightarrow S$ as codomain, and satisfies

$q_* s_* = p_*$

i.e., an intensive quality of X has the same number of points as X.

THEOREM 2. *Any category of cohesion satisfying reasonable completeness conditions has a canonical intensive quality s whose codomain is the subcategory $s^* : L \longrightarrow E$ consisting of those X for which the map $p_* X \longrightarrow p_! X$ is an isomorphism. Moreover, s^* has a left adjoint $s_!$ and a coproduct-preserving right adjoint s_*.*

PROOF. Since $p^!$ exists, p_* preserves colimits, so the subcategory (where "there is just one point in any piece") is closed under colimits; therefore by completeness the coreflection s_* exists. Because $p_!$ preserves finite products (as does p_*) s^* is also closed under finite products. Thus 1 belongs to the subcategory and therefore by extensivity s_* preserves coproducts. From the Nullstellensatz it follows that this subcategory is closed under arbitrary subobjects and arbitrary products and is hence by completeness epi-reflective; that is, the reflection $s_!$ exists and $X \longrightarrow s^* s_! X$ is epimorphic. In fact, the reflection can be constructed as the pushout of the adjunction map along the basic epimorphism $p_* \longrightarrow p_!$. ∎

The canonical intensive quality defined on a topos of cohesion is valued again in a topos L because these values are coalgebras for a lex comonad. As the construction shows in examples, $s_!$ (unlike its underlying $p_!$) does not preserve products, but it is another example of extensive quality. I further distinguish the two aspects s_* and $s_!$ of the substance as *rarefied* vs. *condensed*. Substance s can be considered as a "function" on E with values in a topos L, but the condensed substance $s_!$ can be viewed as a family of distributions on E parameterized by L.

A helpful metaphor views the two aspects as the result of partial observation of the same space (or sample of material), under extreme conditions of hot vs. cold; the canonical "cooling" map $s_* X \longrightarrow s_! X$ gives further partial information about the specific nature of the substance of X. The rarefied substance of X is more precisely the adjunction map $s^* s_* X \longrightarrow X$ and the condensed substance of X is the other adjunction $X \longrightarrow s^* s_! X$, and the cooling map is the composite of those. (Recall that s^* is full and faithful.)

IV. Non-cohesion within cohesion via constancy on infinitesimals

Most of the examples of Cantor-Galois-Grothendieck abstraction $p : E \longrightarrow S$ actually arise in the following concrete way. A map $i : T_0 \longrightarrow T$ in E gives rise to the subcategory $p^* : S \longrightarrow E$ consisting of all those objects Y for which Y^i is an isomorphism (for example $Y = Y^T$ for a given pointed object T). If i (or T) has suitable properties, then not only do the further adjoints exist, but clauses (b) and (c) hold, as well as the obvious (a). Then some of the A for which $Y = Y^A$ for all Y in S may form a subcanonical site of definition for E over S. The Galois connection arising from the constancy relation "$Y = Y^X$" expresses lack of internal cohesion and variation for Y relative to X. In the examples studied in synthetic differential geometry (including the smooth, analytic, and algebraic cases) it seems clear that whereas cohesion makes (connected) variation possible, the objectification of motion (as "amazingly tiny" spaces) "generates" the model E for cohesion over the resulting background S. Note that if T has just one point and also just one piece, then of course T lies in the subcategory L. Objects T of nilpotent quantities, and those satisfying the ATOM property that $(\ )^T$ has a right adjoint, are fundamental, but even without those properties the spaces in L may be considered to have a weak infinitesimal nature and surprisingly often even generate in an "infinitesimal" sense the whole topos E.

V. The example of reflexive graphs and their atomic numbers

For example, let M be the four-element monoid of endomaps of a two-element set and let E be the topos of right actions of M on finite sets S. Because M has suitable idempotents, there are four functors p that together have the properties required of a category of cohesion. The spaces X are reversible reflexive graphs, whereas the corresponding canonical intensive quality is valued in the subcategory L of graphs consisting entirely of loops. (That is, the objects in L are the actions of an involution and a central idempotent.) When X is in a rarefied condition, the interactions between its points can be neglected, but after replacing X by its subobject of substance $s^*s_*X \longrightarrow X$, the self-interactions at each point remain; these are measured by counting the n loops, n_0 of which are their own reverse. The non-trivial self-interactions could be considered as virtual particles, of which $n - n_0$ are paired off and $n_0 - 1$ are neutral; these "atomic numbers" illustrate the kind of qualitative information retained in passing from an object to its mere rarefied substance. By contrast, the condensed substance of an object X consists of giant "atoms" whose new self-interactions involve all the mutual interactions within each piece of X; in particular, within a connected X all trivial loops are collapsed to one trivial loop upon cooling. The cooling map from rarefied to condensed also indicates which elements of its codomain were already self-interactions before cooling.

Another intensive quality, that depends on the canonical one of substance and partially measures it, is obtained by "superheating" until only the $n_0 - 1$ neutral virtual particles remain, namely by composing s with the geometric morphism from L to the topos of actions of a single idempotent. There is an adjoint "supercooling".

VI. Sufficient cohesion and the Grothendieck condition

The above locally finite example illustrates two further important features. These features of very general notions of cohesive spatiality are also present in the smooth toposes of synthetic differential geometry.

Definition 5. *A category of cohesion (i.e. a functor satisfying conditions (a), (b) and (c)) is* sufficiently cohesive *if*
(d) for every X there exists a monic map $X \longrightarrow Y$ with Y contractible *in the sense that $p_!(Y^A) = 1$ for all A (i.e. with Y terminal in the Hurewicz category).*

Proposition 3. *If E over S is both sufficiently cohesive and a quality type, then S is inconsistent.*

Proof. Let $X \longrightarrow Y$ be monic and let Y be connected. The natural map from points to pieces is an isomorphism for both X and Y, hence $p_!(X)$ is a subobject of 1. For example, taking $X = p^!(B)$, we have that $B = p_*X$ is a subobject of 1 for any object B of the base category S. ∎

Proposition 4. *A topos of cohesion is sufficiently cohesive iff the truth-value object is connected, and also iff all injective objects are connected.*

Proof. In a topos, if an injective object is embedded in a connected object, then it is a retract of that object and hence connected itself. Conversely, assume all injective objects are connected. Any object can be embedded in an injective, for example its partial-map representor. The partial map representor Y is not only connected, but actually contractible, because it has a pointed action of the connected monoid with zero formed by the truth-value object under conjunction; hence Y^A also has such an action, which implies that Y^A is also connected. ∎

Corollary. *If 2 is injective in the base S, then a cohesive $p : E \longrightarrow S$ is sufficiently cohesive iff $p^!(2)$ is connected.*

Proof. Note that any connected bi-pointed object can be used to define homotopies and that two maps homotopic in that way will induce the same map on the object of pieces. If a connected bi-pointed object moreover satisfies Grothendieck's condition that the two points have empty equalizer, then the characteristic function of one will map the other to false, permitting the construction of a homotopy between the identity map of the truth-value object and the constantly false endomap, by using conjunction. Thus the truth-value object is contractible if there exists such a strictly bi-pointed connected object, and so E will be sufficiently cohesive over S. If $p^!(2)$ is connected, it is such an object. Conversely, if 2 is injective in S, then $p^!(2)$ is injective in E, so sufficient cohesion implies that $p^!(2)$ is connected. ∎

Note that the injectivity of 2 in S implies that the truth-value object of S is decomposable, a kind of non-cohesion. A sufficiently cohesive $E \longrightarrow S$ is never an equivalence.

Proposition 5. *The topos of reversible reflexive graphs is sufficiently cohesive.*

Proof. The interval graph is connected and has two distinct points, so Grothendieck's condition applies. A simple picture shows that a graph can be augmented to make it connected; the above argument implies that we can even make it contractible. ∎

Many examples suggest that a Grothendieck topos should be sufficiently cohesive (i.e. satisfy all of (a) (b) (c) (d) over sets) if any subcanonical site needs to have several idempotents. (Note that non-subcanonical sites without idempotents can be found for any topos as McLarty [9] points out.) In Grothendieck's condition, two points $1 \longrightarrow I$ with empty equalizer (yet with connected codomain I) compose with $I \longrightarrow 1$ to yield two distinct idempotents. By contrast, a topos of pure variation has a subcanonical site with no idempotents at all (as in proposition 2), whereas the fundamental quality type consists of the actions of just one idempotent. The distinction between the three classes (sufficient cohesion, quality type, pure variation) may be determined by the structure of the idempotents in subcanonical sites; the results of [4] suggest that sufficient cohesion and pure variation very rarely hold for the same p.

VII. Weak generation of a subtopos by a quotient topos

Euler observed that real magnitudes are ratios between infinitesimals, and I have argued in [6] that his observation is conversely a useful definition of the one-dimensional continuum as a retract of T^T where T is an infinitesimal continuum in a cartesian-closed category. Because sheaf subtoposes of a topos are always closed under exponentials, we are led to broaden the usual covering-based notion of "generating" to obtain a notion of "weakly generated" topos.

Definition 6. *Given a connected morphism $s : E \longrightarrow L$ of toposes, let j in E be the strongest localness operator for which every s^*Y (for Y in L) is a j-sheaf. If j is actually the identity map on the truth-value space, then E is* weakly generated *by s. (A sufficient condition for weak generation is that exponentials of values of s^* are adequate in Isbell's sense [7].)*

Proposition 6. *The cohesive topos of reversible reflexive graphs is infinitesimally generated, that is, weakly generated by its substance.*

Proof. L is the topos of actions on sets of a three-element commutative monoid; if A is the one-vertex graph obtained as s^* of the usual generator of L, then the exponential A^A contains as a retract the interval graph I. E has no subtoposes that contain I, so E is weakly generated by L. ∎

References

[1] Bunge, Marta, and Funk, Jonathon, *Singular Coverings of Toposes,* Springer Lecture Notes in Mathematics 1890, Springer Verlag (2006).

[2] Gabriel, Peter and Zisman, Michel, *Calculus of Fractions and Homotopy Theory,* Ergebnisse der Mathematik, Band 35, Springer Verlag 1967.

[3] Johnstone, Peter T., and Moerdijk, Ieke, Local maps of toposes, Proceedings London Mathematical Society **58** (1989), 281-305.

[4] Lawvere, F. William, Categories of spaces may not be generalized spaces as exemplified by directed graphs. Revista Colombiana de Matemáticas, **XX** (1986) 179-186. Also Reprints in Theory and Applications of Categories, No. 9 (2005), 1-7.

[5] Lawvere, F. William, Some thoughts on the future of category theory, Proceedings of Como 1990, Springer Lecture Notes in Mathematics 1488 (1991), 1-13, Ed. A. Carboni, M.-C. Pedicchio, G. Rosolini.

[6] Lawvere, F. William, Euler's continuum functorially vindicated, dedicated to John Bell for his 60th birthday (2006), preprint. `http://www.buffalo.edu/~wlawvere`

[7] Lawvere, F. William, John Isbell's adequate subcategories, Topological Commentary, vol. 11, (2006). `http://at.yorku.ca/t/o/p/d/65.htm`

[8] Lawvere, F. William, Are homotopy types the same as infinitesimal skeleta? Special Session on Category Theory organized by Richard Wood, `http://www.camel.math.ca/CMS/Events/summer 98/s98-abs/`

[9] McLarty, C., Every Grothendieck topos has a one-way site, *Theory and Applications of Categories*, **16**,123-126

SUNY at Buffalo
244 Mathematics Building
Buffalo, N.Y. 14260
Email: `wlawvere@buffalo.edu`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/19/3/19-03.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at `http://www.tac.mta.ca/tac/` and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at `http://www.tac.mta.ca/tac/`.