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A Note on Intersecting Graphs.

A. Bretto

Abstract

A set X of vertices in a graph $G = (V; E)$ is intersecting if every two vertices of X are adjacent or have a common neighbor in V. In this note we present the intersecting graphs, graphs generated by an intersecting set, and we study some of their properties.

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1 Introduction

In [4] we have introduced the notion of intersecting set to study the Helly property of the neighborhoods of graphs. These sets lead us to define intersecting graphs. These graphs seem interesting because they contain the classes of trivial perfect graphs split graphs (without pendant vertices) for example . Our goal in this article, is to give some basic properties of graphs generated by an intersecting set.

The general terminology concerning graphs and hypergraphs is similar to [2, 3]. All graphs in this paper are both finite and undirected.

Universit Jean Monnet, dpartement de Mathmatiques-Informatique
Equipe d'ingénierie de la vision (UMR 5516)
Groupe de Combinatoire et Traitement d'Image.
Site G.I.A.T Industries
3, rue Javelin Pagnon BP 505, 42007 Saint-Etienne cedex 1 FRANCE
E-mail: bretto@@vision.univ-st-etienne.fr

One will consider that these graphs are simple –graphs with no loops or multiple edges–and all graphs will be considered as connected with no isolated vertex. We denote graphs as $G = (V; E)$. Given a graph G, we denote the *neighborhood* of a vertex x by $\Gamma(x)$, i.e. the set formed by all the vertices adjacent to x is defined by:

$$\Gamma(x) = \{y \in V, \{x, y\} \in E\}.$$

The number of neighbors of x is the *degree* of x (we denoted by dx). If $dx = 1$, the vertex x is *pendant.*
A *chain* in a graph is a sequence of distinct edges one following one another, and the number of edges is the *length* of this chain. We denote by P_4 the chain induced by four distinct vertices (i.e. three distinct edges).
We will say that a graph G *does not contain an induced* P_4 in the two following cases:

- any chain has a length two at most.
- if a chain has four distinct vertices $x_1,\ x_2,\ x_3,\ x_4$ then $\{x_1, x_3\}$ or $\{x_2, x_4\}$ are edges.

If the extremities of P_4 are adjacent or have a common vertex we will say that P_4 is *closed.* A *cycle* is a chain such that the first vertex and the last vertex are the same. We denote a cycle with a length equal to n by C_n. Let C_n be a cycle, a *chord* of C_n is an edge linking two nonconsecutive vertices of C_n.
A graph G is *triangulated* if every cycle C_n with $n \geq 4$ contains a chord.
The square of $G = (V, E)$ is the graph $G^2 = (V, E^2)$ such that:
$\{x; y\} \in E^2$ if and only if either $\{x; y\} \in E$ or there exists $z \in V$ such that $\{x; z\} \in E$ and $\{z; y\} \in E$.
$G' = (V'; E')$ is a *subgraph* of G when it is a graph satisfying $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$ then G' is a *spanning subgraph.*
An *induced subgraph*(generated by A) $G(A) = (A; U)$, with $A \subseteq V$ and $U \subseteq E$ is a subgraph such that: for $x, y \in A$, when $\{x; y\} \in E$ implies $\{x; y\} \in U$.
A graph in which each pair of distinct vertices is adjacent is a *complete graph.*
By *clique* we mean a complete subgraph of G maximal for this property.
A graph $G = (V, E)$ is bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge joins a vertex of V_1 to a vertex V_2. We will denote a bipartite graph by $G(V_1, V_2)$.
A *vertex independant set* is a set of vertices S such that no two of which are adjacent.
A graph is *trivially perfect* [12] if and only if it contains no induced subgraph isomorphic to C_4 or P_4.
A *split graph* [11] is a graph whose one can partitioning the set of vertices into two sets A and B whose A generates a clique and B is an independent set.
The *eccentricity* of vertex x is the distance to a vertex farthest from x.
The *diameter* of G is the maximum of eccentricity of a vertex v, and the *radius* of G is the minimum eccentricity.

A *hypergraph* H on a finite set $\mathbf{S}$ is a family $(E_i)_{i\in I}$, $I=\{1,2,\ldots,n\} and \quad n \in \mathbb{N}$, of non-empty subsets of $\mathbf{S}$ called hyperedges with

$$\bigcup_{i\in I} E_i = \mathbf{S}$$

Let us denote : $H=(\mathbf{S};(E_i)_{i\in I})$
For $x\in\mathbf{S}$, a *star* of H (with the center x) is the set of hyperedges which contains x, and is called $H(x)$. The *degree of* x is the cardinality of the star $H(x)$. The *line-graph* $L(H)$ of a hypergraph H is a graph whose vertices are in one to one correspondence with the hyperedges of H. Two vertices of $L(H)$ are adjacent if and only if the corresponding hyperedges have a non empty intersection.
A hypergraph has the *Helly property* if each family of hyperedges intersecting two by two (intersecting family) has a non-empty intersection (belonging to a star).
A hypergraph is *arbored* if it has the Helly property and if its line-graph is triangulated.

Let $G=(S;E)$ be a graph; we can associate a hypergraph, called the *neighborhood hypergraph*:

$$H_G = (\mathbf{S},(E_x = \{x\}\cup\Gamma(x))).$$

2 Intersecting graphs

Let $G=(V,E)$ be a graph. A set $X\subseteq V$ is *intersecting* if every pair of vertices $\{x,y\}$ from X is adjacent or has at least one common neighbor in V. The set of neighbor vertices (not belonging to X) will be called the *intersection set*, it will be denoted by $\mathbb{I}(X)$. A graph $G=(V,E)$ is *radiant* if V is intersecting.
A graph $G=(V,E)$ is *intersecting* if there exists an intersecting $X\subseteq V$ such that $G=G(X\cup\mathbb{I}(X))$.

In [] we have proved the following result:

Some properties of the graphs generated by the intersecting sets will be presented .

Proposition 1. *Let $X\subseteq V$ be an intersecting set from $G=(V;E)$ and $G'=G(X\cup\mathbb{I}(X))$. The three following assertions are equivalent.*

i G' is a tree

ii G' has a pendant vertex, and does not contain C_3 and C_4.

iii G' does not contain C_3 and an induced P_4.

Proof. $(i) implies (ii)$. It is obvious.
$(ii) implies (i)$. Let x be a pendant vertex and u be the vertex adjacent to x. Then $x\in X$, because every $x\in\mathbb{I}(X)$ has $dx\geq 2$. If u is pendant, we are done. If u is not pendant, let y be a vertex adjacent to u.Then $y\in X$ since otherwise

there would exist a vertex $v \in X$ ($v \neq u$) adjacent to y so u, v, y would be a C_3; contradiction. If y is pendant we are done. Otherwise there would exist a vertex z adjacent to y:
if $z \in \mathbb{I}(X)$, there exists $t \in X$ adjacent to z, and u, y, z, t is a C_4 (resp is a C_3 if $t = u$).
If $z \in X$, then z, u, y is a C_3; contradiction. Consequently y is pendant and $G' = G(X \cup \mathbb{I}(X))$ is a tree.
$(i)implies(iii)$. G' is a tree, thus it does not contain C_3. Let x be a pendant vertex of G' and u the vertex adjacent to it. Any vertex of G' (except u) is adjacent to u; consequently G' does not contain an induced P_4.
$(iii)implies(ii)$. We will suppose that G' has at least four vertices, otherwise it is obvious. Let a, b, c, d be a C_4. These vertices give rise to a P_4, by hypothesis $\{a, c\}$ or $\{b, d\}$ are edges, consequently G' contains a C_3.
Let x be a vertex of G' such that $dx \geq 2$. x has at least two neighbors u, v. u, v are not adjacent (because this will yield a C_3). If u is pendant it is over, otherwise there would exist a vertex a adjacent to u. Consequently a, u, x, v is a P_4 and $\{a, x\}$ or $\{u, v\}$ are edges. So this yields a C_3: contradiction so u is pendant. □

Corollary 1. *Let G be an intersecting graph. If G contains a pendant vertex then there exists an intersecting set X such that $G(X)$ is radiant.*

Proof. The prove of this result can be deduce from the prouf above. □

Theorem 1. *Let $G = (V; E)$ be a graph and $X \subseteq V$ an intersecting set. $G' = G(X \cup \mathbb{I}(X))$ is radiant if and only if any P_4 from G' is closed.*

Proof. If $x, y \in X$ or x, y are adjacent it is over, otherwise two cases will arise:

- firstly $x \in X$, $y \in \mathbb{I}(X)$. Then y has at least two neighbors $u, v \in X$. If u (resp v) is adjacent to x then x, y have a common vertex, it is over. If x is not adjacent to u, there exists v adjacent to x and u. So x, v, u, y is a P_4. By hypothesis there exists t adjacent to x and y, and these vertices are a common neighbor.

- secondly $x, y \in \mathbb{I}(X)$.There exists two vertices u, v belonging to X adjacent respectively to x and y. If u, v are adjacent one has P_4 and we will proceed as above. If u, v are not adjacent there exists z adjacent to u, v. So y, v, z, u is a P_4. By hypothesis P_4 is closed. Suppose y, u are adjacent, then x, y have u as common neighbor. Suppose that y, u have c as common neighbor, y, c, u, x is a P_4. Consequently y is adjacent to x or has a common neighbor. So $G' = G(X \cup \mathbb{I}(X))$ is radiant.

Suppose $G' = G(X \cup \mathbb{I}(X))$ is radiant. Let P_4 be a chain with four distinct vertices, necessarily this one is closed. □

Corollary 2. *Let G be a connected graph such that any P_4 is closed. H_G (its associated neighborhood hypergraph) has the Helly property if and only if H_G is arbored.*

Proof. If H_G is arbored then H_G has the Helly property. Suppose that H_G has the Helly property. By hypothesis P_4 is closed and from theorem 2, G is radiant, consequently $L(G) = G^2$ is a clique, thus it is triangulated. So H_G is arbored. □

Lemma 1. *Let $G = (V, E)$ be a graph. If G is intersecting then the diameter of G is four at most and the radius is three at most.*

Proof. G being intersecting, there exists an intersecting set $X \subseteq V$ such that $G = G(X \cup \mathbb{I}(X))$.

(1) If $x, y \in X$, x, y are adjacent or have a common neighbor at least.

(2) If $x \in X$ and $y \in \mathbb{I}(X)$, y has two neighbors in X at least. Let u, v be two of these neighbors, $\{x, u\}$ and (resp $\{x, v\}$) are adjacent or have a common neighbor, consequently the distance between x, y is four at most.

(3) If $x, y \in \mathbb{I}(X)$, these two vertices have respectively two neighbors in X at most. By hypothesis these vertices are adjacent or have a common neighbor. So the distance between x, y is four at most.

One can conclude that the diameter is four at most.
From (2) we can conclude that the radius of G is three at most. □

Theorem 2. *Let $G = G(X \cup \mathbb{I}(X))$ be an intersecting graph without C_4 (we will suppose that the cardinality of X is three at least). G does not contain C_3 and C_5 if and only if G is bipartite ($G = G(X, \mathbb{I}(X))$).*

Proof. Let $x, y \in X$. Suppose that x, y are adjacent, the cardinality of X being three at least, there exists $z \in X$ with $z \neq x$ and $z \neq y$. X being intersecting, we have three cases:

- z is adjacent to both x and y. One have a cycle C_3, which is a contradiction.
- z is adjacent to x and has a common neighbor with y, that leads to a cycle C_4.
- z has a common neighbor with both x and y, so G contains either C_3 or C_5.

Consequently x cannot be adjacent to y. So X is an independant set.

Let $x, y \in \mathbb{I}(X)$, a, b and u, v denote two neighbors of x and y respectively. Let z be another vertex of X, and suppose that x, y are adjacent, then two cases arise:

- x and y have a common neighbor, without losing generality we can suppose that $a = u$, it give rise to a cycle C_3.
- x and y have not got a common neighbor, from lemma 1 the diameter of G is four at most, so there exists a cycle C_5. Contradiction.

x and y are not adjacent and $\mathbb{I}(X)$ is an independant set.

Suppose that $G = G(X, \mathbb{I}(X))$, it is obvious that G does not contain C_3 and C_5. $\square$

To illustrate these graphs, it is easy to prove that the trivially perfect graphs and the split graphs without pendant vertex are intersecting graphs.

Problem 1. *Giving other classes of well known graphs which are intersecting.*

Trivially perfect graphs and split graphs are perfect graphs, which leads us to:

Problem 2. *Charaterizing the smallest class of perfect intersecting graphs which contains the classes given above.*

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