Theory and Applications of Categories, Vol. 11, No. 10, 2003, pp. 215–251.

SOME ALGEBRAIC APPLICATIONS OF GRADED CATEGORICAL GROUP THEORY

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Abstract. The homotopy classification of graded categorical groups and their homomorphisms is applied, in this paper, to obtain appropriate treatments for diverse crossed product constructions with operators which appear in several algebraic contexts. Precise classification theorems are therefore stated for equivariant extensions by groups either of monoids, or groups, or rings, or rings-groups or algebras as well as for graded Clifford systems with operators, equivariant Azumaya algebras over Galois extensions of commutative rings and for strongly graded bialgebras and Hopf algebras with operators. These specialized classifications follow from the theory of graded categorical groups after identifying, in each case, adequate systems of factor sets with graded monoidal functors to suitable graded categorical groups associated to the structure dealt with.

1. Introduction

Graded categorical groups provide a suitable categorical setting for the treatment of an extensive list of subjects with recognized mathematical interest. Let us briefly recall that, if Γ is a group, then a Γ-graded categorical group is a groupoid $\mathbb{G}$ equipped with a grading functor $\mathrm{gr} : \mathbb{G} \to \Gamma$ and with a graded monoidal structure, by graded functors $\otimes : \mathbb{G} \times_\Gamma \mathbb{G} \to \mathbb{G}$ and $I : \Gamma \to \mathbb{G}$, such that for each object X, there is an object X' with a 1-graded arrow $X \otimes X' \to I$ (see Section 2 for the details). These graded categorical groups were originally considered by Fröhlich and Wall in [20] to study Brauer groups in equivariant situations (see also [18, 19, 21]). Indeed, the principal examples in that context are either the Picard, or the Azumaya or the Brauer Γ-graded categorical groups defined by a commutative ring on which the group Γ acts by automorphisms. Several other examples of graded categorical groups that deal with algebraic problems are considered in this paper, but there are also interesting instances arising in algebraic topology (see [4, 6]). Furthermore, in the same way which was dealt with by Sinh the ungraded case [36], homotopy classifications theorems for graded categorical groups and their homomorphisms have been shown in [6]. From these general results derive then the utility of graded categorical group theory either in equivariant homotopy theory or group

Financially supported by DGI of Spain and FEDER, Project: BFM2001-2886

Received by the editors 2002-06-26 and, in revised form, 2003-05-07.

Transmitted by Ieke Moerdijk. Published on 2003-05-22.

2000 Mathematics Subject Classification: 18D10, 20J06, 20M10, 20M50, 16S35, 16W50, 16H05, 16W30..

Key words and phrases: graded categorical group, graded monoidal functor, equivariant group cohomology, crossed product, factor set, monoid extension, group extension, ring-group extension, Clifford system, Azumaya algebra, graded bialgebra, graded Hopf algebra.

extensions with operators. In this article, our goal is to state several theorems solving a selected list of classification problems of equivariant algebraic nature and to prove them by using also the above quoted abstract results on the homotopy classification of graded categorical groups and their homomorphisms.

The equivariant classification results we obtain along the paper mainly concern commutative rings on which an action by automorphisms from a group Γ is given. These so-called Γ-rings are a classical subject of investigation in Algebra as either the 14th Hilbert's problem and the invariant theory, or even the more recent equivariant algebraic K-theory, make clear (cf. [31]). More precisely, the most striking Γ-rings we consider as a matter of study are central simple Γ-algebras over a field or, more generally, central separable Γ-algebras over a commutative ring k, whose classes are indeed the elements of the equivariant Brauer group of k (see [17, 18, 19, 20, 21]). Azumaya Γ-algebras are also in the heart of Long's Brauer groups [28] (cf. [2, chapter VI]). If R is a commutative Γ-ring and R/k is a Galois extension, with Galois group G, splitting a central separable Γ-algebra A, then A represents, in the equivariant Brauer group, the same element than a strongly G-graded Γ-algebra whose 1-component is R, that is, a (G, R)-graded Γ-Clifford system. This is the reason why we pay attention to these last kind of Γ-rings. Moreover, in the particular case that the Picard group of R vanishes, then the structure of such Azumaya Γ-algebras is precisely that of a Γ-equivariant crossed product algebra of R by G and this justifies its separate study in Section 5. When the ground ring $k = \mathbb{Z}$ then, the possibility of offering an unified treatment of that theory together with the one of extensions of groups with operators motivates the study about Γ-RINGS-GROUPS we carry out in Section 4.

The plan of this paper is briefly as follows. In the second section we summarize what will be needed in the paper regarding graded categorical groups theory and also on the equivariant cohomology theory of groups with operators studied in [5]. Theorem 2.2, in the second section, shows a classification of the homotopy classes of graded monoidal functors between graded categorical groups with discrete domain. It is of special relevance because suitable specializations of it lead to the classification of several kinds of systems of factor sets for the diverse types of "crossed product constructions with operators" studied throughout the subsequent sections.

In the third section we deal with the general problem of equivariant perfect Rédei extensions of monoids by groups when a group of operators Γ is acting on both structures (also called equivariant monoid coextensions of groups). This problem includes that of group extensions with operators studied in [6, 37]. We develop here an equivariant factor set theory for equivariant extensions of Γ-monoids by Γ-groups. Since these equivariant factor sets become graded monoidal functors from the discrete graded categorical group associated with the group to the *holomorph graded categorical group* of the monoid, we can deduce the desired cohomological classification for equivariant perfect extensions of a Γ-monoid by a Γ-group from the categorical results.

The next section is devoted to the classification of extensions of RINGS-GROUPS with operators. Our results extend to the equivariant case the theory developed by Hacque in

[24, 25] and they are directly applicable both to the theory of Γ-group extensions and to the theory of crossed product Γ-rings. We explain a theory of mixed crossed products with operators that reveals how graded monoidal functors, from discrete graded categorical groups to the graded categorical groups of holomorphisms of Γ-RINGS-GROUPS, are the most suitable systems of data to construct the survey of all equivariant extensions of any RING-GROUP with operators by a group with operators.

In the following section we focus our attention on the classification of graded Clifford systems (also called strongly graded algebras) with operators, which were introduced by Dade [12, 13] to develop Clifford's theory axiomatically. The key in obtaining that classification lies in the fact that giving a graded Γ-Clifford system is equivalent to giving a graded monoidal functor from discrete graded categorical groups to Γ-graded Picard categorical groups of Γ-algebras. Our results for equivariant graded Clifford systems are then used in three subsections, where we deal respectively with equivariant crossed product algebras, equivariant central graded Clifford systems and Azumaya algebras with operators over Galois extensions of commutative rings as well. In the two last cases, the classification theorems are reformulated in terms of certain equivariant cohomology exact sequences.

The last section is dedicated to the classification of strongly graded Γ-bialgebras and strongly graded Hopf Γ-algebras. Our results here parallel the non-equivariant case studied in [8, 3] and they are obtained, from the general theory outlined in the second section, in a similar way to that previously run for equivariant Clifford systems.

2. On graded categorical groups

The beginning of this section is devoted to recalling the definition and some basic facts concerning the cohomology of groups with operators and graded categorical groups. Then, we will derive, from general results established in [6], a classification theorem for homotopy classes of graded monoidal functors between graded categorical groups whose domain is discrete. This result is what is needed for the applications set forth in the next sections.

Hereafter Γ is a fixed group. Let us recall that a Γ-group G means a group G enriched with a left Γ-action by automorphisms, and that a Γ-equivariant module over a Γ-group G is a Γ-module A, that is, an abelian Γ-group, endowed with a G-module structure such that ${}^{\sigma}({}^{x}a) = {}^{({}^{\sigma}x)}({}^{\sigma}a)$ for all $\sigma \in \Gamma$, $x \in G$ and $a \in A$ [5, Definition 2.1]. The abelian groups of Γ-equivariant derivations from a Γ-group G into equivariant G-modules define a left-exact functor $\mathrm{Der}_{\Gamma}(G, -)$ from the category of equivariant G-modules to the category of abelian groups, whose right derived functors lead to the equivariant cohomology functors $H^*_{\Gamma}(G, -)$. More specifically, the cohomology groups of a Γ-group G with coefficients in an equivariant G-module A are [5, (3)]:

$$H^n_{\Gamma}(G, A) = \big(R^{n-1}\mathrm{Der}_{\Gamma}(G, -)\big)(A), \quad n \geq 1,$$

and $H^0_{\Gamma}(G, A) = 0$.

We shall recall from [5] that the cohomology groups $H^n_\Gamma(G,A)$, for $n \leq 3$, can be computed as the cohomology groups of the Whitehead [37] truncated cochain complex

$$0 \to C^1_\Gamma(G,A) \xrightarrow{\partial} C^2_\Gamma(G,A) \xrightarrow{\partial} Z^3_\Gamma(G,A) \to 0\,,$$

in which $C^1_\Gamma(G,A)$ consists of normalized maps $c : G \to A$, $C^2_\Gamma(G,A)$ consists of normalized maps $c : G^2 \cup (G \times \Gamma) \to A$ and $Z^3_\Gamma(G,A)$ consists of all normalized maps $c : G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \to A$ satisfying the following 3-cocycle conditions:

$$\begin{aligned}
&c(x,y,zt) + c(xy,z,t) = {}^x c(y,z,t) + c(x,yz,t) + c(x,y,z)\,,\\
&{}^\sigma c(x,y,z) + c(xy,z,\sigma) + c(x,y,\sigma) = c({}^\sigma x, {}^\sigma y, {}^\sigma z) + {}^{({}^\sigma x)} c(y,z,\sigma) + c(x,yz,\sigma)\,,\\
&{}^\sigma c(x,y,\tau) + c({}^\tau x, {}^\tau y, \sigma) + c(x,\sigma,\tau) + {}^{({}^{\sigma\tau} x)} c(y,\sigma,\tau) = c(x,y,\sigma\tau) + c(xy,\sigma,\tau)\,,\\
&{}^\sigma c(x,\tau,\gamma) + c(x,\sigma,\tau\gamma) = c(x,\sigma\tau,\gamma) + c({}^\gamma x,\sigma,\tau)\,,
\end{aligned}$$

for $x,y,z,t \in G$, $\sigma,\tau,\gamma \in \Gamma$. For each $c \in C^1_\Gamma(G,A)$, the coboundary ∂c is given by

$$\begin{aligned}
(\partial c)(x,y) &= {}^x c(y) - c(xy) + c(x)\,,\\
(\partial c)(x,\sigma) &= {}^\sigma c(x) - c({}^\sigma x)\,,
\end{aligned}$$

and for $c \in C^2_\Gamma(G,A)$, ∂c is given by

$$\begin{aligned}
(\partial c)(x,y,z) &= {}^x c(y,z) - c(xy,z) + c(x,yz) - c(x,y)\,,\\
(\partial c)(x,y,\sigma) &= {}^\sigma c(x,y) - c({}^\sigma x, {}^\sigma y) - {}^{({}^\sigma x)} c(y,\sigma) + c(xy,\sigma) - c(x,\sigma)\,,\\
(\partial c)(x,\sigma,\tau) &= {}^\sigma c(x,\tau) - c(x,\sigma\tau) + c({}^\tau x,\sigma).
\end{aligned}$$

Next, we regard the group Γ as a category with exactly one object, say $*$, where the morphisms are the members of Γ and the composition law is the group operation: $* \xrightarrow{\sigma} * \xrightarrow{\tau} * = * \xrightarrow{\tau\sigma} *$.

A Γ-*grading* on a category $\mathbb{G}$ [20] is a functor $\mathrm{gr} : \mathbb{G} \to \Gamma$. For any morphism f in $\mathbb{G}$ with $\mathrm{gr}(f) = \sigma$, we refer to σ as the *grade* of f, and we say that f is an σ-*morphism.* The grading is *stable* if, for any object X of $\mathbb{G}$ and any $\sigma \in \Gamma$, there exists an isomorphism $X \xrightarrow{\sim} Y$ with domain X and grade σ; in other words, the grading is a cofibration in the sense of Grothendieck [23]. Suppose $(\mathbb{G},\mathrm{gr})$ and $(\mathbb{H},\mathrm{gr})$ are stably Γ-graded categories. A *graded functor* $F : (\mathbb{G},\mathrm{gr}) \to (\mathbb{H},\mathrm{gr})$ is a functor $F : \mathbb{G} \to \mathbb{H}$ preserving grades of morphisms. If $F' : (\mathbb{G},\mathrm{gr}) \to (\mathbb{H},\mathrm{gr})$ is also a graded functor, then a *graded natural equivalence* $\theta : F \to F'$ is a natural equivalence of functors such that all isomorphisms $\theta_X : FX \xrightarrow{\sim} F'X$ are of grade 1.

For a Γ-graded category $(\mathbb{G},\mathrm{gr})$, we write $\mathbb{G} \times_\Gamma \mathbb{G}$ for the subcategory of the product category $\mathbb{G} \times \mathbb{G}$ whose morphisms are those pairs of morphisms of $\mathbb{G}$ with the same grade; this has an obvious grading, which is stable if and only if gr is.

A Γ-*graded monoidal category* [20] (see [34, Chapter I, §4.5] for the general notion of fibred monoidal category) $\mathbb{G} = (\mathbb{G},\mathrm{gr},\otimes,I,\mathbf{a},\mathbf{l},\mathbf{r})$, is a stably Γ-graded category $(\mathbb{G},\mathrm{gr})$ together with graded functors

$$\otimes : \mathbb{G} \times_\Gamma \mathbb{G} \to \mathbb{G}\,, \qquad I : \Gamma \to \mathbb{G}\,,$$

and graded natural equivalences

$$\mathbf{a} : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)\,,$$
$$\mathbf{l} : I\,\mathrm{gr}(-) \otimes - \xrightarrow{\sim} \mathrm{id}_{\mathbb{G}}, \quad \mathbf{r} : - \otimes I\,\mathrm{gr}(-) \xrightarrow{\sim} \mathrm{id}_{\mathbb{G}}\,,$$

such that for any objects $X, Y, Z, T \in \mathbb{G}$, the following two coherence conditions hold:

$$\mathbf{a}_{X,Y,Z\otimes T}\ \mathbf{a}_{X\otimes Y,Z,T} = (X \otimes \mathbf{a}_{Y,Z,T})\ \mathbf{a}_{X,Y\otimes Z,T}\ (\mathbf{a}_{X,Y,Z} \otimes T)\,,$$
$$(X \otimes \mathbf{l}_Y)\ \mathbf{a}_{X,I,Y} = \mathbf{r}_X \otimes Y\,.$$

If $\mathbb{G}, \mathbb{H}$ are Γ-graded monoidal categories, then a *graded monoidal functor*

$$F = (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}\,,$$

consists of a graded functor $F : \mathbb{G} \to \mathbb{H}$, natural isomorphisms of grade 1,

$$\Phi = \Phi_{X,Y} : FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)\,,$$

and an isomorphism of grade 1 (natural with respect to the elements $\sigma \in \Gamma$)

$$\Phi_* : I \xrightarrow{\sim} FI$$

(where $I = I(*)$) such that, for all objects $X, Y, Z \in \mathbb{G}$, the following coherence conditions hold:

$$\Phi_{X,Y\otimes Z}\ (FX \otimes \Phi_{Y,Z})\ \mathbf{a}_{FX,FY,FZ} = F(\mathbf{a}_{X,Y,Z})\ \Phi_{X\otimes Y,Z}\ (\Phi_{X,Y} \otimes FZ)\,, \tag{1}$$
$$F(\mathbf{r}_X)\ \Phi_{X,I}\ (FX \otimes \Phi_*) = \mathbf{r}_{FX}\,, \quad F(\mathbf{l}_X)\ \Phi_{I,X}\ (\Phi_* \otimes FX) = \mathbf{l}_{FX}\,.$$

Suppose $F' : \mathbb{G} \to \mathbb{H}$ is also a graded monoidal functor. A *homotopy* (or *graded monoidal natural equivalence*) $\theta : F \to F'$ *of graded monoidal functors* is a graded natural equivalence $\theta : F \xrightarrow{\sim} F'$ such that, for all objects $X, Y \in \mathbb{G}$, the following coherence conditions hold:

$$\Phi'_{X,Y}\ (\theta_X \otimes \theta_Y) = \theta_{X\otimes Y}\ \Phi_{X,Y}\,, \quad \theta_I\ \Phi_* = \Phi'_*\,. \tag{2}$$

For later use, we state here the lemma below [6, Lemma 1.1].

2.1. Lemma. *Every graded monoidal functor $F = (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ is homotopic to a graded monoidal functor $F' = (F', \Phi', \Phi'_*)$ with $F'I = I$ and $\Phi'_* = \mathrm{id}_I$.*

A Γ-graded monoidal category, $\mathbb{H} = (\mathbb{H}, \mathrm{gr}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ is said to be a Γ-*graded categorical group* if every morphism is invertible; that is, $\mathbb{H}$ is a groupoid, and for any object X there is an object X' with a 1-morphism $X \otimes X' \to I$. In subsequent sections several examples of Γ-graded categorical groups will be discussed. However, we shall describe now an elementary example, the *discrete* Γ-*graded categorical group* defined for any Γ-group G, which will be frequently used in this paper.

Let G be a Γ-group. Then, the discrete Γ-graded categorical group $\underline{\mathrm{dis}}^{\Gamma} G$ has the elements of G as objects and their morphisms $\sigma : x \to y$ are the elements $\sigma \in \Gamma$ with

${}^{\sigma}x = y$. Composition is multiplication in Γ and the grading $\mathrm{gr} : \underline{\mathrm{dis}}_{\Gamma} G \to \Gamma$ is the obvious map $\mathrm{gr}(\sigma) = \sigma$. The graded tensor product is given by

$$(x \xrightarrow{\sigma} y) \otimes (x' \xrightarrow{\sigma} y') = (xx' \xrightarrow{\sigma} yy')\,,$$

and the graded unit $I : \Gamma \to \underline{\mathrm{dis}}^{\Gamma} G$ by

$$I(* \xrightarrow{\sigma} *) = (1 \xrightarrow{\sigma} 1)\,;$$

the associativity and unit isomorphisms are identities.

Each Γ-graded categorical group $\mathbb{H}$ has two associated *homotopy groups*, which we shall describe below (see [6, Proposition 1.2] for details):

$\pi_0\mathbb{H}$ = *the* Γ*-group of* 1*-isomorphism classes of the objects in* $\mathbb{H}$, where multiplication is induced by tensor product, $[X]\,[Y] = [X \otimes Y]$, and the Γ-action is defined by ${}^{\sigma}[X] = [Y]$ whenever there exists a morphism $X \to Y$ in $\mathbb{H}$ of grade σ.

$\pi_1\mathbb{H}$ = *the* Γ*-equivariant* $\pi_0\mathbb{H}$*-module of 1-automorphism in* $\mathbb{H}$ *of the unit object* I, where the operation is composition, the Γ-action is ${}^{\sigma}u = I(\sigma)\; u\; I(\sigma)^{-1}$, and the structure of $\pi_0\mathbb{H}$-module is as follows : if $[X] \in \pi_0\mathbb{H}$ and $u \in \pi_1\mathbb{G}$, then ${}^{[X]}u$ is determined by the formula

$$\delta(u) = \gamma({}^{[X]}u),$$

where

$$\mathrm{Aut}_1(X) \xleftarrow[\sim]{\delta} \pi_1\mathbb{H} \xrightarrow[\sim]{\gamma} \mathrm{Aut}_1(X) \tag{3}$$

are the isomorphisms defined respectively by $\delta(u) = \mathbf{r}\,(X \otimes u)\,\mathbf{r}^{-1}$ and $\gamma(u) = \mathbf{l}\,(u \otimes X)\,\mathbf{l}^{-1}$.

Thus, for example, if G is any Γ-group, then $\pi_0\underline{\mathrm{dis}}^{\Gamma} G = G$, as a Γ-group, while $\pi_1\underline{\mathrm{dis}}^{\Gamma} G = 0$.

For each Γ-group G and each Γ-graded categorical group $\mathbb{H}$, let

$$[\underline{\mathrm{dis}}^{\Gamma} G, \mathbb{H}]$$

denote the set of homotopy classes of graded monoidal functors from $\underline{\mathrm{dis}}^{\Gamma} G$ to $\mathbb{H}$. There is a canonical map

$$\pi_0 : [\underline{\mathrm{dis}}^{\Gamma} G, \mathbb{H}] \longrightarrow \mathrm{Hom}_{\Gamma}(G, \pi_0\mathbb{H}) \qquad [F] \mapsto \pi_0 F\,, \tag{4}$$

where $\mathrm{Hom}_{\Gamma}(G, \pi_0\mathbb{H})$ is the set of equivariant homomorphisms from the Γ-group G to the Γ-group $\pi_0\mathbb{H}$. We refer to a Γ-group homomorphism $\rho : G \to \pi_0\mathbb{H}$ as *realizable* whenever it is in the image of the above map π_0, that is, if $\rho = \pi_0 F$ for some graded monoidal functor $F : \underline{\mathrm{dis}}^{\Gamma} G \to \mathbb{H}$. The map (4) produces a partitioning

$$[\underline{\mathrm{dis}}^{\Gamma} G, \mathbb{H}] = \bigsqcup_{\rho} \, [\underline{\mathrm{dis}}^{\Gamma} G, \mathbb{H}; \rho]\,,$$

where, for each $\rho \in \mathrm{Hom}_{\Gamma}(G, \pi_0\mathbb{H})$, $[\underline{\mathrm{dis}}^{\Gamma} G, \mathbb{H}; \rho] = \pi_0^{-1}(\rho)$ is the set of *homotopy classes of realizations of* ρ.

When an equivariant homomorphism $\rho : G \to \pi_0\mathbb{H}$ is specified, it is possible that there is no graded monoidal functor that realizes ρ, and this leads to a problem of obstructions which (even in a more general form) is solved in [6, Theorem 3.2] by means of a 3-dimensional equivariant group cohomology class $\mathrm{Obs}(\rho)$. Next we outline this theory for this particular case.

Let $\rho : G \to \pi_0\mathbb{H}$ be any given Γ-group homomorphism. Then, $\pi_1\mathbb{H}$ is, via ρ, a Γ-equivariant G-module and so the equivariant cohomology groups $H^n_{\Gamma}(G, \pi_1\mathbb{H})$ are defined. The *obstruction cohomology class*

$$\mathrm{Obs}(\rho) \in H^3_{\Gamma}(G, \pi_1\mathbb{H}) \tag{5}$$

is represented by any 3-cocycle $c^{\rho} \in Z^3_{\Gamma}(G, \pi_1\mathbb{H})$ built as follows. For each $x \in G$, let us choose an object $F(x) \in \rho(x)$, with $F(1) = I$. Since $\rho(xy) = \rho(x)\rho(y)$, for each $x, y \in G$, we can select a 1-morphism $\Phi_{x,y} : F(x) \otimes F(y) \to F(xy)$, with $\Phi_{1,y} = \mathbf{l}$ and $\Phi_{x,1} = \mathbf{r}$. Furthermore, since $\rho({}^{\sigma}x) = {}^{\sigma}\rho(x)$, for each $x \in G$ and $\sigma \in \Gamma$, we can also select an σ-morphism $\Phi_{x,\sigma} : F(x) \to F({}^{\sigma}x)$, with $\Phi_{1,\sigma} = I(\sigma)$ and $\Phi_{x,1} = id$. Then, the map

$$c^{\rho} : G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \longrightarrow \pi_1\mathbb{H} \tag{6}$$

is determined, for all $x, y, z \in G$ and $\sigma, \tau \in \Gamma$, by the commutativity of the diagrams in $\mathbb{H}$,

$$\begin{tikzcd}
\big(F(x) \otimes F(y)\big) \otimes F(z) \arrow[r, "\Phi_{x,y} \otimes id"] \arrow[d, "\mathbf{a}"'] & F(xy) \otimes F(z) \arrow[r, "\Phi_{xy,z}"] & F(xyz) \arrow[d, dashed, "\gamma\left(c^{\rho}(x,y,z)\right)"] \\
F(x) \otimes \big(F(y) \otimes F(z)\big) \arrow[r, "id \otimes \Phi_{y,z}"] & F(x) \otimes F(yz) \arrow[r, "\Phi_{x,yz}"] & F(xyz)\ ,
\end{tikzcd}$$

$$\begin{tikzcd}
F(xy) \arrow[rr, "\Phi_{xy,\sigma}"] \arrow[d, dashed, "\gamma\left(c^{\rho}(x,y,\sigma)\right)"'] & & F({}^{\sigma}x\,{}^{\sigma}y) \arrow[d, "\Phi^{-1}_{{}^{\sigma}x,{}^{\sigma}y}"] \\
F(xy) \arrow[r, "\Phi^{-1}_{x,y}"] & F(x) \otimes F(y) \arrow[r, "\Phi_{x,\sigma} \otimes \Phi_{y,\sigma}"] & F({}^{\sigma}x) \otimes F({}^{\sigma}y)\ ,
\end{tikzcd}$$

$$\begin{tikzcd}
F(x) \arrow[r, "\Phi_{x,\tau}"] \arrow[d, dashed, "\gamma\left(c^{\rho}(x,\sigma,\tau)\right)"'] & F({}^{\tau}x) \arrow[d, "\Phi_{{}^{\tau}x,\sigma}"] \\
F(x) \arrow[r, "\Phi_{x,\sigma\tau}"] & F({}^{\sigma\tau}x)\ ,
\end{tikzcd}$$

where the γ's are the isomorphisms defined in (3). The fact that c^{ρ} is an equivariant 3-cocycle, whose cohomology class

$$\mathrm{Obs}(\rho) = [c^{\rho}] \in H^3_{\Gamma}(G, \pi_1\mathbb{H})$$

only depends on ρ, follows from [6, Theorem 3.2]. Also, as a consequence of this theorem, the following is true:

2.2. Theorem. *Let G be a Γ-group and let $\mathbb{H}$ be a Γ-graded categorical group. Then,*

(i) A Γ-group homomorphism $\rho : G \to \pi_0\mathbb{H}$ is realizable, that is, $[\underline{\mathrm{dis}}^\Gamma G, \mathbb{H}; \rho] \neq \emptyset$, if, and only if, its obstruction $\mathrm{Obs}(\rho)$ vanishes.

(ii) If $\mathrm{Obs}(\rho) = 0$, then there is a bijection

$$[\underline{\mathrm{dis}}^\Gamma G, \mathbb{H}; \rho] \cong H^2_\Gamma(G, \pi_1\mathbb{H}). \tag{7}$$

The bijection (7), which is non-natural, can be described as follows (see [6] for details). By hypothesis, and using Lemma 2.1, there is a strictly unitary graded monoidal functor $(F, \Phi) : \underline{\mathrm{dis}}^\Gamma G \to \mathbb{H}$ realizing the equivariant homomorphism ρ. Then, once we have fixed such a (F, Φ), the bijection (7) associates to any equivariant 2-cohomology class $[c] \in H^2_\Gamma(G, \pi_1\mathbb{H})$, represented by a 2-cocycle $c : G^2 \cup (G \times \Gamma) \to \pi_1\mathbb{H}$, the homotopy class of the strictly unitary graded monoidal functor $(c \cdot F, c \cdot \Phi) : \underline{\mathrm{dis}}^\Gamma G \to \mathbb{H}$, where $c \cdot F$ is the graded functor sending a morphism $\sigma : x \to y$ of $\underline{\mathrm{dis}}^\Gamma G$ to the dotted morphism in the commutative diagram in $\mathbb{H}$

$$\begin{tikzcd} F(x) \arrow[rr, dashed, "(c\cdot F)(\sigma)"] \arrow[dr, "F(\sigma)"'] & & F(y) \\ & F(y) \arrow[ur, "\gamma\left(c(x,\sigma)\right)"'] & \end{tikzcd},$$

(γ is the isomorphism (3)), and the natural isomorphisms $(c \cdot \Phi)_{x,y}$, for $x, y \in G$, are the dotted morphisms in the commutative diagrams

$$\begin{tikzcd} F(x) \otimes F(y) \arrow[rr, dashed, "(c\cdot\Phi)_{x,y}"] \arrow[dr, "\Phi_{x,y}"'] & & F(xy) \\ & F(xy) \arrow[ur, "\gamma\left(c(x,y)\right)"'] & \end{tikzcd}.$$

3. Equivariant extensions of monoids by groups

If M is any monoid and G is a group, then a *perfect Schreier extension of M by G* is a sequence of monoids and homomorphisms

$$\mathcal{E} : \ M \overset{i}{\rightarrowtail} N \overset{p}{\twoheadrightarrow} G \tag{8}$$

which is exact in the sense that i is injective, p is surjective and $i(M) = p^{-1}(1)$ and, moreover, such that the congruence on N defined by p is perfect, that is, $p^{-1}(x)p^{-1}(y) = p^{-1}(xy)$, for all $x, y \in G$. Observe that, since $1 \in p^{-1}(x)p^{-1}(x^{-1})$ for all $x \in G$, the monoid N contains at least one unit u_x such that $p(u_x) = x$. Hence, since the map of multiplication by u_x establishes a bijection $i(M) \cong p^{-1}(x)$, it follows that $M \cong i(M)$ is a normal divisor of N (cf. [22]) whose congruence on N coincides with the one defined by p. Therefore, $N/M \cong G$.

The study of Schreier extensions of monoids goes back to Redei [33], and a precise classification theorem for perfect Schreier extensions of monoids by groups was stated in [7, 4.2]. Below, our goal is to study equivariant extensions of monoids by groups when a given group of operators is acting on both structures.

If Γ is a fixed group of operators, then a Γ-*monoid* is a monoid endowed with a Γ-action by automorphisms. An *equivariant perfect Schreier extension* (hereafter, an *equivariant extension* for short) of a Γ-monoid M by a Γ-group G is a perfect Schreier extension $\mathcal{E}$ as in (8) such that N is also a Γ-monoid and maps i and p are equivariant. Thus, M can be identified with an equivariant normal divisor of N and $N/M \cong G$ as Γ-groups. Two such extensions $\mathcal{E}$ and $\mathcal{E}'$ are *equivalent* whenever there exists an equivariant isomorphism $\mathbf{g} : N \to N'$ such that $\mathbf{g}i = i'$ and $p'\mathbf{g} = p$, and we denote by

$$\mathrm{Ext}^{\Gamma}(G, M)$$

the set of equivalence classes of equivariant extensions of the Γ-monoid M by the Γ-group G. This set is pointed by the class of the Γ-monoid product $M \times G$.

Next, we shall explain an *equivariant factor set* theory for equivariant extensions of Γ-monoids M by Γ-groups G. Then, an equivariant factor set for an equivariant extension will be an appropriate set of data to rebuild (by an appropriate *crossed product* construction) the equivariant extension up to equivalence. Furthermore, as we will see later, these equivariant factor sets are essentially the same as graded monoidal functors from the discrete Γ-graded categorical group $\underline{\mathrm{dis}}^{\Gamma}G$ to the Γ-*graded categorical group of holomorphims* of the Γ-monoid M, $\underline{\mathrm{Hol}}^{\Gamma}(M)$. Thanks to that observation, we will able to use Theorem 2.2 in this context. To do so, we begin by describing the graded categorical group $\underline{\mathrm{Hol}}^{\Gamma}(M)$ and then we will state Theorem 3.1 below, in which proof we include the precise notion of equivariant factor set.

Let M be any Γ-monoid. Then, the group $\mathrm{Aut}(M)$ of all monoid automorphisms of M is a Γ-group under the diagonal Γ-action, $f \mapsto {}^{\sigma}\!f$, where ${}^{\sigma}\!f : m \mapsto {}^{\sigma}f({}^{\sigma^{-1}}m)$, and the map

$$C : M^* \to \mathrm{Aut}(M),$$

sending each unit u of M to the inner automorphism given by conjugation with u, $C_u : m \mapsto umu^{-1}$, is a Γ-group homomorphism. The *holomorph* Γ-graded categorical group of the Γ-monoid M, $\underline{\mathrm{Hol}}^{\Gamma}(M)$, has the elements of the Γ-group $\mathrm{Aut}(M)$ as objects, and a morphism of grade $\sigma \in \Gamma$ is a pair $(u, \sigma) : f \to g$, where $u \in M^*$, with ${}^{\sigma}\!f = C_u g$. The composition of morphisms is given by

$$f \xrightarrow{(u,\sigma)} g \xrightarrow{(v,\tau)} h \;=\; f \xrightarrow{({}^{\tau}u\,v,\tau\sigma)} h\ , \tag{9}$$

the graded tensor product is

$$\big(f \xrightarrow{(u,\sigma)} g\big) \otimes \big(f' \xrightarrow{(u',\sigma)} g'\big) \;=\; ff' \xrightarrow{(u\,g(u'),\sigma)} gg'\ , \tag{10}$$

and the graded unit $I:\Gamma\to\underline{\mathrm{Hol}}^{\Gamma}(M)$ is defined by $I(\sigma)=\mathrm{id}_M\xrightarrow{(1,\sigma)}\mathrm{id}_M$. The associativity and unit constraints are identities.

3.1. Theorem. (Equivariant crossed products theory for monoids with operators) *For any Γ-monoid M and any Γ-group G, there is a bijection*

$$\Sigma:\big[\underline{\mathrm{dis}}^{\Gamma}G,\underline{\mathrm{Hol}}^{\Gamma}(M)\big]\cong\mathrm{Ext}^{\Gamma}(G,M)\,, \tag{11}$$

between the set of homotopy classes of graded monoidal functors from $\underline{\mathrm{dis}}^{\Gamma}G$ to $\underline{\mathrm{Hol}}^{\Gamma}(M)$ and the set of equivalence classes of equivariant extensions of M by G.

Proof. By Lemma 2.1, every graded monoidal functor is homotopic to a given (F,Φ,Φ_*) in which $FI=I$ and $\Phi_*=\mathrm{id}_I$, and so we can restrict our attention to this kind of graded monoidal functors, $(F,\Phi):\mathrm{dis}^{\Gamma}G\to\underline{\mathrm{Hol}}^{\Gamma}(M)$. Then, let us start by observing that the data describing such a graded monoidal functor consist of a pair of maps (f,φ), where

$$f:G\longrightarrow\mathrm{Aut}(M)\ ,\qquad \varphi:G^2\cup(G\times\Gamma)\longrightarrow M^*\,, \tag{12}$$

such that we write $F(x)=f_x,\ F\big(\,x\xrightarrow{\sigma}{}^{\sigma}x\,\big)=\ f_x\xrightarrow{(\varphi(x,\sigma),\sigma)}f_{{}^{\sigma}x}$ and $\Phi_{x,y}=\ f_xf_y\xrightarrow{(\varphi(x,y),1)}f_{xy}$, for all $x,y\in G$ and $\sigma\in\Gamma$. When we attempt to write the conditions of (F,Φ) being a graded monoidal functor in terms of (f,φ), then we find the following conditions for (f,φ):

$$f_1=id,\quad \varphi(x,1)=1=\varphi(1,y)\,, \tag{13}$$

$$f_xf_y=C_{\varphi(x,y)}\,f_{xy}\ ,\ \ {}^{\sigma}f_x=C_{\varphi(x,\sigma)}\,f_{{}^{\sigma}x}\,, \tag{14}$$

$$\varphi(x,y)\,\varphi(xy,z)=f_x(\varphi(y,z))\,\varphi(x,yz)\,, \tag{15}$$

$${}^{\sigma}\varphi(x,y)\,\varphi(xy,\sigma)=\varphi(x,\sigma)\ f_{{}^{\sigma}x}\big(\varphi(y,\sigma)\big)\,\varphi({}^{\sigma}x,{}^{\sigma}y)\,, \tag{16}$$

$$\varphi(x,\sigma\tau)={}^{\sigma}\varphi(x,\tau)\ \varphi({}^{\tau}x,\sigma)\,, \tag{17}$$

for all $x,y,z\in G$, $\sigma,\tau\in\Gamma$. To prove this in full several verifications are needed, but they are straightforward: conditions (14) say that $(\varphi(x,y),1)$ and $(\varphi(x,\sigma),\sigma)$ are, respectively, morphisms in $\underline{\mathrm{Hol}}^{\Gamma}M$; (15) expresses the coherence condition (1), while (16) means that the isomorphisms $\Phi_{x,y}$ are natural and (17) that F preserves the composition of morphisms. The normalization condition (13) says that F preserves both identities and the unit object.

We call an *equivariant factor set*, or a non-abelian 2-cocycle of the Γ-group G with coefficients in the Γ-monoid M, any pair (f,φ) as in (12) satisfying the conditions (13)-(17). We should stress that when $\Gamma=\mathbf{1}$ (the trivial group), a factor set is just the same as a Redei system of factor sets for a Schreier extension of the monoid M by the group G [33, 16]. In particular, when M is a group, we have precisely a Schreier factor system for a group extension [35]. Furthermore, if M and G are any two Γ-groups, then such a pair (f,φ) is a Schreier system of factor sets for an equivariant group extension of M by G [6].

Suppose now that (f',φ') describes another graded monoidal functor $(F',\Phi'):\underline{\mathrm{dis}}^{\Gamma}G\to \underline{\mathrm{Hol}}^{\Gamma}M$. Then, any homotopy $\theta:F\to F'$ is given by a map

$$g:G\to M^*,$$

such that one writes $\theta_x=\ f_x\xrightarrow{(g(x),1)}f'_x$ for all $x\in G$. In terms of map g, the conditions for θ to be a homotopy are:

$$g(1)=1 \tag{18}$$

$$f_x=C_{g(x)}\,f'_x\,, \tag{19}$$

$$\varphi(x,y)\ g(xy)=g(x)\ f'_x\big(g(y)\big)\ \varphi'(x,y)\,, \tag{20}$$

$$\varphi(x,\sigma)\ g({}^{\sigma}x)=\ {}^{\sigma}g(x)\ \varphi'(x,\sigma)\,, \tag{21}$$

for all $x,y\in G$, $\sigma\in\Gamma$. Condition (19) expresses that θ_x is a morphism in $\underline{\mathrm{Hol}}_{\Gamma}(M)$ from f_x to f'_x, (21) is the naturalness of θ and (20) and (18) say that the coherence conditions (2) hold.

We now are ready to prove bijection (11).

Every equivariant factor set (f,φ) as in (12) gives rise to an *equivariant crossed product extension* of M by G,

$$\Sigma(f,\varphi):M\overset{i}{\rightarrowtail}M\times_{f,\varphi}G\overset{p}{\twoheadrightarrow}G\,, \tag{22}$$

in which $M\times_{f,\varphi}G$ is Rédei's Schreier product monoid defined in [33]; that is, it is the cartesian product set $M\times G$, with multiplication given by $(m,x)(n,y)=(m\,f_x(n)\,\varphi(x,y),xy)$. This multiplication is associative and unitary thanks to equalities (14), (16) and (13). The Γ-action on $M\times_{f,\varphi}G$ is given by ${}^{\sigma}(m,x)=({}^{\sigma}m\,\varphi(x,\sigma),{}^{\sigma}x)$ which, owing to (14), (16) and (17), satisfies the required conditions in order for $M\times_{f,\varphi}G$ to be a Γ-monoid. The Γ-equivariant monoid homomorphisms i and p are given by $i(m)=(m,1)$ and $p(m,x)=x$. Since $(m,xy)=(m\varphi(x,y)^{-1},x)(1,y)\in p^{-1}(x)p^{-1}(y)$, we see that $p^{-1}(xy)=p^{-1}(x)p^{-1}(y)$, and therefore the congruence is perfect. Thus, $\Sigma(f,\varphi)$ is an equivariant extension of the Γ-monoid M by the Γ-group G.

Suppose (f',φ') is another equivariant factor set. If there is a Γ-monoid isomorphism, say $\mathbf{g}:M\times_{f,\varphi}G\cong M\times_{f',\varphi'}G$, establishing an equivalence between the corresponding equivariant crossed product extensions of M by G, then we can write $\mathbf{g}$ in the form $\mathbf{g}(m,x)=\mathbf{g}(m,1)\mathbf{g}(1,x)=(m,1)(g(x),x)=(mg(x),x)$ for a map $g:G\to M^*$. Since $\mathbf{g}((1,x)(m,1))=\mathbf{g}(f_x(m),x)=(f_x(m)\,g(x),x)$, while $\mathbf{g}(1,x)\,\mathbf{g}(m,1)=(g(x)\,f'_x(m),x)$, it follows that $f_x=C_{g(x)}\,f'_x$; that is, (19) holds. Analogously, we see that (20) follows from the equality $\mathbf{g}((1,x)(1,y))=\mathbf{g}(1,x)\mathbf{g}(1,y)$ and (21) from the equality $\mathbf{g}({}^{\sigma}(1,x))={}^{\sigma}\mathbf{g}(1,x)$. Therefore, g defines a homotopy between (the graded monoidal functors defined by) (f',φ') and (f,φ). Conversely, if (f,φ) and (f',φ') are made homotopic by a $g: G\to M^*$, thus satisfying (18)-(21), then they lead to isomorphic equivariant crossed product extensions, just by the map $\mathbf{g}:(m,x)\mapsto (mg(x),x)$, as we can see by retracing our steps.

Finally, we prove that any equivariant extension of M by G, $\mathcal{E}: M \overset{i}{\rightarrowtail} N \overset{p}{\twoheadrightarrow} G$, has an *associated equivariant factor set*, that is, it is equivalent to an equivariant crossed product like (22), for some equivariant factor set (f,φ). To do so, there is no loss of generality in assuming that i is the inclusion map. Let us choose then, for each $x\in G$, a unit $u_x \in p^{-1}(x)\cap N^*$. We take $u_1 = 1$. Hence, the maps $m\mapsto mu_x$ establish bijections $M = p^{-1}(1) \cong p^{-1}(x)$, for all $x\in G$. Since $p(u_xu_y) = xy = p(u_{xy})$ and $p({}^\sigma u_x) = {}^\sigma x = p(u_{{}^\sigma x})$, it follows that there are unique elements $\varphi(x,y),\varphi(x,\sigma)\in M$, such that $u_x\,u_y = \varphi(x,y)\,u_{xy}$ and ${}^\sigma u_x = \varphi(x,\sigma)\,u_{{}^\sigma x}$, for all $x,y\in G$ and $\sigma\in\Gamma$. Note that $\varphi(x,y),\varphi(x,\sigma)\in M^*$ since the u_x are units. Moreover, each $x\in G$ induces an automorphism f_x of M, $f_x: m\mapsto u_x\,m\,u_x^{-1}$, and this pair of maps

$$f: G\to \mathrm{Aut}(M),\quad \varphi: G^2\cup (G\times\Gamma)\to M^*\,,$$

is actually an equivariant factor set. Indeed, conditions (13) and (14) follow immediately from the definition of (f,φ). To observe the remaining conditions (15)-(17), note that every element of N has a unique expression of the form $m\,u_x$, with $m\in M$ and $x\in G$. Because $u_x\,m = f_x(m)\,u_x$, it follows that the Γ-monoid structure of N can be described, in terms of the Γ-group structure of G, the Γ-monoid structure of M and the pair (f,φ), by $(m\,u_x)\,(m'u_y) = m\,f_x(m')\,\varphi(x,y)\,u_{xy}$ and ${}^\sigma(m\,u_x) = {}^\sigma\!m\;\varphi(x,\sigma)\;u_{{}^\sigma x}$. Then, (15) follows from the associative law $u_x(u_yu_z) = (u_xu_y)u_z$ in M, (16) follows from the equality ${}^\sigma(u_xu_y) = {}^\sigma u_x{}^\sigma u_y$, while (17) is a consequence of the equality ${}^\tau({}^\sigma u_x) = {}^{\tau\sigma}u_x$. Hence, (f,φ) defines an equivariant factor set, whose associated equivariant crossed product extension $\Sigma(f,\varphi)$ is equivalent to $\mathcal{E}$ by the existence of the Γ-monoid isomorphism $M\times_{f,\varphi}G\cong N$, $(m,x)\mapsto m\,u_x$. ∎

Let us now observe that, for any Γ-monoid M, the homotopy groups of the Γ-graded categorical group $\underline{\mathrm{Hol}}^\Gamma(M)$ are:

$\pi_0\underline{\mathrm{Hol}}^\Gamma(M) = \mathrm{Coker}\big(M^*\xrightarrow{C}\mathrm{Aut}(M)\big) = \mathrm{Aut}(M)/\mathrm{In}(M) = \mathrm{Out}(M)$, the group of outer automorphisms of M, which is a Γ-group with action ${}^\sigma[f] = [{}^\sigma\!f]$, where ${}^\sigma\!f: m\mapsto {}^\sigma\!f({}^{\sigma^{-1}}m)$,

$\pi_1\underline{\mathrm{Hol}}^\Gamma(M) = \mathrm{Ker}\big(M^*\xrightarrow{C}\mathrm{Aut}(M)\big) = \mathrm{Z}(M)^*$ the abelian group of units in the centre of the monoid M, which is a Γ-submonoid of M and a Γ-equivariant $\mathrm{Out}(N)$-module with action ${}^{[f]}u = f(u)$, $f\in\mathrm{Aut}(M)$, $u\in \mathrm{Z}(M)^*$.

Then, owing to bijection (11) and map (4), each equivalence class $[\mathcal{E}]$ of an equivariant extension $\mathcal{E}: M\rightarrowtail N\twoheadrightarrow G$ of the Γ-monoid M by the Γ-group G, determines a Γ-equivariant group homomorphism

$$\rho_{[\mathcal{E}]}: G\to \mathrm{Out}(M)\,,$$

by $\rho_{[\mathcal{E}]}(x) = [f_x]$, where (f,φ) is any equivariant factor set for $\mathcal{E}$. A pair (M,ρ), where M is a Γ-monoid and $\rho : G \to \mathrm{Out}(M)$ is a Γ-group homomorphism, is termed an *equivariant G-kernel* for an equivariant monoid extension (or an *equivariant collective character* of G in M, cf. [14, 25, 7]), and an *equivariant extension of an equivariant G-kernel* (M,ρ) is an equivariant extension $\mathcal{E}$ of M by G that *realizes* ρ, that is, with $\rho_{[\mathcal{E}]} = \rho$. We denote by

$$\mathrm{Ext}^{\Gamma}\big(G,(M,\rho)\big)$$

the set of equivalence classes of such equivariant extensions of (M,ρ). Hence, there is a canonical partition

$$\mathrm{Ext}^{\Gamma}(G,M) = \bigsqcup_{\rho} \mathrm{Ext}^{\Gamma}\big(G,(M,\rho)\big)\,.$$

Since, for each equivariant homomorphism $\rho : G \to \mathrm{Out}(M)$, the bijection (11) restricts to a bijection $\Sigma : \big[\underline{\mathrm{dis}}^{\Gamma}G, \underline{\mathrm{Hol}}^{\Gamma}(M);\rho\big] \cong \mathrm{Ext}^{\Gamma}\big(G,(M,\rho)\big)$, Theorem 2.2 *i*) shows, in this particular case, a necessary and sufficient condition for the set $\mathrm{Ext}^{\Gamma}\big(G,(M,\rho)\big)$ to be non-empty in terms of a 3-dimensional equivariant cohomology class (5)

$$\mathrm{Obs}(M,\rho) \in H^3_{\Gamma}(G, \mathrm{Z}(M)^*)\,,$$

of the Γ-group G with coefficients in the equivariant G-module $\mathrm{Z}(M)^*$, where the G-action is via ρ (i.e., ${}^{x}u = {}^{\rho(x)}u$). We refer to this invariant as the *obstruction* of the equivariant G-kernel, and it is not hard to see that the general construction (6) of an equivariant 3-cocycle c^{ρ}, representing in this case the cohomology class $\mathrm{Obs}(M,\rho)$, particularizes as follows. In each automorphism class $\rho(x)$, $x \in G$, let us choose an automorphism f_x of M; in particular, we select $f_1 = \mathrm{id}_M$. Since $f_x f_y \in \rho(xy)$ and $f_{{}^{\sigma}x} \in {}^{\sigma}\rho(x)$, for $x,y \in G$ and $\sigma \in \Gamma$, we can select elements $\varphi(x,y), \varphi(x,\sigma) \in M^*$, such that $f_x f_y = C_{\varphi(x,y)}\, f_{xy}$ and ${}^{\sigma}f_x = C_{\varphi(x,\sigma)}\, f_{{}^{\sigma}x}$, with $\varphi(x,1) = 1 = \varphi(1,y) = \varphi(1,\sigma)$. Then, the pair of maps

$$f : G \to \mathrm{Aut}(M), \quad \varphi : G^2 \cup (G \times \Gamma) \to M^*\,,$$

satisfies conditions (13) and (14), although (15)-(17) need not be satisfied. The measurement of such a lack is precisely given by the equivariant 3-cocyle

$$c^{\rho} : G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \longrightarrow \mathrm{Z}(M)^*$$

determined by the equations:

$$\begin{aligned}
&c^{\rho}(x,y,z)\ \varphi(x,y)\ \varphi(xy,z) = f_x(\varphi(y,z))\,\varphi(x,yz)\,,\\
&c^{\rho}(x,y,\sigma)\ {}^{\sigma}\varphi(x,y)\ \varphi(xy,\sigma) = \varphi(x,\sigma)\ f_{{}^{\sigma}x}(\varphi(y,\sigma))\ \varphi({}^{\sigma}x,{}^{\sigma}y)\,,\\
&c^{\rho}(x,\sigma,\tau)\ \varphi(x,\sigma\tau) = {}^{\sigma}\varphi(x,\tau)\ \varphi({}^{\tau}x,\sigma)\,,
\end{aligned} \tag{23}$$

for $x,y,z \in G$, $\sigma,\tau \in \Gamma$.

If $\mathrm{Obs}(M,\rho) = 0$, so that there exists an equivariant crossed product extension $\Sigma(f,\varphi)$ of M by G that realizes ρ, then Theorem 2.2 *ii*) states the existence of a bijection

$H^2_\Gamma\big(G, \mathrm{Z}(M)^*\big) \cong \mathrm{Ext}^\Gamma\big(G,(M,\rho)\big)$. In this case, such a bijection associates to any equivariant 2-cohomology class represented by a 2-cocycle $c : G^2 \cup (G\times\Gamma) \to \mathrm{Z}(M)^*$, the equivalence class of the equivariant crossed product monoid extension $\Sigma(f, c\cdot\varphi)$, where $(c\cdot\varphi)(x,y) = c(x,y)\,\varphi(x,y)$ and $(c\cdot\varphi)(x,\sigma) = c(x,\sigma)\,\varphi(x,\sigma)$, for all $x, y \in G$ and $\sigma\in\Gamma$.

In summary, we have the following classification theorem for equivariant perfect Schreier extensions of a Γ-monoid by a Γ-group.

3.2. Theorem. *Let M be a Γ-monoid and G a Γ-group.*

(i) There is a canonical partition $\mathrm{Ext}^\Gamma(G,M) = \bigsqcup_\rho \mathrm{Ext}^\Gamma\big(G,(M,\rho)\big)$*, where, for each Γ-equivariant homomorphism $\rho : G \to \mathrm{Out}(M)$,* $\mathrm{Ext}_\Gamma\big(G,(M,\rho)\big)$ *is the set of classes of equivariant extensions of the equivariant G-kernel (M,ρ).*

(ii) Each equivariant G-kernel (M,ρ) invariably determines a 3-dimensional cohomology class $\mathrm{Obs}(M,\rho) \in H^3_\Gamma\big(G,\mathrm{Z}(M)^*\big)$.

(iii) An equivariant G-kernel (M,ρ) is realizable, that is, $\mathrm{Ext}^\Gamma\big(G,(M,\rho)\big) \neq \emptyset$*, if and only if its obstruction* $\mathrm{Obs}(M,\rho)$ *vanishes.*

(iv) If the obstruction of an equivariant G-kernel, (M,ρ), vanishes, then there is a bijection $\mathrm{Ext}^\Gamma\big(G,(M,\rho)\big) \cong H^2_\Gamma\big(G,\mathrm{Z}(M)^*\big)$.

To end this section, we should remark that the treatment of group extensions with operators as expounded in [6] (cf. [37]), is particularized from the above by requiring simply that the Γ-monoid M be a Γ-group.

4. Equivariant extensions of RINGS-GROUPS

Let us briefly recall from [25] that a RING-GROUP is a pair $[R,M]$ consisting of a unitary ring R together with a subgroup M of the group R^* of all units in R. A morphism of RINGS-GROUPS $f : [R,M] \to [S,N]$ is then a ring homomorphism $f : R \to S$ such that $f(M) \subseteq N$. If $[R,M]$ is any RING-GROUP and G is a group, then an *extension of* $[R,M]$ *by* G is a pair

$$\mathcal{E} = \big([R,M] \overset{i}{\rightarrowtail} [S,N], N \overset{p}{\twoheadrightarrow} G\big)\ , \tag{24}$$

in which i is an injective RING-GROUP morphism and p is a surjective group homomorphism, such that the following conditions hold:

i) The action of N on S by inner automorphisms is restricted to an action of N on R.

ii) $M \overset{i}{\rightarrowtail} N \overset{p}{\twoheadrightarrow} G$ is an extension of groups.

iii) S decomposes as a direct sum of subgroups $S = \oplus_{x\in G} S_x$, where, for each $x\in G$, $S_x = Rp^{-1}(x)R$ is the (R,R)-subbimodule of S generated by the set of elements of N mapped by p to x.

For any group M, isomorphism classes of extensions of the RING-GROUP $[\mathbb{Z}(M), M]$ by a group G are in bijection with isomorphism classes of group extensions of M by G; while for any ring R, isomorphism classes of extensions of the RING-GROUP $[R, R^*]$ by a group G are in bijection with isomorphism classes of crossed product rings of R by G [25, Example 1.6].

Here, we deal with RINGS-GROUPS with operators and their equivariant extensions by a group with the same operators. Let Γ be any fixed group. Then a Γ-RING-GROUP is a RING-GROUP $[R, M]$ on which Γ acts by RING-GROUP automorphisms, that is, a homomorphism $\Gamma \to \mathrm{Aut}[R, M]$ is given. Note that R is then a Γ-ring and M is an equivariant subgroup of the Γ-group of units R^*.

Given a Γ-RING-GROUP $[R, M]$ and a Γ-group G, then an *equivariant extension* of $[R, M]$ by G is an extension $\mathcal{E}$ of the RING-GROUP $[R, M]$ by the group G, say (24), in which $[S, N]$ is endowed with a Γ-action, that is, $[S, N]$ is a Γ-RING GROUP, such that both maps $i : R \rightarrowtail S$ and $p : N \twoheadrightarrow G$ are equivariant. We denote by

$$\mathrm{Ext}^{\Gamma}\big(G, [R, M]\big)$$

the set of isomorphism classes of Γ-equivariant extensions of $[R, M]$ by G.

Next we show a cohomological solution for the problem of classifying all equivariant RING-GROUP extensions for any prescribed pair $\big(G, [R, M]\big)$. The treatment is parallel to the known theory [24, 25] in the case in which $\Gamma = 1$, the trivial group. However, the proofs here bear only an incidental similarity with those by Hacque, since we derive the results on equivariant RING-GROUP extensions from the results obtained on the classification of graded categorical groups, that is, from Theorem 2.2.

Our conclusions can be summarized as follows.

If $[R, M]$ is a Γ-RING-GROUP, then the group $\mathrm{Aut}[R, M]$ of all RING-GROUP automorphisms of $[R, M]$ is also a Γ-group under the diagonal Γ-action, ${}^{\sigma}\!f : r \mapsto {}^{\sigma}\!f({}^{\sigma^{-1}}r)$, and the map $C : M \to \mathrm{Aut}[R, M]$ sending each element $m \in M$ into the inner automorphism given by conjugation with m, $C_m : r \mapsto mrm^{-1}$, is a Γ-group homomorphism. Then, the *centre-group* [24, Definition 2.3] of $[R, M]$, $\mathrm{Z}[R, M] = \mathrm{Ker}(C)$, and the group of *outer automorphisms* $\mathrm{Out}[R, M] = \mathrm{Aut}[R, M]/\mathrm{In}[R, M] = \mathrm{Coker}(C)$, are both Γ-groups. Furthermore, $\mathrm{Z}[R, M]$ is a Γ-equivariant $\mathrm{Out}[R, M]$-module with action ${}^{[f]}m = f(m)$.

If (24) is any equivariant extension of the Γ-RING-GROUP $[R, M]$ by the Γ-group G, then the assignment to each $n \in N$ of the operation of conjugation by n in S restricted to R, that is, the mapping $n \mapsto \rho_n \in \mathrm{Aut}[R, M]$ such that $i\rho_n(r) = n\, i(r)\, n^{-1}$, $r \in R$, induces an equivariant homomorphism (which depends only on the isomorphism class of $\mathcal{E}$),

$$\rho_{[\mathcal{E}]} : G \to \mathrm{Out}[R, M].$$

A pair $\big([R, M], \rho : G \to \mathrm{Out}[R, M]\big)$, where $[R, M]$ is a Γ-RING-GROUP and ρ is an equivariant homomorphism of groups, is termed an *equivariant collective character* of the Γ-group G in $[R, M]$ (cf. [25, Section 2]).

We state the following theorem:

4.1. Theorem. *Let $[R, M]$ be a Γ-RING-GROUP and let G be a Γ-group.*

(i) There is a canonical partition

$$\mathrm{Ext}^{\Gamma}\big(G, [R, M]\big) = \bigsqcup_{\rho} \mathrm{Ext}^{\Gamma}\big(G, [R, M]; \rho\big)\,,$$

where, for each equivariant collective character $\rho : G \to Out[R, M]$, $\mathrm{Ext}^{\Gamma}\big(G, [R, M]; \rho\big)$ is the set of isomorphism classes of equivariant extensions $\mathcal{E}$ that realize ρ*, that is, such that $\rho_{[\mathcal{E}]} = \rho$.*

(ii) Each equivariant collective character invariably determines a 3-dimensional equivariant cohomology class

$$\mathrm{T}(\rho) \in H^3_{\Gamma}\big(G, \mathrm{Z}[R, M]\big)\,,$$

of G with coefficients in the centre-group of the RING-GROUP (with respect to the Γ-equivariant G-module structure on $\mathrm{Z}[R, M]$ obtained via ρ). This invariant is called the "Teichmüller obstruction" of ρ.

(iii) An equivariant collective character ρ is realizable, that is, $\mathrm{Ext}^{\Gamma}\big(G, [R, M]; \rho\big) \neq \emptyset$, if and only if its obstruction vanishes, that is, $T(\rho) = 0$.

(iv) If the obstruction of an equivariant collective character $\rho : G \to \mathrm{Out}[R, M]$ vanishes, then there is a bijection

$$\mathrm{Ext}^{\Gamma}\big(G, [R, M]; \rho\big) \cong H^2_{\Gamma}\big(G, \mathrm{Z}[R, M]\big)\,.$$

As we will show later, Theorem 4.1 is a specialization of Theorem 2.2 for the particular Γ-graded categorical group $\mathbb{H} = \underline{\mathrm{Hol}}^{\Gamma}[R, M]$, the *holomorph* graded categorical group of the Γ-RING-GROUP $[R, M]$, which is defined below similarly to the holomorph graded categorical group of a Γ-monoide (see Section 3). The objects of $\underline{\mathrm{Hol}}^{\Gamma}[R, M]$ are the elements of the Γ-group $\mathrm{Aut}[R, M]$. A morphism of grade $\sigma \in \Gamma$ from f to g is a pair $(m, \sigma) : f \to g$, where $m \in M$, with ${}^{\sigma}\!f = C_m\, g$. The composition of morphisms is given by equality (9), the graded tensor product is given by equality (10) and the graded unit $I : \Gamma \to \underline{\mathrm{Hol}}^{\Gamma}[R, M]$ is defined by $I(\sigma) = \mathrm{id}_{R} \xrightarrow{(1,\sigma)} \mathrm{id}_{R}$. The associativity and unit constraints are identities.

We develop next the device of *mixed crossed products* for equivariant extensions of RINGS-GROUPS by groups, such as Hacque did in [25] in the non-equivariant case. These constructions allow us to show how the graded monoidal functors $\underline{\mathrm{dis}}^{\Gamma} G \to \underline{\mathrm{Hol}}^{\Gamma}[R, M]$ are the appropriate systems of data to construct the manifold of all equivariant extensions of $[R, M]$ by G.

4.2. Theorem. (Structure of equivariant extensions of RINGS-GROUPS) *For any Γ-RING-GROUP $[R, M]$ and Γ-group G, there is a bijection*

$$\Sigma : \big[\underline{\mathrm{dis}}^{\Gamma} G, \underline{\mathrm{Hol}}^{\Gamma}[R, M]\big] \cong \mathrm{Ext}^{\Gamma}\big(G, [R, M]\big)\,, \tag{25}$$

between the set of homotopy classes of graded monoidal functors from $\underline{\mathrm{dis}}^\Gamma G$ *to* $\underline{\mathrm{Hol}}^\Gamma[R,M]$ *and the set of equivalence classes of equivariant extensions of* $[R,M]$ *by* G.

Proof. This is quite parallel to the proof of Theorem 3.1, to which we refer for some details. First, observe that the data describing a strictly unitary graded monoidal functor $(F,\Phi):\underline{\mathrm{dis}}^\Gamma G\to\underline{\mathrm{Hol}}^\Gamma[R,M]$ consist of a pair of maps (f,φ), where

$$f:G\to\mathrm{Aut}[R,M],\quad \varphi:G^2\cup(G\times\Gamma)\to M,$$

such that we write $F\big(\sigma:x\to y\big)=(\varphi(x,\sigma),\sigma):f_x\to f_y$ and $\Phi_{x,y}=(\varphi(x,y),1):f_xf_y\to f_{xy}$, for all $x,y\in G$ and $\sigma\in\Gamma$. Then, the conditions of (F,Φ) being a graded monoidal functor in terms of (f,φ), are precisely those given by the five equations (13)-(17). A pair (f,φ) describing a graded monoidal functor from $\underline{\mathrm{dis}}^\Gamma G$ to $\underline{\mathrm{Hol}}^\Gamma[R,M]$, is what we call an *equivariant factor set*, or a non-abelian 2-cocycle of a Γ-group G with coefficients in a Γ-RING-GROUP $[R,M]$. Further, if (f',φ') is another factor set describing the graded monoidal functor $(F',\Phi'):\underline{\mathrm{dis}}^\Gamma G\to\underline{\mathrm{Hol}}^\Gamma[R,M]$, then any homotopy $\theta:F\to F'$ is given by a map $g:G\to M$, such that one writes $\theta_x=(g(x),1):f_x\to f'_x$, for all $x\in G$. In terms of this map g, the conditions for θ being a homotopy are those described by the four equations (18)-(21).

Let us stress that when $\Gamma=\mathbf{1}$, the trivial group, then a factor set is exactly a Hacque's *mixed system of factor sets* for a RING-GROUP extension of $[R,M]$ by G [25]. Particularly, if $M=R^*$, then such a pair (f,φ) is just a factor set for a crossed product ring of R by G (cf. [29]). Furthermore, for M and G any two Γ-groups, if one considers the Γ-RING-GROUP $[R,M]=[\mathbb{Z}(M),M]$, then a pair (f,φ) is a Schreier system of factor sets for an equivariant group extension of M by G (cf. [6]).

Every equivariant factor set (f,φ) provides a Γ-equivariant RING-GROUP extension of $[R,M]$ by G,

$$\Sigma(f,\varphi):\left([R,M]\overset{i}{\rightarrowtail}[R\star_{f,\varphi}G,M\times_{f,\varphi}G],\ M\times_{f,\varphi}G\overset{p}{\twoheadrightarrow}G\right),\tag{26}$$

which we term an *equivariant mixed crossed product extension.* In it, the crossed product ring $R\star_{f,\varphi}G=\oplus_x R\times\{x\}$ is the free left R-module with basis $\{(1,x),\,x\in G\}$, with multiplication according to the rule $(r,x)\,(s,y)=(r\,f_x(s)\,\varphi(x,y),xy)$, and the crossed product group $M\times_{f,\varphi}G$ is the subgroup $M\times G\subseteq\big(R\star_{f,\varphi}G\big)^*$. The Γ-action on $[R\star_{f,\varphi}G,M\times_{f,\varphi}G]$ is given by ${}^\sigma(r,x)=({}^\sigma r\,\varphi(x,\sigma),{}^\sigma x)$ and the equivariant homomorphisms i and p are defined by $i(r)=(r,1)$ and $p(r,x)=x$, respectively. Note that, for each $x\in G$, $Rp^{-1}(x)R=R\times\{x\}$.

If (f',φ') is another equivariant factor set such that there is an equivariant RING-GROUP isomorphism, say $\mathbf{g}:[R\star_{f',\varphi'}G,M\times_{f',\varphi'}G]\to[R\star_{f,\varphi}G,M\times_{f,\varphi}G]$, establishing an isomorphism between the corresponding equivariant mixed crossed product extensions $\Sigma(f',\varphi')$ and $\Sigma(f,\varphi)$, then we can write $\mathbf{g}(r,x)=(rg(x),x)$ for a map $g:G\to M$ satisfying the equalities (18)-(21). And conversely, the existence of a map $g:G\to M$

satisfying such equations implies that $\Sigma(f',\varphi')$ and $\Sigma(f,\varphi)$ are isomorphic equivariant RING-GROUP extensions, just by the map $\mathbf{g}:(r,x)\mapsto(r\,g(x),x)$. Hence, the equivariant mixed crossed product construction induces an injective map $\Sigma:\big[\underline{\mathrm{dis}}^\Gamma G,\underline{\mathrm{Hol}}^\Gamma[R,M]\big]\longrightarrow \mathrm{Ext}^\Gamma\big(G,[R,M]\big)$.

To complete the proof of the theorem, it only remains to prove that every equivariant extension of $[R,M]$ by G, say $\mathcal{E}=\big([R,M]\overset{i}{\rightarrowtail}[S,N],N\overset{p}{\twoheadrightarrow}G\big)$, has an associated equivariant factor set, that is, it is isomorphic to an equivariant mixed crossed product extension $\Sigma(f,\varphi)$ for some (f,φ) as above. To do so, we assume that i is the inclusion map, and, for each $x\in G$, let us choose a representative $u_x\in N$, with $u_1=1$. Then, there are unique elements $\varphi(x,y),\varphi(x,\sigma)\in M$, such that $u_x\,u_y=\varphi(x,y)\,u_{xy}$ and ${}^{\sigma}u_x=\varphi(x,\sigma)\,u_{{}^\sigma x}$. Moreover, each $x\in G$ induces an automorphism f_x of $[R,M]$, $f_x:r\mapsto u_x\,r\,u_x^{-1}$. The pair of maps so defined $(f:G\to\mathrm{Aut}[R,M],\varphi:G^2\cup(G\times\Gamma)\to M)$ is actually an equivariant factor set, which we recognize as an equivariant factor set for the given equivariant RING-GROUP extension by the existence of the Γ-equivariant RING-GROUP isomorphism $\mathbf{g}:[R\star_{f,\varphi}G,M\times_{f,\varphi}G]\to[S,N]$, $\sum_x(r_x,x)\mapsto\sum_x r_x\,u_x$. It is straightforward to see that $\mathbf{g}$ is an equivariant RING-GROUP homomorphism. To prove that it is actually an isomorphism, let us recall that $S=\oplus_x S_x$ with $S_x=Rp^{-1}(x)R$. Since for each $x\in G$, $p^{-1}(x)=u_xM=Mu_x$, we have $S_x=Ru_xMR=u_xRMR=u_xR=Ru_x$. It follows that the set $\{u_x,\,x\in G\}$ is a basis of S as a left R-module and therefore the map $\mathbf{g}:R\star_{f,\varphi}G\to S$ is a bijection, which clearly restricts to a bijection between $M\times_{f,\varphi}G$ and N. ∎

The bijection (25) is all one needs to obtain, from Theorem 2.2, the classification of equivariant RING-GROUP extensions as stated in Theorem 4.1. Indeed, for any Γ-RING-GROUP $[R,M]$, the homotopy groups of the Γ-graded categorical group $\underline{\mathrm{Hol}}^\Gamma[R,M]$ are

$$\pi_0\underline{\mathrm{Hol}}^\Gamma[R,M]=\mathrm{Coker}\big(M\overset{C}{\longrightarrow}\mathrm{Aut}[R,M]\big)=\mathrm{Out}[R,M],$$

$$\pi_1\underline{\mathrm{Hol}}^\Gamma[R,M]=\mathrm{Ker}\big(M\overset{C}{\longrightarrow}\mathrm{Aut}[R,M]\big)=\mathrm{Z}[R,M],$$

and the Teichmüller obstruction $\mathrm{T}(\rho)\in H^3_\Gamma\big(G,\mathrm{Z}[R,M]\big)$ in part (ii) of Theorem 4.1, of any equivariant collective character $\rho:G\to\mathrm{Out}[R,M]$, is defined to be precisely the cohomology class $\mathrm{Obs}(\rho)$ in (5) for the particular graded categorical group here considered $\mathbb{H}=\underline{\mathrm{Hol}}^\Gamma[R,M]$.

Moreover, for any factor set (f,φ) describing a graded monoidal functor $F:\underline{\mathrm{dis}}^\Gamma G\to\underline{\mathrm{Hol}}^\Gamma[R,M]$, we have $\pi_0F=\rho_{[\Sigma(f,\varphi)]}$. Therefore, bijection (25) provides, by restriction, bijections $\Sigma:\big[\underline{\mathrm{dis}}^\Gamma G,\underline{\mathrm{Hol}}^\Gamma[R,M];\rho\big]\cong\mathrm{Ext}^\Gamma\big(G,[R,M];\rho\big)$, for any equivariant collective character $\rho:G\to\mathrm{Out}[R,M]$. Hence, parts (iii) and (iv) of Theorem 4.1 follow from Theorem 2.2.

For any given equivariant collective character $\rho:G\to\mathrm{Out}[R,M]$, the general construction of a representative equivariant 3-cocycle c^ρ of the Teichmüller obstruction $\mathrm{T}(\rho)$ in (6) works in this case as follows. In each automorphism class $\rho(x)$, $x\in G$, let us choose an automorphism f_x of $[R,M]$; in particular, select $f_1=\mathrm{id}_R$. Then, for $x,y\in G$

and $\sigma \in \Gamma$, we can select elements $\varphi(x,y), \varphi(x,\sigma) \in M$ such that $f_x f_y = C_{\varphi(x,y)}\, f_{xy}$ and ${}^{\sigma}f_x = C_{\varphi(x,\sigma)}\, f_{{}^{\sigma}x}$, with $\varphi(x,1) = 1 = \varphi(1,y) = \varphi(1,\sigma)$. Then, the equivariant 3-cocycle

$$c^{\rho} : G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \longrightarrow \mathrm{Z}[R,M]$$

is determined by the three equations (23).

If $\mathrm{T}(\rho) = 0$, so that there exists an equivariant mixed crossed product extension $\Sigma(f,\varphi)$ that realizes ρ, then the bijection $\mathrm{Ext}^{\Gamma}\big(G,[R,M];\rho\big) \cong H^2_{\Gamma}\big(G,\mathrm{Z}[R,M]\big)$ associates to any 2-cohomology class represented by a 2-cocycle $c : G^2 \cup (G \times \Gamma) \to \mathrm{Z}[R,M]$, the isomorphism class of the equivariant mixed crossed product $\Sigma(f, c{\cdot}\varphi)$, where $(c{\cdot}\varphi)(x,y) = c(x,y)\,\varphi(x,y)$ and $(c\cdot\varphi)(x,\sigma) = c(x,\sigma)\,\varphi(x,\sigma)$, for all $x,y \in G$ and $\sigma \in \Gamma$.

As a final comment here, we shall stress that, similarly as in the non-equivariant case (cf. [25]), the theory presented of equivariant RING-GROUP extensions particularizes on the one hand to the theory of equivariant extensions of groups with operators as stated in [6] (just by considering the RINGS-GROUPS with operators $[\mathbb{Z}(M), M]$ for any group with operators M), and, on the other hand, to the theory of crossed product rings with operators (simply by considering the RINGS-GROUPS with operators $[R, R^*]$ for any ring with operators R).

5. Graded Clifford systems with operators

Throughout this section, k is a unitary commutative ring and all algebras are unitary k-algebras. Further, Γ is a fixed group of operators and a Γ-algebra is an algebra provided with a Γ-action by automorphisms $(\sigma, r) \mapsto {}^{\sigma}r$.

A *graded Clifford system* (over k) is a triple $\big(S, \{S_x\}_{x\in G}, G\big)$, where S is an algebra, G is a group and $\{S_x\}_{x\in G}$ is a family of k-submodules of S, one for each $x \in G$, such that $S = \bigoplus_{x\in G} S_x$ and $S_x S_y = S_{xy}$ for all $x,y \in G$. A *morphism* of graded Clifford systems is a pair $(h,\alpha) : \big(S, \{S_x\}_{x\in G}, G\big) \to \big(S', \{S'_x\}_{x\in G'}, G'\big)$, where $h : S \to S'$ is a homomorphism of algebras and $\alpha : G \to G'$ is a homomorphism of groups, such that $h(S_x) \subseteq S'_{\alpha(x)}$ for all $x \in G$. Graded Clifford systems were introduced and applied by E. Dade [12] to develop Clifford's theory axiomatically, but they have also been extensively studied with the terminology of *strongly graded algebras* [13] or with that of *generalized crossed product algebras* [32]. Our goal in this section is the classification of graded Clifford systems $\underline{S}$ on which a group of operators Γ acts by automorphisms; that is, a homomorphism $\Gamma \to \mathrm{Aut}\big(S, \{S_x\}_{x\in G}, G\big)$ is given. In such a case, we refer to $\big(S, \{S_x\}_{x\in G}, G\big)$ as a *graded* Γ*-Clifford system.*

Whenever $\big(S, \{S_x\}_{x\in G}, G\big)$ is a graded Γ-Clifford system, then G is a Γ-group and S is a Γ-algebra. Furthermore, the 1-component S_1 is a Γ-subalgebra of S and then it is natural to see S as an equivariant extension of the Γ-algebra S_1 by the Γ-group G. More precisely, we establish the notion below:

5.1. Definition. *Let R be a Γ-algebra and G a Γ-group. A* (G,R)-graded Γ-Clifford system *is a pair $(\underline{S}, j_S)$, where $\underline{S} = \big(S, \{S_x\}_{x\in G}, G\big)$ is a graded Γ-Clifford system such that* *the* Γ*-action on* G *is the given one, that is,* ${}^{\sigma}S_x = S_{{}^{\sigma}x}$*, and* $j_{_S} : R \cong S_1$ *is an isomorphism of* Γ*-algebras.*

The most basic example is the group algebra $R(G) = \bigoplus_{x\in G} R \times \{x\}$, which is a Γ-algebra with the action ${}^{\sigma}(r,x) = ({}^{\sigma}r, {}^{\sigma}x)$, but also, equivariant crossed products of Γ-algebras by Γ-groups (considered in Subsection 5.4), yield examples of (G,R)-graded Γ-Clifford systems.

If $(\underline{S}, j_S)$ and $(\underline{T}, j_T)$ are two (G,R)-graded Γ-Clifford systems, by a *morphism* $h : (\underline{S}, j_S) \to (\underline{T}, j_T)$ we mean a Γ-equivariant homomorphism of algebras $h : S \to T$, so that $h({}^{\sigma}s) = {}^{\sigma}h(s)$ for $\sigma \in \Gamma$ and $s \in S$, which is grade-preserving, in the sense that $h(S_x) \subseteq T_x$ for all $x \in G$, and such that $hj_{_S} = j_{_T}$. It follows from [13, Corollary 2.10] that such a morphism is always an isomorphism and

$$\mathrm{Cliff}_k^{\Gamma}(G,R)$$

will denote the set of isomorphism classes of (G,R)-graded Γ-Clifford systems, for any prescribed Γ-algebra R and Γ-group G.

The general results about abstract graded categorical groups can be used now to obtain an appropriate treatment of the equivariant graded Clifford systems. The key for such a treatment lies in the observation we will detail below that giving a (G,R)-graded Γ-Clifford system is essentially the same as giving a Γ-graded monoidal functor from the discrete Γ-graded categorical group $\underline{\mathrm{dis}}^{\Gamma}G$ to the Γ*-graded Picard categorical group*, of the Γ-algebra R, denoted here by

$$\underline{\mathrm{Pic}}_k^{\Gamma}(R)\,,$$

which is an example of Γ-graded categorical group originally considered by Fröhlich and Wall in [20]. This graded categorical group, canonically built from any Γ-algebra R, has, as objects, the invertible (R,R)-bimodules over k. A morphism $P \to P'$ in $\underline{\mathrm{Pic}}_k^{\Gamma}(R)$, of grade $\sigma \in \Gamma$, is a pair (φ,σ), where $\varphi : P \xrightarrow{\sim} P'$ is an isomorphism of k-modules with $\varphi(rp) = {}^{\sigma}r\,\varphi(p)$ and $\varphi(pr) = \varphi(p)\,{}^{\sigma}r$ for all $r \in R$ and $p \in P$. The composition of (φ,σ) with another morphism $(\varphi',\sigma') : P' \to P''$ is given by $(\varphi',\sigma')(\varphi,\sigma) = (\varphi'\varphi, \sigma'\sigma)$. The graded monoidal product $\otimes : \underline{\mathrm{Pic}}_k^{\Gamma}(R) \times_{\Gamma} \underline{\mathrm{Pic}}_k^{\Gamma}(R) \to \underline{\mathrm{Pic}}_k^{\Gamma}(R)$ is defined by the usual tensor product of bimodules

$$(P \xrightarrow{(\varphi,\sigma)} P') \otimes (Q \xrightarrow{(\psi,\sigma)} Q') \;=\; (P \otimes_R Q \xrightarrow{(\varphi\otimes_R\psi,\sigma)} P' \otimes_R Q')\,,$$

and the graded unit $I : \Gamma \to \underline{\mathrm{Pic}}_k^{\Gamma}(R)$ is defined by $I(\sigma) = (\sigma,\sigma) : R \to R$, $r \mapsto {}^{\sigma}r$. The associativity and unit 1-graded constraints are the usual isomorphisms for the tensor product of bimodules.

Now, suppose that a Γ-group G and a Γ-algebra R are given. Then, a strictly unitary Γ-graded monoidal functor $(F,\Phi) : \underline{\mathrm{dis}}^{\Gamma}G \to \underline{\mathrm{Pic}}_k^{\Gamma}(R)$ consists of: i) a family of invertible (R,R)-bimodules F_x, $x \in G$; ii) isomorphisms of k-modules

$$F_{x,\sigma} : F_x \xrightarrow{\sim} F_{{}^{\sigma}x}, \quad \sigma \in \Gamma,\ x \in G, \tag{27}$$

such that $F_{x,\sigma}(r\,a) = {}^{\sigma}r\, F_{x,\sigma}(a)$ and $F_{x,\sigma}(a\,r) = F_{x,\sigma}(a)\, {}^{\sigma}r$, for any $r \in R$ and $a \in F_x$; and iii) isomorphisms of (R,R)-bimodules

$$\Phi_{x,y} : F_x \otimes_R F_y \xrightarrow{\sim} F_{xy}, \quad x, y \in G, \tag{28}$$

such that the following conditions hold:

$$\begin{aligned}
&F_1 = R, \quad F_{x,1} = \mathrm{id}_{F_x}, \quad F_{1,\sigma} = \mathrm{id}_R, \quad \Phi_{x,1} : a \otimes r \mapsto a\,r, \;\; \Phi_{1,y} : r \otimes a \mapsto r\,a,\\
&F_{x,\tau\sigma} = F_{{}^{\sigma}x,\tau}\, F_{x,\sigma}\,,\\
&\Phi_{x,yz}\,(\mathrm{id}_{F_x} \otimes_R \Phi_{y,z}) = \Phi_{xy,z}\,(\Phi_{x,y} \otimes_R \mathrm{id}_{F_z})\,,\\
&F_{xy,\sigma}\, \Phi_{x,y} = \Phi_{{}^{\sigma}x,{}^{\sigma}y}\,(F_{x,\sigma} \otimes_R F_{y,\sigma})\,.
\end{aligned}$$

Thus, such a graded monoidal functor should be termed an *equivariant factor set*, since when $\Gamma = 1$, this is exactly a factor set in the sense of Kanzaki [27]. Indeed, for any graded monoidal functor (F,Φ), the family of the invertible (R,R)-bimodules F_x, $x \in G$, together with the isomorphisms of bimodules $\Phi_{x,y}$, $x, y \in G$, is actually a Kanzaki factor set and so (F,Φ) gives rise to a *generalized crossed product algebra* in his sense; that is, $\Sigma(F,\Phi) = \bigoplus_{x\in G} F_x$, where the product of the elements $a \in F_x$ and $b \in F_y$ is defined by $a\,b = \Phi_{x,y}(a \otimes b)$. Moreover, thanks to the isomorphisms $F_{x,\sigma}$, the group Γ acts on $\Sigma(F,\Phi)$ by the rule ${}^{\sigma}a = F_{x,\sigma}(a)$ if $a \in F_x$. In this way, $\underline{\Sigma}(F,\Phi) = \big(\Sigma(F,\Phi), \{F_x\}, G\big)$, together with the identity $R = F_1$, is a (G,R)-graded Γ-Clifford system, which we refer to as the *generalized crossed product (G,R)-graded Γ-Clifford system with equivariant factor set (F,Φ)*.

Suppose now that $F' = (F',\Phi') : \underline{\mathrm{dis}}_{\Gamma} G \to \underline{\mathrm{Pic}}^{\Gamma}_{k}(R)$ is another strictly unitary graded monoidal functor. Then, a homotopy $\theta : (F,\Phi) \to (F',\Phi')$ consists of a family of (R,R)-bimodule isomorphisms $\theta_x : F_x \xrightarrow{\sim} F'_x$, $x \in G$, satisfying that $\theta_1 = \mathrm{id}_R$, $\theta_{{}^{\sigma}x}\, F_{x,\sigma} = F'_{x,\sigma}\, \theta_x$ and $\theta_{xy}\, \Phi_{x,y} = \Phi'_{x,y}\,(\theta_x \otimes_R \theta_y)$, for all $x, y \in G$ and $\sigma \in \Gamma$. Therefore, it is clear that giving such a homotopy is equivalent to giving an isomorphism of Γ-algebras, $\Sigma(F,\Phi) \xrightarrow{\sim} \Sigma(F',\Phi')$, $a \mapsto \theta_x(a)$, $a \in F_x$, which establishes an isomorphism of (G,R)-graded Γ-Clifford systems $\theta : (\underline{\Sigma}(F,\Phi), \mathrm{id}_R) \xrightarrow{\sim} (\underline{\Sigma}(F',\Phi'), \mathrm{id}_R)$.

Moreover, any (G,R)-graded Γ-Clifford system $(\underline{S}, j_S)$, $\underline{S} = \big(S, \{S_x\}_{x\in G}, G\big)$, is actually isomorphic to a generalized crossed product (G,R)-graded Γ-Clifford system for a certain equivariant factor set (F,Φ). Indeed, up to an isomorphism, we can assume that $R = S_1$ and $j_S = \mathrm{id}_R$. By [32, Proposition 2.5], for any $x, y \in G$, the canonical multiplication map $\Phi_{x,y} : S_x \otimes_R S_y \cong S_{xy}$, $a \otimes b \mapsto a\,b$, is a (R,R)-bimodule isomorphism and each component S_x is an invertible (R,R)-bimodule. Furthermore, the Γ-action on S determines, for all $x \in G$ and $\sigma \in \Gamma$, isomorphisms of k-modules $F_{x,\sigma} : S_x \cong S_{{}^{\sigma}x}$, $a \mapsto {}^{\sigma}a$, and it is straightforward to see that, in this way, we have a strictly unitary graded monoidal functor $(F,\Phi) : \underline{\mathrm{dis}}^{\Gamma} G \to \underline{\mathrm{Pic}}^{\Gamma}_{k}(R)$, with $F_x = S_x$ for each $x \in G$, whose associated generalized crossed product (G,R)-graded Γ-Clifford system $(\underline{\Sigma}(F,\Phi), \mathrm{id}_R)$ is exactly $(\underline{S}, \mathrm{id}_R)$.

All in all, we state the following:

5.2. Theorem. *For any Γ-algebra R and any Γ-group G, the mapping carrying any strictly unitary graded monoidal functor to the associated generalized crossed product (G,R)-graded Γ-Clifford system, induces a bijection*

$$\underline{\Sigma} : \big[\underline{\mathrm{dis}}^{\Gamma} G, \underline{\mathrm{Pic}}_k^{\Gamma}(R)\big] \cong \mathrm{Cliff}_k^{\Gamma}(G,R)\,, \tag{29}$$

between the set of homotopy classes of graded monoidal functors from $\underline{\mathrm{dis}}^{\Gamma}G$ to $\underline{\mathrm{Pic}}_k^{\Gamma}(R)$ and the set of isomorphism classes of (G,R)-graded Γ-Clifford systems.

Now, let us remark that, for any Γ-algebra R, the homotopy groups of the Γ-graded categorical group $\underline{\mathrm{Pic}}_k^{\Gamma}(R)$ are:

$\pi_0\underline{\mathrm{Pic}}_k^{\Gamma}(R) = \mathrm{Pic}_k(R)$, the Γ-group of isomorphism classes $[P]$ of invertible (R,R)-bimodules over k, where the multiplication is induced by the tensor product, $[P]\,[Q] = [P\otimes_R Q]$, and the Γ-action is given by

$$^{\sigma}[P] = [{}_{\sigma}P_{\sigma}]\,, \tag{30}$$

where ${}_{\sigma}P_{\sigma}$ denotes the invertible (R,R)-bimodule which is the same k-module as P with R-actions $r\cdot p = {}^{\sigma^{-1}}\!r\,p$ and $p\cdot r = p\,{}^{\sigma^{-1}}\!r$, for $r\in R$ and $p\in P$;

$\pi_1\underline{\mathrm{Pic}}_k^{\Gamma}(R) = Z(R)^*$, the Γ-equivariant $\mathrm{Pic}_k(R)$-module of all units in the center of R. The Γ-action is given by restriction of the one on R and the action of $\mathrm{Pic}_k(R)$ on $Z(R)^*$ is given by the restriction to $Z(R)^*$ of Bass' homomorphism $\mathrm{Pic}_k(R)\to\mathrm{Aut}_k(Z(R))$ [1], that is, for each $[P]\in\mathrm{Pic}_k(R)$ and $u\in Z(R)^*$, ${}^{[P]}u$ is the element in $Z(R)^*$ uniquely determined by the equality ${}^{[P]}u\,p = p\,u$ for all $p\in P$.

Every (G,R)-graded Γ-Clifford system $(\underline{S} = \big(S,\{S_x\}_{x\in G},G\big), j_S)$ induces a Γ-equivariant homomorphism (which depends only on its isomorphism class $[\underline{S}]$)

$$\chi_{[\underline{S},j_S]} : G\to\mathrm{Pic}_k(R), \quad x\mapsto [S_x]\ , \quad x\in G\ .$$

Hence, there is a canonical map

$$\chi : \mathrm{Cliff}_k^{\Gamma}(G,R) \longrightarrow \mathrm{Hom}_{\Gamma}(G,\mathrm{Pic}_k(R))\ , \tag{31}$$

where $Hom_{\Gamma}(G,\mathrm{Pic}_k(R))$ is the set of Γ-equivariant group homomorphisms of G into $\mathrm{Pic}_k(R)$, which clearly makes this diagram commutative:

$$\begin{tikzcd}
\big[\underline{\mathrm{dis}}^{\Gamma}G, \underline{\mathrm{Pic}}_k^{\Gamma}(R)\big] \arrow[rr, "\underline{\Sigma}"] \arrow[dr, "\pi_0"'] & & \mathrm{Cliff}_k^{\Gamma}(G,R) \arrow[dl, "\chi"] \\
& \mathrm{Hom}_{\Gamma}(G,\mathrm{Pic}_k(R))\,. &
\end{tikzcd} \tag{32}$$

We say that any Γ-equivariant group homomorphism $\rho : G\to\mathrm{Pic}_k(R)$ is an *equivariant generalized collective character* of the Γ-group G in the Γ-algebra R, and we say that a

(G,R)-graded Γ-Clifford system $(\underline{S}, j_S)$ *realizes* such an equivariant generalized collective character ρ whenever $\chi_{[\underline{S},j_S]} = \rho$. The map χ provides a partitioning

$$\mathrm{Cliff}_k^\Gamma(G,R) = \bigsqcup_\rho \mathrm{Cliff}_k^\Gamma(G,R;\rho)\,,$$

where $\mathrm{Cliff}_k^\Gamma(G,R;\rho)$ is the set isoclasses of (G,R)-graded Γ-Clifford systems that realize each Γ-equivariant generalized collective character $\rho : G \to \mathrm{Pic}_k(R)$. Due to the commutativity of triangle (32), bijection (29) restricts to bijections $\underline{\Sigma} : \big[\underline{\mathrm{dis}}^\Gamma G, \underline{\mathrm{Pic}}_k^\Gamma(R);\rho\big] \cong \mathrm{Cliff}_k^\Gamma(G,R;\rho)$, and then Theorem 2.2 *i*) shows here a necessary and sufficient condition for the set $\mathrm{Cliff}_k^\Gamma(G,R;\rho)$ to be non-empty in terms of a 3-dimensional equivariant cohomology class (5)

$$\mathrm{T}(\rho) \in H^3_\Gamma(G, \mathrm{Z}(R)^*) \tag{33}$$

of the Γ-group G with coefficients in the equivariant G-module $\mathrm{Z}(R)^*$, where the G-action is via ρ (i.e., ${}^x u = {}^{\rho(x)}u$). We refer to this invariant as the *Teichmüller obstruction* of ρ since its construction, outlined below, has a clear precedent in the *Teichmüller cocycle map* $Br_G(R) \to H^3(G,R^*)$, defined for any finite Galois field extension R/k with group G. The general construction (6) of an equivariant 3-cocycle c^ρ representing the cohomology class $\mathrm{T}(\rho)$ particularizes as follows. In each isomorphism class $\rho(x)$, $x \in G$, let us choose an invertible (R,R)-bimodule P_x; in particular, select $P_{_1} = R$. Since ρ is a Γ-equivariant homomorphism, we can select (R,R)-bimodule isomorphisms $\Phi_{x,y} : P_x \otimes_R P_y \cong P_{xy}$ and $\Phi_{x,\sigma} : {}_\sigma(P_x)_\sigma \cong P_{{}^\sigma x}$ for every $x,y \in G$ and $\sigma \in \Gamma$, chosen as the canonical ones whenever either x,y or σ are identities. For any three elements $x,y,z \in G$, the isomorphisms of (R,R)-bimodules $\Phi_{xy,z}(\Phi_{x,y} \otimes_R \mathrm{id}_{P_z})$ and $\Phi_{x,yz}(\mathrm{id}_{P_x} \otimes_R \Phi_{y,z})$ from $P_x \otimes_R P_y \otimes_R P_z$ to P_{xyz} need not coincide, but then there is (cf. [9, Lemma 3.1] for example) a unique element $c^\rho(x,y,z) \in Z(R)^*$ such that

$$c^\rho(x,y,z)\ \Phi_{xy,z}\big(\Phi_{x,y}(p_x \otimes p_y) \otimes p_z\big) = \Phi_{x,yz}\big(p_x \otimes \Phi_{y,z}(p_y \otimes p_z)\big),$$

for all $p_x \in P_x$, $p_y \in P_y$ and $p_z \in P_z$. Analogously, for any $x,y \in G$ and $\sigma,\tau \in \Gamma$, there are unique elements $c^\rho(x,y,\sigma)$ and $c^\rho(x,\sigma,\tau)$ in $Z(R)^*$ such that

$$\begin{aligned}
&c^\rho(x,y,\sigma)\ \Phi_{{}^\sigma x,{}^\sigma y}\big(\Phi_{x,\sigma}(p_x) \otimes \Phi_{y,\sigma}(p_y)\big) = \Phi_{xy,\sigma}\big(\Phi_{x,y}(p_x \otimes p_y)\big)\\
&c^\rho(x,\sigma,\tau)\ \Phi_{x,\sigma\tau}(p_x) = \Phi_{{}^\tau x,\sigma}\big(\Phi_{x,\tau}(p_x)\big).
\end{aligned}$$

The resulting map is exactly the equivariant 3-cocycle

$$c^\rho : G^3 \cup (G^2 \times \Gamma) \cup (G \times \Gamma^2) \longrightarrow Z(R)^*. \tag{34}$$

If $\mathrm{T}(\rho) = 0$, then Theorem 2.2 *ii*) states the existence of a bijection

$$H^2_\Gamma\big(G, \mathrm{Z}(R)^*\big) \cong \mathrm{Cliff}_k^\Gamma\big(G,R;\rho\big)\,, \tag{35}$$

that can be described as follows. Once we have chosen any (G,R)-graded Γ-Clifford system $\big((S,\{S_x\}_{x\in G},G), j_S\big)$ that realizes ρ, then the bijection associates to any equivariant 2-cohomology class represented by a 2-cocycle $c : G^2 \cup (G \times \Gamma) \to \mathrm{Z}(R)^*$ the class of the

(G,R)-graded Γ-Clifford system $\big((c{\star}S,\{S_x\}_{x\in G},G),j_S\big)$, where $c{\star}S$ is the same k-module as S with product $s_x\star s_y=c(x,y)s_xs_y$ and Γ-action $\sigma\star s_x=c(x,\sigma)\,{}^{\sigma}x$, for all $s_x\in S_x$, $s_y\in S_y$ and $\sigma\in\Gamma$.

In summary, we have the following classification theorem:

5.3. Theorem. *Let R be a Γ-algebra and G a Γ-group.*

(i) There is a canonical partition $\mathrm{Cliff}_k^\Gamma(G,R)=\bigsqcup_\rho \mathrm{Cliff}_k^\Gamma(G,R;\rho)$*, where ρ varies in the range of the set of Γ-equivariant generalized collective characters of G in R.*

(ii) Each Γ-equivariant generalized collective character ρ invariably determines a 3-dimensional cohomology class $\mathrm{T}(\rho)\in H^3_\Gamma\big(G,\mathrm{Z}(R)^*\big)$ *of G with coefficients in the group of units of the centre of R (with respect to the equivariant G-module structure on $Z(R)^*$ obtained via ρ).*

(iii) An equivariant generalized collective character $\rho:G\to\mathrm{Pic}_k(R)$ *is realizable, that is,* $\mathrm{Cliff}_k^\Gamma(G,R;\rho)\neq\emptyset$*, if, and only if, its Teichmüller obstruction vanishes.*

(iv) If the obstruction of an equivariant generalized collective character, ρ, vanishes, then there is a bijection

$$\mathrm{Cliff}_k^\Gamma(G,R;\rho)\cong H^2_\Gamma\big(G,\mathrm{Z}(R)^*\big)\,.$$

In the three next subsections we shall emphasize the interest of Theorem 5.3 by explaining some of its applications.

5.4. Equivariant crossed product algebras. If R is any Γ-algebra and G is a Γ-group, then a Γ-*equivariant crossed product algebra of R by G* is a (G,R)-graded Γ-Clifford system $\big((S,\{S_x\}_{x\in G},G),j:R\cong S_1\big)$ such that every component S_x, $x\in G$, is a free left R-module of rank one. Let

$$\mathrm{Ext}_k^\Gamma(G,R)$$

denote the set of isomorphism classes of equivariant crossed product algebras of R by G. Thus, $\mathrm{Ext}_k^\Gamma(G,R)\subseteq\mathrm{Cliff}_k^\Gamma(G,R)$, and we shall characterize this subset of $\mathrm{Cliff}_k^\Gamma(G,R)$ by means of *equivariant collective characters* as follows.

Let us recall Bass' group exact sequence [1, Chap. II, (5.4)]

$$1\to\mathrm{Z}(R)^*\to R^*\xrightarrow{C}\mathrm{Aut}_k(R)\xrightarrow{\delta}\mathrm{Pic}_k(R)\,,\tag{36}$$

in which C maps each unit u of R to the inner automorphism $C_u:r\mapsto u\,r\,u^{-1}$, and δ carries each algebra automorphism f of R to the class of the invertible (R,R)-bimodule R_f, the underlying left R-module R with the right R-action via f, that is, $r\cdot r'=r\,f(r')$. Actually, since R is a Γ-algebra, sequence (36) is of Γ-groups, where Γ acts both on $\mathrm{Z}(R)^*$ and R^* by restriction, on $\mathrm{Aut}_k(R)$ by the diagonal action, that is, ${}^{\sigma}f:r\mapsto{}^{\sigma}(f({}^{\sigma^{-1}}r))$, and the Γ-action on $\mathrm{Pic}_k(R)$ is the one considered before, that is, ${}^{\sigma}[P]=[{}_{\sigma}P_{\sigma}]$, where ${}_{\sigma}P_{\sigma}$

is the same k-module as P with R-actions $r\cdot p = {}^{\sigma^{-1}}\! r\, p$ and $p\cdot r = p\, {}^{\sigma^{-1}}\! r$. To see that δ is a Γ-equivariant map, let us observe that the map $r\mapsto {}^{\sigma^{-1}}\! r$ establishes an isomorphism of (R,R)-bimodules $R_{{}^{\sigma}\! f}\cong {}_{\sigma}(R_f)_{\sigma}$, for all $f\in \mathrm{Aut}_k(R)$ and $\sigma\in\Gamma$, and so, $\delta({}^{\sigma}\! f)={}^{\sigma}\delta(f)$. Then, there is an induced Γ-group embedding

$$\mathrm{Out}_k(R)\overset{\delta}{\hookrightarrow}\mathrm{Pic}_k(R)\,,$$

of the group of outer automorphisms of the algebra R, $\mathrm{Out}_k(R)=\mathrm{Aut}_k(R)/\mathrm{In}(R)$, into the Picard group $\mathrm{Pic}_k(R)$. By [1, Chap. II, (5.3)] we know the image of δ, namely

$$\mathrm{Im}(\delta)=\big\{[P]\in\mathrm{Pic}_k(R)\mid P\cong R \text{ as left } R\text{-modules}\big\}\,,$$

and therefore we have a cartesian square

$$\begin{tikzcd} \mathrm{Ext}_k^{\Gamma}(G,R) \arrow[r, hook, "in"] \arrow[d] & \mathrm{Cliff}_k^{\Gamma}(G,R) \arrow[d, "\chi"'] \\ \mathrm{Hom}_{\Gamma}\big(G,\mathrm{Out}_k(R)\big) \arrow[r, hook, "\delta_*"] & \mathrm{Hom}_{\Gamma}\big(G,\mathrm{Pic}_k(R)\big)\,, \end{tikzcd} \tag{37}$$

where χ is the map (31). If any Γ-equivariant homomorphism

$$\rho: G\to \mathrm{Out}_k(R)$$

is termed a Γ-*equivariant collective character* of the Γ-group G in the Γ-algebra R, then equivariant crossed product algebras of R by G are exactly those (G,R)-graded Γ-Clifford systems that realize equivariant collective characters. The equivariant collective character

$$\chi_{[\underline{S},j]}: G\to\mathrm{Out}_k(R)$$

realized by any equivariant crossed product algebra of R by G, $\big((S,\{S_x\}_{x\in G},G),j:R\cong S_1\big)$, is built as follows. Since each component S_x is an invertible (R,R)-bimodule isomorphic to R as left R-module, there must exist a $f_x\in\mathrm{Aut}_k(R)$ and an isomorphism of (R,R)-bimodules over k, $\psi_x:R_{f_x}\cong S_x$, and therefore $\chi_{[\underline{S},j]}(x)=[f_x]$. Observe that if we write $u_x=\psi_x(1)$, then $S_x=Ru_x=u_xR$. Further, from $S_xS_{x^{-1}}=R\,1=S_{x^{-1}}S_x$ it follows that $u_xRu_{x^{-1}}=R\,1=u_{x^{-1}}Ru_x$ and therefore there exist $a,b\in R$ such that $1=u_xau_{x^{-1}}=u_{x^{-1}}bu_x$, whence each u_x is a unit of S. Since $r\,u_x=\psi_x(r)=\psi_x\big(1\cdot f_x^{-1}(r)\big)=u_x\,f_x^{-1}(r)$, we conclude that $f_x(r)=u_x\,r\,u_x^{-1}$ for all $r\in R$ and $x\in G$. Hence, the equivariant collective character $\chi_{[\underline{S},j]}$ maps every $x\in G$ to the class of the automorphism of R given by conjugation by u_x in S.

Equivariant factor sets, or strictly unitary Γ-graded monoidal functors $(F,\Phi):\underline{\mathrm{dis}}^{\Gamma}G\to\underline{\mathrm{Pic}}_k^{\Gamma}(R)$, for equivariant crossed product algebras of R by G have a particulary simple reformulation (well-known for $\Gamma=1$, cf. [29]), namely, they consist of pairs of maps (f,φ), where

$$f:G\longrightarrow \mathrm{Aut}_k(R)\ ,\qquad \varphi: G^2\cup(G\times\Gamma)\longrightarrow R^*\,, \tag{38}$$

subject to the conditions (13)-(17). Moreover, two such factor sets (f,φ) and (f',φ') are equivalent, that is, they define homotopic graded monoidal functors if, and only if, there is a map $g: G\to R^*$ such that the equalities (18)-(21) hold. By the bijection (29), the equivariant crossed product algebra defined by a factor set (38) is $R\star_{f,\varphi}G=\bigoplus_x R\times\{x\}$, the free left R-module with basis $\{(1,x),\, x\in G\}$, with multiplication according to the rule $(r,x)\,(s,y)=(r\,f_x(s)\,\varphi(x,y),xy)$, on which Γ acts by ${}^\sigma(r,x)=({}^\sigma r\,\varphi(x,\sigma),{}^\sigma x)$.

Since the square (37) is cartesian, Theorem 5.3 gives the following classification theorem for equivariant crossed product algebras.

5.5. Theorem. *Let R be a Γ-algebra and G a Γ-group.*

(i) There is a canonical partition $\mathrm{Ext}^\Gamma_k(G,R)=\bigsqcup_\rho \mathrm{Ext}^\Gamma_k\big(G,R;\rho\big)$, where, for each Γ-equivariant collective character $\rho: G\to \mathrm{Out}_k(R)$, $\mathrm{Ext}^\Gamma_k\big(G,R;\rho\big)$ is the set of classes of equivariant crossed products of R by G that realize ρ.

(ii) Each equivariant collective character $\rho: G\to\mathrm{Out}_k(R)$ determines a 3-dimensional cohomology class $\mathrm{T}(\rho)\in H^3_\Gamma\big(G,\mathrm{Z}(R)^*\big)$, of G with coefficients in the equivariant G-module of the units in the centre of R (with respect to the Γ-equivariant G-module structure on $\mathrm{Z}(R)^*$ obtained via ρ). This invariant is called the "Teichmüller obstruction" of ρ.

(iii) An equivariant collective character ρ is realizable, that is, $\mathrm{Ext}^\Gamma_k\big(G,R;\rho\big)\neq\emptyset$, if and only if its obstruction vanishes.

(iv) If the obstruction of an equivariant collective character, ρ, vanishes, then there is a bijection $\mathrm{Ext}^\Gamma_k\big(G,R;\rho\big)\cong H^2_\Gamma\big(G,\mathrm{Z}(R)^*\big)$.

Remarks.

i) The equivariant action of the Γ-group $\mathrm{Out}_k(R)$ on the Γ-module $\mathrm{Z}(R)^*$ in Theorem 5.5 is given by ${}^{[f]}u=f(u)$. Indeed, since ${}^{\delta[f]}u\,1={}^{[R_f]}u\,1=1\cdot u=1\,f(u)=f(u)$, the action is the one obtained via the embedding $\delta:\mathrm{Out}_k(R)\hookrightarrow\mathrm{Pic}_k(R)$.

ii) The construction (34) of an equivariant 3-cocycle $c^\rho\in\mathrm{T}(\rho)$, for any given equivariant generalized collective character, particularizes to any equivariant collective character as follows. Let us choose an automorphism $f_x\in\rho(x)$ for each $x\in G$, with $f_1=\mathrm{id}_R$. Then, we select normalized maps $(f: G\to\mathrm{Aut}_k(R),\ \varphi: G^2\cup(G\times\Gamma)\to R^*)$ such that $f_xf_y=C_{\varphi(x,y)}\,f_{xy}$ and ${}^\sigma f_x=C_{\varphi(x,\sigma)}\,f_{{}^\sigma x}$, for all $x,y\in G$ and $\sigma\in\Gamma$. The equivariant 3-cocycle

$$c^\rho: G^3\cup(G^2\times\Gamma)\cup(G\times\Gamma^2)\longrightarrow \mathrm{Z}(R)^*$$

is then determined by the three equations (23).

Finally, we should comment that for any Γ-ring R and any Γ-group G,

$$\mathrm{Ext}^\Gamma_{\mathbb{Z}}(G,R)=\mathrm{Ext}^\Gamma\big(G,[R,R^*]\big)\,,$$

as one can easily check by seeing that an equivariant factor set for an equivariant crossed product ring of R by G is the same as an equivariant factor set for an equivariant RING-GROUP extension of $[R, R^*]$ by G. Hence, Theorem 5.5 for $k=\mathbb{Z}$ is the same as Theorem 4.1 restricted to those RINGS-GROUPS in which the group is the full group of units.

5.6. Central graded Clifford systems with operators. This example is motivated by Dade's classification of graded Clifford systems over a field k for which the 1-component is 1-dimensional over k. In [12, Section 14], he shows that such a (G,k)-graded Clifford system is essentially the same as a central group extension of k^* by group G. In other words, there is a natural bijection $H^2(G,k^*)\cong \mathrm{Cliff}_k(G,k)$ between the 2nd cohomology group of G with coefficients in the trivial G-module k^* and the set of isomorphism classes of (G,k)-graded Clifford systems over k.

Suppose R to be any *commutative* Γ-ring and G a Γ-group. A (G,R)-graded Γ-Clifford system $(\underline{S},j)=\big((S,\{S_x\}_{x\in G},G),j:R\cong S_1\big)$ is termed *central* whenever S_1 is central in S. Let

$$\mathrm{Cliff}^{\Gamma}_{\mathrm{cen}}(G,R)$$

denote the subset of $\mathrm{Cliff}^{\Gamma}_{\mathbb{Z}}(G,R)$ consisting of isomorphism classes of central (G,R)-graded Γ-Clifford systems. Observe that, when $\Gamma=1$, $\mathrm{Cliff}^{1}_{\mathrm{cen}}(G,R)=\mathrm{Cliff}_R(G,R)$.

The set $\mathrm{Cliff}^{\Gamma}_{\mathrm{cen}}(G,R)$ is actually an abelian group under the operation

$$[\underline{S},j][\underline{S}',j']=\big[\big(S\otimes S',\{S_x\otimes_R S'_x\}_{x\in G},G\big),j\otimes j'\big]\,,$$

where $S\otimes S'=\bigoplus_{x\in G}(S_x\otimes_R S'_x)$, with multiplication $(s_x\otimes s'_x)(s_y\otimes s'_y)=s_xs_y\otimes s'_xs'_y$ and Γ-action ${}^{\sigma}(s_x\otimes s'_x)={}^{\sigma}s_x\otimes {}^{\sigma}s'_x$, and $j\otimes j'$ is the composition $R\cong R\otimes_R R\cong S_1\otimes_R S'_1$. The class of the group ring $R(G)=\bigoplus_{x\in G}R\times\{x\}$, on which Γ-acts by ${}^{\sigma}(r,x)=({}^{\sigma}r,{}^{\sigma}x)$, is the identity element of $\mathrm{Cliff}^{\Gamma}_{\mathrm{cen}}(G,R)$, and the inverse is given by $[\underline{S},j]^{-1}=\big[(S^{op},\{S_{x^{-1}}\}_{x\in G},G),j:R\cong S_1\big]$, where S^{op} is the opposite ring of S.

Below, as a consequence of Theorem 5.3, we obtain a generalization of the aforementioned Dade's theorem. Let $H^n_{\Gamma}(G,R^*)$ denote the nth cohomology group of the Γ-group G with coefficients in the Γ-equivariant trivial G-module R^* and recall from [5, Theorem 3.3] that $H^2_{\Gamma}(G,R^*)$ is isomorphic to the abelian group of classes of equivariant central extensions of R^* by G.

5.7. Theorem. *Let R be a commutative Γ-ring and G a Γ-group. There exists an exact sequence of abelian groups*

$$0\longrightarrow H^2_{\Gamma}(G,R^*)\xrightarrow{\ \Sigma\ }\mathrm{Cliff}^{\Gamma}_{\mathrm{cen}}(G,R)\xrightarrow{\ \chi\ }\mathrm{Hom}_{\Gamma}(G,\mathrm{Pic}_R(R))\xrightarrow{\ \mathrm{T}\ }H^3_{\Gamma}(G,R^*)\,,\qquad(39)$$

in which the homomorphism Δ is defined by means of the equivariant crossed product construction, that is,

$$\Sigma[\varphi]=\big[(R\star_{\varphi}G,\{R\times\{x\}\}_{x\in G},G),j:R\cong R\times\{1\}\big]\,,$$

where $R\star_\varphi G$ is the twisted group ring $R\star_\varphi G = \bigoplus_{x\in G} R\times\{x\}$ whose multiplication is given by $(r,x)(r',x') = (rr'\psi(x,x'), xx')$ and on which Γ-acts by ${}^\sigma(r,x) = ({}^\sigma r\psi(x,\sigma), {}^\sigma x)$. The homomorphisms χ and T are given, respectively, by restricting the equivariant generalized collective character map (31) and the Teichmüller obstruction map (33).

Proof. It is straightforward to check that the maps Σ, χ and T in the sequence are certainly group homomorphisms. As for the exactness in the first two points, let us consider the bijection (35) when one chooses the Γ-group ring $R(G)$, in the bijection, as a representative (G,R)-graded Γ-Clifford system that realizes the trivial equivariant generalized collective character. In this case, the bijection simply says that $H^2_\Gamma\big(G,R^*\big) \overset{\Sigma}{\cong} \mathrm{Cliff}^\Gamma_\mathbb{Z}\big(G,R;0\big)$. In addition, since a (G,R)-graded Γ-Clifford system that realizes the zero collective character is necessarily central, $\mathrm{Cliff}^\Gamma_\mathbb{Z}\big(G,R;0\big) = \mathrm{Ker}(\chi)$, whence the exactness in the first two points of the sequence follows. Part (*iii*) of Theorem 5.3 directly gives the exactness in the remaining point. ∎

When R is a field, but also when R is any commutative algebra with $\mathrm{Pic}_R(R) = 0$ (for instance a principal ideal domain, a local algebra, ...), then $\mathrm{Cliff}^\Gamma_{\mathrm{cen}}(G,R) \cong H^2_\Gamma(G,R^*)$, and Dade's result follows. We should also remark that the construction of the above exact sequence (39) is quite similar to the corresponding part of the seven-term exact sequence of Chase-Harrison-Rosenberg about the Auslander-Goldman-Brauer group relative to a Galois extension of commutative rings [11] (cf. with exact sequence (45) in the next subsection, also with Miyashita's generalized seven term exact sequences [30]).

5.8. Equivariant Azumaya algebras over Galois extensions with operators. Throughout this paragraph, R is a commutative k-algebra such that $R \supseteq k$ is a Galois extension with finite Galois group $G \subseteq \mathrm{Aut}_k(R)$ and, in addition, we assume that R is a Γ-algebra through a given homomorphism $\Gamma \to G$. The Galois group G is then a Γ-group by the diagonal action $(\sigma, x) \mapsto {}^\sigma x$, where ${}^\sigma x(r) = {}^\sigma\!\big(x({}^{\sigma^{-1}}r)\big)$.

An *R/k-Azumaya algebra* [26] is a pair (A,j) consisting of an Azumaya k-algebra (i.e. central separable) A and a maximal commutative embedding $j : R \hookrightarrow A$. When A is endowed with a Γ-action by algebra automorphisms such that j is Γ-equivariant, then the pair (A,j) is called a *Γ-equivariant R/k-Azumaya algebra.* Two such algebras, (A,j) and (A',j') are isomorphic whenever there exists a Γ-equivariant isomorphism of algebras $\phi : A \cong A'$ which respects the embeddings of R, that is, $\phi j = j'$. Let

$$\widehat{Br}^{\,\Gamma}(R/k)$$

denote the corresponding set of isomorphism classes.

If (A,j) is any equivariant R/k-Azumaya algebra and, for each $x \in G$, we let

$$A_x = \{a \in A \mid x(r)\,a = a\,r \text{ for all } r \in R\}\,. \tag{40}$$

Then, by [27, Proposition 3] or [2, Theorem I 2.15], it holds that $A=\oplus_{x\in G}A_x$, $A_xA_y=A_{xy}$ for all $x,y\in G$ and $A_1=j(R)\cong R$. Moreover, for any $\sigma\in\Gamma$ and $x\in G$,

$$\begin{aligned}{}^{\sigma}\!A_x &= \{{}^{\sigma}a \mid x(r)\,a=a\,r,\ r\in R\}=\{a\mid x(r)\,{}^{\sigma^{-1}}\!a\;=\;{}^{\sigma^{-1}}\!a\,r,\ r\in R\}\\ &=\{a\mid x({}^{\sigma^{-1}}\!r)\,{}^{\sigma^{-1}}\!a\;=\;{}^{\sigma^{-1}}\!a\,{}^{\sigma^{-1}}\!r,\ r\in R\}=\{a\mid {}^{\sigma}\big(x({}^{\sigma^{-1}}\!r)\big)\,a=a\,r,\ r\in R\}\\ &=\{a\mid {}^{\sigma}x(r)\,a=a\,r,\ r\in R\}=A_{{}^{\sigma}x}\,.\end{aligned}$$

Hence, $\big((A,\{A_x\}_{x\in G},G),j:R\cong A_1\big)$ is a (G,R)-graded Γ-Clifford system over k.

And conversely, if $\big((A,\{A_x\}_{x\in G},G),j:R\cong A_1\big)$ is any (G,R)-graded Γ-Clifford system over k such that

$$x(r)\,a_x=a_x\,r\qquad\text{for all } x\in G, r\in R \text{ and } a_x\in A_x\,, \tag{41}$$

then it follows from [27, Proposition 2] that (A,j) is an equivariant R/k-Azumaya algebra.

Thus, *equivariant R/k-Azumaya algebras are the same as (G,R)-graded Γ-Clifford systems such that equalities* (41) *hold.* Next we shall characterize the subset $\widehat{Br}^{\Gamma}(R/k)\subseteq \mathrm{Cliff}_k^{\Gamma}(G,R)$ by means of the generalized collective character map (31). To do so, let us recall Bass' split exact sequence of groups [1, II, (5.4)]

$$1\to \mathrm{Pic}_R(R)\xrightarrow{in}\mathrm{Pic}_k(R)\underset{\delta}{\overset{h}{\rightleftarrows}}\mathrm{Aut}_k(R)\to 1\ , \tag{42}$$

in which $\delta: f\mapsto[R_f]$ is the homomorphism already recalled in sequence (36), and h maps the class of any invertible (R,R)-bimodule $[P]\in\mathrm{Pic}_k(R)$ to the automorphism $h([P])=h_P\in\mathrm{Aut}_k(R)$ such that $h_P(r)\,p=p\,r$ for all $r\in R$ and $p\in P$. Let

$$\xi:\mathrm{Pic}_k(R)\longrightarrow\mathrm{Pic}_R(R),\qquad [P]\mapsto[\xi P]\,, \tag{43}$$

be the map that assigns to each class of an invertible (R,R)-bimodule P the class of the (R,R)-bimodule ξP which is the same P as left R-module but with right R-action $p\cdot r=rp$. Thus, $\xi[P]=[P\otimes_R R_{h_P^{-1}}]=[P][R_{h_P}]^{-1}=[P]\big(\delta h([P])\big)^{-1}$.

Sequence (42) is actually of Γ-groups. Indeed, when we recalled sequence (36), we showed that δ is a Γ-equivariant homomorphism. Homomorphism h is also equivariant since, from equalities $h_P({}^{\sigma^{-1}}\!r)\,p=p\,{}^{\sigma^{-1}}\!r$ in P, it follows that the equalities ${}^{\sigma}h_P({}^{\sigma^{-1}}\!r)\cdot p=p\cdot r$ hold in ${}_{\sigma}P_{\sigma}$, which means that ${}^{\sigma}h_P=h_{{}_{\sigma}P_{\sigma}}$. Thus, ${}^{\sigma}h([P])=h({}^{\sigma}[P])$ according to the Γ-action (30) on $\mathrm{Pic}_k(R)$. Hence, the abelian group $\mathrm{Pic}_R(R)$ is a Γ-equivariant $\mathrm{Aut}_k(R)$-module, on which Γ acts by (30), and $\mathrm{Aut}_k(R)$ acts in a similar way by

$${}^{f}[P]=[{}_fP_f]\,,$$

where ${}_fP_f\cong R_f\otimes_R P\otimes_R R_{f^{-1}}$ is the same k-module as P with $r\cdot p=f^{-1}(r)\,p=p\,f^{-1}(r)=p\cdot r$, for $r\in R$ and $p\in P$. In particular, since G is a Γ-subgroup of $\mathrm{Aut}_k(R)$, $\mathrm{Pic}_R(R)$ is also a Γ-equivariant G-module.

Now, if $\rho : G \to \mathrm{Pic}_k(R)$, say $\rho(x) = [A_x]$, is any equivariant generalized collective character of the Γ-group G in the Γ-algebra R, then the equalities (41) hold if and only if $h\rho = in : G \hookrightarrow \mathrm{Aut}_k(R)$. Moreover, since map (43) induces an isomorphism of abelian groups [5, Proposition 2.5]

$$\{\rho \in \mathrm{Hom}_{\Gamma}(G, \mathrm{Pic}_k(R)) \mid h\rho(x) = x,\ x \in G\} \cong \mathrm{Der}_{\Gamma}(G, \mathrm{Pic}_R(R)), \quad \rho \mapsto \xi\,\rho\,,$$

where $\mathrm{Der}_{\Gamma}(G, \mathrm{Pic}_R(R))$ is the abelian group of Γ-equivariant derivations from G into $\mathrm{Pic}_R(R)$, we conclude that there is a cartesian square

$$\begin{tikzcd} \widehat{Br}^{\Gamma}(R/k) \arrow[r, hook, "in"] \arrow[d, "\hat{\chi}"'] & \mathrm{Cliff}_k^{\Gamma}(G,R) \arrow[d, "\chi"] \\ \mathrm{Der}_{\Gamma}\big(G, \mathrm{Pic}_R(R)\big) \arrow[r, hook] & \mathrm{Hom}_{\Gamma}\big(G, \mathrm{Pic}_k(R)\big)\,, \end{tikzcd} \tag{44}$$

where χ is the map (31) and $\hat{\chi} : [A, j] \mapsto \hat{\chi}_{[A,j]}$, where $\hat{\chi}_{[A,j]}(x) = [\xi A_x]$ for each $x \in G$. Hence, *equivariant R/k-Azumaya algebras are the same as (G,R)-graded Γ-Clifford systems over k that realize equivariant derivations from G into* $\mathrm{Pic}_R(R)$.

The set $\widehat{Br}^{\Gamma}(R/k)$ has a canonical structure of abelian group under the operation

$$[A, j][A', j'] = [A \otimes A', j \otimes j']\,,$$

where $A \otimes A' = \bigoplus_{x\in G} \xi A_x \otimes_R A'_x$, in which, for each $x \in G$, A_x and A'_x are defined by (40) and ξA_x by (43), with multiplication given by the chain of isomorphisms:

$$\begin{aligned} \xi A_x \otimes_R A'_x \otimes_R \xi A_y \otimes_R A'_y &\cong \xi A_x \otimes_R {}_x(\xi A_y)_x \otimes_R A'_x \otimes_R A'_y \\ &\cong \xi(A_x \otimes_R A_y) \otimes_R A'_x \otimes_R A'_y \cong \xi A_{xy} \otimes_R A'_{xy}\,, \end{aligned}$$

that is, $(a_x \otimes a'_x)(a_y \otimes a'_y) = a_x a_y \otimes a'_x a'_y$. The Γ-action is given by ${}^{\sigma}(a_x \otimes a'_x) = {}^{\sigma}a_x \otimes {}^{\sigma}a'_x$, and $j \otimes j'$ is the composition $R \cong R \otimes_R R \cong A_1 \otimes_R A'_1 = \xi A_1 \otimes_R A'_1$. The class of the skew group algebra $R \star G = \bigoplus_{x\in G} R \times \{x\}$, whose multiplication is given by $(r, x)(r', y) = (r\,x(r'), xy)$ and on which Γ-acts by ${}^{\sigma}(r, x) = ({}^{\sigma}r, {}^{\sigma}x)$, gives the identity element of $\widehat{Br}^{\Gamma}(R/k)$, and the inverse is given by

$$[A, j]^{-1} = [A^{op}, j]\,,$$

where A^{op} is the opposite Γ-algebra of A (observe that, for any $x \in G$, $A^{op}_x = A_{x^{-1}}$ with $r \cdot a_{x^{-1}} = a_{x^{-1}}\, r = x^{-1}(r)\, a_{x^{-1}}$ and $a_{x^{-1}} \cdot r = r\, a_{x^{-1}}$, and then, $A \otimes A^{op} \cong R \star G$ by the map $\sum_x a_x \otimes b_{x^{-1}} \mapsto \sum_x (a_x b_{x^{-1}}, x)$).

Then, we have:

5.9. THEOREM. *Let R/k be a Galois extension of commutative rings with finite Galois group $G \subseteq Aut_k(R)$ and suppose that an action of a group Γ on R by automorphisms in G is given. Then, there exists an exact sequence of abelian groups*

$$0 \longrightarrow H^2_{\Gamma}(G, R^*) \xrightarrow{\ \Sigma\ } \widehat{Br}^{\Gamma}(R/k) \xrightarrow{\ \hat{\chi}\ } \mathrm{Der}_{\Gamma}\big(G, \mathrm{Pic}_R(R)\big) \xrightarrow{\ \hat{\mathrm{T}}\ } H^3_{\Gamma}(G, R^*)\,, \tag{45}$$

in which the homomorphism Σ *is defined by means of the equivariant crossed product construction, that is*

$$\Sigma[\varphi] = \left[R \star_{in,\varphi} G, j : R \cong R \times \{1\}\right],$$

where $R \star_{in,\varphi} G$ *is the crossed product algebra* $R \star_{in,\varphi} G = \bigoplus_{x\in G} R \times \{x\}$*, whose multiplication is given by* $(r,x)(r',x') = (r\,x(r')\,\varphi(x,x'), xx')$ *and on which* Γ*-acts by* ${}^{\sigma}(r,x) = ({}^{\sigma}r\,\varphi(x,\sigma), {}^{\sigma}x)$*. The homomorphism* $\hat{\chi}$ *is that given in* (44) *and* $\hat{\mathrm{T}}$ *is the Teichmüller obstruction map* (33) *restricted to* $\mathrm{Der}_{\Gamma}\big(G, \mathrm{Pic}_R(R)\big)$*; that is, for any* Γ*-derivation* $d : G \to \mathrm{Pic}_R(R)$*,* $\hat{\mathrm{T}}(d) = \mathrm{T}(\rho_d)$*, where* $\rho_d : G \to \mathrm{Pic}_k(R)$ *is the equivariant homomorphism given by* $\rho_d(x) = d(x)[R_x]$*.*

Proof. To check that maps Σ, $\hat{\chi}$ and $\hat{\mathrm{T}}$ are group homomorphisms is routine. The exactness in the two first points means that Σ establishes a bijection between $H^2_{\Gamma}(G, R^*)$ and $\mathrm{Ker}(\hat{\chi})$, but this is just bijection (35),

$$H^2_{\Gamma}\big(G, R^*\big) \overset{\Delta}{\cong} \mathrm{Cliff}^{\Gamma}_k\big(G, R; \rho_0\big) = \chi^{-1}(\rho_0) = \mathrm{K}er(\hat{\chi}),$$

when one chooses the skew Γ-group algebra $R \star G$, in the bijection, as the representative (G, R)-graded Γ-Clifford system that realizes the equivariant generalized collective character ρ_0 corresponding to the zero derivation $0 : G \to \mathrm{Pic}_k(R)$. Part (*iii*) of Theorem 5.3 directly gives the exactness in the remaining point. ∎

Exact sequence (45) in the above theorem extends the sequence in [10, Theorem 2.1].

6. Strongly graded bialgebras and Hopf algebras with operators

Throughout this section, k is a unitary commutative ring and all algebras and coalgebras are over k. Recall that a bialgebra (R, Δ, ϵ) is an algebra R enriched with a coalgebra structure such that the comultiplication $\Delta : R \to R \otimes_k R$ and the counit $\epsilon : R \to k$ are algebra maps.

A *strongly graded bialgebra* [8, Definition 6.4]) $\big((S, \Delta, \epsilon), \{S_x\}_{x\in G}, G\big)$ is a graded Clifford system over k in which S is endowed with a bialgebra structure such that every component S_x is a subcoalgebra. A *morphism* of strongly graded bialgebras is a morphism (h, α) between the underlying graded Clifford systems such that h is also a coalgebra homomorphism. Strongly graded bialgebras were classified in [8, Section 6] as a specific application of the general treatment of graded monoidal categories.

Hereafter, Γ is a fixed group of operators and a Γ-bialgebra means a bialgebra provided with a Γ-action by bialgebra automorphisms $(\sigma, r) \mapsto {}^{\sigma}r$. Our goal in this section is the classification of strongly graded bialgebras on which a group of operators Γ acts by automorphisms, that is, the classification of *strongly graded* Γ*-bialgebras*.

Similarly to what happens with graded Γ-Clifford systems, if $\big((S, \Delta, \epsilon), \{S_x\}_{x\in G}, G\big)$ is a strongly graded Γ-bialgebra, then G is a Γ-group, (S, Δ, ϵ) is a Γ-bialgebra and its 1-component S_1 is a Γ-subbialgebra. Then, we shall regard $\big((S, \Delta, \epsilon), \{S_x\}_{x\in G}, G\big)$ as an equivariant extension of the Γ-bialgebra (S_1, Δ, ϵ) by the Γ-group G. More precisely, if (R, Δ, ϵ) is any Γ-bialgebra and G is a Γ-group, then a *strongly* $(G, (R, \Delta, \epsilon))$*-graded* Γ*-bialgebra* is a pair $(\underline{S}, j_S)$, where $\underline{S} = \big((S, \Delta, \epsilon), \{S_x\}_{x\in G}, G\big)$ is a strongly graded Γ-bialgebra, such that the Γ-action on G is the given one, that is, ${}^\sigma S_x = S_{{}^\sigma x}$ for all $x \in G$ and $\sigma \in \Gamma$, and $j_S : (R, \Delta, \epsilon) \cong (S_1, \Delta, \epsilon)$ is an isomorphism of Γ-bialgebras.

The most striking example is, for (R, Δ, ϵ) any given Γ-bialgebra, the group bialgebra $R(G) = \bigoplus_{x\in G} R \times \{x\}$ (where the multiplication is given by $(r, x)(r', x') = (rr', xx')$ and the comultiplication, using Sweedler's notation, by $\Delta(r, x) = \sum_{(r)}(r_{(1)}, x)\otimes(r_{(2)}, x)$, which is a Γ-bialgebra with the action ${}^\sigma(r, x) = ({}^\sigma r, {}^\sigma x)$. However, equivariant crossed products of (R, Δ, ϵ) and G also yield examples of equivariant strongly graded bialgebras. Note that the underlying graded algebra of a strongly graded Γ-bialgebra is a graded Γ-Clifford system as defined in Section 5.

Two strongly $(G, (R, \Delta, \epsilon))$-graded Γ-bialgebras, $(\underline{S}, j_S)$ and $(\underline{T}, j_T)$, are *isomorphic* if there exists a Γ-equivariant isomorphism of Γ-bialgebras $h : (S, \Delta, \epsilon) \cong (T, \Delta, \epsilon)$ which is grade-preserving. We denote by

$$\mathrm{Ext}^\Gamma_k\big(G, (R, \Delta, \epsilon)\big)$$

the set of isomorphism classes of strongly $(G, (R, \Delta, \epsilon))$-graded Γ-bialgebras.

The classification of strongly graded Γ-bialgebras parallels the non-equivariant case (i.e., when Γ is the trivial group) studied in [8]. Let (R, Δ, ϵ) be any Γ-bialgebra. Then, the group

$$\mathrm{Pic}_k(R, \Delta, \epsilon)$$

of isomorphism classes of invertible (R, R)-coalgebras [8, Definition 6.3] is a Γ-group with action given by ${}^\sigma[P] = [{}_\sigma P_\sigma]$, where ${}_\sigma P_\sigma$ is the invertible (R, R)-coalgebra which is the same coalgebra as P but with (R, R)-actions $r\cdot p = {}^{\sigma^{-1}}r\, p$ and $p\cdot r = p\; {}^{\sigma^{-1}}r$ for $r \in R$ and $p \in P$. Note that, since every invertible (R, R)-coalgebra is an invertible (R, R)-bimodule, there is an obvious equivariant homomorphism, from the Γ-group $\mathrm{Pic}_k(R, \Delta, \epsilon)$ to the Γ-group $\mathrm{Pic}_k(R)$ of the underlying Γ-algebra as considered in Section 5, which is neither injective nor surjective in general. Moreover, the centre of the bialgebra $Z(R, \Delta, \epsilon)$ [8, Definition 6.2], which is the multiplicative submonoid of R consisting of all group-like elements of R as a coalgebra that belong to the centre of R as an algebra, that is,

$$Z(R, \Delta, \epsilon) = \{u \in R \mid \ \Delta(u) = u \otimes u,\ \epsilon(u) = 1,\ \text{and } ur = ru \text{ for all } r \in R\}\,,$$

is a Γ-monoid in the obvious way. Therefore, the group of units of $Z(R, \Delta, \epsilon)$, denoted by $Z(R, \Delta, \epsilon)^*$, is a Γ-module. In fact, $Z(R, \Delta, \epsilon)^*$ is actually a Γ-equivariant $\mathrm{Pic}_k(R, \Delta, \epsilon)$-module, where the action of $\mathrm{Pic}_k(R, \Delta, \epsilon)$ on $Z(R, \Delta, \epsilon)^*$ (see [8]) is determined by the equalities ${}^{[P]}u\, p = p\, u$, for any invertible (R, R)-coalgebra P, any $u \in Z(R, \Delta, \epsilon)^*$ and $p \in P$.

Every strongly $(G, (R, \Delta, \epsilon))$-graded Γ-bialgebra $(\underline{S} = \big((S, \Delta, \epsilon), \{S_x\}_{x\in G}, G\big), j_S)$ induces a Γ-equivariant homomorphism (which depends only on its isomorphism class $[\underline{S}, j_S]$)

$$\chi_{[\underline{S}, j_S]} : G \to \mathrm{Pic}_k(R, \Delta, \epsilon), \quad x \mapsto [S_x]\ , \quad x \in G\ ,$$

and so there is a canonical map

$$\chi : \mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon)) \longrightarrow \mathrm{Hom}_\Gamma(G,\mathrm{Pic}_k(R,\Delta,\epsilon))\ .$$

We call any Γ-equivariant group homomorphism $\rho : G \to \mathrm{Pic}_k(R,\Delta,\epsilon)$ an *equivariant generalized collective character* of the Γ-group G in the Γ-bialgebra (R,Δ,ϵ). We also say that a strongly $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebra $(\underline{S},j_S)$ *realizes* such an equivariant generalized collective character ρ whenever $\chi_{[\underline{S},j_S]} = \rho$. Map χ provides a partitioning

$$\mathrm{Ext}_k^\Gamma\big(G,(R,\Delta,\epsilon)\big) = \bigsqcup_\rho \mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon);\rho)\,,$$

where, for each Γ-equivariant homomorphism $\rho : G \to \mathrm{Pic}_k(R,\Delta,\epsilon)$, $\mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon);\rho)$ is the set of isoclasses of strongly $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebras that realize ρ.

Now we state the following:

6.1. Theorem. *(Classification of strongly graded bialgebras with operators)* *Let (R,Δ,ϵ) be a Γ-bialgebra and G a Γ-group.*

(i) Each Γ-equivariant generalized collective character ρ invariably determines a 3-dimensional cohomology class $\mathrm{T}(\rho) \in H^3_\Gamma\big(G,\mathrm{Z}(R,\Delta,\epsilon)^*\big)$ of G with coefficients in the equivariant G-module of all group-like central units of (R,Δ,ϵ) (with respect to the G-module structure on $\mathrm{Z}(R,\Delta,\epsilon)^*$ obtained via ρ). This invariant is called the obstruction of ρ.

(ii) An equivariant generalized collective character ρ is realizable, that is, the set $\mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon);\rho)$ is non-empty if, and only if, its obstruction vanishes.

(iii) If the obstruction of an equivariant generalized collective character, ρ, vanishes, then there is a bijection

$$\mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon);\rho) \cong H^2_\Gamma\big(G,\mathrm{Z}(R,\Delta,\epsilon)^*\big)\,.$$

As in the examples studied in the previous sections, this classification theorem for the strongly graded bialgebras with operators will follow from the general theory in Section 2, once we are able to identify, up to isomorphisms, strongly graded Γ-bialgebras with Γ-graded monoidal functors from the discrete Γ-graded categorical group $\underline{\mathrm{dis}}^\Gamma G$ to the Γ-*graded Picard categorical group* of a Γ-bialgebra (R,Δ,ϵ)

$$\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)\,,$$

which is defined in a similar way to the Γ-graded categorical group $\underline{\mathrm{Pic}}_k^\Gamma(R)$ of the underlying Γ-algebra R. The objects of $\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)$ are the invertible (R,R)-coalgebras and the morphisms are pairs (φ,σ) where φ is an isomorphism of coalgebras. Composition and the graded tensor product are defined much as for $\underline{\mathrm{Pic}}_k^\Gamma(R)$, in such a way that, omitting the coalgebra structure, one has a graded monoidal functor $\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon) \to \underline{\mathrm{Pic}}_k^\Gamma(R)$.

Then we have:

6.2. Theorem. *For any Γ-bialgebra (R,Δ,ϵ) and any Γ-group G, there is a bijection*

$$\underline{\Sigma} : \big[\underline{\mathrm{dis}}^\Gamma G, \underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)\big] \cong \mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon))\,, \tag{46}$$

between the set of homotopy classes of graded monoidal functors from $\underline{\mathrm{dis}}^\Gamma G$ to $\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)$ and the set of isomorphism classes of strongly $\big(G,(R,\Delta,\epsilon)\big)$-graded Γ-bialgebras.

Proof. It is parallel to the proof for Theorem 5.2 and so we will omit some straightforward details. First we observe that a strictly unitary Γ-graded monoidal functor $(F,\Phi) : \underline{\mathrm{dis}}^\Gamma G \to \underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)$ is the same as a strictly unitary Γ-graded monoidal functor $(F,\Phi) : \underline{\mathrm{dis}}^\Gamma G \to \underline{\mathrm{Pic}}_k^\Gamma(R)$ in which every F_x is an invertible (R,R)-coalgebra and the isomorphisms (27) and (28) are of (R,R)-coalgebras. Such a graded monoidal functor (F,Φ) is now termed an *equivariant factor set* of the Γ-group G in the Γ-bialgebra (R,Δ,ϵ) and it gives rise to a *generalized crossed product bialgebra* $\Sigma(F,\Phi) = \bigoplus_{x\in G} F_x$ which is the direct sum (= coproduct) coalgebra of the coalgebras F_x, where the product of the elements $a \in F_x$ and $b \in F_y$ is defined by $a\,b = \Phi_{x,y}(a\otimes b)$. Further, owing to the isomorphisms $F_{x,\sigma}$, group Γ acts on $\Sigma(F,\Phi)$ by the rule ${}^\sigma a = F_{x,\sigma}(a)$ if $a \in F_x$. In this way, $\underline{\Sigma}(F,\Phi) = \big((\Sigma(F,\Phi),\Delta,\epsilon),\{F_x\},G\big)$, together with the identity $R = F_1$, is a strongly $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebra.

If $F' = (F',\Phi') : \underline{\mathrm{dis}}^\Gamma G \to \underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)$ is another strictly unitary graded monoidal functor, then it is straightforward to see that giving a homotopy $\theta : (F,\Phi) \to (F',\Phi')$ is equivalent to giving an isomorphism of Γ-bialgebras, $\Sigma(F,\Phi) \xrightarrow{\sim} \Sigma(F',\Phi')$, $a \mapsto \theta_x(a)$, $a \in F_x$, which establishes an isomorphism of strongly $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebras $\theta : (\underline{\Sigma}(F,\Phi),\mathrm{id}_R) \xrightarrow{\sim} (\underline{\Sigma}(F',\Phi'),\mathrm{id}_R)$.

Moreover, any strongly $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebra is actually isomorphic to a generalized crossed product $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebra for a certain equivariant factor set (F,Φ). In fact, given any $(\underline{S},j_{{}_S})$, $\underline{S} = \big((S,\Delta,\epsilon),\{S_x\}_{x\in G},G\big)$, we can assume, up to isomorphism, that $R = S_1$ and $j_{{}_S} = \mathrm{id}_R$. For any $x,y \in G$, the canonical multiplication map $\Phi_{x,y} : S_x \otimes_R S_y \to S_{xy}$, $\ a\otimes b \mapsto a\,b$ is a (R,R)-coalgebra isomorphism and each component S_x is an invertible (R,R)-coalgebra [8, Proposition 6.4 and Corollary 6.1]. Furthermore, the Γ-action on S determines, for all $x \in G$ and $\sigma \in \Gamma$, isomorphisms of coalgebras $F_{x,\sigma} : S_x \to S_{{}^\sigma x}$, $a \mapsto {}^\sigma a$, and it is plain to see that $(F,\Phi) : \underline{\mathrm{dis}}^\Gamma G \to \underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)$, with $F_x = S_x$ for each $x \in G$, is a strictly unitary graded monoidal functor whose associated generalized crossed product $(G,(R,\Delta,\epsilon))$-graded Γ-bialgebra $(\underline{\Sigma}(F,\Phi),\mathrm{id}_R)$ is exactly $(\underline{S},\mathrm{id}_R)$. ∎

The classification of strongly $(G,(R,\Delta,\epsilon))$-graded bialgebras with operators stated in Theorem 6.1 is now a consequence of Theorem 2.2 and the above bijection (46). Indeed, for any Γ-bialgebra (R,Δ,ϵ), the homotopy groups of the Γ-graded categorical group $\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon)$ are $\pi_0\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon) = \mathrm{Pic}_k(R,\Delta,\epsilon)$ and $\pi_1\underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon) = Z(R,\Delta,\epsilon)^*$ and the obstruction $\mathrm{T}(\rho) \in H^3_\Gamma\big(G,\mathrm{Z}(R,\Delta,\epsilon)^*\big)$ in part (i) of Theorem 6.1, of any equivariant generalized collective character $\rho : G \to \mathrm{Pic}_k(R,\Delta,\epsilon)$, is defined in the same way as (33) for equivariant generalized collective characters of Γ-groups in Γ-algebras.

Furthermore, for any equivariant factor set (F,Φ) of the Γ-group G in the Γ-bialgebra (R,Δ,ϵ), $\pi_0 F = \chi_{[\Sigma(F,\Phi),\mathrm{id}_R]}$, and then bijection (46) provides, by restriction, bijections $\underline{\Sigma} : \big[\underline{\mathrm{dis}}^\Gamma G, \underline{\mathrm{Pic}}_k^\Gamma(R,\Delta,\epsilon);\rho\big] \cong \mathrm{Ext}_k^\Gamma(G,(R,\Delta,\epsilon);\rho)$ for any equivariant generalized character $\rho : G \to \mathrm{Pic}_k(R,\Delta,\epsilon)$. Hence, parts (*ii*) and (*iii*) of Theorem 6.1 follow from Theorem 2.2.

By [3, Theorem 1.1], Theorem 6.1 applies to the classification of *strongly graded Hopf* Γ-*algebras over a field k with 1-component of finite dimension*, that is, strongly graded Γ-bialgebras $\big((S,\Delta,\epsilon),\{S_x\}_{x\in G},G\big)$ where (S,Δ,ϵ) is a Hopf algebra whose antipode map $\xi : S \to S$ satisfies that $\xi(S_x) \subseteq S_{x^{-1}}$ for all $x \in G$, and S_1 is finite dimensional over k. Indeed, when (R,Δ,ϵ) is any finite dimensional Hopf Γ-algebra, then every strongly graded Γ-bialgebra whose 1-component is isomorphic to (R,Δ,ϵ) is necessarily a strongly graded Hopf Γ-algebra and, therefore, for any Γ-group G, $\mathrm{Ext}_k^\Gamma\big(G,(R,\Delta,\epsilon)\big)$ is exactly the set of isomorphism classes of strongly $(G,(R,\Delta,\epsilon))$-graded Hopf Γ-algebras.

References

[1] H. Bass, Algebraic K-theory, Benjamin, New York, 1968.

[2] S. Caenepeel, F. Van Oystaeyen, Brauer groups and the cohomology of graded rings, *Monographs and Textbooks in Pure and Applied Mathematics* **121** Marcel Dekker, Inc., New York, 1988.

[3] A.M. Cegarra, On the classification of strongly graded Hopf algebras, *J. Algebra* **251** (2002), 358-370.

[4] A.M. Cegarra, L. Fernández, Cohomology of cofibred categorical groups, *J. Pure Appl. Algebra* **143** (1999), 107-154.

[5] A.M. Cegarra, J.M. García-Calcines, J.A. Ortega, Cohomology of groups with operators, *Homology, Homotopy and its Applications* **4 (1)** (2002), 1-23.

[6] A.M. Cegarra, J.M. García-Calcines, J.A. Ortega, On graded categorical groups and equivariant group extensions, *Canadian J. of Math.* **54 (5)** (2002), 970-997.

[7] A.M. Cegarra, A.R. Garzón, A.R. Grandjean, Graded extensions of categories *J. Pure Appl. Algebra* **154** (2000), 117-141.

[8] A.M. Cegarra, A.R. Garzón, J.A. Ortega, Graded extensions of monoidal categories, *J. Algebra* **241** (2001), 620-657.

[9] A.M. Cegarra, A.R. Garzón, Obstructions to Clifford system extensions of algebras, *Proc. Indian Acad. Sci. Math. Sci.* **111** (2001), 151-161.

[10] A.M. Cegarra, A.R. Garzón, Equivariant group cohomology and Brauer group *Bull. Belg. Math. Soc. Simon Stevin*, accepted article pending of publication.

[11] S.U. Chase, D.K. Harrison, A. Rosenberg, Galois theory and Galois cohomology of commutative rings, *Mem. Amer. Math. Soc.* **52** (1965), 15-33.

[12] E.C. Dade, Compounding Clifford's theory, *Ann. Math.* **91** (1970), 236-290.

[13] E.C. Dade, Group graded rings and modules, *Math. Z.* **174** (1980), 241-262.

[14] S. Eilenberg, S. Mac Lane, Cohomology theory in abstract groups. II, Group Extensions with a non-Abelian Kernel, *Ann. Math.* **48** (1947), 326-341.

[15] S. Eilenberg, S. Mac Lane, Cohomology and Galois theory I, normality of algebras and Teichmüller's cocycle, *Trans. Amer. Mac. Soc.* **64** (1948), 1-20.

[16] I. Fleischer, Monoid extension theory, *J. Pure Appl. Algebra* **21** (1981), 151-159.

[17] A. Fröhlich, C.T.C. Wall, Foundations of equivariant algebraic K-theory, *Springer Lecture Notes in Math.* **108** (1969), 12-27.

[18] A. Fröhlich, C.T.C. Wall, Equivariant Brauer groups in algebraic number theory, Bull. Soc. Math. France Mém. No. **25**, Soc. Math. France, Paris, 1971.

[19] A. Fröhlich, C.T.C. Wall, Generalizations of the Brauer group I, (1971), preprint.

[20] A. Fröhlich, C.T.C. Wall, Graded monoidal categories, *Compos. Math.* **28** (1974), 229-285.

[21] A. Fröhlich, C.T.C. Wall, Equivariant Brauer groups, *Contemp. Math.* **272** (2000), 57-71.

[22] H. Grassmann, On Schreier extensions of finite semigroups, Algebraic theory of semigroups (Proc. Sixth Algebraic Conf., Szeged, 1976), *Colloq. Math. Soc. Jnos Bolyai*, **20**, North-Holland, Amsterdam-New York (1979), 219-224.

[23] A. Grothendieck, Catégories fibrées et déscente, (SGA I, expos VI), *Springer Lecture Notes in Math.* **224** (1971), 145-194.

[24] M. Hacque, Cohomologies des anneaux-groupes, *Comm. Algebra* **18 (11)** (1990), 3933-3997.

[25] M. Hacque, Produits croisés mixtes: Extensions des groupes et extensions d'anneaux, *Comm. Algebra* **19 (7)** (1991), 1993-2049.

[26] A. Hattori, On groups $H^n(S,G)$ and the Brauer group of commutative rings, *Sci. Papers College Gen. Ed. Univ. Tokyo* **28** (1978/79), 1-20.

[27] T. Kanzaki, On generalized crossed products and Brauer group, *Osaka J. Math.* **5** (1968), 175-188.

[28] F.W. Long, A generalization of the Brauer group of graded algebras, *Proc. London Math. Soc.* **29** (1974), 237-256.

[29] M. Lorenz, Primitive ideals in crossed products and rings with finite group actions, *Math. Z.* **158** (1978), 285-294.

[30] Y. Miyashita, An exact sequence associated with a generalized crossed product, *Nagoya Math. J.* **49** (1973), 21-51.

[31] S. Montgomery, editor, Group actions on rings, *Contemp. Math.* **43**, 1985.

[32] C. Nastasescu, F. Van Oystaeyen, On strongly graded rings and crossed products, *Com. Algebra* **10** (1982), 2085-2106.

[33] L. Rédei, Die Verallgemeinerung der Schreierschen Erweiterungstheorie. *Acta Sci. Math. Szeged* **14** (1952), 252-273.

[34] N. Saavedra, Catégories Tannakiennes, *Springer Lecture Notes in Math.* **265** (1972).

[35] O. Schreier, Über die Erweiterung von Gruppen I, *Monatsh. Math. Phys.* **34** (1926), 165-180.

[36] Sinh, H.X. Gr-catégories. Université Paris VII, Thése de doctorat, **1975**.

[37] J.H.C. Whitehead, On group extensions with operators, *Quat. J. Math. Oxford* **2** (1950), 219-228.

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