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SIMULTANEOUSLY REFLECTIVE AND COREFLECTIVE SUBCATEGORIES OF PRESHEAVES

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ABSTRACT. It is proved that any category $\mathcal{K}$ which is equivalent to a simultaneously reflective and coreflective full subcategory of presheaves $[\mathcal{A}^{op}, \mathsf{Set}]$, is itself equivalent to the category of the form $[\mathcal{B}^{op}, \mathsf{Set}]$ and the inclusion is induced by a functor $\mathcal{A} \longrightarrow \mathcal{B}$ which is surjective on objects. We obtain a characterization of such functors.

Moreover, the base category Set can be replaced with any symmetric monoidal closed category $\mathcal{V}$ which is complete and cocomplete, and then analogy of the above result holds if we replace categories by $\mathcal{V}$-categories and functors by $\mathcal{V}$-functors.

As a consequence we are able to derive well-known results on simultaneously reflective and coreflective categories of sets, Abelian groups, etc.

1. Introduction

A full subcategory $\mathcal{K}$ of $\mathcal{L}$, which is simultaneously reflective and coreflective in $\mathcal{L}$, inherits many pleasant properties of $\mathcal{L}$, for example, the computation of limits and colimits. In the literature, various instances of categories $\mathcal{L}$ were studied: for example the category of all topological spaces and continuous maps in [9] or the category Ab of all Abelian groups and their homomorphisms in [2]. The latter paper characterized full subcategories of Ab, which arise as simultaneously reflective and coreflective, as precisely the categories of (right) R-modules for certain commutative rings R with a unit.

The approach we take in this paper is motivated by results proved in [2]. The category Ab of Abelian groups bears a symmetric monoidal closed structure and therefore gives rise to the theory of categories enriched over Ab. Any full embedding

$$J : \mathcal{K} \longrightarrow \mathsf{Ab}$$

is then, in fact, an Ab-functor: hom-sets of $\mathcal{K}$ are Abelian groups and actions of J on these groups are group homomorphisms. Moreover, if J possesses both left and right adjoints L and R, then both these functors can be given a structure of Ab-functors as well. Thus, the adjoint situation

$$L \dashv J \dashv R$$

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is an adjoint situation in the realm of Ab-categories and Ab-functors. Any ring R with a unit corresponds precisely to a one-object Ab-category $\mathcal{R}$ and the category of right R-modules Mod-R is the Ab-category $[\mathcal{R}^{op}, \mathsf{Ab}]$ of Ab-functors from $\mathcal{R}^{op}$ to Ab and Ab-natural transformations. Similarly, viewing Abelian groups as right Z-modules (where Z denotes the ring of integers) allows us to consider Ab as the category of the form $[\mathcal{Z}^{op}, \mathsf{Ab}]$, where $\mathcal{Z}$ is a one-object Ab-category corresponding to Z.

Therefore, the result of Section 1.2 in [2] can be restated as follows:

> *Every full sub-Ab-category $\mathcal{K}$ of $[\mathcal{Z}^{op}, \mathsf{Ab}]$ which is simultaneously reflective and coreflective in $[\mathcal{Z}^{op}, \mathsf{Ab}]$ is equivalent to an Ab-category $[\mathcal{R}^{op}, \mathsf{Ab}]$ for some one-object Ab-category $\mathcal{R}$.*

Such reformulation allows us to consider a general question:

If we replace Ab by a general symmetric monoidal closed category $\mathcal{V}$ and a one-object Ab-category $\mathcal{Z}$ by a general small $\mathcal{V}$-category $\mathcal{A}$, what do simultaneously reflective and coreflective full sub-$\mathcal{V}$-categories of $[\mathcal{A}^{op}, \mathcal{V}]$ look like?

The answer is similar to the case of $\mathcal{V} = \mathsf{Ab}$: such categories are, up to equivalence, precisely those full sub-$\mathcal{V}$-categories of the form $[\mathcal{B}^{op}, \mathcal{V}]$ and the inclusion is induced by a $\mathcal{V}$-functor $P : \mathcal{A} \longrightarrow \mathcal{B}$, that is, the inclusion is the $\mathcal{V}$-functor

$$[P^{op}, 1_{\mathcal{V}}] : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$$

of "restriction along P^{op}". We call $\mathcal{V}$-functors P with the property that $[P^{op}, 1_{\mathcal{V}}]$ is fully faithful *connected* (see 3.3 below) and give several characterizations of them.

Among others, connected functors will be proved to be exactly the representably fully faithful functors (Condition 4, Proposition 3.6). Such morphism in general 2-categories have already been studied by John Gray, Ross Street and other authors, see [12] or [4]. In [5], Brian Day calls such functors Cauchy dense.

2. Preliminaries

We use the notation and terminology of [10]. We fix a locally small symmetric monoidal closed category $\mathcal{V}_o$, which is complete and cocomplete. The tensor product in $\mathcal{V}_o$ is denoted by $\otimes$, its unit by I. The "internal hom" is denoted by $[\_, \_]$. Thus, for every object A in $\mathcal{V}_o$, there is an adjunction

$$\_ \otimes A \dashv [A, \_] : \mathcal{V}_o \longrightarrow \mathcal{V}_o$$

The category $\mathcal{V}_o$ therefore serves as a base category for enriched category theory. Especially, there is an enriched category — denoted by $\mathcal{V}$ — which has the same objects as $\mathcal{V}_o$ and for which $\mathcal{V}(A, B) = [A, B]$.

In what follows, whenever we say category, functor, natural transformation, etc., we *always* mean $\mathcal{V}$-category, $\mathcal{V}$-functor, $\mathcal{V}$-natural transformation. Non-enriched categories, functors, natural transformations, etc., will be called *ordinary.* Each $\mathcal{V}$-category $\mathcal{X}$ has an *underlying ordinary category* $\mathcal{X}_o$ having the same objects as $\mathcal{X}$ and with hom-sets defined by

$$\mathcal{X}_o(X, Y) = \mathcal{V}_o(I, \mathcal{X}(X, Y))$$

For a small category $\mathcal{D}$ and functors $D : \mathcal{D}^{op} \longrightarrow \mathcal{V}$ and $F : \mathcal{D} \longrightarrow \mathcal{K}$ a *colimit of F weighted by D* is an object $D * F$ together with an isomorphism

$$\mathcal{K}(D * F, K) \cong [\mathcal{D}^{op}, \mathcal{V}](D, \mathcal{K}(F_-, K))$$

natural in K. A dual notion is that of a *limit of $F : \mathcal{D} \longrightarrow \mathcal{K}$ weighted by $D : \mathcal{D} \longrightarrow \mathcal{V}$*.

For a small category $\mathcal{A}$ we denote by $[\mathcal{A}^{op}, \mathcal{V}]$ the category of all functors $H : \mathcal{A}^{op} \longrightarrow \mathcal{V}$ as objects and the hom-objects in $[\mathcal{A}^{op}, \mathcal{V}]$ are defined by the end (a special weighted limit):

$$[\mathcal{A}^{op}, \mathcal{V}](H, G) = \int_A [HA, GA]$$

This weighted limit exists. In fact, assumptions on $\mathcal{V}_o$ entail that $\mathcal{V}$ is complete and cocomplete, thus, so is $[\mathcal{A}^{op}, \mathcal{V}]$, weighted limits and colimits there being formed pointwise.

3. Connected Functors

In the present section we characterize those functors $P : \mathcal{A} \longrightarrow \mathcal{B}$ for which the functor

$$[P^{op}, 1_{\mathcal{V}}] : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$$

is fully faithful and we show that, up to equivalence, these are precisely all simultaneously reflective and coreflective full subcategories of $[\mathcal{A}^{op}, \mathcal{V}]$.

We begin with the following well-known assertion (see, for example, Theorem 4.50 of [10] or Theorem 6.7.7 in the second volume of [3] for the proof):

3.1. Lemma. *Suppose that $P : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor between small categories. Then the functor*

$$\begin{aligned} [P^{op}, 1_{\mathcal{V}}] \quad : \quad & [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}] \\ & H \mapsto H \cdot P^{op} \end{aligned}$$

has both left and right adjoints.

A left adjoint of $[P^{op}, 1_{\mathcal{V}}]$ assigns to $H : \mathcal{A}^{op} \longrightarrow \mathcal{V}$ a *left Kan extension* of H along P^{op}, denoted by $\mathrm{Lan}_{P^{op}} H$. Analogously, a right adjoint of $[P^{op}, 1_{\mathcal{V}}]$ assigns to $H : \mathcal{A}^{op} \longrightarrow \mathcal{V}$ a *right Kan extension* of H along P^{op}, denoted by $\mathrm{Ran}_{P^{op}} H$.

If, in the notation of the previous lemma, the functor $[P^{op}, 1_{\mathcal{V}}]$ were full and faithful, it would therefore provide us with an example of a subcategory of presheaves which is simultaneously reflective and coreflective.

3.2. Definition. *A functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ between small categories is called* connected*, if the restriction along P^{op}, that is, the functor*

$$[P^{op}, 1_{\mathcal{V}}] : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$$

is full and faithful.

3.3. Remark. The terminology "connected functor" has been proposed by Peter Johnstone in his lecture at PSSL in Cambridge in November 2000 for ordinary functors $P : \mathcal{A} \longrightarrow \mathcal{B}$ for which the induced functor

$$[P, 1_{\mathsf{Set}}] : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{A}, \mathsf{Set}]$$

is full and faithful. It follows, however, that this condition is equivalent to the requirement that

$$[P^{op}, 1_{\mathsf{Set}}] : [\mathcal{B}^{op}, \mathsf{Set}] \longrightarrow [\mathcal{A}^{op}, \mathsf{Set}]$$

is full and faithful (see Proposition 3.6 below or [1]). It is the latter condition which is more convenient for our setting — this is why we have chosen it in our definition.

3.4. Remark. A useful necessary property of a connected functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ is its *density*, that is, the fact that the functor

$$\begin{aligned} \widetilde{P} \quad : \quad & \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathcal{V}] \\ & B \mapsto \mathcal{B}(P\,\text{\_}, B) \end{aligned}$$

is full and faithful. This follows immediately from the commutative triangle

$$\begin{tikzcd} {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[rr, "{[P^{op}, 1_{\mathcal{V}}]}"] & & {[\mathcal{A}^{op}, \mathcal{V}]} \\ & \mathcal{B} \arrow[ul, "Y"] \arrow[ur, "\widetilde{P}"'] & \end{tikzcd}$$

where Y is the (full and faithful) Yoneda embedding.

Density of a functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ can equivalently be expressed by saying that each object B is represented as a weighted colimit $\widetilde{P}B * P$, or, equivalently, that a left Kan extension $\mathrm{Lan}_P P$ of P along itself exists and is isomorphic to the identity functor $1_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ (see Theorem 5.1 of [10]).

3.5. Definition. *A functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ between small categories is called* absolutely dense*, if it is dense and all colimits $\widetilde{P}B * P$ are absolute, that is, they are preserved by any functor.*

An easy characterization of connected functors follows from analyzing a counit of the left Kan extension. For the case of $\mathcal{V} = \mathsf{Set}$ (that is, for ordinary categories) some of the characterizations below have been given in [1].

3.6. Proposition. *For $P : \mathcal{A} \longrightarrow \mathcal{B}$ between small categories the following are equivalent:*

1. *The functor P is connected.*
2. *For every object B of $\mathcal{B}$, the counit*
$$\varepsilon_{YB} : \mathrm{Lan}_{P^{op}}(YB \cdot P^{op}) \longrightarrow YB$$
of $\mathrm{Lan}_{P^{op}}(\_) \dashv [P^{op}, 1_{\mathcal{V}}]$ is an isomorphism, where $Y : \mathcal{B} \longrightarrow [\mathcal{B}^{op}, \mathcal{V}]$ denotes the Yoneda embedding.
3. *The functor P is absolutely dense.*
4. *For every category $\mathcal{X}$, the functor*
$$[P, 1_{\mathcal{X}}] : [\mathcal{B}, \mathcal{X}] \longrightarrow [\mathcal{A}, \mathcal{X}]$$
is full and faithful.
5. *The functor $P^{op} : \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$ is connected, that is, the functor*
$$[P, 1_{\mathcal{V}}] : [\mathcal{B}, \mathcal{V}] \longrightarrow [\mathcal{A}, \mathcal{V}]$$
is full and faithful.
6. *For every pair of functors $W : \mathcal{B}^{op} \longrightarrow \mathcal{V}$ and $D : \mathcal{B} \longrightarrow \mathcal{X}$ a weighted colimit $W * D$ exists if and only if a weighted colimit $(W \cdot P^{op}) * (D \cdot P)$ exists and both colimits are isomorphic.*

Proof. Condition 1 is equivalent to the fact that ε evaluated at any $H : \mathcal{B}^{op} \longrightarrow \mathcal{V}$ is a natural isomorphism. Since any such H can be expressed as a weighted colimit $H * Y$ of representables, and since both $\mathrm{Lan}_{P^{op}}(\_)$ and $[P^{op}, 1_{\mathcal{V}}]$ preserve colimits (see Lemma 3.1), conditions 1 and 2 are equivalent.

2 implies 3. P is dense by considerations in 3.4. To prove that P is absolutely dense, we verify that every functor $F : \mathcal{B} \longrightarrow \mathcal{X}$ preserves weighted colimits $B \cong \widetilde{P}B * P$. In fact, recall that $[P^{op}, 1_{\mathcal{V}}]$ is assumed to be full and faithful and use $\widetilde{P}B = \mathcal{B}(\_, B) \cdot P^{op}$ to deduce the following isomorphisms

$$\begin{aligned}
\mathcal{X}\Big(\widetilde{P}B * F \cdot P, X\Big) &\cong [\mathcal{A}^{op}, \mathcal{V}]\Big(\mathcal{B}(P_, B), \mathcal{X}(F \cdot P_, X)\Big) \\
&\cong [\mathcal{A}^{op}, \mathcal{V}]\Big(\mathcal{B}(_, B) \cdot P^{op}, \mathcal{X}(F_, X) \cdot P^{op}\Big) \\
&\cong [\mathcal{B}^{op}, \mathcal{V}]\Big(\mathcal{B}(_, B), \mathcal{X}(F_, X)\Big) \\
&\cong \mathcal{X}(FB, X)
\end{aligned}$$

natural in every object X in $\mathcal{X}$.

3 implies 4. Since we assume that P is absolutely dense, this means that a left Kan extension $\mathrm{Lan}_P P \cong 1_{\mathcal{B}}$ of P along itself is preserved by any functor $F : \mathcal{B} \longrightarrow \mathcal{X}$. Thus, for any pair $F, G : \mathcal{B} \longrightarrow \mathcal{X}$ of functors we have isomorphisms

$$\begin{aligned} [\mathcal{B},\mathcal{X}](F,G) &\cong [\mathcal{B},\mathcal{X}]\Big(\mathrm{Lan}_P(F\cdot P),G\Big) \\ &\cong [\mathcal{A},\mathcal{X}](F\cdot P, G\cdot P) \end{aligned}$$

where the last isomorphism is induced by precomposing with P. We conclude that condition 4 holds.

Since 4 clearly implies 5, we prove that 5 implies 2. Analogously to the equivalence of conditions 1 and 2 we firstly deduce that condition 5 is equivalent to the fact that the counit of $\mathrm{Lan}_P(\_) \dashv [P, 1_{\mathcal{V}}]$ is an isomorphism, whenever it is evaluated at a representable functor $\mathcal{B}(B', \_)$. Thus, the map

$$\int^A \mathcal{B}(B',PA)\otimes\mathcal{B}(PA,B) \longrightarrow \mathcal{B}(B',B)$$

induced by composition, is an isomorphism, natural in both B and B'. Therefore, condition 2 holds, since the natural isomorphism

$$\int^A \mathcal{B}(\_,PA)\otimes\mathcal{B}(PA,B) \longrightarrow \mathcal{B}(\_,B)$$

is precisely ε_{YB}.

1 implies 6. Consider the following isomorphisms

$$\begin{aligned} \mathcal{X}\Big((W\cdot P^{op} * (D\cdot P),X)\Big) &\cong [\mathcal{A}^{op},\mathcal{V}]\Big(W\cdot P^{op},\mathcal{X}(D\cdot P\,\_,X)\Big) \\ &\cong [\mathcal{A}^{op},\mathcal{V}]\Big(W\cdot P^{op},\mathcal{X}(D\,\_,X)\cdot P^{op}\Big) \\ &\cong [\mathcal{B}^{op},\mathcal{V}]\Big(W,\mathcal{X}(D\,\_,X)\Big) \\ &\cong \mathcal{X}(W*D,X) \end{aligned}$$

natural in X.

6 implies 1. Apply condition 6 to the case when D is the Yoneda embedding $Y : \mathcal{B} \longrightarrow [\mathcal{B}^{op}, \mathcal{V}]$. Since $(W \cdot P^{op}) * (Y \cdot P)$ is isomorphic to $\mathrm{Lan}_{P^{op}}(W \cdot P^{op})$ and since $W * Y$ is isomorphic to W, from

$$(W\cdot P^{op}) * (Y\cdot P) \cong W$$

we conclude that the counit of $\mathrm{Lan}_{P^{op}}(\_) \dashv [P^{op}, 1_{\mathcal{V}}]$ is an isomorphism. Therefore $[P^{op}, 1_{\mathcal{V}}]$ is full and faithful. ∎

3.7. Examples. The above proposition allows one to characterize connected functors for some choices of the base category $\mathcal{V}$.

1. In case when $\mathcal{V}$ is the category Ab of Abelian groups and their homomorphisms, connected functors between one-object categories (that is, homomorphisms of rings with unit) are precisely epimorphisms of rings (see [11]). It holds that the functor $P : R \longrightarrow S$ is an epimomorphism if and only if the canonical map $m : S \underset{R}{\otimes} S \longrightarrow S$, $m(s \otimes s') = ss'$ is a bijection (Proposition 16.3 in [11]). Connected functors do not increase the size of infinite rings. For the classification of connected functors with domain Z, see [2].

   It has been proved in 2.4 of [7] that every epimorphism $f : R \longrightarrow S$ of rings with unit induces a full embedding of categories

$$[P, 1_{\mathcal{X}}] : [\mathcal{S}, \mathcal{X}] \longrightarrow [\mathcal{R}, \mathcal{X}]$$

   for any Ab-category $\mathcal{X}$. (Here, $\mathcal{S}$ and $\mathcal{R}$ denote the one-object Ab-categories corresponding to R and S.) Condition 4 of Proposition 3.6 shows that a converse also holds.

2. In case when $\mathcal{V}$ is the category Set of sets and mappings we obtain the following characterization of connected functors $P : \mathcal{A} \longrightarrow \mathcal{B}$ (see Theorem 2.1 in [1]):

   Firstly, for every arrow $f : B' \longrightarrow B$ in $\mathcal{B}$ define the category $f \mathbin{/\!\!/} P$ as follows:

   - objects are pairs $\langle p : B' \longrightarrow PA, q : PA \longrightarrow B \rangle$ with $q \cdot p = f$
   - morphisms from $\langle p : B' \longrightarrow PA, q : PA \longrightarrow B \rangle$ to $\langle p' : B' \longrightarrow PA', q' : PA' \longrightarrow B \rangle$ are those morphisms $h : A \longrightarrow A'$ such that $Ph \cdot p = p'$ and $Ph \cdot q' = q$.

   Then P is connected if and only if the category $f \mathbin{/\!\!/} P$ is connected for every $f : B' \longrightarrow B$. This can be seen immediately from Proposition 3.6: the functor P is connected if and only if the map

$$\int^{A} \mathcal{B}(B', PA) \times \mathcal{B}(PA, B) \longrightarrow \mathcal{B}(B', B)$$

   induced by composition, is an isomorphism. Thus, the elements of the coend on the left are in one to one correspondence with morphisms $f : B \longrightarrow B'$. This is precisely to say that every category of the form $f \mathbin{/\!\!/} P$ is connected.

   Connected functors between one-object ordinary categories are precisely epimorphisms in the category of monoids (see Corollary 2.2 in [7]). Such an epimorphism may increase the size of monoid M only if M is finite — see [6]. An easy example of a connected functor which is not full is the inclusion of the additive monoid N of natural numbers into the monoid Z of integers.

Connected functors between ordinary categories are not precisely the epimorphisms. It holds that if a functor P is an epimorphism which is one-to-one on objects, then P is connected (Corollary 2.2 of [7]). It is shown in [1] that connected functors $P : \mathcal{A} \longrightarrow \mathcal{B}$ are precisely *lax epimorphisms* in the 2-category Cat of small categories, functors and natural transformations. A functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ is, by definition, a lax epimorphism whenever the functor

$$[P, 1_{\mathcal{X}}] : [\mathcal{B}, \mathcal{X}] \longrightarrow [\mathcal{A}, \mathcal{X}]$$

is full and faithful for every small category $\mathcal{X}$ (see also condition 4 of Proposition 3.6).

The next result shows that connected functors enjoy good closure properties.

3.8. Lemma. *The class of connected functors is closed under composition and tensor products.*

Proof. It is clear that $Q \cdot P : \mathcal{A} \longrightarrow \mathcal{C}$ is a connected functor whenever $P : \mathcal{A} \longrightarrow \mathcal{B}$ and $Q : \mathcal{B} \longrightarrow \mathcal{C}$ are connected functors.

To show that $P \otimes Q : \mathcal{A} \otimes \mathcal{C} \longrightarrow \mathcal{B} \otimes \mathcal{D}$ is a connected functor for connected functors $P : \mathcal{A} \longrightarrow \mathcal{B}$ and $Q : \mathcal{C} \longrightarrow \mathcal{D}$ it suffices to show that

$$P \otimes 1_{\mathcal{C}} : \mathcal{A} \otimes \mathcal{C} \longrightarrow \mathcal{B} \otimes \mathcal{C} \quad \text{and} \quad 1_{\mathcal{B}} \otimes Q : \mathcal{B} \otimes \mathcal{C} \longrightarrow \mathcal{B} \otimes \mathcal{D}$$

are connected functors, because $P \otimes Q = (1_{\mathcal{B}} \otimes Q) \cdot (P \otimes 1_{\mathcal{C}})$.

Connectedness of $P \otimes 1_{\mathcal{C}}$ follows immediately from the commutative diagram

$$\begin{tikzcd}
[\mathcal{B} \otimes \mathcal{C}, \mathcal{V}] \arrow[r, "{[P \otimes 1_{\mathcal{C}}, \mathcal{V}]}"] \arrow[d, "\cong"'] & [\mathcal{A} \otimes \mathcal{C}, \mathcal{V}] \arrow[d, "\cong"] \\
\big[\mathcal{B}, [\mathcal{C}, \mathcal{V}]\big] \arrow[r, "{[P, 1_{[\mathcal{C}, \mathcal{V}]}]}"'] & \big[\mathcal{A}, [\mathcal{C}, \mathcal{V}]\big]
\end{tikzcd}$$

and connectedness of $1_{\mathcal{B}} \otimes Q$ is proved in a similar way. ■

3.9. Remark. Connected functors can also be characterized via profunctors.

Recall that the bicategory Prof of profunctors is defined as follows (see, for example, [3] for details):

1. Objects are small categories.

2. Morphisms from $\mathcal{X}$ to $\mathcal{Y}$ are functors $\mathcal{Y}^{op} \otimes \mathcal{X} \longrightarrow \mathcal{V}$ called *profunctors* (also *bimodules*, or *distributors*) and denoted by $\mathcal{X} \mathrel{-\!\!\circ\!\!\rightarrow} \mathcal{Y}$.

3. 2-cells are natural transformations between the respective functors.

4. For profunctors

$$\varphi : \mathcal{X} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathcal{Y} \quad \text{and} \quad \psi : \mathcal{Y} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathcal{Z}$$

the composite profunctor $\psi \cdot \varphi : \mathcal{X} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathcal{Z}$ has values

$$\Big(\psi \cdot \varphi\Big)(Z, X) = \int^{Y} \psi(Z, Y) \otimes \varphi(Y, X)$$

This composition is associative up to an isomorphism and an identity profunctor $i_{\mathcal{X}} : \mathcal{X} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathcal{X}$ is the hom-functor of the category $\mathcal{X}$.

Recall that every functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ between small categories induces a pair of profunctors

$$P^{\diamond} : \mathcal{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathcal{B} \quad \text{and} \quad P_{\diamond} : \mathcal{B} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathcal{A}$$

where $P^{\diamond}(B, A) = \mathcal{B}(B, PA)$ and $P_{\diamond}(A, B) = \mathcal{B}(PA, B)$. Moreover, there is always an adjunction

$$P^{\diamond} \dashv P_{\diamond}$$

in Prof with counit $P^{\diamond} \cdot P_{\diamond} \longrightarrow i_{\mathcal{B}}$ defined pointwise by the morphism

$$\varepsilon_{B',B} : \int^{A} \mathcal{B}(B', PA) \otimes \mathcal{B}(PA, B) \longrightarrow \mathcal{B}(B', B)$$

which is the unique morphism induced by composition.

Thus, by Proposition 3.6, P is connected if and only if the counit of $P^{\diamond} \dashv P_{\diamond}$ is an isomorphism.

It will be useful later to note that the unit $i_{\mathcal{A}} \longrightarrow P_{\diamond} \cdot P^{\diamond}$ of $P^{\diamond} \dashv P_{\diamond}$ is given pointwise by the morphism

$$\mathcal{A}(A', A) \longrightarrow \int^{B} \mathcal{B}(PA', B) \otimes \mathcal{B}(B, PA)$$

which is the action $P_{A',A} : \mathcal{A}(A', A) \longrightarrow \mathcal{B}(PA', PA)$ of P on hom-objects — here, we use the fact that $\int^{B} \mathcal{B}(PA', B) \otimes \mathcal{B}(B, PA) \cong \mathcal{B}(PA', PA)$.

We now show that one can assume, without loss of generality, that connected functors are surjective on objects. Recall that small categories $\mathcal{X}$ and $\mathcal{Y}$ are called *Morita equivalent*, if the corresponding categories of presheaves $[\mathcal{X}^{op}, \mathcal{V}]$ and $[\mathcal{Y}^{op}, \mathcal{V}]$ are equivalent, or, equivalently, if the categories $\mathcal{X}$ and $\mathcal{Y}$ are equivalent in the bicategory Prof.

3.10. Lemma. *Suppose that $P : \mathcal{A} \longrightarrow \mathcal{B}$ is a connected functor. Factorize P as*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "P"] \arrow[dr, "F"'] & & \mathcal{B} \\ & \mathcal{C} \arrow[ur, "G"'] & \end{tikzcd}$$

with F surjective on objects and G fully faithful. Then F is connected and $\mathcal{C}$, $\mathcal{B}$ are Morita equivalent categories.

PROOF. For every $H, K : \mathcal{B}^{op} \longrightarrow \mathcal{V}$ the following triangle commutes:

$$\begin{tikzcd}[\mathcal{B}^{op},\mathcal{V}](H,K) \arrow[rr, "{[P^{op},1_{\mathcal{V}}]_{H,K}}"] \arrow[dr, "{[G^{op},1_{\mathcal{V}}]_{H,K}}"'] & & [\mathcal{A}^{op},\mathcal{V}](H\cdot P^{op}, K\cdot P^{op}) \\ & [\mathcal{C}^{op},\mathcal{V}](H\cdot G^{op}, K\cdot G^{op}) \arrow[ur, "{[F^{op},1_{V}]_{HG^{op},KG^{op}}}"'] & \end{tikzcd}$$

Since F is surjective on objects, the morphism

$$[F^{op}, 1_{\mathcal{V}}]_{HG^{op},KG^{op}}$$

is a monomorphism (see the proof of Proposition 5.11 of [10]). Since it is also a split epimorphism, we conclude that it is an isomorphism. Thus, also the morphism

$$[G^{op}, 1_{\mathcal{V}}]_{H,K}$$

is an isomorphism and we have shown that G is a fully faithful connected functor. By Remark 3.9 the adjunction of profunctors $G^{\diamond} \dashv G_{\diamond}$ is an adjoint equivalence, thus $\mathcal{C}$ and $\mathcal{B}$ are Morita equivalent categories. As a consequence, the functor F is connected. ∎

The main result of this paper shows that, up to equivalence, there are no other simultaneously reflective and coreflective full subcategories of $[\mathcal{A}^{op}, \mathcal{V}]$ than inclusions of the form

$$[P^{op}, 1_{\mathcal{V}}] : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$$

for a (necessarily connected) functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ which is surjective on objects.

3.11. THEOREM. *Let $\mathcal{A}$ be a small category and let J be a full embedding*

$$J : \mathcal{K} \longrightarrow [\mathcal{A}^{op}, \mathcal{V}].$$

of a simultaneously reflective and coreflective subcategory. Then there exists a connected functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ which is surjective on objects such that, up to equivalence, the embedding J is

$$[P^{op}, 1_{\mathcal{V}}] : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$$

PROOF. Form the composite

$$\mathcal{A} \xrightarrow{Y} [\mathcal{A}^{op}, \mathcal{V}] \xrightarrow{L} \mathcal{K}$$

of the Yoneda embedding and the left adjoint L of J and denote it by E. Then E is a dense functor and $J : \mathcal{K} \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$ is isomorphic to a fully faithful functor

$$\widetilde{E} : K \mapsto \mathcal{K}(E_{-}, K)$$

(see Proposition 5.15 of [10]).

Factorize E as follows:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "E"] \arrow[dr, "P"'] & & \mathcal{K} \\ & \mathcal{B} \arrow[ur, "E'"'] & \end{tikzcd}$$

where P is surjective on objects and E' is fully faithful. Since E is dense, so are P and E' (Theorem 5.13 of [10]) and the triangle

$$\begin{tikzcd} {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[rr, "{[P^{op}, 1_{\mathcal{V}}]}"] & & {[\mathcal{A}^{op}, \mathcal{V}]} \\ & \mathcal{K} \arrow[ul, "\widetilde{E'}"] \arrow[ur, "\widetilde{E}"'] & \end{tikzcd}$$

commutes, because $E' \cdot P = E$.

The functor $\widetilde{E'} : \mathcal{K} \longrightarrow [\mathcal{B}^{op}, \mathcal{V}]$ has a left adjoint $\_ * E'$, sending a functor $H : \mathcal{B}^{op} \longrightarrow \mathcal{V}$ to a colimit $H * E'$ of E' weighted by H, because $\mathcal{K}$ is cocomplete.

It remains to show that the functor $\widetilde{E'}$ is an equivalence. It is clearly full and faithful, since E' is dense. It suffices to show that $\widetilde{E'}$ is essentially surjective on objects. This is done in the following steps:

1. Firstly observe that every object of the form $E'A$ is small-projective in $\mathcal{K}$, that is, the hom-functor
$$\mathcal{K}(E'A, \_) : \mathcal{K} \longrightarrow \mathcal{V}$$
preserves all colimits. Indeed, consider a colimit $W * D$ in $\mathcal{K}$. Then we obtain the following isomorphisms
$$\begin{aligned} \mathcal{K}(E'A, W * D) &\cong \mathcal{K}(LYA, W * D) \\ &\cong [\mathcal{A}^{op}, \mathcal{V}]\Big(YA, \widetilde{E}(W * D)\Big) \\ &\cong W * [\mathcal{A}^{op}, \mathcal{V}](YA, \widetilde{E}D\,\_) \\ &\cong W * \mathcal{K}(E'A, D\,\_) \end{aligned}$$
by using the fact that $L \dashv \widetilde{E}$ and that $\widetilde{E}$ preserves colimits (since $\widetilde{E}$ is a left adjoint).

2. From the above it follows that $\widetilde{E'}$ preserves small colimits.

3. To conclude the proof, take any $H : \mathcal{B}^{op} \longrightarrow \mathcal{V}$. Then the following isomorphisms
$$\widetilde{E'}(H * E') \cong H * \widetilde{E'}E' \cong H * Y \cong H$$
take place, because E' is full and faithful and thus the composite $\widetilde{E'} \cdot E'$ is the Yoneda embedding Y.

■

Since the category $\mathcal{V}$ itself is equivalent to $[\mathcal{I}^{op}, \mathcal{V}]$, where $\mathcal{I}$ is the one-object unit category, as a consequence of Theorem 3.11 we obtain:

3.12. Corollary. *Any full reflective and coreflective subcategory of $\mathcal{V}$ is equivalent to the category of the form $[\mathcal{A}^{op}, \mathcal{V}]$, where $\mathcal{A}$ has one object, that is, it corresponds to a monoid in $\mathcal{V}$.*

3.13. Examples.

1. Any (not necessarily commutative) ring R with a unit can be viewed as a one-object Ab-category $\mathcal{R}$. Theorem 3.11 asserts that full reflective and coreflective subcategories of $[\mathcal{R}^{op}, \mathsf{Ab}]$ are of the form $[\mathcal{S}^{op}, \mathcal{V}]$ for a one-object Ab-category $\mathcal{S}$, that is, a ring S with a unit. Moreover, the inclusion $[\mathcal{S}^{op}, \mathcal{V}] \longrightarrow [\mathcal{R}^{op}, \mathcal{V}]$ is induced by an epimorphism of rings $f : R \longrightarrow S$ (see Example 3.7). Thus, full reflective and coreflective subcategories of right R-modules are up to equivalence precisely categories of right S-modules for ring-epimorphic images S of R.

2. More in general, given a commutative theory, that is, a variety $\mathcal{V}$ of algebras in which all operations are homomorphisms, we can produce in a natural way the tensor product with one-generated free algebra as its unit (see [3]). By Theorem 3.11 we can represent full reflective and coreflective subcategories of the presheaf categories $[\mathcal{A}^{op}, \mathcal{V}]$, for the resulting enriched category $\mathcal{V}$, by connected functors. Among the many instances of this construction one can find the varieties of medial groupoids (see, for example, [8]), commutative semigroups, abelian groups.

3. From Theorem 3.11 we can conclude that, up to equivalence, Set does not have any simultaneously reflective and coreflective full subcategories than Set itself. This can be seen as follows:

   Let $\mathcal{I}$ be a monoid on one-element set, considered as a category. Then $[\mathcal{I}^{op}, \mathsf{Set}]$ is equivalent to Set and any reflective and coreflective subcategory $J : \mathcal{K} \longrightarrow \mathsf{Set}$ must be, up to equivalence, of the form $[P^{op}, 1_{\mathsf{Set}}] : [\mathcal{B}^{op}, \mathsf{Set}] \longrightarrow [\mathcal{I}^{op}, \mathsf{Set}]$, where $P : \mathcal{I} \longrightarrow \mathcal{B}$ is an epimorphism of monoids (see Example 3.7).

   Since, by Remark 3.4, the functor P is dense, the monoid $\mathcal{B}$ must be isomorphic to the monoid of $\mathcal{I}$-equivariant maps of $\mathcal{B}$ to itself. Since every map is $\mathcal{I}$-equivariant, it follows that $\mathcal{B}$ must be isomorphic to $\mathcal{I}$.

By Remark 3.9, a functor $P : \mathcal{A} \longrightarrow \mathcal{B}$ is connected if and only if the adjunction $P^\diamond \dashv P_\diamond$ of the corresponding profunctors is a "full reflection", that is, if the counit of this adjunction is an isomorphism. Applying Theorem 3.11 we now show that every full reflection of profunctors is generated by a connected functor. More precisely, the following holds:

3.14. Proposition. *Suppose that $\varphi : \mathcal{B} \multimap \mathcal{A}$ and $\psi : \mathcal{A} \multimap \mathcal{B}$ are profunctors with $\psi \dashv \varphi$ and such that the counit $\psi \cdot \varphi \longrightarrow i_{\mathcal{B}}$ is an isomorphism. Then there is a category $\mathcal{C}$ and a connected functor $P : \mathcal{A} \longrightarrow \mathcal{C}$ such that $\mathcal{C}$ is Morita equivalent to $\mathcal{B}$ and the adjunction $\psi \dashv \varphi$ is, up to this equivalence, the adjunction $P^\diamond \dashv P_\diamond$.*

Proof. Put $\mathrm{tr}(\varphi) : \mathcal{B} \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$ and $\mathrm{tr}(\psi) : \mathcal{A} \longrightarrow [\mathcal{B}^{op}, \mathcal{V}]$ to be the functors

$$\mathrm{tr}(\varphi) : B \mapsto \varphi(\_, B) \quad \text{and} \quad \mathrm{tr}(\psi) : A \mapsto \psi(\_, A)$$

and consider the functors

$$\_ * \mathrm{tr}(\varphi) : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}] \quad \text{and} \quad \_ * \mathrm{tr}(\psi) : [\mathcal{A}^{op}, \mathcal{V}] \longrightarrow [\mathcal{B}^{op}, \mathcal{V}]$$

By assumption, $\_ * \mathrm{tr}(\psi) \dashv \_ * \mathrm{tr}(\varphi)$ and the counit of this adjunction is an isomorphism. Since, clearly, the functor $\_ * \mathrm{tr}(\varphi)$ preserves colimits, it has a right adjoint. Thus, the functor

$$\_ * \mathrm{tr}(\varphi) : [\mathcal{B}^{op}, \mathcal{V}] \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$$

is a full embedding of a simultaneously reflective and coreflective subcategory. The rest follows by applying Theorem 3.11. ∎

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