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CHANGE OF BASE FOR RELATIONAL VARIABLE SETS

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ABSTRACT. Following [5], a relational variable set on a category B is a lax functor $B \to \mathrm{Rel}$, where Rel is the category of sets and relations. Change-of-base functors and their adjoints are considered for certain categories of relational variable sets and applied to construct the simplification of a dynamic set (in the sense of [11]).

1. Introduction

A *relational variable set* on a category B is a lax functor $B \to \mathrm{Rel}$, where Rel is the locally partially-ordered 2-category of sets and relations. Also called relational presheaves [8], specification structures [1], and dynamic sets [11], these lax functors have played a role in the study of modal and basic predicate logic [4, 5], concurrency [1], automata [8], and spatio-temporal databases [11].

For a category Rel^B of relational variable sets and a functor $p: E \to B$, we consider adjoints to the change-of-base functor $p^*: \mathrm{Rel}^B \to \mathrm{Rel}^E$. We will see that p^* always has a left adjoint Σ_p, while the existence of a right adjoint Π_p is related to the exponentiability of p in the category Cat of small categories and functors.

In "Powerful functors" [12], Street describes exponentiability in Cat via an equivalence (due to Bénabou) between the slice 2-category Cat/B and a 2-category of normal lax functors $m: B^{op} \to \mathrm{Mod}$, where Mod denotes the bicategory whose objects are small categories and the hom category $\mathrm{Mod}(A, B)$ is the functor category $\mathrm{Set}^{B^{op}\times A}$, and a normal lax functor strictly preserves identities. In particular, $p: E \to B$ is exponentiable if and only if the corresponding $m_E: B^{op} \to \mathrm{Mod}$ is a pseudofunctor, and the latter readily translates into the Giraud-Conduché [6, 3] factorization lifting condition for exponentiability in Cat.

A modification of Bénabou's result yields an equivalence between Rel^B and the category Cat_f/B of faithful functors over B. It turns out that a faithful functor $p: E \to B$ is exponentiable in Cat_f/B if and only if the corresponding relational variable set $B \to \mathrm{Rel}$ is a non-unitary (i.e., not necessarily identity preserving) functor, or equivalently a certain weak factorization lifting property (WFLP) holds. Thus, when $p: E \to B$ is faithful, using the general relationship between exponentiability of p and the existence of Π_p, it follows that $\Pi_p: \mathrm{Rel}^E \to \mathrm{Rel}^B$ exists if and only if p satisfies WFLP, and this generalizes to the case when p is not assumed to be faithful.

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We begin with the introduction of the category Rel^B of relational variable sets and morphisms, and its equivalence to Cat_f/B. In §4, we consider adjoints to the change-of-base functor $p^*\colon \mathrm{Rel}^B \to \mathrm{Rel}^E$, and their relationship to the exponentiability of p. We conclude, in §5, with an application to the construction of the simplification of a dynamic set with respect to a change in time domain (in the sense of [11]).

2. Relational Variable Sets

Let Rel denote the locally partially-ordered 2-category of sets and relations, i.e., $\mathrm{Rel}(X, Y)$ is the poset of relations $R \subseteq X \times Y$, with the identity morphism on X given by the diagonal $\Delta \subseteq X \times X$ and composition by the usual relation composites. To distinguish relations from functions, elements of $\mathrm{Rel}(X, Y)$ will be denoted by $R\colon X \mathrel{-\!\!\!+\!\!\!\rightarrow} Y$. The composite of $R\colon X \mathrel{-\!\!\!+\!\!\!\rightarrow} Y$ and $S\colon Y \mathrel{-\!\!\!+\!\!\!\rightarrow} Z$ will be written $S \circ R$, and abbreviated as SR.

A *relational variable set* or Rel-*set* on a category B consists of a set X_b, for every object b of B, and a relation $X_\beta\colon X_b \mathrel{-\!\!\!+\!\!\!\rightarrow} X_{b'}$, for every morphism $\beta\colon b \to b'$ of B, satisfying

(RS1) $\Delta_{X_b} \subseteq X_{\mathrm{id}_b}$

(RS2) $X_{\beta'} X_\beta \subseteq X_{\beta'\beta}$

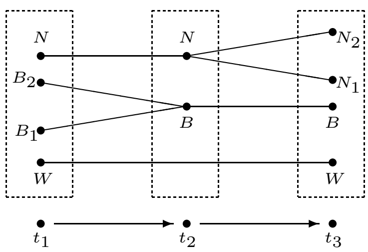
for all objects b and for all morphisms $\beta\colon b \to b'$ and $\beta'\colon b' \to b''$ of B. Writing $x \to_\beta x'$ for the infix form of $(x, x') \in X_\beta$, and $x \to_b x'$ when β is the identity morphism on b, these conditions become

(RS1*) $x \to_b x$, for all $x \in X_b$

(RS2*) $x \to_\beta x', x' \to_{\beta'} x'' \Rightarrow x \to_{\beta'\beta} x''$

Note that a Rel-set is just a *lax functor* or *morphism of bicategories*, in the sense of [7] or [2], respectively. Of course, X is a functor if and only if the containments in (RS1) and (RS2) are equalities.

An example (in the spirit of [11]) of a Rel-set as a model of a data base changing over time is given by



Here, the objects of the category $t_1 \to t_2 \to t_3$ are time values (perhaps, certain years), the elements of X_t represent daily flights to three airports (say, Newark, Boston, and Washington) with subscripts used to indicate multiple flights.

Among the examples from *Logics for Concurrency* [1] is the Rel-set on $B = \mathrm{Rel}$ with $X_b = \mathcal{P}(b) = \mathrm{Rel}(1, b)$, and $S \to_R T$ whenever $RS \subseteq T$. Note that this example can be generalized to any bicategory B, where 1 is replaced by any fixed object of B.

Also, Rel-sets on the power set $\mathcal{P}(M)$ of a monoid M arise in a categorical approach to automata theory. For details of this and other applications, see [8].

A *morphism* $f: X \to Y$ of Rel-sets on B is an op-lax natural transformation. Thus, f consists of a function $f_b: X_b \to Y_b$, for every object b, such that for every morphism $\beta: b \to b'$ there is a diagram

$$\begin{tikzcd} X_b \arrow[r, "f_b"] \arrow[d, "X_\beta"', "|" marking] \arrow[dr, phantom, "\subseteq"] & Y_b \arrow[d, "Y_\beta", "|" marking] \\ X_{b'} \arrow[r, "f_{b'}"'] & Y_{b'} \end{tikzcd}$$

in Rel, i.e., $x \to_\beta x' \Rightarrow f_b x \to_\beta f_{b'} x'$.

Note that this is the notion of morphism given by Ghilardi and Meloni [4, 5] and Rosenthal [8]. A more general definition of morphism, in which the functions f_b are replaced by relations, is also given in [8]. Abramsky, Gay, and Nagarajan do not consider morphisms of specification structures in [1].

When B is a small category, Rel-sets and their morphisms form a locally small locally preordered 2-category Rel^B with $f \to g$, if $f_b x \to_b g_b x$, for all $b \in B$ and $x \in X_b$, and consequently, $x \to_\beta x' \Rightarrow f_b x \to_\beta g_{b'} x'$, for all $\beta: b \to b'$, $x \in X_b$, and $x' \in X_{b'}$. The symbol "$\to$" is used here for a preorder to distinguish it from a partial order "$\leq$" since both arise on the same set in §5.

3. Rel-Sets and Faithful Functors

In this section, we assume B is a small category and consider a 2-adjunction

$$\mathrm{Rel}^B \underset{\Gamma}{\overset{L}{\leftrightarrows}} \mathrm{Cat}/B$$

which gives rise to an equivalence between Rel^B and the category Cat_f/B of faithful functors over B. In particular, Γ is given by the lax fibration for the Grothendieck construction on a Rel-set.

Recall that Cat/B is the 2-slice category whose objects are functors $p: E \to B$, morphisms are commutative triangles

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

in Cat, and 2-cells are natural transformations $\theta\colon f \to g$ such that $q\theta = \mathrm{id}_p$.

For $p\colon E \to B$ and an object b of B, the *fiber* E_b *of* E *over* b is the subcategory of E consisting of objects over b and morphisms over id_b. Let Lp denote the Rel-set with $(Lp)_b = E_b$ and $e \to_\beta e'$ if there exists $\xi\colon e \to e'$ such that $p\xi = \beta$. As is customary for slice categories, we will often suppress the explicit reference to p and write LE instead of Lp. If $f\colon E \to F$ is a functor over B, then $f_b\colon E_b \to F_b$ defines a morphism $f\colon LE \to LF$, since $e \to_\beta e'$ when there exists $\xi\colon e \to e'$ over $\beta\colon b \to b'$, and hence $f\xi\colon fe \to fe'$ over β, showing that $f_b e \to_\beta f_{b'} e'$. Moreover, $f \to g$ for every 2-cell $\theta\colon f \to g$, since the morphism $\theta_e\colon fe \to ge$ satisfy $q\theta_e = \mathrm{id}_{pe} = \mathrm{id}_b$, and so $f_b e \to_b g_b e$, for all e in $(LE)_b$. Thus, $L\colon \mathrm{Cat}/B \to \mathrm{Rel}^B$ is a 2-functor.

Note that the lax functor $LE\colon B \to \mathrm{Rel}$ need not be a functor. In fact, it is unitary (i.e., identity preserving) precisely when $p\colon E \to B$ has discrete fibers, and it preserves composition when p satisfies the *weak factorization lifting property (WFLP)*

$$\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} e \arrow[rr, "\xi''"] \arrow[dr, dotted, "\xi"'] & & e'' \\ & e' \arrow[ur, dotted, "\xi'"'] & \end{tikzcd}$$
$$\begin{tikzcd} pe \arrow[rr, "p\xi''"] \arrow[dr, "\beta"'] & & pe'' \\ & b' \arrow[ur, "\beta'"'] & \end{tikzcd}$$

i.e., for every morphism ξ'' of E and every factorization $p\xi'' = \beta'\beta$ in B, there exists a factorization $\xi'' = \xi'\xi$ in E such that $p\xi = \beta$ and $p\xi' = \beta'$.

This condition is a weakening of the usual Giraud-Conduché [6, 3] factorization lifting property characterizing exponentiable objects of Cat/B. In particular, it does not require the usual "zigzag" relating any two liftings of the same factorization. However, restricting to posets, $p\colon E \to B$ satisfies WFLP if and only if it is exponentiable in the category *Pos* of posets and order-preserving maps [10]. The fact that WFLP arises (instead of FLP) should be clear from the discussion of exponentiability in the next section.

To define $\Gamma\colon \mathrm{Rel}^B \to \mathrm{Cat}/B$, let X be a Rel-set on B, and consider the category E_X whose objects are pairs (x, b) where b is an object of B and $x \in X_b$, and morphisms $\beta\colon (x, b) \to (x', b')$ are morphisms $\beta\colon b \to b'$ of B such that $x \to_\beta x'$ in X. Then E_X is a category over B via the projection $\Gamma X\colon E_X \to B$ which is, in fact, a faithful functor. A morphism $f\colon X \to Y$ of Rel-sets gives rise to a functor $\Gamma f\colon E_X \to E_Y$ over B defined by $\Gamma f(x, b) = (f_b x, b)$ and $\Gamma f(\beta) = \beta$. Note that every functor $g\colon E_X \to E_Y$ over B is of this form, for given such a g, define $f_b\colon X_b \to Y_b$ by $f_b(x) = \pi_1 g(x, b)$ and $f(\beta) = \beta$. Then f is a morphism of Rel-sets, since

$$x \to_\beta x' \Rightarrow \beta\colon (x, b) \to (x', b') \Rightarrow g\beta\colon (f_b x, b) \to (f_{b'} x', b') \Rightarrow f_b x \to_\beta f_{b'} x'$$

and $\Gamma f = g$. A 2-cell $f \to g$ induces a natural transformation $\theta\colon \Gamma f \to \Gamma g$ over B given by $\mathrm{id}_b\colon (f_b x, b) \to (g_b x, b)$ since $f_b x \to_b g_b x$, for all b. Thus, $\Gamma\colon \mathrm{Rel}^B \to \mathrm{Cat}/B$ is a 2-functor whose image is the full subcategory of Cat_f/B of faithful functors over B.

To see that L is left adjoint to Γ, define $\varepsilon\colon L\Gamma \to \mathrm{id}$ and $\eta\colon \mathrm{id} \to \Gamma L$, as follows. Given a Rel-set X, note that

$$(L\Gamma X)_b = (E_X)_b = \{(x,b) | x \in X_b\}$$

Then the projections $(\varepsilon_X)_b\colon (L\Gamma X)_b \to X_b$ define a morphism $\varepsilon_X\colon L\Gamma X \to X$, which is clearly a 2-natural isomorphism. Likewise, given $p\colon E \to B$ in Cat/B, ΓLE is the category whose objects are pairs (e,b) where $e \in LE_b$, i.e., $pe = b$, and morphisms $\beta\colon (e,b) \to (e',b')$ are morphisms $\beta\colon b \to b'$ such that $e \to_\beta e'$ in LE, i.e., there exists $\xi\colon e \to e'$ in E with $p\xi = \beta$. Then $\eta_E\colon E \to \Gamma LE$, given by $\eta_E(e) = (e, pe)$ and $\eta_E(\xi) = p\xi$, gives rise to a 2-natural transformation $\eta\colon p \to Lp$, such that η_E is full and surjective on objects, in any case, and faithful if and only if p is faithful. Moreover, the adjunction identities easily follow. Therefore, the following has been established:

3.1. Theorem. *There is a 2-adjunction*

$$\mathrm{Rel}^B \underset{\Gamma}{\overset{L}{\leftrightarrows}} \mathrm{Cat}/B$$

which induces an equivalence between the locally preordered 2-category Rel^B *and the slice 2-category* Cat_f/B *of faithful functors over* B. *As a result,* Cat_f/B *is a reflective subcategory of* Cat/B *with reflection* ΓL.

Note that the equivalence $\mathrm{Rel}^B \simeq \mathrm{Cat}_f/B$ is essentially that of Bénabou mentioned in the introduction and described by Street in [12]. But, the "*op*" does not appear here since the morphisms of Rel are opposite those of Mod. Also, normal (i.e., strictly identity-preserving) lax functors $B^{op} \to \mathrm{Mod}$ are considered there in order to obtain *all* of Cat/B. Normality does not arise here since the identity morphisms of Rel are the diagonals. In fact, normal lax functors $B \to \mathrm{Rel}$ correspond to faithful functors $E \to B$ whose fiber E_b are discrete.

Since Cat_f/B is a reflective subcategory of Cat/B, it is closed under limits. Using the equivalence with Rel^B, we get:

3.2. Corollary. *Limits exist and are computed point-wise in* Rel^B.

In the case where B a preordered set (in particular, a poset), Cat_f/B is the category Pr/B of preordered sets over B. Thus, Theorem 3.1 gives rise to the following equivalence which was used by Ghilardi and Meloni [4] in their relational semantics.

3.3. Corollary. *If B is a preordered set, then* $\mathrm{Rel}^B \simeq \mathrm{Pr}/B$.

4. Change of Base and Exponentiability

A functor $p\colon E \to B$ induces a 2-functor $p^*\colon \mathrm{Rel}^B \to \mathrm{Rel}^E$, given by $(p^*X)_e = X_{pe}$ with $x \to_\xi x'$ whenever $x \to_{p\xi} x'$ in X, and $(p^*f)_e = f_{pe}$, since $f \to f'$ implies $p^*f \to p^*f'$. A straightforward calculation shows that p^* has a left adjoint Σ_p given by $(\Sigma_p Y)_b = \coprod_{pe=b} Y_e$ with $y \to_\beta y'$ whenever $y \to_\xi y'$ for some ξ such that $p\xi = \beta$, and $(\Sigma_p g)_b = \coprod_{pe=b} g_e$.

In this section, we will show that p^* has a right adjoint if and only if p satisfies the weak factorization lifting property (WFLP) introduced in the previous section, and then establish a connection to exponentiability in Cat_f/B.

To begin, one easily shows that pulling back along p preserves faithful functors, i.e., if $q: F \to B$ is faithful, then so is the projection $E \times_B F \to E$ in the pullback diagram

$$\begin{tikzcd} E \times_B F \arrow[r] \arrow[d] & F \arrow[d, "q"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

and that $p^*: \mathrm{Rel}^B \to \mathrm{Rel}^E$ corresponds (via the equivalence of Theorem 3.1) to the pullback functor $\mathrm{Cat}_f/B \to \mathrm{Cat}_f/E$, also denoted by p^*. Moreover, by uniqueness of adjoints, $\Sigma_p: \mathrm{Rel}^E \to \mathrm{Rel}^B$ corresponds to the functor obtained by first composing with p and then reflecting, i.e.,

$$\mathrm{Cat}_f/E \xrightarrow{\Sigma_p} \mathrm{Cat}/B \xrightarrow{\widehat{\ }} \mathrm{Cat}_f/B$$

4.1. THEOREM. *The following are equivalent for a functor $p: E \to B$.*

(a) $p^*: \mathrm{Rel}^B \to \mathrm{Rel}^E$ has a right 2-adjoint

(b) $p^*: \mathrm{Cat}_f/B \to \mathrm{Cat}_f/E$ has a right 2-adjoint

(c) p satisfies the weak factorization lifting property (WFLP)

(d) The lax functor $Lp: B \to \mathrm{Rel}$ is a non-unitary functor

PROOF. Since (a)⇔(b) and (c)⇔(d) in any case, it suffices to prove (c)⇒(a) and (b)⇒(c).

For (c)⇒(a), suppose p satisfies WFLP, and define a 2-functor

$$\Pi_p: \mathrm{Rel}^E \to \mathrm{Rel}^B$$

as follows. Given Y in Rel^E, let $(\Pi_p Y)_b$ denote the set of functions

$$\sigma: E_b \to \coprod_{pe=b} Y_e$$

such that $\sigma e \in Y_e$ and $\sigma e_1 \to_\iota \sigma e_2$ for all $\iota: e_1 \to e_2$ over id_b, and define $\sigma \to_\beta \sigma'$ if $\sigma e \to_\xi \sigma' e'$, for all $\xi: e \to e'$ over $\beta: b \to b'$.

Then (RS1*) holds, since $\sigma \to_b \sigma$, for all $\sigma \in (\Pi_p Y)_b$, by definition. For (RS2*), suppose $\sigma \to_\beta \sigma'$ and $\sigma' \to_{\beta'} \sigma''$, where $\beta: b \to b'$ and $\beta': b' \to b''$. To see that $\sigma \to_{\beta''} \sigma''$ for $\beta'' = \beta'\beta$, suppose $\xi'': e \to e''$ over β''. Applying WFLP, there exists a factorization $\xi'' = \xi'\xi$ such that $p\xi = \beta$ and $p\xi' = \beta'$. Then $\sigma e \to_\xi \sigma' e'$ and $\sigma' e' \to_{\xi'} \sigma'' e''$, and so $\sigma e \to_{\xi''} \sigma'' e''$, since Y is a Rel-set over E. Therefore, $\sigma \to_{\beta''} \sigma''$, and it follows that $\Pi_p Y$ is a Rel-set over B.

A morphism $g\colon Y \to Z$ of Rel^E induces a function

$$(\Pi_p g)_b\colon (\Pi_p Y)_b \to (\Pi_p Z)_b$$

for each b, which takes σ to the composite

$$E_b \xrightarrow{\sigma} \coprod_{pe=b} Y_e \xrightarrow{\coprod g_e} \coprod_{pe=b} Z_e$$

A straightforward calculation shows that this is a morphism of Rel^B.

To see that a 2-cell $g \to g'$ gives rise to $\Pi_p g \to \Pi_p g'$, we must show that $(\Pi_p g)_b\sigma \to_b (\Pi_p g')_b\sigma$, for all $\sigma \in (\Pi_p Y)_b$, i.e., $g_{e_1}\sigma e_1 \to_\iota g'_{e_2}\sigma e_2$, for all $\iota\colon e_1 \to e_2$ over id_b. Now, $\sigma e_1 \to_\iota \sigma e_2$ since $\sigma \in (\Pi_p Y)_b$, and so $g_{e_1}\sigma e_1 \to_\iota g_{e_2}\sigma e_2$ since g is a morphism. Also, $g_{e_2}\sigma e_2 \to_{e_2} g'_{e_2}\sigma e_2$, since $g \to g'$, and so $g_{e_1}\sigma e_1 \to_\iota g'_{e_2}\sigma e_2$ follows from transitivity in Z.

It remains to show that p^* is left adjoint to Π_p. For the counit $\varepsilon\colon p^*\Pi_p \to \mathrm{id}$, note that $(p^*\Pi_p Y)_e = (\Pi_p Y)_{pe}$, where elements are functions

$$\sigma\colon E_{pe} \to \coprod_{pe'=pe} Y_{e'}$$

such that $\sigma e' \in Y_{e'}$ and $\sigma e'_1 \to_{\iota'} \sigma e'_2$ for all $\iota'\colon e'_1 \to e'_2$ over id_{pe}. Then one can show that the evaluation map $(\varepsilon_Y)_e\colon (\Pi_p Y)_{pe} \to Y_e$ given by $(\varepsilon_Y)_e\sigma = \sigma e$, defines a morphism of Rel^E. To define the unit $\eta\colon \mathrm{id} \to \Pi_p p^*$, note that $(\Pi_p p^* X)_b$ is the set of functions

$$\sigma\colon E_b \to \coprod_{pe=b} X_{pe}$$

such that $\sigma e \in X_{pe}$ and $\sigma e_1 \to_\iota \sigma e_2$ for all $\iota\colon e_1 \to e_2$ over id_b. Then the function $(\eta_X)_b\colon X_b \to (\Pi_p p^* X)_b$, which takes x to the constant function at x

$$\lceil x \rceil\colon E_b \to \coprod_{pe=b} X_{pe}$$

defines a morphism of Rel^B. One checks that the adjunction identity holds, to complete the proof of (c)$\Rightarrow$(a).

For (b)$\Rightarrow$(c), suppose $p^*\colon \mathrm{Cat}_f/B \to \mathrm{Cat}_f/E$ has a right adjoint. To see that p satisfies WFLP, suppose $\xi''\colon e \to e''$ in E and $p\xi'' = \beta'\beta$, where $\beta\colon pe \to b'$ and $\beta'\colon b' \to pe''$. Then the composite $\beta'\beta$ gives rise to a pushout in Cat_f/B of the form

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d] & \mathbf{2} \arrow[d] \arrow[ddr, "\beta'"] & \\
\mathbf{2} \arrow[r] \arrow[drr, "\beta"'] & \mathbf{3} \arrow[dr, "\beta'\beta"] & \\
& & B
\end{tikzcd}$$

where $\mathbf{2}$ and $\mathbf{3}$ are the categories $0 \to 1$ and $0 \to 1 \to 2$, respectively.

Since p^* preserves pushouts (being a left adjoint), we get a corresponding pushout in Cat_f/E

$$\begin{tikzcd}
E \times_B \mathbf{1} \arrow[r] \arrow[d] & E \times_B \mathbf{2} \arrow[d] \arrow[ddr, "p^*\beta'"] & \\
E \times_B \mathbf{2} \arrow[r] \arrow[drr, "p^*\beta"'] & E \times_B \mathbf{3} \arrow[dr] & \\
& & E
\end{tikzcd}$$

Since Cat_f/E is a reflective subcategory of Cat/E, pushouts are formed in Cat/E and reflected to Cat_f/E. Thus, the pushout $P \to E$ of this diagram can be constructed as follows. Let E_β and $E_{\beta'}$ denote the subcategories of E obtained by identifying $p^*\beta$ and $p^*\beta'$ with their images in E. Then the objects of P are the union of those of E_β and $E_{\beta'}$, and the morphisms are those of E_β and $E_{\beta'}$ together with pairs $(\xi, \xi'): e \to e''$ such that $\xi: e \to e'$ in E_β and $\xi': e' \to e''$ in $E_{\beta'}$, subject to an appropriate equivalence relation. Since $\xi'': e \to e''$ corresponds to a morphism of $E \times_B \mathbf{3}$, and hence one of P over $\beta'\beta$, the desired factorization of ξ'' follows, to complete the proof. ∎

Recall that, for a category $\mathcal{A}$ with binary products, an object X is called *exponentiable* if the functor $X \times -: \mathcal{A} \to \mathcal{A}$ has a right adjoint, and $\mathcal{A}$ is called *cartesian closed* if every object is exponentiable. By Corollary 3.2, Rel^B has products and $X \times Y$ is given by $(X \times Y)_b = X_b \times Y_b$ with $(x, y) \to_\beta (x', y')$ whenever $x \to_\beta x'$ and $y \to_\beta y'$. It turns out that Rel^B is not cartesian closed, and so exponentiable objects are of interest there.

Now, it is well-known that if $\mathcal{A}$ is has pullbacks and $p: E \to B$ is a morphism of $\mathcal{A}$, then the pullback functor $p^*: \mathcal{A}/B \to \mathcal{A}/E$ has a left adjoint (denoted by Σ_p) defined by composition with p. Moreover, p^* has a right adjoint (denoted Π_p) if and only if $p: E \to B$ is exponentiable in $\mathcal{A}/B$ (e.g., see [9]).

Thus, when $p: E \to B$ is faithful, it is an object of Cat_f/B, and so Theorem 4.1 yields:

4.2. Corollary. *The following are equivalent for a faithful functor $p: E \to B$.*

(a) $p^*: \mathrm{Rel}^B \to \mathrm{Rel}^E$ has a right 2-adjoint

(b) $Lp: B \to \mathrm{Rel}$ is 2-exponentiable in Rel^B

(c) $p^*: \mathrm{Cat}_f/B \to \mathrm{Cat}_f/E$ has a right 2-adjoint

(d) $p: E \to B$ is 2-exponentiable in Cat_f/B

(e) p satisfies WFLP

(f) $Lp: B \to \mathrm{Rel}$ is a non-unitary functor

∎

Using the equivalence $\mathrm{Cat}_f/B \simeq \mathrm{Rel}^B$, one obtains:

4.3. COROLLARY. *The following are equivalent for a* Rel*-set X on B.*

(a) X is 2-exponentiable in Rel^B

(b) Given $x \to_{\beta''} x''$ and a factorization $\beta'' = \beta'\beta$, there exists x' such that $x \to_\beta x'$ and $x' \to_{\beta'} x''$

(c) The lax functor $X\colon B \to \mathrm{Rel}$ is a non-unitary functor

■

Also, the equivalence $\mathrm{Rel}^B \simeq \mathrm{Pr}/B$ of Corollary 3.3 yields:

4.4. COROLLARY. *The following are equivalent for $p\colon E \to B$ in* Pr/B.

(a) $p^*\colon \mathrm{Rel}^B \to \mathrm{Rel}^E$ has a right 2-adjoint

(b) $Lp\colon B \to \mathrm{Rel}$ is 2-exponentiable in Rel^B

(c) $p^*\colon \mathrm{Pr}/B \to \mathrm{Pr}/E$ has a right 2-adjoint

(d) $p\colon E \to B$ is 2-exponentiable in Pr/B

(e) p satisfies WFLP, i.e., if $e \le e''$ in E and $pe \le b' \le pe''$ in B, there exists $e' \in E$ such that $e \le e' \le e''$ and $pe' = b'$

(f) The lax functor $Lp\colon B \to \mathrm{Rel}$ is a non-unitary functor

■

5. Application to Granularity

In [11], Stell uses unitary lax functors $X\colon T \to \mathrm{Rel}$, on a (finite) poset T, to model data varying over time. In this context, T is called a *time domain* and X is a *dynamic set*. Thus, a dynamic set is just a unitary Rel-set. The laxity is intended to account for data such as "countries or states which have had multiple episodes of existence through history, such as Austria [11]." A classification structure is then added so that objects of the data base can be further identified, for example, as roads, railways, houses, or by other features.

A *classification structure* is a preordered set $(\Phi, \to)$ together with a partial order $\le$ (denoted by $\sqsupseteq$ in [11]) on Φ. In practice, $\phi \le \phi'$ indicates that ϕ' is a more general class than ϕ, e.g., a building is more general than a house. And, $\to$ signifies that one class can evolve into another, e.g., a child can become an adult. The structure is assumed to satisfy

(C1) For all $\phi \to \phi'$, there exists ψ with $\phi \le \psi$ and $\phi' \le \psi$.

(C2) Any set of elements in the same connected component of $(\Phi, \leq)$ has a least upper bound.

(C3) These least upper bounds preserve $\to$, in the sense that, if some element of one such set A can evolve into an element of another set B, then $\operatorname{lub} A \to \operatorname{lub} B$.

A *classification* of a Rel-set X is given by a family of functions $\lambda_t\colon X_t \to \Phi$, $t \in T$, such that

$$\lambda_t x \to \lambda_{t'} x', \quad \text{for all } x \to x' \ x \in X_t, x' \in X_{t'},$$

where the subscript has been omitted on $x \to_\beta x'$ since there is at most one morphism $t \to t'$ in T. Thus, a classification on X is just a morphism $\lambda\colon X \to T^*\Phi$ in Rel^T, where $T^*\Phi$ is the image of Φ under the functor

$$\mathrm{Pr} \simeq \mathrm{Rel}^1 \xrightarrow{T^*} \mathrm{Rel}^T$$

and T denotes the unique morphism $T \to 1$. Then a *classified dynamic set* (in the sense of [11]) is just a unitary classified Rel-set.

Loss of detail in the time domain is represented in [11] via a *simplification from T to S*, which is a span

$$\begin{tikzcd} & U \arrow[dl, "p"'] \arrow[dr, "q"] & \\ T & & S \end{tikzcd}$$

of order-preserving maps, where p is injective and q is surjective. In [11], the author constructs the *simplification of a classified dynamic set* X over T to one over S. Properties (C1)–(C3) play a crucial role in obtaining the classification in the unitary case.

In what follows, we obtain this construction in stages. For general Rel-sets, we get a simplification from T to S via the functor

$$\mathrm{Rel}^T \xrightarrow{p^*} \mathrm{Rel}^U \xrightarrow{\Sigma_q} \mathrm{Rel}^S$$

To obtain the simplification of a dynamic set (in the sense of [11]), we first introduce a unitary reflection, and then we adapt the construction to the classified case.

Let Dyn^T denote the full subcategory of Rel^T consisting of dynamic sets, and define $\bar{(\ )}\colon \mathrm{Rel}^T \to \mathrm{Dyn}^T$ as follows. Given a Rel-set X, let $\sim_t$ (or simply $\sim$) denote the equivalence relation on X_t generated by $\to$, and let $\bar{X}_t = X_t/\sim_t$. For $t \leq t'$, $\overline{x} \in \bar{X}_t$, and $\overline{x}' \in \bar{X}_{t'}$, define $\bar{x} \to \bar{x}'$ if

$$x \to x_1 \sim_{t_1} x_1' \to \cdots \to x_n \sim_{t_n} x_n' \to x'$$

for some $x_1, x_1', \ldots, x_n, x_n'$. Then $\to$ is well-defined and makes $\bar{X}$ into a dynamic set on T. Note that $\bar{X}$ would not necessarily be unitary if T were merely a preordered set. If $f\colon X \to Y$ is a morphism of Rel-sets on T, then $\bar{f}_t\colon \bar{X}_t \to \bar{Y}_t$, given by $\bar{f}_t(\bar{x}) = \overline{f_t x}$, is well-defined and provides a Rel-set morphism $\bar{f}\colon \bar{X} \to \bar{Y}$, since f is order-preserving.

5.1. Proposition. *The functor* $\bar{(\ )}\colon \mathrm{Rel}^T \to \mathrm{Dyn}^T$ *is left 2-adjoint to the inclusion.*

Proof. With the unit $\eta_X\colon X \to \bar{X}$ given by $\eta_X(x) = \bar{x}$ and the counit by the "identity" functor, the adjunction identities easily follow. ∎

Now, suppose $T \xleftarrow{p} U \xrightarrow{q} S$ is any span of posets. Then

$$\mathrm{Dyn}^T \xrightarrow{p^*} \mathrm{Rel}^U \xrightarrow{\Sigma_q} \mathrm{Rel}^S \xrightarrow{\bar{(\ )}} \mathrm{Dyn}^S$$

gives a simplification functor for (non-classified) dynamic sets. If p is injective, then p^* preserves dynamic sets, since unitary Rel-sets on T correspond to posets with discrete fibers over T, and so this simplification functor becomes

$$\mathrm{Dyn}^T \xrightarrow{p^*} \mathrm{Dyn}^U \xrightarrow{\Sigma_q} \mathrm{Rel}^S \xrightarrow{\bar{(\ )}} \mathrm{Dyn}^S$$

Next, we consider classified Rel-sets. Let $\mathrm{Rel}^T /\!\!/ \Phi$ denote the category whose objects are classified Rel-sets on T, i.e., $\lambda\colon X \to T^*\Phi$ in Rel^T, with morphisms given by triangles

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\lambda"'] & \leq & Y \arrow[dl, "\mu"] \\ & T^*\Phi & \end{tikzcd}$$

i.e., morphisms $f\colon X \to Y$ in Rel^T such that $\lambda_t x \leq \mu_t f_t x$, for all $x \in X_t$.

Given a morphism $p\colon U \to T$ of posets, it is easy to show that the adjunction $\Sigma_p \dashv p^*$ restricts to

$$\mathrm{Rel}^U /\!\!/ \Phi \underset{p^*}{\overset{\Sigma_p}{\rightleftarrows}} \mathrm{Rel}^T /\!\!/ \Phi$$

Thus, for a span $T \xleftarrow{p} U \xrightarrow{q} S$, we get a simplification functor for classified Rel-sets

$$\mathrm{Rel}^T /\!\!/ \Phi \xrightarrow{p^*} \mathrm{Rel}^U /\!\!/ \Phi \xrightarrow{\Sigma_q} \mathrm{Rel}^S /\!\!/ \Phi$$

Note that (C1)–(C3) were not needed for this general (not necessarily unitary) case which was not considered in [11].

Now, let $\mathrm{Dyn}^T /\!\!/ \Phi$ be the full subcategory of $\mathrm{Rel}^T /\!\!/ \Phi$ consisting of classified dynamic sets. Then the reflection $\bar{(\ )}\colon \mathrm{Rel}^T \to \mathrm{Dyn}^T$ extends to the classified case, as follows. Given a classified Rel-set $\lambda\colon X \to T^*\Phi$, using (C1) and (C2), define $\bar{\lambda}_t\colon \bar{X}_t \to \Phi$ by $\bar{\lambda}_t \bar{x} = \mathrm{lub}\{\lambda_t a | a \sim x\}$. Then, by property (C3), $\bar{\lambda}$ is a classification on $\bar{X}$, i.e.,

$$\bar{x} \to \bar{x}' \;\Rightarrow\; \bar{\lambda}_t \bar{x} \to \bar{\lambda}_{t'} \bar{x}'$$

for all $x \in X_t$, $x' \in X_{t'}$, and $t \leq t'$. To see that $\bar{(\ )}\colon \mathrm{Rel}^T /\!\!/ \Phi \to \mathrm{Dyn}^T /\!\!/ \Phi$ is a functor, suppose $f\colon X \to Y$ is a morphism of classified Rel-sets. Then so is $\bar{f}$, that is,

$$\begin{tikzcd} \bar{X} \arrow[rr, "\bar{f}"] \arrow[dr, "\bar{\lambda}"'] & \leq & \bar{Y} \arrow[dl, "\bar{\mu}"] \\ & T^*\Phi & \end{tikzcd}$$

since $\bar{\lambda}_t \bar{x} = \mathrm{lub}\{\lambda_t a | a \sim x\} \leq \mathrm{lub}\{\mu_t f_t a | a \sim x\} \leq \mathrm{lub}\{\mu_t f_t a | f_t a \sim f_t x\} \leq \mathrm{lub}\{\mu_t b | b \sim f_t x\} = \bar{\mu}_t \bar{f}_t \bar{x}$.

5.2. LEMMA. *The functor* $(\bar{\ }) \colon \mathrm{Rel}^T /\!/ \Phi \to \mathrm{Dyn}^T /\!/ \Phi$ *is left adjoint to the inclusion.*

PROOF. It suffices to show that the unit $\eta_X \colon X \to \bar{X}$ (given by $\eta_X(x) = \bar{x}$ in the proof of Proposition 5.1) is a morphism of $\mathrm{Dyn}^T /\!/ \Phi$. But, this is clear, since $\lambda_t x \leq \mathrm{lub}\{\lambda_t a | a \sim x\} = \bar{\lambda}_t \bar{x} = \bar{\lambda}_t \eta_t x$. ∎

Thus, we get:

5.3. THEOREM. *The simplification of classified dynamic sets relative to the span* $T \xleftarrow{p} U \xrightarrow{q} S$ *(in the sense of [11]) is given by*

$$\mathrm{Dyn}^T /\!/ \Phi \xrightarrow{p^*} \mathrm{Rel}^U /\!/ \Phi \xrightarrow{\Sigma_q} \mathrm{Rel}^S /\!/ \Phi \xrightarrow{(\bar{\ })} \mathrm{Dyn}^S /\!/ \Phi$$

We conclude with a consideration of right adjoints to the simplification functors.

5.4. THEOREM. *If* $T \xleftarrow{p} U \xrightarrow{q} S$ *is a span of preordered sets, then the simplification functor* $\mathrm{Rel}^T \xrightarrow{p^*} \mathrm{Rel}^U \xrightarrow{\Sigma_q} \mathrm{Rel}^S$ *has a right adjoint if and only if* p^* *does, i.e.,* p *is a WFLP map.*

PROOF. Consider the corresponding composite $\mathrm{Pr}/T \xrightarrow{p^*} \mathrm{Pr}/U \xrightarrow{\Sigma_q} \mathrm{Pr}/S$. By [9, Proposition 1.1], a functor $F \colon \mathrm{Pr}/T \to \mathrm{Pr}/S$ has a right adjoint if and only if

$$\mathrm{Pr}/T \xrightarrow{F} \mathrm{Pr}/S \xrightarrow{\Sigma_S} \mathrm{Pr}$$

does, and so the desired result follows. ∎

5.5. THEOREM. *If* $T \xleftarrow{p} U \xrightarrow{q} S$ *is a span of posets and* p *is an injective WFLP map, then the simplification functor*

$$\mathrm{Dyn}^T \xrightarrow{p^*} \mathrm{Dyn}^U \xrightarrow{\Sigma_q} \mathrm{Rel}^S \xrightarrow{(\bar{\ })} \mathrm{Dyn}^S$$

has a right 2-adjoint.

PROOF. Since $\Sigma_q \colon \mathrm{Rel}^U \to \mathrm{Rel}^S$ and $(\bar{\ }) \colon \mathrm{Rel}^S \to \mathrm{Dyn}^S$ have right adjoints and q^* preserves dynamic sets, it suffices to show that p^* has a right adjoint. Now, $p^* \colon \mathrm{Rel}^T \to \mathrm{Rel}^U$ does by Corollary 4.4, since p satisfies WFLP, and one can show using the description in the proof of Theorem 4.1, that Π_p preserves dynamic sets, since p is injective, and the desired result follows. ∎

For the classified case, the situation is more complicated. In particular, Π_p cannot be easily adapted unless we impose conditions on Φ which may not make sense for the intended interpretation [11] of the relations on Φ. For example, taking $T = \{0, 1\}$ and p to be the inclusion of $U = \{1\}$, one can show that Φ would need an element ϕ_0 such that $\phi \leq \phi_0$ and $\phi_0 \to \phi$, for all $\phi \in \Phi$, i.e., a class that is more general than and could evolve into any other class. This can be seen using $Y = T^*\Phi$ and the description of $\Pi_p Y$ in Theorem 4.1. On the other hand, using the inclusion of $U = \{0\}$ in T, one can show that Φ would need an element ϕ_1 such that $\phi \leq \phi_1$ and $\phi \to \phi_1$, for all $\phi \in \Phi$, i.e., ϕ_1 would be a class that is more general than any other class and such that any class could evolve into ϕ_1.

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