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DYNAMIC OLIGOPOLIES WITH PRODUCTION FACTORS

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Abstract. A dynamic single-product oligopoly is introduced with continuous time scales that includes production factors used by the firms. The existence of the Nash–Cournot equilibrium is first established, and then the local and global asymptotical stability of interior equilibria are proven under realistic conditions.

1. Introduction

Since the pioneering work of Cournot (1838) has been published a large number of researchers have investigated the different aspects of oligopoly markets. The existence and uniqueness of the Nash–Cournot equilibrium was the first area that resulted in a series of work. Okuguchi (1976) gave a comprehensive summary of relevant results on single-product models without and with product differentiation. Multi-product models and related results are summarized in Okuguchi and Szidarovszky (1999), and also several model variants are introduced and examined including models with market saturation, labor-managed oligopolies, rent-seeking games, oligopsonies among others. Most of the models consider N-firms competing on the market in which the sale price is a function of the total output of the industry, and the boundedness of the available resources is taken into account by bounded feasible output sets. Oligopsonies include the competition of the firms in the labor and capital markets as well. A slightly more general model has been introduced and examined in Szidarovszky and Okuguchi (2001) in which multiple production factors were considered in a single-product oligopoly. The asymptotical stability of the equilibrium was examined under discrete time scales. In this paper the continuous time scale counterpart of the same model will be examined. The paper develops as follows. The next section introduces the basic model and the dynamic system, and the asymptotical behavior of the state trajectory will then be investigated.

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2. The Basic Model and Dynamic System

Consider an N-firm oligopoly with single-product firms without product differentiation, and assume that the firms use M factors of production. Perfectly competitive product market and an oligopsonistic factor market are assumed. The profit of firm k is given as

$$\varphi_k(\mathbf{l}_1, \dots, \mathbf{l}_n) = pf_k(\mathbf{l}_k) - \mathbf{l}_k^T \mathbf{w}(\mathbf{L}) \tag{1}$$

where p is the competitive product price, f_k is the production of firm k, $\mathbf{l}_k$ is its M-element production factor, $\mathbf{w}$ is the M-element factor price vector, and $\mathbf{L} = \sum_{i=1}^{N} \mathbf{l}_i$ is the total production factor usage of the industry. For the sake of mathematical simplicity we assume that

(i) $f_k(\mathbf{l}_k) = \mathbf{l}_k^T \mathbf{C}_k \mathbf{l}_k + \mathbf{c}_k^T \mathbf{l}_k + \gamma_k \quad$ for all k,

and

(ii) $\mathbf{w}(\mathbf{L}) = \mathbf{A}L + \mathbf{a}$,

where $\mathbf{c}_k$ and $\mathbf{a}$ are M-element vectors, $\mathbf{C}_k$ is an $M \times M$ symmetric matrix, $\mathbf{A}$ is an $M \times M$ matrix, and γ_k is a scalar. The available amount of the production factors is usually limited; therefore we assume that for all k,

(iii) $\mathbf{l}_k \in S_k$,

where S_k is a convex, bounded, and closed subset of R_+^M.

The Cournot–Nash equilibrium of this N-person noncooperative game is a vector $(\mathbf{l}_1^*, \dots, \mathbf{l}_N^*)$ such that

(a) $\mathbf{l}_k^* \in S_k$ for all k,

and

(b) for all k and $\mathbf{l}_k \in S_k$,

$$\varphi_k(\mathbf{l}_1^*, \dots, \mathbf{l}_N^*) \geq \varphi_k(\mathbf{l}_1^*, \dots, \mathbf{l}_{k-1}^*, \mathbf{l}_k, \mathbf{l}_{k+1}^*, \dots, \mathbf{l}_N^*). \tag{2}$$

Theorem 1. *Assume that $\mathbf{C}_k$ is negative definite for all k, and $\mathbf{A} + \mathbf{A}^T$ is positive semidefinite. Then there is at least one Cournot–Nash equilibrium.*

Proof. Since $\mathbf{C}_k$ is negative definite, f_k is concave for all k. The assumption that $\mathbf{A} + \mathbf{A}^T$ is positive semidefinite implies that w is monotonic in $\mathbf{L}$ in the sense that for all feasible $\mathbf{L}$ and $\mathbf{L}'$,

$$(\mathbf{L} - \mathbf{L}')^T \big(w(\mathbf{L}) - w(\mathbf{L}')\big) \geq 0$$

(see for example, Ortega and Rheinboldt, 1970). Therefore Lemma 3.1.1 of Okuguchi and Szidarovszky (1999, pp. 23–24) implies that $\mathbf{l}_k^T \mathbf{w}(\mathbf{L})$ is convex in $\mathbf{l}_k$. Therefore φ_k is concave in $\mathbf{l}_k$. Since it is continuous, and for all k, S_k is a convex, bounded, and closed subset of a finite dimensional Euclidean space, the Nikaido–Isoda theorem (see Nikaido and Isoda, 1955) implies the existence of an equilibrium. □

If the production factor usages are at the equilibrium, then it is the interest of all firms to keep the equilibrium, since none of the firms can increase its profit by unilaterally changing its production factor use. Otherwise it is usually assumed that the firms change their usages proportionally to their marginal profits. Notice that the gradient of φ_k with respect to $\mathbf{l}_k$ is the following:

$$2\mathbf{C}_k p\mathbf{l}_k + p\mathbf{c}_k - (\mathbf{A} + \mathbf{A}^T)\mathbf{l}_k - \mathbf{A}\sum_{i\neq k}\mathbf{l}_i - \mathbf{a},$$

therefore the dynamic model can be formulated as

$$\dot{\mathbf{l}}_k = \mathbf{K}_k \left(2\mathbf{C}_k p\mathbf{l}_k + p\mathbf{c}_k - (\mathbf{A} + \mathbf{A}^T)\mathbf{l}_k - \mathbf{A}\sum_{i\neq k}\mathbf{l}_i - a \right), \tag{3}$$

where $\mathbf{K}_k$ is a diagonal matrix with positive diagonal elements for all k. Under the assumptions of Theorem 1, a vector $(\mathbf{l}_1^*, \dots, \mathbf{l}_N^*)$ is an equilibrium of dynamic system (3) if and only if it is an interior Nash–Cournot equilibrium of the static N-person game.

3. Stability Analysis

The main result of this section can be stated as follows.

Theorem 2. *Under the conditions of Theorem 1, the interior Nash–Cournot equilibrium is asymptotically stable.*

Proof. The coefficient matrix of system (3) has the special structure $\mathbf{K}H$ with

$$\mathbf{H} = \begin{pmatrix} 2\mathbf{C}_1 p - (\mathbf{A} + \mathbf{A}^T) & -\mathbf{A} & \cdots & -\mathbf{A} \\ -\mathbf{A} & 2\mathbf{C}_2 p - (\mathbf{A} + \mathbf{A}^T) & \cdots & -\mathbf{A} \\ \vdots & \vdots & \ddots & \\ -\mathbf{A} & -\mathbf{A} & \cdots & 2\mathbf{C}_N p - (\mathbf{A} + \mathbf{A}^T) \end{pmatrix}$$

and

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & & & \\ & \mathbf{K}_2 & & \\ & & \ddots & \\ & & & \mathbf{K}_N \end{pmatrix}.$$

Since $\mathbf{K}$ is positive definite, it is sufficient to show that $\mathbf{H} + \mathbf{H}^T$ is a negative definite matrix, and then Theorem 4.9 of Szidarovszky and Bahill (1998, pp. 213–214) implies that all eigenvalues of $\mathbf{K}H$ has negative real parts.

Notice that

$$\mathbf{H} + \mathbf{H}^T = \begin{pmatrix} 4\mathbf{C}_1 p & & & \\ & 4\mathbf{C}_2 p & & \\ & & \ddots & \\ & & & 4\mathbf{C}_N p \end{pmatrix} + \begin{pmatrix} -2(\mathbf{A} + \mathbf{A}^T) & -(\mathbf{A} + \mathbf{A}^T) & \cdots & -(\mathbf{A} + \mathbf{A}^T) \\ -(\mathbf{A} + \mathbf{A}^T) & -2(\mathbf{A} + \mathbf{A}^T) & \cdots & -(\mathbf{A} + \mathbf{A}^T) \\ & & \ddots & \\ -(\mathbf{A} + \mathbf{A}^T) & -(\mathbf{A} + \mathbf{A}^T) & \cdots & -2(\mathbf{A} + \mathbf{A}^T) \end{pmatrix}. \tag{4}$$

The first term is negative definite, since $\mathbf{C}_k$ is negative definite for all k, and $p > 0$. The second term is the Kronecker product of $\mathbf{A} + \mathbf{A}^T$ and $-(\mathbf{I} + \mathbf{E})$, where $\mathbf{I}$ is the $M \times M$ identity matrix and $\mathbf{E}$ is the $M \times M$ matrix, all elements of which are equal to unity. It is well known (see, for example Okuguchi and Szidarovszky, 1999, p. 69) that the eigenvalues of $-(\mathbf{I} + \mathbf{E})$ are -1 and $-M - 1$, both are negative. The eigenvalues of $\mathbf{A} + \mathbf{A}^T$ are nonnegative, since $\mathbf{A} + \mathbf{A}^T$ is positive semidefinite. It is also known (see for example, Ròzsa, 1974) that the eigenvalues of the Kronecker product of two matrices equal the products of the eigenvalues of the two factors. Since this matrix is symmetric, it is negative semidefinite. Hence $\mathbf{H} + \mathbf{H}^T$ is the sum of a negative definite matrix and a negative semidefinite matrix, which is negative definite with real and negative eigenvalues. $\square$

In the case of time invariant linear systems local and global asymptotical stability are equivalent to each other. Therefore the interior equilibrium of system (3) is globally asymptotically stable implying its uniqueness.

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